J. Plasma Physics (2005), *vol.* ?, *part* ?, *pp.* ??.
DOI: 10.1017/S0000000000000000 Printed in the United Kingdom

Stepped pressure profile equilibria in cylindrical plasmas via partial Taylor relaxation

M. J. Hole[1], S. R. Hudson[2] and R. L. Dewar[1]
[1] Res. School of Phys. Sci. and Eng., Australian National University, ACT 0200, Australia.
2 Princeton Plas. Phys. Lab., P.O. Box 451, Princeton, New Jersey 08543, U.S.A.

(Received 17 July 2005 and in revised form ??)

Abstract. One of the most important advances in magnetically confined fusion plasma physics has been the discovery of high confinement regimes, where at sufficiently high heating power, the plasma self-organizes to produce internal transport barriers (ITB's). A possible explanation for the existence of such barriers is that they represent constrained minimum energy states, where the plasma within the barrier satisfies ideal MHD, and the plasma between barriers is in a Taylor relaxed state. In this work we develop a multiple interface variational formulation, comprising multiple Taylor-relaxed plasma regions, each of which are separated by an ideal MHD barrier. Application to a plasma cylinder is a generalization of the analysis of the treatment of Kaiser and Uecker, Q. Jl. Mech. Appl. Math.,57(1), 2004, who treated the single interface in cylindrical geometry. Expressions for the equilibrium field are generated, and equilibrium states computed. Unlike other Taylor relaxed equilibria, for the equilibria investigated here, only the plasma core necessarily has reverse magnetic shear. We show the existence of tokamak like equilibria, with increasing safety factor and stepped-pressure profiles.

1. Introduction

Taylor relaxation describes plasma with small but finite resistivity and viscosity, in which the magnetic field has evolved to a minimum energy state, subject to the conservation of magnetic helicity and toroidal flux, and the presence of a perfectly conducting wall. In such plasma states the pressure gradient is zero, and the magnetic field **B** satisfies the Beltrami equation [1]

$$\nabla \times \mathbf{B} = \mu \mathbf{B} \tag{1.1}$$

with the Lagrange multiplier μ below some critical value μ_T, which depends only on the vessel. By introducing ideal MHD barriers between different Taylor relaxed states, equilibria with stepped pressure profiles can be constructed. Subject to the imposed constraints, these new equilibria are in a relaxed or minimum energy state, and so may explain, *ab initio* the existence of transport barriers in toroidal magnetic confinement experiments. Other theories exist to describe the subsequent formation of transport barriers (*e.g.* shear flow suppression of turbulence [2] or chaotic magnetic field line dynamics [3]). The stepped pressure profile model developed here also offers a possible solution to the long-standing 3D equilibrium existence problem [4]. Our working builds principally upon a variational model developed by Spies *et al*[5], which comprised a plasma/vacuum/conducting wall system. In Spies [5] the theory is applied to a plasma slab equilibrium, with boundary conditions designed to simulate a torus. Later analysis by Spies [6] extended the plasma model to include finite pressure. More recently, Kaiser and Uecker [7] analyzed the finite pressure model in cylindrical geometry.

2. Multiple Interface Plasma Vacuum Model

In this work, we generalize the analysis of Kaiser and Uecker [7] to an arbitrary number N of Taylor relaxed states, each separated by an ideal MHD barrier. The system is enclosed by a vacuum, and encased in a perfectly conducting wall. For such a system, the energy functional can be written

$$W = U - \sum_{i=1}^{N} \mu_i H_i/2 - \sum_{i=1}^{N} \nu_i M_i \tag{2.1}$$

Setting the first variation to zero yields the following set of equations :

$$\mathscr{P}_i; \nabla \times \mathbf{B} = \mu_i \mathbf{B},\ p_i = \text{const.}, \tag{2.2}$$
$$\mathscr{I}_i; \quad \mathbf{n}\cdot\mathbf{B} = 0, \quad < p_i + 1/2B^2 >= 0, \tag{2.3}$$
$$\mathscr{V}; \quad \nabla \times \mathbf{B} = 0, \quad \nabla \cdot \mathbf{B} = 0 \tag{2.4}$$
$$\mathscr{W}; \quad \mathbf{n}\cdot\mathbf{B} = 0 \tag{2.5}$$

where $\mathscr{P}_i, \mathscr{I}_i$ are the i'th plasma region and interface (or ideal MHD barrier), and $\mathscr{V}, \mathscr{W}$ are the vacuum region and wall, respectively. Also, μ_i is the Lagrange multiplier in each region, p_i the pressure in each region, $\mathbf{n}$ a unit vector normal to the plasma interface, and $< x >= x_{i+1} - x_i$ denotes the change in quantity x across the interface $\mathscr{I}_i$. The boundary conditions on $\mathbf{n}\cdot\mathbf{B}$ arise because each interface and the conducting wall is assumed to have infinite conductivity. In turn, these imply the following flux constraints $\psi^t_{\mathscr{R}} = \text{const.}$ and $\Psi^p_V = \text{const}$ during Taylor relaxation : where the subscripts $\mathscr{R}$ are labels for each region, V denotes the vacuum only, and the superscripts p, t label the fluxes as poloidal and toroidal, respectively. Given the vessel with boundary $\mathscr{W}$, the interfaces $\mathscr{I}_i$, and the magnetic field $\mathbf{B}$, Eqs. (2.2)-(2.5) constitute a free boundary problem for p_i.

3. Cylindrical Equilibria

In this section cylindrically symmetric equilibrium solutions are generated. A cylindrical co-ordinate system is used (r, θ, z), with equilibrium variations permitted only in the radial direction. Following Kaiser we use the normalization of plasma-vacuum boundary $r = 1$, and assume that the cylinder is periodic in the z direction, with periodicity L. In this system, solutions to Eqs. (2.2) - (2.5) can be written in vector notation $\mathbf{B} = \{B_r(r), B_\theta(r), B_z(r)\}$ as

$$\mathscr{P}_1 : \mathbf{B} = \{0, \quad k_1 J_1(\mu_1 r), \quad k_1 J_0(\mu_1 r)\}, \tag{3.1}$$
$$\mathscr{P}_i : \mathbf{B} = \{0, k_i J_1(\mu_i r) + d_i Y_1(\mu_i r), k_i J_1(\mu_i r) + d_i Y_1(\mu_i r)\}, \tag{3.2}$$
$$\mathscr{V} : \mathbf{B} = \{0, \quad B^V_\theta/r, \quad B^V_z\}, \tag{3.3}$$

where $k_i, d_i \in \Re$, and J_0, J_1 and Y_0, Y_1 are Bessel functions of the first kind of order 0, 1, and second kind of order 0, 1, respectively. The terms B^V_θ and B^V_z are constants. The constant d_1 is zero in the plasma core $\mathscr{P}_1$, because the Bessel functions $Y_0(\mu_1 r)$ and $Y_1(\mu_1 r)$ have a simple pole at $r = 0$ [8].

The equilibrium problem can be prescribed by the $4N+2$ parameters describing the magnetic field profile and the radial position of the barriers. That is,

$$\{k_1, ..., k_N, d_2, ..., d_N, \mu_1, ..., \mu_N, r_1, ...r_{N-1}, r_w, B^V_\theta, B^V_z\} \tag{3.4}$$

where r_i are the radial positions of the N ideal MHD barriers, and r_w is the radial position of the conducting wall. Equivalently, the equilibrium can be constrained by the safety factors and magnetic fluxes. That is, the $4N+2$ quantities

$$\{\Psi^t_1,...,\Psi^t_N,\Psi^p_1,...,\Psi^p_N,\Psi^t_V,\Psi^p_V,q^i_1,...,q^i_N,q^o_1,...,q^o_N\} \tag{3.5}$$

where q^i_i and q^o_i are the safety factor on the inside and outside of each interface. In cylindrical geometry the safety factor expands as

$$q^i_i = \frac{2\pi r_i}{L}\frac{B_{z,i}(r_i)}{B_{\theta,i}(r_i)}, \qquad q^o_i = \frac{2\pi r_i}{L}\frac{B_{z,i+1}(r_i)}{B_{\theta,i+1}(r_i)}, \tag{3.6}$$

In the core, we note that the function $rJ_0(\mu_1 r)/J_1(\mu_r)$ has positive radial derivative regardless of the value of μ_1, and so the plasma core will necessarily exhibit reverse magnetic shear. The toroidal and poloidal fluxes compute as follows:

$$\Psi^t_i = \int_{r_{i-1}}^{r_i} B_z(r) r d\theta dr = \frac{2\pi}{\mu_i}\left[k_i r J_1(r\mu_i) + d_i r Y_1(r\mu_i)\right]^{r_i}_{r_{i-1}}, \tag{3.7}$$

$$\Psi^p_i = \int_{r_{i-1}}^{r_i} B_\theta(r) L dr = \frac{2\pi}{\mu_i}\left[k_i J_0(r\mu_i) + d_i Y_0(r\mu_i)\right]^{r_i}_{r_{i-1}}. \tag{3.8}$$

Finally, in the vacuum region, the fluxes compute as

$$\Psi^t_V = B^V_\theta L ln(r_w), \qquad \Psi^p_V = B^V_z \pi(r_w^2 - 1) \tag{3.9}$$

In both formulations, the plasma pressure can be expressed in terms of the field strength B at the barriers.

Equations (3.6)-(3.9) form a mapping from the magnetic field profile factors and interface positions, constraints (3.4), to the safety factors and magnetic fluxes, Eq. (3.5). An analytic form for the inverse mapping is not simply available. To compute the **b** field and interface positions, given the safety factor and magnetic flux, we have used the method of least squares. Starting in the core, Ψ^p_1, Ψ^t_1 and q^i_1 can be solved for k_1, μ_1, r_1. In the plasma body, Ψ^p_i, Ψ^t_i and q^i_i, q^o_i can be solved for k_i, d_i, μ_i, r_i. Finally, the position of the conducting wall r_w is found by solving the equation

$$r_w^2 - 1 - \left(\frac{\Psi^t_V}{\Psi^p_V}\frac{B^V_\theta}{B^V_z}\right) L \ln r_W = 0. \tag{3.10}$$

In a full 3D equilibrium problem, the barriers must be placed on flux surfaces, introducing an additional irrationality constraints on q at each interface. In this cylindrically symmetric problem however, all points in the domain lie on flux surfaces, and so q is everywhere rational. Figure 1 shows an example with 5 ideal barriers (shown as the solid contours in panel (a)). The plasma-vacuum interface is the lighter solid line at $r_w = 1.5$. This particular example has been chosen with no change in the q across the interfaces, and hence no surface currents. The equilibrium is described by the constraints: $k_i = \{0.22, 0.25, 0.29, 0.31, 0.35\}, d_i = \{0.0, -0.010, -0.019, -0.050, -0.060\}$, $r_i = \{0.2, 0.4, 0.6, 0.8, 1.0\}, r_w = 1.5, B_{V,\theta} = 0.24$ T, $B_{V,z} = 0.40$ T. Figure 1 demonstrates existence of multi-interface, tokamak-like solutions, which do not require the existence of surface currents. By increasing the number of interfaces, the pressure can be approximated arbitrarily close to an experimental profile.

4. Conclusions

We have formulated a model for equilibria that comprise multiple Taylor-relaxed plasma regions, each of which is separated by an ideal MHD barrier of zero width. The system is enclosed by a vacuum region, and encased by a perfectly conducting wall. For these equilibria, the safety factor in the core necessarily decreases monotonically. For regions outside of the innermost ideal barrier, solutions can be constructed with increasing safety factors, and decreasing pressures. A tokamak like example of a multiple-interface equilibria is provided. These equilibria exhibit many of the same qualities observed in high performance H-mode discharges. In a following publication, the stability of the multiple interface equilibria presented here will be studied.

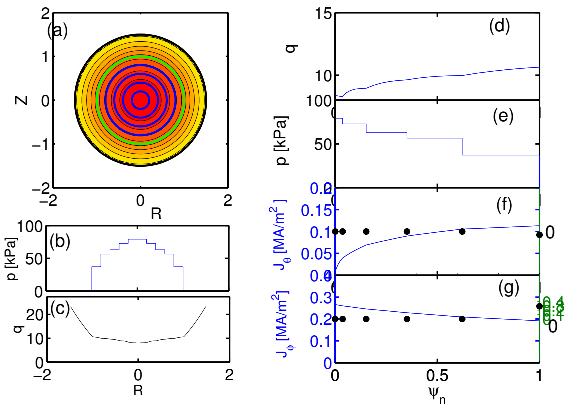

Figure 1. Example of a stepped-pressure plasma profile, with five ideal MHD barriers, showing : (a) a contour plot of the poloidal flux ψ_p, (b) the pressure profile p, (c) the safety factor q versus position. Panels (d)-(f) show p, q, and toroidal, and poloidal current densities J_θ, J_ϕ as a function of normalized poloidal flux, where ψ_n is zero at the core, and unity at the plasma-vacuum interface. In panels (e) and (f) the solid points are the surface currents in each interface, and take the right axis.

Acknowledgments The authors would like to acknowledge the support of the Australian Research Council, through grant DP0452728.

References

[1] J. B. Talyor. Relaxation and magnetic reconnection in plasmas. *Rev. Mod. Phys.*, 58(3):741–763, 1986.

[2] K. H. Burrel. Effects of $\mathbf{E} \times \mathbf{B}$ velocity shear and magnetic shear on turbulence and transport in magnetic confinement devices. *Phys. Plas.*, 4(5):1499–1518, 1997.

[3] J. H. Misguich. Dynamics of chaotic magnetic lines: Intermittency and noble internal transport barriers in the tokamap. *Phys. Plas.*, 8(5):2132–2138, 2001.

[4] H. Grad. Toroidal Containment of a Plasma. *Phys. Fluids*, 10(1):137–154, 1967.

[5] G. O. Spies, D. Lortz, and R. Kaiser. Relaxed plasma-vacuum systems. *Phys. Plas.*, 8(8):3652–3663, 2001.

[6] G. O. Spies. Relaxed plasma-vacuum systems with pressure. *Phys. Plas.*, 10(7):3030–3031, 2003.

[7] R. Kaiser and H. Uecker. Relaxed plasma-vacuum states in cylinders. *Q. Jl. Mech. Appl. Math.*, 57(1):1–17, 2004.

[8] M. Abromowitz and I. A. Stegun. *Handbook of Mathematical Functions*. Dover, 1972.