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STICKELBERGER IDEALS AND DIVISOR CLASS NUMBERS*

Abstract. Let K/k be a finite abelian extension of function fields with Galois group G. Using the Stickelberger elements associated to K/k studied by J. Tate, P. Deligne and D. Hayes, we construct an ideal I in the integral group ring $\mathbb{Z}[G]$ relative to the extension K/k, whose elements annihilate the group of divisor classes of degree zero of K and whose rank is equal to the degree of the extension. When K/k is a (wide or narrow) ray class extension, we compute the index of I in $\mathbb{Z}[G]$, which is equal to the divisor class number of K up to a trivial factor.

1. Introduction.

Let k be a global field. Let K/k be a finite abelian extension with Galois group $G = G_K$. The Stickelberger elements associated to the extension K/k have been studied extensively. These elements enter into the formulation of the well-known Brumer-Stark conjecture. When K/k is an extension of function fields, J. Tate [T] first proved that these Stickelberger elements annihilate the divisor classes of degree zero of K (i.e. the Brumer part of the conjecture) by using the action of G_K on the Jacobian of K. P. Deligne and D. Hayes independently proved the whole conjecture by different method. In fact, their results are more precise than the function field analogue of the Brumer-Stark conjecture, see [Chap.V, T] or [Th.1.1, H2]. In this paper, using these Stickelberger elements, we define an ideal in the integral group ring $\mathbb{Z}[G]$ relative to the extension K/k, and call it the Stickelberger ideal of K. When K/k is a function field extension, the elements in the ideal have the remarkable property that they annihilate the divisor class group of degree zero of K, and the rank of the ideal is equal to the degree of the extension K/k. Furthermore when K/k is a (wide or narrow) ray class extension of function fields, we compute the index of the Stickelberger ideal in the group ring $\mathbb{Z}[G]$, which is equal to the divisor class number of K up to a simple constant factor. Now we state the results precisely.

Let k be a global function field with constant field $\mathbb{F}_q$ of q elements, and let ∞ be a fixed place of k with degree 1. Let k_∞ be the completion of k at ∞. Let $\mathbb{A}$ be the Dedekind ring of functions in k which are holomorphic away from ∞. We fix a sign function sgn: $k_\infty \longrightarrow \mathbb{F}_q$ with $\mathrm{sgn}(0) = 0$ (cf. [Def.4.1, H2]). Let $\mathfrak{m}$ be a non-zero integral $\mathbb{A}$-ideal. Let $K = K_\mathfrak{m}$ be the cyclotomic extension of the triple $(k, \infty$, sgn) of conductor $\mathfrak{m}$, which is the narrow ray class field of the triple modulo $\mathfrak{m}$. If in particular $\mathfrak{m} = \mathfrak{e}$, the unit ideal, then $K_\mathfrak{e}$ is the narrow Hilbert class field of the triple. The cyclotomic extension K is generated over $K_\mathfrak{e}$ by the $\mathfrak{m}$-torsion points of Hayes' sgn-normalized rank one Drinfeld modules. Let $G = G_\mathfrak{m} = \mathrm{Gal}(K/k)$. Let $G_\mathfrak{e} = \mathrm{Pic}(\mathbb{A})$ be the Picard group of $\mathbb{A}$ and let $N = N_\mathfrak{m}$ be the subgroup of $G_\mathfrak{e}$ generated by the Artin symbols $\tau_\mathfrak{p} = (\mathfrak{p}, K_\mathfrak{e}/k)$ with $\mathfrak{p} \mid \mathfrak{m}$. Let $t = [G_\mathfrak{e} : N]$ be the index. Let s be the number of distinct prime divisors of $\mathfrak{m}$. Let $R = \mathbb{Z}[G]$ and let I be the Stickelberger ideal of K. For a function field F, let $h(F)$ be the divisor class number of F. We have

Theorem 1. $\quad [R : I] = (q-1)^{t(2^{s-1}-1)} h(K).$

Let $K^+ = K^+_\mathfrak{m}$ be the maximal subfield of K in which the place ∞ splits completely. Then K^+ is the ray class field of (k, ∞) modulo $\mathfrak{m}$. Since $\deg \infty = 1$, we have $K^+_\mathfrak{e} = K_\mathfrak{e}$. Let $G^+ = G^+_\mathfrak{m} = \mathrm{Gal}(K^+/k)$ and let $R^+ = \mathbb{Z}[G^+]$. Let I^+ be the Stickelberger ideal of K^+.

Theorem 2. *We have*

$$[R^+ : I^+] = (q-1)^b h(K^+)$$

where $b = 0$ *if* $s = 0, 1$ *and* $b = t(2^{s-2} - 1)$ *if* $s \geq 2$. *Here* $s = 0$ *means* $\mathfrak{m} = \mathfrak{e}$.

The ideal I^+ is a subideal of I under the corestriction map. We call it the real part of I. This part is closely related to the extended cyclotomic units of K we studied in [Y1]. Our main method of

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computing the indices is the same as that we used in [Y1], that is, we factor the indices as the product of some lattice indices introduced by Sinnott in [Si], which we compute individually. A key step is to use Anderson's remarkable results [Th.5.2.3, An] on the Galois module structure of the sign-cohomology of the universal ordinary distribution associated to a global function field. The results in this paper and in [Y1, Y3] affirm Anderson's claim in [Sect.1, An] that "more number theoretic applications of complete cohomology can be expected".

In the rest of the introduction, we mention some history of Stickelberger ideals and compare our results with the classical ones. The notion of Stickelberger ideal was first introduced by Iwasawa [Iw] in 1962. Using the well-known Stickelberger elements, Iwasawa defines an ideal in the integral Galois group ring of a cyclotomic number field of prime power conductor, whose elements annihilate the ideal class group of the cyclotomic field (Stickelberger's Theorem), but whose rank is only half that of the group ring. In fact, it is an ideal of the minus part of the group ring, with the same rank. Iwasawa shows that in this prime power conductor case the index of the Stickelberger ideal in the minus part of the group ring is equal to the relative class number of the cyclotomic field, see [Iw] or [Sect. 6.4, Wa]. This result complements Kummer's famous formula expressing the class number of the maximal real subfield of a cyclotomic field of prime power conductor as the index of cyclotomic units, see [Th.8.2, Wa]. In 1978, Sinnott [Si] extends the results of both Iwasawa and Kummer to a general cyclotomic field by introducing the powerful device of cohomology to the computation of the indices.

In the function field case, Hayes (cf. [H1]) develops the theory of sign-normalized rank one elliptic modules and sets up explicit class field theory for a function field (also called cyclotomic extensions). In this theory, the present author works out Kummer-Sinnott's unit index formula in [Y1] and Iwasawa-Sinnott's index formula for the Stickelberger ideal in [Y3]. We point out that in the two cases of the rational number field and of a function field above, the cyclotomic units and Stickelberger ideals both arise from ray class partial zeta functions. By Weil's theorem, partial zeta functions of a function field are rational functions of q^{-s}. This makes it possible to construct a bigger Stickelberger ideal in the function field case compared to the number field case. In the cyclotomic case of function fields, roughly speaking, the Stickelberger ideal defined in this paper can be regarded as a composition of the real part I^+ and the relative part I^-, where I^+ is the image of the extended cyclotomic units [Y1] under the logarithm map and I^- is defined in [Y3]. So it has the same rank as the Galois group ring and its index is essentially equal to the full (divisor) class number. It is the relative part I^- that is the analogue of Iwasawa-Sinnott's definition. We introduce I^- in [Y3] only for the purpose to give a result analogous with Iwasawa-Sinnott's. In the classical cyclotomic case, there is no real part I^+ since the wide ray class zeta functions of the rational number field take irrational values at $s = 0$. Thus the Stickelberger ideal in the classical case has smaller rank than that of the group ring, and its index is only equal to the relative class number. We think that the definition of the Stickelberger ideal in this paper is more natural than that in [Y3]. In fact, the present definition applies to any finite abelian extension of global fields. In the function field case we do not need to fix a place and a sign function first. Arguably, the result here is more beautiful than those in [Si] and in [Y3]. As Iwasawa points out in [Iw], such an index-class number formula suggests the existence of deeper group theoretical relations between the (divisor) class group and the factor group of the Galois group ring modulo the Stickelberger ideal. In this direction, Euler systems form a powerful device. We refer the reader to Rubin's work [R1] and [R2].

2. Stickelberger ideal and its rank.

In this section, we recall the definition of Stickelberger elements and their basic properties. Using these elements we define the Stickelberger ideal. We also compute its rank in the function field case.

Let k be an algebraic number field or a global function field. Let K/k be a finite abelian extension with Galois group G. We denote M_k the set of all places of k, $T_\infty = T_\infty(k)$ the set of archimedean places of k, W_k the number of unity roots in k. Let T be a finite non-empty subset of M_k which contains T_∞ as well as all the places which ramified in K. For a place $v \in M_k \setminus T$, let σ_v be the Frobenius automorphism associated to v and let Nv be the norm of v. We define a function for $\mathrm{Re}(s) > 1$

$$\theta_{K/k,T}(s) = \prod_v (1 - \sigma_v^{-1} Nv^{-s})^{-1} = \sum_{\sigma \in G} Z(s, \sigma)\sigma^{-1} \qquad (v \in M_k \setminus T) \tag{2.1}$$

with values in $\mathbb{C}[G]$, where $Z(s,\sigma) = \sum_v Nv^{-s}$, the summation is over all $v \in M_k \setminus T$ such that $\sigma_v = \sigma$. It is well-know that this function can be extended to the whole complex plane, and is well defined except at $s=1$. Let $\theta_{K,T}=\theta_{K/k,T}(0)$. We know that $\theta_{K,T}\in\mathbb{Q}[G]$ and $W_K\theta_{K,T}\in\mathbb{Z}[G]$. For detail, we refer to [S] and [DR] in number field case and to [H2] or [W] in function field case. The element $\omega_{K,T}=W_K\theta_{K,T}$ is called the Stickelberger element of K/k relative to T. In function field case it annihilates the group of divisor classes of degree zero of K. In fact, Deligne proved a more precise result than this (cf. [Th.1.1, H2]).

Let χ be a character of G with complex values. We extend the definition of χ to $\mathbb{C}[G]$ linearly. Let $L_T(0,\chi)$ be the Artin L-function associated to χ with Euler factors for the places in T removed. Then we have $\chi(\theta_{K,T})=L_T(0,\bar{\chi})$, where $\bar{\chi}$ is the inverse of χ. The element $\theta_{K,T}$ is determined uniquely by this equality for all characters of G. By considering the order of zero of $L_T(s,\chi)$ at $s=0$, we see that in number field case these Stickelberger elements are 0 unless k is totally real and K is a CM field.

Suppose that E/k is a finite abelian overextension of K/k with Galois group G_E. The restriction of automorphisms from E to K induces a ring homomorphism $\mathrm{Res}_{E/K}:\mathbb{Q}[G_E]\longrightarrow\mathbb{Q}[G_K]$. The corestriction map is defined as follows

$$\mathrm{Cor}_{E/K}:\mathbb{Q}[G_K]\longrightarrow\mathbb{Q}[G_E],\qquad \sigma\mapsto\sum_{\tau\mapsto\sigma}\tau.$$

For a subset M of G_E, we set $s(M)=\sum_{\sigma\in M}\sigma$. The corestriction map induces an isomorphism $\mathbb{Q}[G_K]\simeq s(\mathrm{Gal}(E/K))\mathbb{Q}[G_E]$. For $\alpha\in\mathbb{Q}[G_E]$ and $\beta\in\mathbb{Q}[G_K]$, we have

$$\mathrm{Res}_{E/K}\mathrm{Cor}_{E/K}(\beta)=[E:K]\beta\quad\text{and}\quad\mathrm{Cor}_{E/K}\mathrm{Res}_{E/K}(\alpha)=s(\mathrm{Gal}(E/K))\alpha. \tag{2.2}$$

Now we assume that K/k is ramified at least at one non-archimedean place if it is an extension of function fields. Let T_K denote the set of non-archimedean places of k which are ramified in K. Let $T_0=T_\infty\cup T_K$. It is non-empty. Let $\theta_K=\theta_{K,T_0}$. This element is uniquely determined by the set T_K. We have

Lemma 2.1. *If E and K have the same set of ramified non-archimedean places, then* $\mathrm{Res}_{E/K}(\theta_E)=\theta_K$.

The definition of Stickelberger ideal is divided into two parts, the ramified part and the unramified part. We now define them respectively. Let K^0/k be the maximal subextension of K/k in which all non-archimedean places are unramified.

Ramified part: Assume $K\neq K^0$ in function field case. For every non-empty set T with $T_\infty\subseteq T\subseteq T_K\cup T_\infty$, let K_T/K^0 be the maximal subextension of K/K^0 in which exactly the non-archimedean places in T are ramified. Then $\theta_T=\mathrm{Cor}_{K/K_T}(\theta_{K_T})$ is an element in $\mathbb{Q}[G]$. Let S^{ra} be the G-submodule of $\mathbb{Q}[G]$ generated by θ_T with all T satisfying the condition above. We mention that if we remove the "maximal" condition in the definition of K_T/K^0, then the element θ_T defined in the same way is in S^{ra} by Eq. (2.2). Let $I^{\mathrm{ra}}=S^{\mathrm{ra}}\cap\mathbb{Z}[G]$. We call it the ramified part of the Stickelberger ideal of K/k. When $K=K^0$ in function field case, there is no such T. Thus we set $S^{\mathrm{ra}}=\{0\}$ in this case.

Unramified part: For a non-archimedean place v, let K_v^0/K^0 be a finite extension of K^0 in which only v is ramified and that K_v^0/k is abelian. Let $\theta_v=\mathrm{Cor}_{K/K^0}\mathrm{Res}_{K_v^0/K^0}(\theta_{K_v^0})$. It is an element in $\mathbb{Q}[G]$. Notice that this element is only dependent on v, not on K_v^0, by Lemma 2.1. If for some v there does not exist such extension K_v^0/K^0, we set $\theta_v=0$. Let S^{un} be the G-submodule of $\mathbb{Q}[G]$ generated by θ_v with all non-archimedean places v and by $\theta=(1/W_K)s(G)$. Let $I^{\mathrm{un}}=S^{\mathrm{un}}\cap\mathbb{Z}[G]$. We call it the unramified part of the Stickelberger ideal of K/k. The idea of the definition of the unramified part comes from Hayes' structure of unramified elliptic units in [H4].

Definition 2.2. Let $S=S^{\mathrm{un}}+S^{\mathrm{ra}}$ and let $I=S\cap\mathbb{Z}[G]$. We call I the Stickelberger ideal of the extension K/k.

Notice that we may not have $I=I^{\mathrm{un}}+I^{\mathrm{ra}}$. We remark that the definition of Stickelberger ideal in number field case is not meaningful unless k is a totally real field and K is a CM field. By Deligne's result, the elements in I annihilate the group of divisor classes of degree zero of K in function field case.

We now begin to calculate the rank of the Stickelberger ideal. The notations are as above. Let K/k be a finite abelian extension of function fields with Galois group $G=G_K$. Let $I=I_K$ be the Stickelberger ideal of K. Let $\hat{G}$ denote the character group of G with complex values. For $1\neq\chi\in\hat{G}$, let $L_k(s,\chi)$ be the Artin L-function associated to χ. We have the following well-known analytic class number formula

$$h(K)=h(k)\prod_{1\neq\chi\in\hat{G}}L_k(0,\chi).$$

Thus $L_k(0,\chi)\neq0$ if $\chi\neq1$.

Proposition 2.3. *Suppose that k is a function field. Then* $\operatorname{rank} I = [K : k]$.

Proof. Since $W_K S \subseteq I \subset \mathbb{Z}[G]$ and $\operatorname{rank} S = \dim_{\mathbb{C}} \mathbb{C} \otimes S$, we have

$$\operatorname{rank} I = \operatorname{rank} S = \#\{\chi \in \hat{G} \mid \chi(S) \neq 0\}.$$

We now show that $\chi(S) \neq 0$ for all $\chi \in \hat{G}$. We first assume that χ is ramified. Let T be the set of places of k on which χ ramifies. It is non-empty. Let θ_T be the element in S defined above. We have

$$\chi(\theta_T) = [K : K_T] \cdot \chi(\theta_{K_T}) = [K : K_T] \cdot L_k(0, \bar{\chi}) \neq 0.$$

Next we suppose that χ is unramified. If χ is trivial, we have $\chi(s(G)) = |G| \neq 0$. If $\chi \neq 1$, let v be a place such that $\chi(v) \neq 1$. Let θ_v be the element defined above. We have

$$\chi(\theta_v) = [K : K^0] \cdot (1 - \bar{\chi}(v)) \cdot L_k(0, \bar{\chi}) \neq 0.$$

This completes the proof.

3. Cyclotomic extensions of a function field.

¿From now on, we assume that k is a global function field. And from now on, we fix a place ∞ of k and fix a sign function sgn of k_∞. The other notations are as in the introduction. Let Ω be the completion of an algebraic closure of k_∞. A rank one Drinfeld $\mathbb{A}$-module ρ is called sgn-normalized if the coefficients of ρ_x are in $K_{\mathfrak{e}}$ for all $x \in \mathbb{A}$ and coefficient of the highest term of ρ_x is equal to $\operatorname{sgn}(x)$. Let $h = h(\mathbb{A})$ be the ideal class number of $\mathbb{A}$. Since $\deg \infty = 1$, we also have $h = h(k)$. We know that there are h sgn-normalized rank one Drinfeld $\mathbb{A}$-modules. Let ρ be one such $\mathbb{A}$-module and let $x \in \mathbb{A} \backslash \mathbb{F}_q$. Then the Hilbert class field $K_{\mathfrak{e}}$ is the extension of k generated by the coefficients of ρ_x. For an integral ideal $\mathfrak{m} \neq \mathfrak{e}$, let $\Lambda^{\rho}_{\mathfrak{m}}$ be the set of $\mathfrak{m}$-torsion points of ρ in Ω. Then cyclotomic extension $K = K_{\mathfrak{m}}$ of k is the extension over $K_{\mathfrak{e}}$ generated by $\Lambda^{\rho}_{\mathfrak{m}}$. Let $J = \operatorname{Gal}(K_{\mathfrak{m}}/H_{\mathfrak{m}}) \simeq \mathbb{F}_q^*$, which is the decomposition group and inertia subgroup at ∞. For the details, we refer the reader to [Part II, H1]. The sets of ramified places in K^+ and in K are $T_{K^+} = \{\mathfrak{p} \mid \mathfrak{p} \mid \mathfrak{m}\}$ and $T_K = T_{K^+} \cup \{\infty\}$ respectively. Notice that $K_{\mathfrak{e}} = K_{\mathfrak{e}}^+$ is the maximal unramified extension both in K and in K^+.

Let S be the G-submodule of $\mathbb{Q}[G]$ corresponding to $K = K_{\mathfrak{m}}$ defined in last section. We now divide S into two parts. For a non-empty subset T of T_K, the element θ_T is defined as in section 2. We regard the unramified part S^{un} defined in section 2 as θ_ϕ, where ϕ is the empty set. Let S^+ (resp. S^-) be the G-submodule of $\mathbb{Q}[G]$ generated by θ_T with $T \subseteq T_{K^+}$ or $T = \{\infty\}$ (resp. $\infty \in T \subseteq T_K$ but $T \neq \{\infty\}$). The reason that we divide $\theta_{\{\infty\}}$ in S^+, not in S^-, will becomes clear in the following lemma 4.2. We call S^+ and S^- the real part and relative part of S, respectively. Notice that the unramified part θ_ϕ belongs to the real part S^+. Let $I^\pm = S^\pm \cap \mathbb{Z}[G]$ respectively. Notice that we do have $S = S^+ + S^-$, but we may not have $I = I^+ + I^-$. By Deligne's results [Th.1.1, H2], the elements in I^- actually annihilate the divisor class group of K, and thus the ideal class group of K. We will see that the element $\theta_{\{\infty\}}$ actually belongs to θ_ϕ. Thus we have $I^+ = \operatorname{Cor}_{K/K^+}(I_1^+)$ by the definitions, where I_1^+ is the Stickelberger ideal of the extension K^+/k. In this paper we compute the indices of I^+ and I in the whole group rings $\mathbb{Z}[G^+]$ and $\mathbb{Z}[G]$ respectively. Here we regard $\mathbb{Z}[G^+]$ as a subring of $\mathbb{Z}[G]$ by the corestriction map. In the rest of this section we will elucidate the structure of S^+ and S^- by means of torsion points of sgn-normalized rank one Drinfeld $\mathbb{A}$-modules and of partial zeta functions of k.

If $0 \neq x \in k$ satisfies $\operatorname{sgn}(x) = 1$, we call x is positive and write $x \gg 0$. We also write $||x|| = N(x\mathbb{A})$, the norm of $x\mathbb{A}$. Let $\mathfrak{a}, \mathfrak{f}$ be integral $\mathbb{A}$-ideals. For $\operatorname{Re}(s) > 1$, we define

$$Z_{\mathfrak{f}}^{+}(s, \mathfrak{a}) = N\mathfrak{a}^{-s} \sum_{-1 \neq x \in \mathfrak{a}^{-1}\mathfrak{f}} ||1 + x||^{-s},$$

and

$$Z_{\mathfrak{f}}^{-}(s, \mathfrak{a}) = N\mathfrak{a}^{-s} \sum_{\substack{x \in \mathfrak{a}^{-1}\mathfrak{f} \\ 1 + x \gg 0}} ||1 + x||^{-s}.$$

Notice that $Z_{\mathfrak{f}}^{+}(s, \mathfrak{a})$ and $Z_{\mathfrak{f}}^{-}(s, \mathfrak{a})$ are only dependent on the wide and the narrow ray classes of $\mathfrak{a}$ modulo $\mathfrak{f}$ respectively. It is well-known by Weil's theorem that they are rational functions of q^{-s} over $\mathbb{Q}$, and is holomorphic except for a simple pole at $s = 1$. In [Sect.2, Y2] we showed that these functions satisfy distribution relations in the sense of B. Mazur. For $\mathfrak{f} \mid \mathfrak{m}, \mathfrak{f} \neq \mathfrak{e}$ we set

$$\theta_{\mathfrak{f}}^{+} = \frac{1}{\log q} \sum_{\mathfrak{a}} \frac{\mathrm{d}}{\mathrm{d}s} Z_{\mathfrak{f}}^{+}(0, \mathfrak{a}) \sigma_{\mathfrak{a}}^{-1} \quad \text{and} \quad \theta_{\mathfrak{f}}^{-} = \sum_{\mathfrak{a}} Z_{\mathfrak{f}}^{-}(0, \mathfrak{a}) \sigma_{\mathfrak{a}}^{-1},$$

where the summations are over all representatives of $G_{\mathfrak{m}}$, and $\sigma_{\mathfrak{a}} = (\mathfrak{a}, K/k)$ is the Artin morphism of the extension K/k associated to the ideal $\mathfrak{a}$.

Let T be a set such that $\{\infty\} \subsetneqq T \subseteq T_K$. Let $\mathfrak{f}$ be the maximal factor of $\mathfrak{m}$ which is divisible by all finite primes in T. Notice that $\mathfrak{f} \neq \mathfrak{e}$. The cyclotomic field $K_{\mathfrak{f}}$ is the maximal subextension K_T of K/k in which all places in T are ramified. By the definition in section 2, we have $\theta_{K_T} = \sum_{\mathfrak{a}} Z_{\mathfrak{f}}^{-}(0, \mathfrak{a}) \sigma_{\mathfrak{a}}^{-1}$, the summation is over all representatives of $G_{\mathfrak{f}}$ and $\sigma_{\mathfrak{a}} = (\mathfrak{a}, K_{\mathfrak{f}}/k)$. Thus $\theta_T = \mathrm{Cor}_{K/K_T}(\theta_{K_T}) = \theta_{\mathfrak{f}}^{-}$. We get

Lemma 3.1. *As a G-module, S^{-} is generated by $\theta_{\mathfrak{f}}^{-}$ with $\mathfrak{f} \mid \mathfrak{m}, \mathfrak{f} \neq \mathfrak{e}$.*

Now we consider the real part. The case $T = \{\infty\}$ is easy to deal with. Write $\theta = s(G)/(q-1) (\in \theta_{\phi})$. As the discussion above, we have $\theta_{\{\infty\}} = \theta_{\mathfrak{e}}^{-} = -\theta$, where the second equality follows from [Prop.6.1, H2]. To describe the real part clearly, we need to recall the torsion points of sgn-normalized rank one Drinfeld modules and the definition of logarithm map. Let $\xi(\mathfrak{a}) \in \Omega$ be the ξ-invariant of ideal $\mathfrak{a}$, which is characterized by the condition that the lattice ($\mathbb{A}$-submodule of Ω) $\xi(\mathfrak{a})\mathfrak{a}$ corresponds to some sgn-normalized $\mathbb{A}$-module, say ρ. Let $e_{\mathfrak{a}}(x)$ be the Drinfeld exponential function associated to ideal $\mathfrak{a}$. Let $\lambda_{\mathfrak{a}} = \xi(\mathfrak{a}) e_{\mathfrak{a}}(1)$. Assume that $\mathfrak{a}$ is an integral ideal. Then $\lambda_{\mathfrak{a}}$ is a $\mathfrak{a}$-torsion point of ρ. In fact, it is a generator of the set $\Lambda_{\mathfrak{a}}^{\rho}$ of $\mathfrak{a}$-torsion points of ρ. For $\mathfrak{f} \neq \mathfrak{e}$ and $\mathfrak{a}$ coprime to $\mathfrak{f}$, Hayes showed that $Z_{\mathfrak{f}}^{+}(0, \mathfrak{a}) = 0$, see [Prop.6.1, H2], and that $\frac{\mathrm{d}}{\mathrm{d}s} Z_{\mathfrak{f}}^{+}(0, \mathfrak{a}) = \frac{\log q}{q-1} \cdot v_{\infty}((\lambda_{\mathfrak{f}}^{s(J)})^{\sigma_{\mathfrak{a}}})$, see [Ths.6.1 and 5.1, H2], where $\sigma_{\mathfrak{a}} = (\mathfrak{a}, K_{\mathfrak{f}}^{+}/k)$. Recall that the logarithm map $l : K^{*} \longrightarrow \mathbb{Q}[G]$ is defined for $x \in K^{*}$ by

$$l(x) = \sum_{\sigma \in G} v_{\infty}(x^{\sigma}) \sigma^{-1},$$

where v_{∞} is the extension to Ω of the normalized valuation of k_{∞} at ∞.

Now let T be a non-empty subset of $T_{K^{+}}$ and let $\mathfrak{f}$ be the maximal factor of $\mathfrak{m}$ which is divisible by all primes in T. We know that $K_{\mathfrak{f}}^{+}$ is the maximal subfield K_T of K in which only the places in T are ramified. By Eq. (2.1), we have

$$(1 - q^{-s}) \theta_{K_{\mathfrak{f}}^{+}/k, T}(s) = \sum_{\mathfrak{a}} Z_{\mathfrak{f}}^{+}(s, \mathfrak{a}) \sigma_{\mathfrak{a}}^{-1},$$

where $\mathfrak{a}$ runs over representatives of $G_{\mathfrak{f}}^{+}$ and $\sigma_{\mathfrak{a}} = (\mathfrak{a}, K_{\mathfrak{f}}^{+}/k)$. We get via l'Hopital that

$$\theta_{K_T} = \theta_{K_{\mathfrak{f}}^{+}/k, T}(0) = \frac{1}{\log q} \sum_{\mathfrak{a}} \frac{\mathrm{d}}{\mathrm{d}s} Z_{\mathfrak{f}}^{+}(0, \mathfrak{a}) \sigma_{\mathfrak{a}}^{-1} = \frac{1}{q-1} \sum_{\mathfrak{a}} v_{\infty}((\lambda_{\mathfrak{f}}^{s(J)})^{\sigma_{\mathfrak{a}}}) \sigma_{\mathfrak{a}}^{-1},$$

where $\mathfrak{a}$ runs over all representatives of $G_{\mathfrak{f}}^{+}$. Thus $\theta_T = \mathrm{Cor}_{K/K_T}(\theta_{K_T}) = l(\lambda_{\mathfrak{f}})$. Let P be the subgroup of K^{*} generated by $\lambda_{\mathfrak{f}}$ with $\mathfrak{f} \mid \mathfrak{m}$ and $\mathfrak{f} \neq \mathfrak{e}$. It is the group of cyclotomic numbers of K introduced in [Def.1.1, Y1]. We have showed that the ramified part of S^{+} is equal to $l(P)$. Next we consider the unramified part.

The maximal unramified subfield of K is $K_{\mathfrak{e}}^{+}$. Let v be a place of k. For $v = \infty$, since there does not exist finite extension of $K_{\mathfrak{e}}^{+}$ which is abelian over k and in which only ∞ is ramified (notice that $\deg \infty = 1$), we have $\theta_{\infty} = 0$. Now we assume that $v = \mathfrak{p}$ is a finite place. $K_{\mathfrak{p}}^{+}$ is an abelian extension of k in which only $\mathfrak{p}$ is ramified. As above, we have

$$\theta_{K_{\mathfrak{p}}^{+}} = \frac{1}{q-1} \sum_{\mathfrak{a}} v_{\infty}((\lambda_{\mathfrak{p}}^{s(J)})^{\sigma_{\mathfrak{a}}}) \sigma_{\mathfrak{a}}^{-1},$$

and thus

$$\mathrm{Res}_{K_{\mathfrak{p}}^{+}/K_{\mathfrak{e}}^{+}}(\theta_{K_{\mathfrak{p}}^{+}}) = \frac{1}{q-1} \sum_{\mathfrak{a}} v_{\infty}(N_{K_{\mathfrak{p}}/K_{\mathfrak{e}}^{+}}(\lambda_{\mathfrak{p}})^{\tau_{\mathfrak{a}}}) \sigma_{\mathfrak{a}}^{-1},$$

where in the first equality $\sigma_{\mathfrak{a}} = (\mathfrak{a}, K_{\mathfrak{p}}^{+}/k)$ and $\mathfrak{a}$ runs over all representatives of $G_{\mathfrak{p}}^{+}$, and in the second equality $\tau_{\mathfrak{a}} = (\mathfrak{a}, K_{\mathfrak{e}}^{+}/k)$ and $\mathfrak{a}$ runs over those of $G_{\mathfrak{e}}^{+}$. Using the fact that $N_{K_{\mathfrak{p}}/K_{\mathfrak{e}}}(\lambda_{\mathfrak{p}}) = \xi(\mathbb{A})/\xi(\mathfrak{p})$, see [Sect.1, H4], we have

$$\theta_{\mathfrak{p}} = \mathrm{Cor}_{K/K_{\mathfrak{e}}^{+}}\mathrm{Res}_{K_{\mathfrak{p}}^{+}/K_{\mathfrak{e}}^{+}}(\theta_{K_{\mathfrak{p}}^{+}}) = l(\xi(\mathbb{A})/\xi(\mathfrak{p}))/(q-1).$$

Let $\bar{P}$ be the G-submodule of K^* generated by $\lambda_{\mathfrak{f}}^{s(J)}$ with $\mathfrak{f} \mid \mathfrak{m}, \mathfrak{f} \neq \mathfrak{e}$ and by $\zeta(\mathbb{A})/\zeta(\mathfrak{p})$ with all primes $\mathfrak{p}$. Notice that $\lambda_{\mathfrak{f}}^{s(J)} = -\lambda_{\mathfrak{f}}^{q-1}$, see [Eq. 4.13, H2]. Furthermore, we claim that $\theta \in l(\bar{P})/(q-1)$. Let k_+ be the subset of k of positive elements. By [Sect.2, H4], we have $k_+ \subseteq \bar{P} \cap k$. Since the GCD of all $\deg x$ with $0 \neq x \in \mathbb{A}$ is equal to $\deg \infty = 1$, we have

$$l(\bar{P})/(q-1) \supset l(k_+)/(q-1) = \theta v_{\infty}(k_+) = \mathbb{Z}\theta \ni \theta.$$

This proved the claim. We have showed

Lemma 3.2. $S^+ = l(\bar{P})/(q-1)$.

4. Preparations for computing the indices.

In this section, we make some preparation for the calculation of the indices of the Stickelberger ideals. We begin this section by recalling the definition of lattice index.

Let Y be a $\mathbb{Q}$-subspace of $\mathbb{Q}[G]$. A lattice in Y is a finitely generated subgroup of Y with the maximal rank. Let L and L' be two lattices in Y. There exists a nonsingular linear transformation $A : Y \longrightarrow Y$ such that $A(L) = L'$. The lattice index is defined to be $(L : L') = |\det A|$. Sinnott has given some properties of lattice indices [Lems.1.1 and 6.1, Si]. Here we mention one more which will be used later. The proof is clear.

Lemma 4.1. *Assume that $L = L_1 \oplus L_2$ and $L' = L_1' \oplus L_2'$ are two lattices in $Y = Y_1 \oplus Y_2$ and that L_i, L_i' are lattices in Y_i for $i = 1, 2$. Then*

$$(L : L') = (L_1 : L_1')(L_2 : L_2').$$

For a G-module M, we denote by M_0 the G-submodule of M of elements killed by $s(G)$, and by M^G the submodule of M fixed by G. Let $e^+ = s(J)/(q-1)$ and let $e^- = 1 - e^+$. Then $e^+e^- = 0$.

Lemma 4.2. *$e^+S^+ = S^+$ and $e^-S^- = S^-$. Thus $S = S^+ \oplus S^-$.*

Proof. Since $S^+ \subset s(J)\mathbb{Q}[G]$, the first equality is obvious. We now show the second. For $\mathfrak{f} \mid \mathfrak{m}, \mathfrak{f} \neq \mathfrak{e}$, by [Prop.6.1, H2] we have

$$\begin{aligned} s(J)\theta_{\mathfrak{f}}^{-} &= \sum_{\alpha}\sum_{\mathfrak{a}} Z_{\mathfrak{f}}^{-}(0,\mathfrak{a})\sigma_{(1+\alpha)\mathfrak{a}}^{-1} = \sum_{\mathfrak{a}}(\sum_{\alpha} Z_{\mathfrak{f}}^{-}(0,(1+\alpha)\mathfrak{a})\sigma_{\mathfrak{a}}^{-1} \\ &= \sum_{\mathfrak{a}}(\sum_{x \in \mathfrak{a}^{-1}\mathfrak{f}} ||1+x||^{-s})|_{s=0}\sigma_{\mathfrak{a}}^{-1} = 0, \end{aligned}$$

where α runs over all representatives of principal ideals $(1+\alpha)\mathbb{A}$ with $\alpha \in \mathfrak{m}$ modulo principal ideals $(1+\alpha)\mathbb{A}$ with $\alpha \in \mathfrak{m}$ and $1 + \alpha \gg 0$. We complete the proof by Lem.3.1.

Lemma 4.3. $(S^+)^G = S^G = \mathbb{Z}\theta$.

Proof. Let $e_1 = s(G)/|G|$. It is easy to check that $S^G = S \cap e_1 S$, which implies the first equality by the last lemma. Clearly $l(k_+) \subseteq l(\bar{P} \cap k) \subseteq l(\bar{P})^G$. Conversely, let $l(\alpha) \in l(\bar{P})^G$, where $\alpha \in \bar{P}$. Then $(\sigma - 1)l(\alpha) = l(\alpha^{\sigma-1}) = 0$ for all $\sigma \in G$. Since $\alpha^{\sigma-1}$ is a positive unit, we get $\alpha^{\sigma-1} = 1$ for all $\sigma \in G$ and thus $\alpha \in k_+$. Here we used the fact that each element in $\bar{P}$ is totally positive [Cor.4.16, H2]. We showed $l(k_+) = l(\bar{P})^G$. Thus we have

$$(S^+)^G = l(k_+)/(q-1) = \mathbb{Z}\theta.$$

This completes the proof.

We now introduce some notations. Let $\chi \in \hat{G}$ be with conductor $\mathfrak{f}_{\chi}$, and let $\mathfrak{a}$ be an integral $\mathbb{A}$-ideal. We define $\chi(\mathfrak{a})$ as follows. If $(\mathfrak{a}, \mathfrak{f}_{\chi}) \neq \mathfrak{e}$, we define $\chi(\mathfrak{a}) = 0$. Otherwise, we set $\chi(\mathfrak{a}) = \chi(\sigma_{\mathfrak{a}})$, where

$\sigma_{\mathfrak{a}} = (\mathfrak{a}, K_{\mathfrak{f}_\chi}/k)$ is the Artin symbol. For prime $\mathfrak{p}$, let $\bar{\sigma}_{\mathfrak{p}}$ be the unique element in $\mathbb{Q}[G]$ such that $\chi(\bar{\sigma}_{\mathfrak{p}}) = \bar{\chi}(\mathfrak{p})$ for all $\chi \in \hat{G}$. For $\mathfrak{f} \mid \mathfrak{m}$, let $I_{\mathfrak{f}} = \mathrm{Gal}(K/K_{\mathfrak{f}})$. Let V be the G-submodule of $\mathbb{Q}[G]$ generated by

$$\alpha_{\mathfrak{f}} = s(I_{\mathfrak{f}}) \prod_{\mathfrak{p} \mid \mathfrak{f}} (1 - \bar{\sigma}_{\mathfrak{p}})$$

with $\mathfrak{f} \mid \mathfrak{m}$, $\mathfrak{f} \neq \mathfrak{e}$. We also set $U = V + s(I_{\mathfrak{e}})R$ and $U' = (q-1)V + s(I_{\mathfrak{e}})R$, where $R = \mathbb{Z}[G]$. We mention that U is the level $\mathfrak{m}$ group of the Iwasawa distribution associated to k, see [Sect.3, Y2]. For convenience of the reader, we list two results from [Y1] and [An] in the follows. They are important in the computation of the indices. The following result is the corollary 3.3 in [Y1].

Lemme 4.4. $[e^+U_0 : e^+U_0'] = (q-1)^{[K^+:k]-t}$.

Let $H^i(J, U)$ denote the i-th Tate cohomology of the J-module U. It is a G-module. The author determined the G-module structure conditionally following the ideas of Sinnott's [Sect.5, Si]. Anderson [An] invented a remarkable method and determined it completely. Recall that s is the number of distinct prime factors of $\mathfrak{m}$.

Lemma 4.5 ([Th.5.2.3, An]). *For all i, we have the following G-equivariant isomorphism*

$$H^i(J, U) \simeq (\mathbb{Z}/(q-1)[G_{\mathfrak{e}}/N])^{2^{s-1}}.$$

A character χ of G is called real if $\chi(J) = 1$. Such χ induces a character of G^+. If χ is not real, we call it non-real. Let $\hat{G}^+$ and $\hat{G}^-$ denote the sets of real characters and of non-real characters of G respectively. Let e_χ be the idempotent element associated to χ. We set

$$\omega^+ = \sum_{1 \neq \chi \in \hat{G}^+} L_k(0, \bar{\chi}) e_\chi \quad \text{and} \quad \omega^- = \sum_{\chi \in \hat{G}^-} L_k(0, \bar{\chi}) e_\chi.$$

Clearly $e^+\omega^+ = \omega^+$ and $e^-\omega^- = \omega^-$. We have

Lemma 4.6. $(1 - e_1)S^+ = \omega^+ e^+ U_0'$ *and* $(1 - e_1)S^- = S^- = \omega^- e^- U$.

Proof. The first equality follows from (Prop.4.1 and Lem.4.2, [Y1]). For a non-real character χ of G of conductor $\mathfrak{f}_\chi$, the L-function $L(s, \chi)$ of χ does not contain the Euler factor at ∞. Using the partial zeta function, we get

$$L_k(0, \chi) = \sum_{\mathfrak{a}} \bar{\chi}(\mathfrak{a}) Z_{\mathfrak{f}_\chi}^-(0, \mathfrak{a}),$$

where $\mathfrak{a}$ runs over all representatives of $G_{\mathfrak{f}_\chi}$. We claim that $\theta_{\mathfrak{f}}^- = \omega^- \alpha_{\mathfrak{f}}$ for all $\mathfrak{f} \mid \mathfrak{m}, \mathfrak{f} \neq \mathfrak{e}$, which will implies the second equality.

If $\chi \in \hat{G}$ is real, we have $\chi(\theta_{\mathfrak{f}}^-) = \chi(\omega^- \alpha_{\mathfrak{f}}) = 0$. Now assume that χ is non-real. If the conductor $\mathfrak{f}_\chi$ of χ does not divide $\mathfrak{f}$, there exists $\sigma \in I_{\mathfrak{f}}$ such that $\chi(\sigma) \neq 1$ and $\sigma\theta_{\mathfrak{f}}^- = \theta_{\mathfrak{f}}^-$. Thus $\chi(\theta_{\mathfrak{f}}^-) = 0$. If $\mathfrak{f}_\chi \mid \mathfrak{f}$, by [Prop.3.1, Y2] we have $\chi(\theta_{\mathfrak{f}}^-) = \frac{|G_{\mathfrak{m}}|}{|G_{\mathfrak{f}}|} \cdot \prod_{\mathfrak{p} \mid \mathfrak{f}} (1 - \bar{\chi}(\mathfrak{p})) \cdot L(0, \bar{\chi})$.

On the other hand, for non-real χ, we have $\chi(\omega^-) = L(0, \bar{\chi})$ and $\chi(\bar{\sigma}_{\mathfrak{p}}) = \bar{\chi}(\mathfrak{p})$, and $\chi(s(I_{\mathfrak{f}})) = \sum_{\alpha \in I_{\mathfrak{f}}} \chi(\sigma)$, which is equal to 0 if $\mathfrak{f}_\chi \nmid \mathfrak{f}$ and to $|G_{\mathfrak{m}}|/|G_{\mathfrak{f}}|$ otherwise. We showed $\chi(\theta_{\mathfrak{f}}^-) = \chi(\omega^- \alpha_{\mathfrak{f}})$ for all $\chi \in \hat{G}$. This is the claim.

Finally we study the relations of S with I and S^+ with I^+. Let $\mathfrak{a}$ be an integral $\mathbb{A}$-ideal coprime to $\mathfrak{m}$. Let $\sigma_{\mathfrak{a}} = (\mathfrak{a}, K_{\mathfrak{e}}/k)$ be the Artin symbol. By [Equ.4.2, H3] and [Sect.1, H4], we have

$$\frac{1}{q-1} v_\infty((\xi(\mathbb{A})/\xi(\mathfrak{p}))^{\sigma_{\mathfrak{a}}}) = \frac{1}{q-1} (v_\infty(\xi(\mathfrak{a}^{-1})) - v_\infty(\xi(\mathfrak{a}^{-1}\mathfrak{p}))) \equiv \frac{\deg \mathfrak{p}}{q-1} \bmod \mathbb{Z},$$

and thus

$$l(\xi(\mathbb{A})/\xi(\mathfrak{p}))/(q-1) \in I^+ + \mathbb{Z}\theta. \tag{4.1}$$

¿From the proof of [Prop.6.1, H2] and via a simple calculation, we get, for $\mathfrak{f} \mid \mathfrak{m}$ and $\mathfrak{f} \neq \mathfrak{e}$,

$$v_\infty(\lambda_{\mathfrak{f}}^{\sigma_{\mathfrak{a}}}) = \frac{1}{\log q} \cdot \frac{\mathrm{d}}{\mathrm{d}s} Z_{\mathfrak{f}}^{+}(s, \mathfrak{a})|_{s=0} \equiv \frac{-1}{q-1} \bmod \mathbb{Z},$$

where $\sigma_{\mathfrak{a}} = (\mathfrak{a}, K_{\mathfrak{f}}, /k)$, which shows

$$l(\lambda_{\mathfrak{f}}) \in I^+ + \mathbb{Z}\theta. \tag{4.2}$$

For large M, by the proof of [Prop.6.1, H2], we have

$$Z_{\mathfrak{f}}^{-}(0, \mathfrak{a}) \equiv \Big(\sum_{\substack{0 \ll x \in \mathfrak{a}^{-1}\mathfrak{f} \\ \deg x \geq M}} ||x||^{-s} \Big)|_{s=0} \equiv \frac{1}{q-1} \Big(\sum_{\substack{x \in \mathfrak{a}^{-1}\mathfrak{f} \\ \deg x \geq M}} ||x||^{-s} \Big)|_{s=0} \equiv \frac{-1}{q-1} \bmod \mathbb{Z}.$$

This gives us

$$\theta_{\mathfrak{f}}^{-} \in I + \mathbb{Z}\theta. \tag{4.3}$$

¿From Eqs. (4.1-4.3) we get

Lemma 4.7. $S^+ = I^+ + \mathbb{Z}\theta$ and $S = I + \mathbb{Z}\theta$.

5. The index of the real part.

In this section we compute the index of I^+ in $R^+ (= s(J)R)$. We leave the reader to check that each lattice index appeared in this and next sections are well defined.

Proof of Theorem 2. We have, by [Lems.1.1 and 6.1, Si],

$$\begin{aligned} [R^+ : I^+] =& (s(J)R : s(J)U)(s(J)U : S^+)(S^+ : I^+) \\ =& (s(J)R : s(J)U)(s(J)U_0 : (1-e_1)S^+)((1-e_1)S^+ : S_0^+) \\ & \times ((q-1)s(G)U : s(G)S^+)(S^+ : I^+). \end{aligned} \tag{5.1}$$

We now compute the indices respectively.

Applying Lem. 4.5 and using the calculation in [Page 64-65, Y1], we have

$$(s(J)R : s(J)U) = \begin{cases} (q-1)^t & \text{if} \quad s = 1 \\ (q-1)^{t2^{s-2}} & \text{if} \quad s > 1. \end{cases} \tag{5a}$$

Since $(s(J)U_0 : (1-e_1)S^+) = (s(J)U_0 : s(J)U_0')(s(J)U_0' : (1-e_1)S^+)$ and since the rank of $s(J)U_0'$ is equal to $[K^+ : k] - 1$, by Lems.4.4 and 4.6, and by analytic class number formula, we get, noting that $h = h(k)$ is the divisor class number of k,

$$\begin{aligned} (s(J)U_0 : (1-e_1)S^+) &= (q-1)^{[K^+:k]-1}(e^+U_0 : e^+U_0')(e^+U_0' : (1-e_1)S^+) \\ &= (q-1)^{1-t} \prod_{1 \neq \chi \in \hat{G}^+} L_k(0, \chi) = (q-1)^{1-t} h(K^+)/h. \end{aligned} \tag{5b}$$

It is easy to show $S_0^+ = S^+ \cap (1-e_1)S^+$, which shows

$$(1-e_1)S^+/S_0^+ \simeq e_1S^+ + S^+/S^+ \simeq e_1S^+/(S^+)^G.$$

Furthermore, $s(G)U = |I_{\mathfrak{e}}|s(G)\mathbb{Z}$ and $|G|/|I_{\mathfrak{e}}| = h$. Thus by Lem.4.3

$$\begin{aligned} & ((1-e_1)S^+ : S_0^+)((q-1)s(G)U : s(G)S^+) \\ =& (e_1S^+ : (S^+)^G)((q-1)|I_{\mathfrak{e}}|s(G)\mathbb{Z} : e^+S^+)(e^+S^+ : s(G)S^+) \\ =& h(s(G)\mathbb{Z} : (S^+)^G)/(q-1) = h/(q-1)^2. \end{aligned} \tag{5c}$$

By Lemma 4.7, we have $S^+/I^+ \simeq \mathbb{Z}\theta/I^+ \cap \mathbb{Z}\theta \simeq \mathbb{Z}/(q-1)\mathbb{Z}$. Thus

$$[S^+ : I^+] = q - 1. \tag{5d}$$

By substituting Eqs.(5a-5d) in Eq.(5.1), we get Theorem 2.

6. The index of the whole ideal.

In this last section we calculate the index of I in R.

Proof of Theorem 1. As in last section, we have, noting that $(e^-U)_0 = e^-U$,

$$\begin{aligned}[R:I] =&(R:e^+R+e^-R)(e^+R+e^-R:e^+U+e^-U)(e^+U+e^-U:U)(U:S)(S:I)\\ =&(R:e^+R+e^-R)(e^+R+e^-R:e^+U+e^-U)(e^+U_0+e^-U:(1-e_1)S)\\ &\times((1-e_1)S:S_0)(s(G)U:s(G)S)(S:I).\end{aligned} \tag{6.1}$$

As above, we compute these indices one by one.

Since $e^+R+e^-R = R+e^+R$, we have $e^+R+e^-R/R \simeq e^+R/s(J)R$. Thus

$$(R:e^+R+e^-R) = (q-1)^{-[K^+:k]}. \tag{6a}$$

By [Page 64, Y1] we have

$$1 = (R:U) = (e^-R:e^-U)(\ker(e^-)|_R : \ker(e^-)|_U) = (e^-R:e^-U)(R^J:U^J).$$

Since $H^0(J,U) = U^J/s(J)U$ and $R^J = s(J)R$, we get, by Lems.4.1 and 4.5,

$$(e^+R+e^-R:e^+U+e^-U) = (e^+R:e^+U)(e^-R:e^-U) = |H^0(J,U)| = (q-1)^{t2^{s-1}}. \tag{6b}$$

By Lems.4.1, 4.2 and 4.6, and by Eq.(5b), we have

$$\begin{aligned}(e^+U_0+e^-U:(1-e_1)S) &= (e^+U_0:(1-e_1)S^+)(e^-U:(1-e_1)S^-)\\ =&(q-1)^{[K^+:k]-t}\prod_{1\neq\chi\in\hat{G}} L_k(0,\bar{\chi}) = (q-1)^{[K^+:k]-t}h(K)/h.\end{aligned} \tag{6c}$$

As in last section we can compute the two other indices. We have

$$((1-e_1)S:S_0)(s(G)U:s(G)S) = h(s(G)\mathbb{Z}:S^G) = h/(q-1). \tag{6d}$$

and

$$[S:I] = q-1. \tag{6e}$$

Substituting Eqs.(6a-6e) in Eq.(6.1), we get Theorem 1.

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