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# STIEFEL-WHITNEY HOMOLOGY CLASSES OF $k$-POINCARÉ-EULER SPACES

Dedicated to Professor Itiro Tamura on his sixtieth birthday

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**1. Introduction and the statement of results.** Let $X$ be a polyhedron. It is said to be totally $n$-dimensional if there exists a locally finite triangulation $K$ of $X$ such that for each $\sigma \in K$, an $n$-dimensional simplex $\tau$ exists in $K$ satisfying $\sigma \prec \tau$ or $\sigma = \tau$. (See Akin [1].) A totally $n$-dimensional polyhedron $X$ is an $n$-dimensional $k$-Euler space if there exist a locally finite triangulation $K$ of $X$ and a subcomplex $L$ of $K$ satisfying the following:

(1) $|L|$ is a totally $(n-1)$-dimensional polyhedron or empty.

(2) The cardinality of $\{\tau \in K | \sigma \prec \tau\}$ is even for every $\sigma$ in $K - L$, whenever $\dim \sigma \geqq n - k$.

(3) The cardinality of $\{\tau \in K | \sigma \prec \tau\}$ is odd for every $\sigma$ in $L$, whenever $\dim \sigma \geqq n - k$.

(4) The cardinality of $\{\tau \in L | \sigma \prec \tau\}$ is even for every $\sigma$ in $L$, whenever $\dim \sigma \geqq n - k - 1$.

We usually denote $\partial X$ instead of $|L|$. If $X$ is an $n$-dimensional $k$-Euler space, then $\partial X$ clearly is an $(n-1)$-dimensional $k$-Euler space. An $n$-dimensional $k$-Euler space $X$ is closed if $X$ is compact and $\partial X$ is empty. If $k \geqq n$, we said $n$-dimensional $k$-Euler spaces to be $n$-dimensional $\boldsymbol{Z}_2$-Euler spaces. (See [10].)

Let $X$ be an $n$-dimensional $k$-Euler space with a triangulation $K$. Then the $i$-th Stiefel-Whitney homology class $s_i(X)$ in $H_i^{\inf}(X, \partial X; \boldsymbol{Z}_2)$ is the homology class determined as the $i$-skeleton $\bar{K}^i$ of the first barycentric subdivision $\bar{K}$ of $K$ for $n - k < i \leqq n$. Here $H_*^{\inf}$ is the homology theory of infinite chains. The Stiefel-Whitney homology classes of $k$-Euler spaces are well defined by Proposition 2.2.

Since an $n$-dimensional differentiable manifold $M$ has a triangulation, the $i$-th Stiefel-Whitney homology class $s_i(M)$ can be defined as above for $0 \leqq i \leqq n$. Whitney [16] announced that the $i$-th Stiefel-Whitney homology class of an $n$-dimensional differentiable manifold $M$ is the Poincaré dual of the $(n-i)$-th Stiefel-Whitney class $w^{n-i}(M)$. Its proof was outlined by Cheeger [5] and given by Halperin and Toledo [6]. Blanton and Schweitzer [2] and Blanton and McCrory [3] gave the proof by using an axiomatic method. Taylor [15] generalized it to the case of $\boldsymbol{Z}_2$-homology manifolds by using the method as in [2]. Matsui [10] studied the case of $\boldsymbol{Z}_2$-Poincaré-Euler spaces in another method.

In this paper, we study the case of $k$-Poincaré-Euler spaces as in [10]. An $n$-dimensional $k$-Euler space $X$ is said to be an $n$-dimensional $k$-Poincaré-Euler space if the cap products $[X]_\cap\colon H^i(X, \boldsymbol{Z}_2) \to H_{n-i}^{\inf}(X, \partial X; \boldsymbol{Z}_2)$ are isomorphisms for $0 \leqq i < k$. Let $X$ be an $n$-dimensional $k$-Poincaré-Euler space. Then there exists a proper embedding $\varphi\colon (X, \partial X) \to (\boldsymbol{R}_+^{n+\alpha}, \partial \boldsymbol{R}_+^{n+\alpha})$ for $\alpha$ sufficiently large, where $\boldsymbol{R}_+^{n+\alpha} = \{(x_1, x_2, \cdots, x_{n+\alpha}) \mid x_{n+\alpha} \geqq 0\}$. (See Hudson [8].) Suppose that $R$ is a regular neighborhood of $X$ in $\boldsymbol{R}_+^{n+\alpha}$. Put $\tilde{R} = R \cap \partial \boldsymbol{R}_+^{n+\alpha}$ and $\bar{R} = \mathrm{cl}(\partial R - \tilde{R})$. Regard $\varphi$ as an embedding from $(X, \partial X)$ to $(R, \tilde{R})$. We also call $(R; \tilde{R}, \bar{R}; \varphi)$ a regular neighborhood of $X$ in $\boldsymbol{R}_+^{n+\alpha}$. Define $U(\varphi)$ in $H^\alpha(R, \bar{R}; \boldsymbol{Z}_2)$ as the Poincaré dual of $\varphi_*[X]$. Then the cup products $U(\varphi)^\cup\colon H^i(R; \boldsymbol{Z}_2) \to H^{i+\alpha}(R, \bar{R}; \boldsymbol{Z}_2)$ are isomorphisms for $0 \leqq i < k$. We call $U(\varphi)$ the Thom class of $(R; \tilde{R}, \bar{R}; \varphi)$. Define cohomology classes $w^i(\varphi)$ by $w^i(\varphi) = \varphi^* \circ (U(\varphi)^\cup)^{-1} \circ Sq^i U(\varphi)$ for $0 \leqq i < k$. Put $w^{(k)}(\varphi) = 1 + w^1(\varphi) + \cdots + w^{k-1}(\varphi)$. Then there exists a unique cohomology class $\tilde{w}(X)$ such that $\tilde{w}(X) \cup w^{(k)}(\varphi) = 1$. Let $\tilde{w}(X) = 1 + \tilde{w}(X)^1 + \cdots + \tilde{w}^n(X)$, where $\tilde{w}^i(X)$ is in $H^i(X; \boldsymbol{Z}_2)$. Define $w^i(X)$ by $w^i(X) = \tilde{w}^i(X)$ for $0 \leqq i < k$. We call $w^i(X)$ the $i$-th Stiefel-Whitney class of a $k$-Poincaré-Euler space $X$ for $0 \leqq i < k$. Define $w^{(k)}(X)$ by $w^{(k)}(X) = 1 + w^1(X) + \cdots + w^{k-1}(X)$.

Let $(R; \tilde{R}, \bar{R}; \varphi)$ be a regular neighborhood of an $n$-dimensional $k$-Poincaré-Euler space $X$ in $\boldsymbol{R}_+^{n+\alpha}$. We will define homomorphisms $(e_\varphi^k)^i\colon \mathfrak{N}_{i+\alpha}(R, \bar{R}) \to \boldsymbol{Z}_2$ and $(\tilde{e}_\varphi^k)^i\colon \mathfrak{N}_{i+\alpha}(R, \bar{R}) \to \boldsymbol{Z}_2$ for $i < k$, where $\mathfrak{N}_{i+\alpha}(R, \bar{R})$ is the unoriented differentiable bordism group. We need the following:

TRANSVERSALITY THEOREM (Rourke and Sanderson [13] and Buoncristiano, Rourke and Sanderson [4]). *Let $M$ and $N$ be PL-manifolds. Suppose that $f\colon (M, \partial M) \to (N, \partial N)$ is a locally flat proper embedding and that $X$ is a subpolyhedron in $N$. If $f(\partial M) \cap X = \varnothing$ or if $(\partial N, \partial N \cap X)$ is collared in $(N, X)$ and $\partial N \cap X$ is block transverse to $f \mid \partial M\colon \partial M \to \partial N$, then there exists an embedding $g\colon M \to N$ ambient isotopic to $f$ relative to $\partial N$ such that $X$ is block transverse to $g$.*

Let $f\colon (M, \partial M) \to (R, \bar{R})$ be in $\mathfrak{N}_{i+\alpha}(R, \bar{R})$. By Transversality Theorem, there exists an embedding $g\colon (M, \partial M) \to (R \times D^\beta, R \times D^\beta)$ for $\beta$ sufficiently large such that $g \cong f \times \{0\}$ and that $(\varphi \times \mathrm{id})(X \times D^\beta)$ is block transverse to $g$. Let $Y = (\varphi \times \mathrm{id})^{-1} \circ g(M)$ and let $\psi\colon Y \to X \times D^\beta$ be the inclusion. If $i < k$, then $Y$ is a closed $\boldsymbol{Z}_2$-Euler space by (1) of Lemma 4.3. Define $(e_\varphi^k)^i(f, M)$ by the modulo 2 Euler number $e(Y)$ of $Y$. Note that $\psi$ has a normal block bundle $\nu$ in $X \times D^\beta$ from (1) of Lemma 4.3. Define $(\tilde{e}_\varphi^k)^i(f, M)$ as $(\tilde{e}_\varphi^k)^i(f, M) = \langle \psi^* w^{(k)}(X \times D^\beta) \cup \tilde{w}(\nu), [Y] \rangle$, where $\tilde{w}(\nu)$ is the cohomology class determined by $w^*(\nu) \cup \tilde{w}(\nu) = 1$. Now define a homomorphism $(o_\varphi^k)^i \colon \mathfrak{N}_{i+\alpha}(R, \bar{R}) \to \boldsymbol{Z}_2$ by $(o_\varphi^k)^i = (\tilde{e}_\varphi^k)^i - (e_\varphi^k)^i$. We can state the main theorem of this paper as follows:

THEOREM. *Let $X$ be an $n$-dimensional $k$-Poincaré-Euler space. Take a regular neighborhood $(R; \tilde{R}, \bar{R}; \varphi)$ of $X$ in $\boldsymbol{R}_+^{n+\alpha}$. Then $[X] \cap w^i(X) = s_{n-i}(X)$ for $i \leqq m$ if and only if $(o_\varphi^k)^i = 0$ for $i \leqq m$, where $m < k$.*

We can apply this theorem to $k$-regular spaces. Let $R$ be a commutative ring with unit. An $n$-dimensional 1-Euler space $X$ is an $n$-dimensional $k$-regular space over $R$ if a triangulation $K$ of $X$ satisfies the following:

(1) For each $\sigma$ in $K - \partial K$, if $\dim \sigma = i$, then $H_j(Lk(\sigma; K); R) = H_j(S^{n-i-1}; R)$ for $j \leqq k - 1$.

(2) For each $\sigma$ in $\partial K$, if $\dim \sigma = i$, then $H_j(Lk(\sigma; K); R) = H_j(pt; R)$ for $j \leqq k - 1$.

(3) For each $\sigma$ in $\partial K$, if $\dim \sigma = i$, then $H_j(Lk(\sigma; \partial K); R) = H_j(S^{n-i-2}; R)$ for $j \leqq k - 1$.

An $n$-dimensional $k$-regular space over $R$ is $R$-orientable if $H_n^{\inf}(X_\alpha, \partial X_\alpha; R) = R$ for each connected component $X_\alpha$ of $X$.

In order to apply our theorem to $k$-regular spaces, we need the following:

PARTIAL POINCARÉ DUALITY THEOREM (Kato [9]). *Let $R$ be a commutative ring with unit. Let $X$ be an $n$-dimensional $k$-regular space over $R$. Suppose that $X$ is $R$-orientable unless $R = \boldsymbol{Z}_2$. Then the cap products $[X]_\cap \colon H^i(X; R) \to H_{n-i}(X, X; R)$ and $[X]_\cap \colon H^i(X, \partial X; R) \to H_{n-i}(X; R)$ are epimorphisms for all $i \leqq k - 1$ or $i \geqq n - k$ and monomorphisms for all $i \leqq k$ or $i \geqq n - k + 1$. Here $H_*$ is the homology theory of infinite chains whenever $H^*$ is the ordinary cohomology theory, or $H_*$ is the ordinary homology theory whenever $H^*$ is the cohomology theory of cochains with compact support.*

In [9], Kato prove this theorem in the case of compact $k$-regular spaces over $\boldsymbol{Z}$. But since we can prove this theorem by using the same method as in [9], we do not repeat the proof here.

By our theorem and Partial Poincaré Duality Theorem, we have the following:

COROLLARY. *Let $X$ be an $n$-dimensional $k$-regular space over $Z_2$. Then $[X]\cap w^i(X) = s_{n-i}(X)$ for all $i < k$.*

In Section 2, we study the Stiefel-Whitney homology classes of $k$-Euler spaces and prove a special product formula for the Stiefel-Whitney homology classes. These are necessary to prove Lemma 5.1. The structure of the bordism group of compact $k$-Euler spaces is given in Proposition 3.1. Lemma 3.1 is necessary to prove Lemma 5.1. In Section 4, we give a characterization of Stiefel-Whitney classes via the unoriented differentiable bordism group. In Section 5, we give a characterization of Stiefel-Whitney homology classes via the unoriented differentiable bordism group. Our theorem follows from Lemmas 4.1 and 5.1.

**2. Stiefel-Whitney homology classes.** The purpose of this section is to show that Stiefel-Whitney homology classes of $k$-Euler spaces is well defined and to prove a special product formula for Stiefel-Whitney homology classes.

In order to prove Propositions 2.2 and 2.3, it is convenient to define $k$-Euler complexes for ball complexes.

A ball complex $K$ (cf. [4]) is totally $n$-dimensional if for each $\sigma$ in $K$ there exists an $n$-dimensional ball $\tau$ in $K$ such that $\sigma \prec \tau$ or $\sigma = \tau$. A totally $n$-dimensional locally finite ball complex $K$ is an $n$-dimensional $k$-Euler complex if there exists a subcomplex $L$ satisfying the same conditions (1), (2), (3) and (4) as in the definition of $k$-Euler spaces in Section 1. We usually denote $\partial K$ instead of $L$. An $n$-dimensional $k$-Euler complex $K$ is said to be closed if $K$ is a finite complex and $\partial K$ is empty. A polyhedron $X$ is an $n$-dimensional $k$-Euler space if there exists an $n$-dimensional $k$-Euler complex $K$ such that $X = |K|$. We usually denote $\partial X$ instead of $|\partial K|$. Such definition of $k$-Euler spaces clearly coincides with that in Section 1.

Let $K$ be a ball complex. The barycentric subdivision $\bar{K}$ of $K$ is defined by $\bar{K} = \{(\sigma_0, \cdots, \sigma_p) \mid \sigma_0 \prec \cdots \prec \sigma_p, \sigma_i \in K\}$. Then $\bar{K}$ can be regarded as a ball complex. Denote the $p$-skeleton of $\bar{K}$ by $\bar{K}^p$. We need the following to prove that Stiefel-Whitney homology classes of $k$-Euler spaces is well defined:

PROPOSITION 2.1. *Let $K$ be an $n$-dimensional $k$-Euler complex. Then $\bar{K}^p$ are $p$-dimensional $(p-n+k)$-Euler complexes such that $\partial\bar{K}^p = \overline{\partial K}^{p-1}$ for $n-k < p \leqq n$.*

In order to prove this proposition, we need the following:

LEMMA 2.1. *Let $K$ be a totally $n$-dimensional locally finite ball* *complex. If* $b \in \bar{K}^{p-1}$, *then the cardinality of* $\{a \in \bar{K} - \bar{K}^p \mid a \succ b\}$ *is even.*

PROOF. If $p = n$, then $\bar{K} - \bar{K}^p$ is empty. Thus we may assume that $p < n$. Let $a = \langle \sigma_0, \cdots, \sigma_s \rangle \in \bar{K} - \bar{K}^p$ and let $b = \langle \tau_0, \cdots, \tau_t \rangle \in \bar{K}^{p-1}$. Then $s > t + 1$. Since the cardinality of $\{\sigma \in K \mid \sigma_0 \prec \sigma \prec \sigma_1\}$ is even for each $\langle \sigma_0, \sigma_1 \rangle \in \bar{K}$, the cardinality $\{a \in \bar{K} - \bar{K}^p \mid a \succ b\}$ is even for $b \in \bar{K}^{p-1}$. q.e.d.

PROOF OF PROPOSITION 2.1. Note that the cardinality of $\{b \in \bar{K} \mid a \prec b\}$ equals the sum of the cardinalities of $\{b \in \bar{K}^p \mid a \prec b\}$ and $\{b \in \bar{K} - \bar{K}^p \mid a \prec b\}$ for $a \in \bar{K}$. By Lemma 2.1, the cardinalities of $\{b \in \bar{K} \mid a \prec b\}$ and $\{b \in \bar{K}^p \mid a \prec b\}$ are congruent modulo 2 for $a \in \bar{K}^{p-1}$. Therefore $\bar{K}^p$ is a $p$-dimensional $(p - n + k)$-Euler complex such that $\partial \bar{K}^p = \overline{\partial \bar{K}^{p-1}}$ for $p > n - k$. q.e.d.

Let $X$ be an $n$-dimensional $k$-Euler space with a ball complex structure $K$. Define the $i$-th Stiefel-Whitney homology classes $s_i(X)$ by $s_i(X) = j_*[|\bar{K}^i|]$ for $n - k < i \leq n$, where $j: |\bar{K}^i| \to X$ are the inclusions. Let $s_{(k)}(X) = s_{n-k+1}(X) + \cdots + s_n(X)$. The Stiefel-Whitney homology classes of $k$-Euler spaces are well defined by the following:

PROPOSITION 2.2. *Let $K$ be an $n$-dimensional $k$-Euler complex and let $L$ be a subdivision of $K$. Then* $(j_K)_*[|\bar{K}^i|] = (j_L)_*[|\bar{L}^i|]$ *for* $n - k < i \leq n$, *where $j_K$ and $j_L$ are the inclusions.*

PROOF. Define an $(n + 1)$-dimensional $k$-Euler complex $W$ and an $n$-dimensional $k$-Euler complex $U$ by $W = (K \times I - K \times \{1\}) \cup (L \times \{1\})$ and $U = (\partial K \times I - \partial K \times \{1\}) \cup (\partial L \times \{1\})$, where $I = \{\{0\}, \{1\}, [0, 1]\}$. We can regard $K$ and $L$ as subcomplexes of $W$ by the identifications $K = K \times \{0\}$ and $L = L \times \{1\}$. Put $\bar{U}^{(i)} = (\bar{U}^i - \partial \bar{U}) \cup \overline{\partial \bar{U}^{i-1}}$. Then $\bar{U}^{(i)}$ is an $i$-dimensional $(i - n + k)$-Euler complex in view of Proposition 2.1. Note that $\bar{K}^i$ and $\bar{L}^i$ are $i$-dimensional $(i - n + k)$-Euler complexes and that $\bar{W}^{i+1}$ is an $(i + 1)$-dimensional $(i - n + k)$-Euler complex such that $\partial \bar{W}^{i+1} = \bar{K}^i \cup \bar{U}^{(i)} \cup \bar{L}^i$ and $\partial \bar{U}^{(i)} = \partial \bar{K}^i \cup \partial \bar{L}^i$ by Proposition 2.1. Hence $(j_K)_*[|\bar{K}^i|] = (j_L)_*[|\bar{L}^i|]$. q.e.d.

The product formula for Stiefel-Whitney homology classes (Halperin and Toledo [7]) may not hold for $k$-Euler spaces, but we need the following to prove Lemma 5.1.

PROPOSITION 2.3. *Let $X$ be an $n$-dimensional $k$-Euler space. Then* $s_i(X) \times [D] = s_{i+1}(X \times D)$ *for* $n - k < i \leq n$, *where* $D = [-1, 1]$.

PROOF. Let $L$ and $\bar{L}$ be ball complexes defined by $L = \{\{-1\}, \{1\}, [-1, 1]\}$ and $\bar{L} = \{\langle -1 \rangle, \langle 1 \rangle, \langle 0 \rangle, \langle -1, 0 \rangle, \langle 1, 0 \rangle\}$. Here $\langle \pm 1 \rangle = \langle \{\pm 1\} \rangle$, $\langle 0 \rangle = \langle [-1, 1] \rangle$ and $\langle \pm 1, 0 \rangle = \langle \{\pm 1\}, [-1, 1] \rangle$. Then $|L| = D = [-1, 1]$

and $\bar{L}$ is the barycentric subdivision of $L$. Let $K$ be a ball complex such that $X = |K|$. Let $\bar{D}$, $c_i$, $\tilde{c}_{i+1}$ and $d_{i+2}$ be chains with $Z_2$-coefficients defined as follows: $\bar{D} = \sum_{\varepsilon=\pm 1} \langle \varepsilon, 0 \rangle$, $c_i = \sum \langle \sigma_0, \cdots, \sigma_i \rangle$, $\tilde{c}_{i+1} = \sum \langle (\sigma_0, \varepsilon), \cdots, (\sigma_p, \varepsilon), (\sigma_p, 0), \cdots, (\sigma_i, 0) \rangle + \sum \langle (\tau_0, \varepsilon), \cdots, (\tau_p, \varepsilon), (\tau_{p+1}, 0), \cdots, (\tau_{i+1}, 0) \rangle$ and $d_{i+2} = \sum [p] \langle (\tau_0, \varepsilon), \cdots, (\tau_p, \varepsilon), (\tau_p, 0), \cdots, (\tau_{i+1}, 0) \rangle$, where $\langle \sigma_0, \cdots, \sigma_i \rangle$ ranges over all $i$-balls of $\bar{K}^i$ while $\langle \tau_0, \cdots, \tau_{i+1} \rangle$ ranges over all $(i+1)$-balls of $\bar{K}^{i+1}$, $0 \leqq p \leqq i+1$ and $\varepsilon = \pm 1$. Here $[p]$ is the class of $p$ modulo 2. Then $\partial d_{i+2} - (\tilde{c}_{i+1} - c_i \times \bar{D}) = \sum [i] \langle (\tau_0, \varepsilon), \cdots, (\tau_{i+1}, \varepsilon) \rangle$. Since $\sum_{\varepsilon=\pm 1} \langle (\tau_0, \varepsilon), \cdots, (\tau_{i+1}, \varepsilon) \rangle$ is exact for each $\langle \tau_0, \cdots, \tau_{i+1} \rangle$, it follows that $\tilde{c}_{i+1} - c_i \times \bar{D}$ is exact. Note that $s_1(D)$, $s_i(X)$ and $s_{i+1}(X \times D)$ coincide with the homology classes defined by chains $\bar{D}$, $c_i$ and $\tilde{c}_{i+1}$, respectively, for $n-k < i \leqq n$. Thus $s_{i+1}(X \times D) = s_i(X) \times [D]$ for $n-k < i \leqq n$. q.e.d.

**3. Bordism groups of $k$-Euler spaces.** Let $\{\mathfrak{B}_n^k, \partial\}$ be the bordism theory of compact $k$-Euler spaces for $k > 0$. Then $\{\mathfrak{B}_n^k, \partial\}$ is a homology theory (See Akin [1].). If $k = \infty$, then $\{\mathfrak{B}_n^k, \partial\}$ is the bordism theory of compact $Z_2$-Euler spaces. (See Akin [1] and Matsui [10].) Let $(A, B)$ be a pair of polyhedra. Define a homomorphism $s_{(k)}: \mathfrak{B}_n^k(A, B) \to H_{n-k+1}(A, B; Z_2) + \cdots + H_n(A, B; Z_2)$ by $s_{(k)}(\varphi, X) = \sum_{i=n-k+1}^{n} \varphi_* s_i(X)$. Then $s_{(k)}$ is well defined by Proposition 2.1. Define a homomorphism $j_{(p,q)}: \mathfrak{B}_n^p(A, B) \to \mathfrak{B}_n^q(A, B)$ by $j_{(p,q)}(\varphi, X) = (\varphi, X)$ for $p \geqq q$. Then the following holds:

PROPOSITION 3.1. *The homomorphisms* $s_{(k)}: \mathfrak{B}_n^k(A, B) \to H_{n-k+1}(A, B; Z_2) + \cdots + H_n(A, B; Z_2)$ *are isomorphisms for* $0 < k \leqq n$. *The homomorphisms* $j_{(p,q)}: \mathfrak{B}_q^p(A, B) \to \mathfrak{B}_p^q(A, B)$ *are surjective for* $p \geqq q$.

PROOF. Put $h_n^{(k)}(A, B) = H_{n-k+1}(A, B; Z_2) + \cdots + H_n(A, B; Z_2)$ for $k > 0$. Define the boundary operator $\partial_n^{(k)}: h_n^{(k)}(A, B) \to h_{n-1}^{(k)}(B)$ as that of the ordinary homology theory. Then $\{h_n^{(k)}, \partial_n^{(k)}\}$ is a homology theory with compact support for $k > 0$. Note that $\{\mathfrak{B}_n^k, \partial\}$ is also a homology theory with compact support and that $s_{(k)}$ is a homomorphism from $\mathfrak{B}_n^k(A, B)$ to $h_n^{(k)}(A, B)$ such that $\partial_n^{(k)} \circ s_{(k)} = s_{(k)} \circ \partial$. Since $h_n^{(k)}(pt) = Z_2$ and $\mathfrak{B}_n^k(pt) = \mathfrak{B}_n(pt) = Z_2$ (cf. [10]) for $n = 0, \cdots, k-1$, and $h_n^{(k)}(pt) = 0$ and $\mathfrak{B}_n^k(pt) = 0$ for $n \geqq k$, where $pt$ is the space of one point, the homomorphism $s_{(k)}$ is an isomorphism. (See Spanier [14].)

Let $\pi: h_n^{(p)}(A, B) \to h_n^{(q)}(A, B)$ be the canonical projection. Note that $s_{(q)} \circ j_{(p,q)} = \pi \circ s_{(p)}$. Since $\pi$ is surjective, so is $j_{(p,q)}$. q.e.d.

Let $\xi = (E(\xi), A, \iota)$ be a $p$-block bundle over a polyhedron $A$. Define $\bar{E}(\xi)$ as the total space of the sphere bundle associated with $\xi$. Then we will define a homomorphism $(e_\xi^k)^i: \mathfrak{B}_{p+i}^k(E(\xi), \bar{E}(\xi)) \to Z_2$ for $i < k$, where $\mathfrak{B}_{p+i}^k(E(\xi), \bar{E}(\xi))$ is the bordism group of compact $k$-Euler spaces. Let $R$ be a regular neighborhood of $A$ in $R^\alpha$. Let $j: A \subset R$ be the inclusion and

$p: R \to A$ be a deformation retraction. Suppose that $p^*\xi = (E(p^*\xi), R, \iota_R)$ is the induced bundle. Then there exist bundle maps $(\bar{j}, j): (E(\xi), A) \to (E(p^*\xi), R)$ and $(\bar{p}, p): (E(p^*\xi), R) \to (E(\xi), A)$. (See Rourke and Sanderson [12].) For each $(\varphi, X)$ in $\mathfrak{B}^k_{p+i}(E(\xi), \bar{E}(\xi))$, there exists an embedding $\tilde{\varphi}: (X, \partial X) \to (E(p^*\xi), \bar{E}(p^*\xi))$ such that $\tilde{\varphi} \cong j \circ \varphi$. By the transversality theorem (see [12]), we may assume that $\tilde{\varphi}(X)$ is block transverse to $\iota_R: R \to E(p^*\xi)$. Let $Y = \tilde{\varphi}^{-1} \circ \iota_R(R)$. Note that the inclusion $Y \subset X$ has a normal block bundle, the total space of which is an $n$-dimensional $k$-Euler space. Then $Y$ is a closed $i$-dimensional $k$-Euler space. Hence $Y$ is a closed $i$-dimensional $\boldsymbol{Z}_2$-Euler space whenever $i < k$. Define $(e^k_\xi)^i(\varphi, X)$ by the modulo 2 Euler number $e(Y)$ of $Y$.

To prove Lemma 5.2, we need the following:

LEMMA 3.1. *Let $\nu = (E, M, \iota)$ be a normal $p$-block bundle of a proper embedding from a compact $q$-dimensional triangulated differentiable manifold $M$ to $D^{p+q} = [-1, 1]^{p+q}$. Let $U_\nu$ be the Thom class of $\nu$. Then $\langle U_\nu \cup (\iota^*)^{-1} w^*(M), \varphi_* s_{(k)}(X) \rangle = (e^k_\nu)^i(\varphi, X)$ for every $(\varphi, X)$ in $\mathfrak{B}^k_{p+i}(E, \bar{E})$ for $i < k$. Here $s_{(k)}(X) = s_{p+i-k+1}(X) + \cdots + s_{p+i}(X)$.*

PROOF. The case $k = \infty$ was proved in [10]. By Proposition 3.1, we may assume that $X$ is a $\boldsymbol{Z}_2$-Euler space. Note that $(e^\infty_\nu)^i(\varphi, X) = (e^k_\nu)^i(\varphi, X)$ for $(\varphi, X)$ in $\mathfrak{B}^\infty_{p+i}(E, \bar{E})$ for $i < k$. Then $\langle U_\nu \cup (\iota^*)^{-1} w^*(M), \varphi_* s_{(k)}(X) \rangle = (e^k_\nu)^i(\varphi, X)$ for $i < k$, in view of the case $k = \infty$. q.e.d.

## 4. A characterization of Stiefel-Whitney classes.

The product formula for Stiefel-Whitney classes (see Milnor [11]) may not hold for $k$-Poincaré-Euler spaces, but we need the following to deduce Lemma 4.1 from Lemma 4.2:

PROPOSITION 4.1. *Let $X$ be an $n$-dimensional $k$-Poincaré-Euler space. Then $w^i(X \times D) = w^i(X) \times 1$ for $0 \leqq i < k$, where $D = [-1, 1]$.*

PROOF. Let $(R; \tilde{R}, \bar{R}; \varphi)$ be a regular neighborhood of $X$ in $\boldsymbol{R}^{n+\alpha}_+$. Let $U(\varphi)$ and $U(\varphi \times \mathrm{id})$ be cohomology classes such that $[R] \cap U(\varphi) = \varphi_*[X]$ and $[R \times D] \cap U(\varphi \times \mathrm{id}) = (\varphi \times \mathrm{id})_*[X \times D]$, where id: $D \to D$ is the identity. Then $U(\varphi \times \mathrm{id}) = U(\varphi) \times 1$. Note that $U(\varphi) \cup (\varphi^*)^{-1} w^i(\varphi) = Sq^i U(\varphi)$ and $U(\varphi \times \mathrm{id}) \cup [(\varphi \times \mathrm{id})^*]^{-1} w^i(\varphi \times \mathrm{id}) = Sq^i U(\varphi \times \mathrm{id})$ for $0 \leqq i < k$. Then $U(\varphi \times \mathrm{id}) \cup [(\varphi \times \mathrm{id})^*]^{-1}(w^i(\varphi) \times 1) = Sq^i U(\varphi \times \mathrm{id})$ for $0 \leqq i < k$. Hence $w^i(\varphi \times \mathrm{id}) = w^i(\varphi) \times 1$ for $0 \leqq i < k$. Thus $w^i(X \times D) = w^i(X) \times 1$ for $0 \leqq i < k$. q.e.d.

Let $(R; \tilde{R}, \bar{R}; \varphi)$ be a regular neighborhood of an $n$-dimensional $k$-Poincaré-Euler space $X$ in $\boldsymbol{R}^{n+\alpha}_+$. Suppose that $(\tilde{e}^k_\varphi)^i: \mathfrak{N}_{i+\alpha}(R, \bar{R}) \to \boldsymbol{Z}_2$ is the homomorphism defined for $i < k$ in Section 1. We need the following to prove our theorem:

LEMMA 4.1. *For every $(f, M)$ in $\mathfrak{N}_{i+\alpha}(R, \bar{R})$, we have* $\langle U(\varphi) \cup (\varphi^*)^{-1} w^{(k)}(X), f_*([M] \cap w^*(M)) \rangle = (\tilde{e}^k_\varphi)^i(f, M)$ *whenever* $i < k$. *Here* $w^{(k)}(X) = 1 + \cdots + w^{k-1}(X)$.

In order to prove Lemma 4.1, we need the following:

LEMMA 4.2. *Let* $f: (M, \partial M) \to (R, \bar{R})$ *be a PL-embedding with a normal block bundle* $\xi$, *where* $M$ *is an* $(i + \alpha)$*-dimensional triangulated differentiable manifold. If* $\varphi(X)$ *is transverse to* $\xi$ *and* $i < k$, *then* $\langle U(\varphi) \cup (\varphi^*)^{-1} w^{(k)}(X), f_*([M] \cap w^*(M)) \rangle = (\tilde{e}^k_\varphi)^i(f, M)$.

In order to prove Lemmas 4.2 and 5.2, we need the following:

LEMMA 4.3. *Let* $(R; \tilde{R}, \bar{R}; \varphi)$ *be a regular neighborhood of an* $n$*-dimensional* $k$*-Poincaré-Euler space* $X$ *in* $\boldsymbol{R}^{n+\alpha}_+$. *Let* $M$ *be an* $(i + \alpha)$*-dimensional triangulated differentiable manifold, where* $0 \leqq i < k$. *Given a PL-embedding* $f: (M, \partial M) \to (R, \bar{R})$ *with a normal block bundle* $\xi = (E, M, f_E)$, *suppose that* $\varphi(X)$ *is transverse to* $\xi$. *Let* $U_\xi$ *be the Thom class of* $\xi$ *and* $j_E: E \to R$ *be the inclusion. Define* $Y = \varphi^{-1} \circ f(M)$ *and* $X_E = \varphi^{-1} \circ j_E(E)$. *Let* $\varphi_E: X_E \to E$ *and* $\psi_M: Y \to M$ *be embeddings defined by* $\varphi_E = j_E^{-1} \circ \varphi$ *and* $\psi_M = f^{-1} \circ (\varphi | Y)$. *Then the following hold*:

(1) *$Y$ is a closed* $\boldsymbol{Z}_2$*-Euler space with a normal block bundle.*

(2) $(f_E)_*([M] \cap f^* U(\varphi)) = (\varphi_E)_*[X_E] \cap U_\xi$.

(3) $[M] \cap f^* U(\varphi) = (\psi_M)_*[Y]$.

PROOF. (1) Clearly $\psi_M^* \xi$ is a normal $(n - i)$-block bundle of $Y$ in $X$. Note that $E$ is an $n$-dimensional $k$-Euler space. Then $Y$ is an $i$-dimensional $k$-Euler space. Hence $Y$ is a $\boldsymbol{Z}_2$-Euler space, since $i < k$. Since $M$ is compact, $Y$ is closed.

(2) Note that $j_E \circ f_E = f$ and $[E] \cap U_\xi = (f_E)_*[M]$. Thus $(f_E)_*([M] \cap f^* U(\varphi)) = ([E] \cap j_E^* U(\varphi)) \cap U_\xi$. If $[E] \cap j_E^* U(\varphi) = (\varphi_E)_*[X_E]$, then $(f_E)_*([M] \cap f^* U(\varphi)) = (\varphi_E)_*[X_E] \cap U_\xi$. Hence we have only to prove $[E] \cap j_E^* U(\varphi) = (\varphi_E)_*[X_E]$. Let $\tilde{\tilde{R}} = \mathrm{cl}(R - j_E(E))$ and let $j_R: (R; \tilde{R}, \bar{R}) \to (R; \tilde{\tilde{R}}, \bar{R})$ be defined by the inclusion. Regard $j_E$ as a map $j_E: (E; \tilde{E}, \bar{E}) \to (R; \bar{R}, \tilde{\tilde{R}})$, where $\tilde{E} = \mathrm{cl}(\partial E - \bar{E})$. Note that $(j_E)_*[E] = (j_R)_*[R]$ and $[R] \cap U(\varphi) = \varphi_*[X]$. Then $(j_E)_*([E] \cap (j_E)^* U(\varphi)) = (j_R)_* \circ \varphi_*[X] = (j_E)_* \circ (\varphi_E)_*[X_E]$. Since $(j_E)_*: H_*(E, \bar{E}; \boldsymbol{Z}_2) \to H_*(R, \tilde{\tilde{R}}; \boldsymbol{Z}_2)$ is an isomorphism, we have $[E] \cap (j_E)^* U(\varphi) = (\varphi_E)_*[X_E]$.

(3) Note that $[X_E] \cap (\varphi_E)_* U_\xi = (\psi_E)_*[Y]$, where $\psi_E: Y \to X_E$ is the inclusion. By (2), we have $(f_E)_*([M] \cap f^* U(\varphi)) = (\varphi_E)_* \circ (\psi_E)_*[Y]$. Note that $\varphi_E \circ \psi_E = f_E \circ \psi_M$ and that $(f_E)_*: H_*(M, \partial M; \boldsymbol{Z}_2) \to H_*(E, \tilde{E}; \boldsymbol{Z}_2)$ is an isomorphism. Then $[M] \cap f^* U(\varphi) = (\psi_M)_*[Y]$. q.e.d.

PROOF OF LEMMA 4.2. We use the notation of Lemma 4.3. By (2) of Lemma 4.3, we have $\langle U(\varphi)\cup(\varphi^*)^{-1}w^{(k)}(X), f_*([M]\cap w^*(M))\rangle = \langle f^*\circ(\varphi^*)^{-1}w^{(k)}(X)\cup w^*(M), (\psi_M)_*[Y]\rangle$. Let $\psi_X\colon Y\to X$ be the inclusion. Note that $f\circ\psi_M = \varphi\circ\psi_X$. Then $\langle U(\varphi)\cup(\varphi^*)^{-1}w^{(k)}(X), f_*([M]\cap w^*(M))\rangle = \langle\psi_X^* w^{(k)}(X)\cup\psi^*(M), [Y]\rangle = \langle\psi_X^* w^{(k)}(X)\cup\psi_M^*\bar{w}(\xi), [Y]\rangle = \langle\psi_X^* w^{(k)}(X)\cup\bar{w}(\psi_M^*\xi), [Y]\rangle$. Thus $\langle U(\varphi)\cup(\varphi^*)^{-1}w^{(k)}(X), f_*([M]\cap w^*(M))\rangle = (\tilde{e}_\varphi^k)^i(f, M)$ by the definition of $(\tilde{e}_\varphi^k)^i$. q.e.d.

PROOF OF LEMMA 4.1. Let $(f, M)$ be in $\mathfrak{N}_{i+\alpha}(R, \bar{R})$. By Transversality Theorem, there exists an embedding $g\colon(M, \partial M)\to(R\times D^\beta, \bar{R}\times D^\beta)$ such that $g\cong f\times\{0\}$ and $(\varphi\times\mathrm{id})(X\times D^\beta)$ is block transverse to $g$. By Lemma 4.2, it follows that $\langle(U(\varphi)\times 1)\cap[(\varphi\times\mathrm{id})^*]^{-1}w^{(k)}(X\times D^\beta), g_*([M]\cap w^*(M))\rangle = (\tilde{e}_\varphi^k)^i(f, M)$. Note that $w^{(k)}(X\times D^\beta) = w^{(k)}(X)\times 1$ by Proposition 4.1. Hence $\langle U(\varphi)\cup(\varphi^*)^{-1}w^{(k)}(X), f_*([M]\cap w^*(M))\rangle = \langle(U(\varphi)\times 1)\cup[(\varphi\times\mathrm{id})^*]^{-1}w^{(k)}(X\times D^\beta), g_*([M]\cap w^*(M))\rangle$. Thus $\langle U(\varphi)\cup(\varphi^*)^{-1}w^{(k)}(X), f_*([M]\cap w^*(M))\rangle = (\tilde{e}_\varphi^k)^i(f, M)$. q.e.d.

A characterization of Stiefel-Whitney classes is given by Lemma 4.1 and the following:

LEMMA 4.4. *Let $(A, B)$ be a pair of polyhedra. Let $\Phi^i$ be in $H^i(A, B; \boldsymbol{Z}_2)$ for $i = 0, 1, \cdots, k-1$. Put $\Phi^{(k)} = \Phi^0 + \cdots + \Phi^{k-1}$. If $\langle\Phi^{(k)}, f_*([M]\cap w^*(M))\rangle = 0$ for every $(f, M)$ in $\mathfrak{N}_*(A, B)$, then $\Phi^{(k)} = 0$.*

PROOF. Since $\langle\Phi^{(k)}, f_*([M]\cap w^*(M))\rangle = \langle\Phi^0, f_*[M]\rangle$ for $(f, M)\in\mathfrak{N}_0(A, B)$, the assumption $\langle\Phi^{(k)}, f_*([M]\cap w^*(M))\rangle = 0$ for every $(f, M)$ implies $\Phi^0 = 0$. Suppose that $\Phi^0 = 0$, $\Phi^1 = 0, \cdots, \Phi^j = 0$. Then $\langle\Phi^{(k)}, f_*([M]\cap w^*(M))\rangle = \langle\Phi^{j+1}, f_*[M]\rangle$ for $(f, M)\in\mathfrak{N}_{j+1}(A, B)$. Hence, if $\langle\Phi^{(k)}, f_*([M]\cap w^*(M))\rangle = 0$ for every $(f, M)$, it follows that $\Phi^{j+1} = 0$. By induction on $j$, we have $\Phi^{(k)} = 0$. q.e.d.

**5. Characterizations of Stiefel-Whitney homology classes.** Let $(R; \tilde{R}, \bar{R}; \varphi)$ be a regular neighborhood of an $n$-dimensional $k$-Poincaré-Euler space $X$ in $\boldsymbol{R}_+^{n+\alpha}$. Suppose that $(e_\varphi^k)^i\colon\mathfrak{N}_{i+\alpha}(R, \bar{R})\to\boldsymbol{Z}_2$ is the homomorphism defined for $i<k$ in Section 1. We need the following to prove our theorem:

LEMMA 5.1. *For every $(f, M)$ in $\mathfrak{N}_{i+\alpha}(R, \bar{R})$, we have $\langle U(\varphi)\cup(\varphi^*)^{-1}\circ([X]_\cap)^{-1}s_{(k)}(X), f_*([M]\cap w^*(M))\rangle = (e_\varphi^k)^i(f, M)$, whenever $i<k$. Here $s_{(k)}(X) = s_{n-k+1}(X) + \cdots + s_n(X)$.*

In order to prove this, we need the following:

LEMMA 5.2. *Let $f\colon(M, \partial M)\to(R, \bar{R})$ be a PL-embedding with a normal block bundle $\xi$, where $M$ is an $(i+\alpha)$-dimensional triangulated differentiable manifold. If $\varphi(X)$ is transverse to $\xi$ and $i<k$, then*

$\langle U(\varphi)\cup(\varphi^*)^{-1}\circ([X]_\cap)^{-1}s_{(k)}(X), f_*([M]\cap w^*(M))\rangle = (e^k_\varphi)^i(f, M)$.

PROOF. We use the notation of Lemma 4.3. By (2) of Lemma 4.3, we have $\langle U(\varphi)\cup(\varphi^*)^{-1}\circ([X]_\cap)^{-1}s_{(k)}(X), f_*([M]\cap w^*(M))\rangle = \langle w^*(M)\cup f^*\circ(\varphi^*)^{-1}\circ([X]_\cap)^{-1}s_{(k)}(X), (f_E)_*^{-1}((\varphi_E)_*[X_E]\cap U_\xi)\rangle$. Note that $j_E\circ f_E = f$. Then $\langle U(\varphi)\cup(\varphi^*)^{-1}\circ([X]_\cap)^{-1}s_{(k)}(X), f_*([M]\cap w^*(M))\rangle = \langle U_\xi\cup(f_E^*)^{-1}w^*(M), ((\varphi_E)_*[X_E])\cap j_E^*\circ(\varphi^*)^{-1}\circ([X]_\cap)^{-1}s_{(k)}(X)\rangle$. Since there exists the following commutative diagram

$$\begin{array}{ccccc} H^{n-i}(X; \boldsymbol{Z}_2) & \xleftarrow{\varphi^*} & H^{n-i}(R; \boldsymbol{Z}_2) & \xrightarrow{j_E^*} & H^{n-i}(E; \boldsymbol{Z}_2) \\ \downarrow [X]_\cup & & & & \downarrow ((\varphi_E)_*[X_E])_\cap \\ H_i^{\inf}(X, \partial X; \boldsymbol{Z}_2) & \xrightarrow{\bar{\varphi}^*} & H_i^{\inf}(R, \mathrm{cl}(R-E); \boldsymbol{Z}_2) & \xrightarrow{(j_E)_*} & H_i^{\inf}(E, \bar{E}; \boldsymbol{Z}_2) \end{array}$$

and since $[X]_\cap$, $\varphi^*$ and $(j_E)_*$ are isomomorphisms for $i<k$, we have $((\varphi_E)_*[X_E])\cap j_E^*\circ(\varphi^*)^{-1}\circ([X]_\cap)^{-1}s_{(k)}(X) = [(j_E)_*]^{-1}\circ\bar{\varphi}_*s_{(k)}(X) = (\varphi_E)_*s_{(k)}(X_E)$. Let $(e^k_\xi)^i\colon \mathfrak{B}_n(E, \bar{E})\to \boldsymbol{Z}_2$ be the homomorphism defined in Section 3. Then $\langle U_\xi\cup(f_E^*)^{-1}w^*(M), (\varphi_E)_*s_{(k)}(X_E)\rangle = (e^k_\xi)^i(\varphi_E, X_E)$ by Lemma 3.1. Note that $(e^k_\varphi)^i(f, M) = (e^k_\xi)^i(\varphi_E, X_E)$ by definition. Thus $\langle U(\varphi)\cup(\varphi^*)^{-1}\circ([X]_\cap)^{-1}s_{(k)}(X), f_*([M]\cap w^*(M))\rangle = (e^k_\varphi)(f, M)$. q.e.d.

PROOF OF LEMMA 5.1. Let $(f, M)$ be in $\mathfrak{N}_{i+\alpha}(R, \bar{R})$. Then there exists an embedding $g\colon (M, \partial M)\to(R\times D^\beta, \bar{R}\times D^\beta)$ such that $g\cong f\times\{0\}$ and $(\varphi\times\mathrm{id})(X\times D^\beta)$ is block transverse to $g$ by Transversality Theorem. By Lemma 5.2, we have $\langle(U(\varphi)\times 1)\cup[(\varphi\times\mathrm{id})^*]^{-1}\circ([X\times D^\beta]_\cap)^{-1}s_{(k)}(X\times D^\beta), g_*([M]\cap w^*(M))\rangle = (e^k_\varphi)^i(f, M)$ for $i<k$. Note that $s_{(k)}(X\times D^\beta) = s_{(k)}(X)\times[D^\beta]$ by Proposition 2.3. Then $\langle U(\varphi)\cup(\varphi^*)^{-1}\circ([X]_\cap)^{-1}s_{(k)}(X), f_*([M]\cap w^*(M))\rangle = \langle(U(\varphi)\times 1)\cup[(\varphi\times\mathrm{id})^*]^{-1}\circ([X\times D^\beta]_\cap)^{-1}s_{(k)}(X\times D^\beta), g_*([M]\cap w^*(M))\rangle$. Thus $\langle U(\varphi)\cup(\varphi^*)^{-1}\circ([X]_\cap)^{-1}s_{(k)}(X), f_*([M]\cap w^*(M))\rangle = (e^k_\varphi)^i(f, M)$ for $i<k$.
q.e.d.

PROOF OF THEOREM. If $[X]\cap w^i(X) = s_{n-i}(X)$ for $i\leqq m$, then $(e^k_\varphi)^i(f, M) = (\tilde{e}^k_\varphi)^i(f, M)$ for $i\leqq m$ by Lemmas 4.1 and 5.1. This means $(o^k_\varphi)^i = 0$ for $i\leqq m$. Conversely, suppose that $(o^k_\varphi)^i = 0$ for $i\leqq m$. By Lemmas 4.1, 4.4 and 5.1, we have $U(\varphi)\cup(\varphi^*)^{-1}w^i(X) = U(\varphi)\cup(\varphi^*)^{-1}\circ([X]_\cap)^{-1}s_{n-i}(X)$ for $i\leqq m$. Since $U(\varphi)\cup(\varphi^*)^{-1}$ and $[X]_\cap$ are isomorphisms for m $<k$, we have $[X]\cap w^i(X) = s_{n-i}(X)$ for i $\leqq m$. q.e.d.

PROOF OF COROLLARY. Note that $k$-regular spaces over $\boldsymbol{Z}_2$ are $k$-Euler spaces by the consideration of the definitions. Then $k$-regular spaces over $\boldsymbol{Z}_2$ are $k$-Poincaré-Euler spaces by Partial Poincaré Duality Theorem. Let $\psi\colon Y\to X\times D^\beta$ be the embedding used to define $(e^k_\varphi)^i$ and $(\tilde{e}^k_\varphi)^i$. Note that $\psi$ has a normal block bundle $\nu$ in $X\times D^\beta$. Then $Y$ is an $i$-dimensional $k$-regular space. Since $Y$ is compact and $i<k$, it follows that $Y$ is a

closed $Z_2$-homology manifold. Hence $\psi^* w^{(k)}(X\times D^\beta) = w^*(Y)\cup w^*(\nu)$. Thus $(o_\varphi^k)^i = 0$ in view of the definition of $(e_\varphi^k)^i$ and $(\tilde{e}_\varphi^k)^i$. Hence $[X]\cap w^i(X) = s_{n-i}(X)$ for $i < k$ by Theorem. q.e.d.

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