THEORY OF RUNAWAY ELECTRONS: STRUCTURE OF THE HIGH ENERGY DISTRIBUTION FUNCTION

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THEORY OF RUNAWAY ELECTRONS: STRUCTURE OF THE HIGH ENERGY DISTRIBUTION FUNCTION

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ABSTRACT

The structure of the high-energy electron tail in a current-carrying, magnetized plasma column is determined self-consistently with the plasma wave turbulence it generates. The theory applies to cases when runaway confinement is good, radial excursions of the magnetic field lines being small. The unstable spectra consist of absolutely unstable $\omega = \omega_{pe}$ waves and convectively unstable $\omega = \omega_{pe} k_{\parallel}/k << \omega_{pe}$ waves. Enhanced dynamic friction resulting from the $\omega_{pe} k_{\parallel}/k$ modes increases with parallel momentum, and cuts off the distribution function at high energies. The convective nature of the modes gives a radial structure to the cutoff, with the highest energies concentrated in the center. Below the cutoff, the distribution function has a small positive slope. Equilibrium is maintained by the ω_{pe} waves which produce the back diffusion flux necessary to offset the electric field acceleration. Five separate asymptotic regions for the tail distribution function are identified and the calculation is carried out to give an explicit solution. Once obtained, the solution is expressed in Lagrangian form to determine the flow paths of particles in momentum space. This clarifies the nature of the steady state.

I. Introduction

When a weak electric field is applied to a plasma, the electron distribution develops a drift, a slight distortion and at very high energies, a runaway electron tail. In the classical runaway theory,[1] the high energy tail extends to infinite momentum (or rather grows indefinitely with time) and if included would produce a divergence in the computed conductivity. The Spitzer-Harm [2] conductivity results by ignoring this part of the current. It works quite well when the runaway confinement is poor, as when large radial excursions of the magnetic field lines occur,[3] or orbit shifts [4] are large. However, there are many practical cases when the runaways are well confined and they can then contribute significantly to the conductivity, radiation, and energy loss processes of the plasma.

We consider an infinitely long, radially finite plasma column immersed in axial magnetic and electric fields. A steady state for the high energy electrons in this situation can be obtained in roughly two different ways. Runaway production can be balanced by some loss mechanism. Experiments are often interpreted with a highly empirical version of this steady state.[5] Alternatively, turbulence resulting from the high energy tail could enhance the dynamic friction on the electrons and prohibit the runaway process, even in the absence of radial loss.[6] The waves that interact with the runaways do not produce significant radial diffusion so that with well formed magnetic surfaces, it is unlikely that the radial loss of runaways determines the steady state. We will assume that the surfaces are well formed. In addition, we assume that the plasma waves convecting radially do not reflect from the edge and cause an absolute instability. These are the principal assumptions of the analysis to be presented in this paper. From them, we develop a self-consistent solution to the kinetic equations for the tail electrons and the waves.

We find that the friction and diffusion forces produced by the turbulent waves spectrum permit the particle distribution function to attain a steady state. Electron runaway to infinite momentum occurs only on the column axis while at other radii, the distribution is cut off by the friction from th unstable waves. This, in fact, makes the designation "runaway" somewhat inappropriate. The present paper is devoted to a detailed derivation of the structure of the high energy distribution function and turbulent wave spectra. A subsequent paper will give some of the consequences, such as: the plasma column conductance and total number of tail electrons, the energy transported across the magnetic field by the plasma waves, and the radiation at ω_{pe}.

The phenomenon of electron runaway was first pointed out by Giovanelli,[7] who observed that since the dynamic friction due to Coulomb collisions decreased at high velocity like v^{-2}, for any electric field there would always be some velocity beyond which collisions could not restrain electrons from accelerating indefinitely. Denoting the friction force, $F = m\bar{v}_e \nu (\bar{v}_e/v)^2$, with $\bar{v}_e = \sqrt{T_e/m}$, and $\nu = 4\pi n e^4 \ln\Lambda / m^2 \bar{v}_e{}^3$, this critical "runaway" velocity is $v_r = v_e\sqrt{E_r/E}$, where $E_r = m\nu\bar{v}_e/e$ is the electric field at which thermal particles runaway.

An actual calculation of the runaway rate requires a determination of the electron distribution function. This started with Spitzer and Harm [2]. They analyzed the Fokker-Planck equation for electrons in a homogeneous, unmagnetized plasma,[8]

$$\frac{\partial f}{\partial t} - \frac{|e|}{m}\bar{E}\,\frac{\partial f}{\partial \bar{v}} = \frac{\partial}{\partial \bar{v}} \cdot \Big\{ \frac{\nu(v)}{2} \Big[v^2 I - \bar{v}\,\bar{v} \Big] \cdot \frac{\partial f}{\partial \bar{v}}$$

$$+ \frac{v^3}{N} \int d^3v' \left(f(\bar{v}')\frac{\partial f}{\partial \bar{v}} - f\,\frac{\partial f}{\partial \bar{v}'} \right) \frac{|\bar{v} - \bar{v}'|^2\, I - (\bar{v} - \bar{v}')(\bar{v} - \bar{v}')}{|\bar{v} - \bar{v}'|^3} \Big] \Big\} \quad (1)$$

where I is the identity tensor and the ions have a Maxwellian distribution. This equation was analyzed in the steady state,[2] neglecting the slow joule heating of the electrons. Their procedure was to expand the distribution function in a power series in the electric field and then to solve the resulting equations order by order using spherical harmonics. This led to the classical resistivity (parallel), $\eta_{SH} = 1.8 \times 10^{-18}\, T_e^{-3/2} \ln\Lambda$ sec. This solution is valid for velocities $v/v_e < (E_r/E)^{1/4}$, and so to be meaningful E/E_r must be small (the limit $E/E_r \gg 1$ was studied by Kovryznik[9]). For velocities above this, their representation of the solution is inappropriate and a different expansion procedure has to be used.[10-15] The primary concern was to determine the flux of electrons into the runaway region, the socalled Runaway Rate. Upon expanding equation (1) for $v \gg v_e$ in the steady state, there results the following linear equation[1]

$$Eu^3\left(\mu f_u + \frac{1-\mu^2}{u} f_\mu\right) = (1 - 1/2u^2)((1-\mu^2)f_\mu)_\mu + (u - 1/u)f_u + f_{uu} \qquad (2)$$

where E is normalized to the runaway field, u is normalized to the thermal velocity and μ is the cosine of the angle between the electric field and the velocity of the particle, subscripts denote derivatives. This is the basic equation of the classical runaway problem.

The first attempt to calculate the flux was made by Dreicer.[10] For $v < v_r$, he assumed that the distribution function was determined predominantly by collisions and hence was isotropic. Equation (2) was expanded in spherical harmonics as in the Spitzer-Harm problem. The rate at which the particles scattered across $v = v_r$ was used to determine the runaway rate numerically.

Gurevich[11] realized that this picture of velocity space was too simplified, that in fact as one approached $\vec{v} \sim \vec{v}_r$ the distribution function was no longer isotropic but would be localized around the electric field direction. He expanded the distribution function near $\vec{v} \sim \vec{v}_r$ and $u >> 1$, using the form $f = \exp\{\phi_0(u) + \phi_1(u)(1 - \mu) + \phi_2(u)(1 - \mu)^2 + \ldots\}$. This was substituted into eq (2) which was then solved order by order in the electric field. However, the match to the distribution function near $\vec{v} \sim \vec{v}_e$ was not performed correctly and an assumption that $\phi_1 = 0$ to the lowest order led to a singularity in the distribution function when $\vec{v} \to \vec{v}_r$. In spite of this, the exponential dependence was correctly determined, only the premultiplicative term was wrong.

Lebedev[12] used a similar approach to Gurevich. He found, however, that there was an internal boundary layer (since the coefficient of $\partial f/\partial u$ vanished at $\mu = 1$, $\vec{v} = \vec{v}_r$) at $\vec{v} = \vec{v}_r$ and also did not set $\phi_1 = 0$ to leading order. However, he did set $\phi_2 = 0$ to leading order. This led to an error in matching to the bulk electron distribution function but did not produce a singularity in the distribution function for $\vec{v} >> \vec{v}_r$. Thus he was able to compute the runaway flux with reasonable accuracy and obtained

$$S_L = 0.36 n\nu(\vec{v}_e)(E_r/E)^{1/4} \exp[-E_r/4E - \sqrt{2E_r/E}] \tag{3}$$

The most rigorous solution to eq (2) was performed by Kruskal and Bernstein.[1] They made no ad hoc assumptions about the distribution function, but found it necessary to introduce five distinct regions for the distribution function. The solutions were matched asymptotically at the transition between the various regions. Their expression for the flux is given by

$$S_{KB} = cn\nu(v_e)(E_r/E)^{3/8}\exp[-E_r/4E - \sqrt{2E_r/E}] \quad (4)$$

The constant (c) is of order one, but not known precisely because the differential equations in two of the regions were unsolved. The details of the Kruskal-Bernstein solution are summarized in a recent paper by Cohen[13] who included impurity ions in the Fokker-Planck equation.

A numerical analysis of the Fokker-Planck eq (1) was performed by Kulsrud et al.[14] They found good agreement with the results of Kruskal-Bernstein if c = 0.35 in eq (4). Comparing the runaway flux with the experimental observations of Van Goeler et al[15] they found that theory predicted runaway rates which were generally larger than the experimental values.

Finally, Connor and Hastie[16] included relativistic effects and impurity ions in the Fokker-Planck equation. They used an asymptotic matching procedure identical to that of Kruskal-Bernstein. The main result introduced by the inclusion of relativistic effects was that, if the electric field was sufficiently small that $v_r = c$ (c = speed of light) then there would be no runaways produced because for relativistic velocities the dynamic friction no longer decreases with momentum. For this effect to come into play one requires $E/E_r \ll T/mc^2$.

For future use, we compute here some of the parameters from the classical collisional solution. Since the tail in general extends to large velocities, a fully relativistic treatment will be used. To proceed, we first define the following perpendicular moments

$$f_{||}(p_{||}) = \int 2\pi p_{\perp} dp_{\perp} \; f(p_{||}, p_{\perp}) \tag{5}$$

$$T_{\perp}(p_{||})f_{||}(p_{||}) = \int 2\pi p_{\perp} dp_{\perp} (p_{\perp}^2/2m) \; f(p_{||}, p_{\perp}) \tag{6}$$

where $(p_{||}, p_{\perp})$ are the parallel and perpendicular momentum respectively. The time rate of change of the density of tail electrons is obtained by integrating the time dependent Fokker-Planck equation over all $p_{\perp}$ (which annihilates the collision operator when $p_{\perp} << p_{||}$ is satisfied) and over $p_{||}$ from $(-\infty, \infty)$. This leads to $\partial n_T/\partial t = eEf_{||}(\infty)$ since $f_{||}(-\infty) = 0$. Since $f_{||}(\infty)$ is obtained from the solution of the kinetic equation, the runaway rate follows. On the other hand, since $f_{||}(p_{||})$ is approximately flat beyond the runaway momentum, we can also determine n_T from this by noting that, $f_{||}(\infty) \simeq f_{||}(p_r) \equiv f_c = n_T/p_e$ where p_e is the thermal momentum. Then

$$n_T/n = 0.35(E_r/E)^{11/8} \exp[-E_r/4E - \sqrt{2E_r/E}] \tag{7}$$

where we used the results of Kruskal-Bernstein together with the constant determined by Kulsrud et al.[14]

Another parameter we shall require is the perpendicular temperature. Once $f_{||}$ is found, it can be obtained from eq (6). In the collisional problem, using the approximately correct formulas of Lebedev, we find that the perpendicular temperature at the runaway momentum is

$$T_{\perp r}/T_e = 2^{1/3}(E_r/E)^{2/3} \tag{8}$$

In order to produce a steady state, Pearson[17] and Bateman[18] included collective effects. They both added a dynamic friction term due to the Cherenhov emission of waves [19] into the classical eq. 2. Neither found substantial alterations of the runaway rate. This is the expected result since in a thermal equilibrium (Maxwellian) plasma the dynamic friction from the waves is smaller than that from collisions by the factor $\ln(v_{\parallel}/v_e)/\ln\Lambda$. An additional problem of this calculation for a stationary, infinite, homogeneous plasma is that the spectral eneregy density of the waves diverges as marginal stability is approached. This situation arises for $\dot{v} > \bar{v}_r$ where the distribution function, $f_{\parallel}$, is flat and Landau damping vanishes.

In the analysis presented in this paper, the parallel distribution function $f_{\parallel}(p_{\parallel})$ is shown (with an enlarged positive slope) in Fig 1, along with the bulk distribution function, for $p_{\parallel} < p_r$, to which it matches. The height of the tail in this notation in $f_c \equiv n_T/p_e$. We will use the results of classical theory for n_T. This does not mean that the analysis hinges on the validity of the classical theory. Rather, the tail distribution function will match to any bulk function which is flat at $p_{\parallel} \sim p_r$ and Gaussian in the perpendicular direction, properties which are fairly universal consequences of the kinetic equation in the vicinity of $p_{\parallel} \sim p_r$. Our results are written in terms of n_T/n, which in this analysis may take on any (small) value. Wave effects become important for $p_{\parallel} > p_r$ where the flattened tail permits instabilities to develop. The unstable plasma wave spectrum splits into two distinct parts, as shown in Fig 2, and detailed in Sec II[20]. For simplicity, we treat the strong magnetic field limit, $\Omega_e >> \omega_{pe}$, in which the plasma wave frequency is $\omega = \omega_{pe} k_{\parallel}/k$, $k_{\parallel}$ being the wave vector component along the magnetic field. That part of the spectrum with $k_{\perp} \simeq 0$ and $\omega \simeq \omega_{pe}$, we refer to as the "$\omega_{pe}$ modes". These waves are driven by a positive slope in $f_{\parallel}$. They have vanishing radial group velocity and when excited are absolutely unstable. In the steady state, their saturation level is determined by marginal stability. The second part of the spectrum, characterized by $k_{\perp} > k_{\parallel}$ and hence $\omega < \omega_{pe}$ is referred to as the "$\omega_{pe}\cos\theta$" modes. They are driven unstable by the anisotropy of the distribution function in the parallel direction through the wave-particle interaction at the first gyroresonance:[21,22] These modes have a large radial group velocity and are saturated by convection out of the unstable region.

The waves contribute additions to the diffusion tensor of the particle kinetic equation according to the well known quasi-linear operator.[23] When the kinetic equation (collisions plus waves) is integrated over $p_{\perp}$ one obtains an equation of the form

$$\frac{\partial}{\partial p_{\parallel}} (eE - F_{\parallel}) f_{\parallel} = \frac{\partial}{\partial p_{\parallel}} D_{\parallel} \frac{\partial}{\partial p_{\parallel}} f_{\parallel} \qquad (9)$$

where $D_{\parallel}$ contains contributions from both spectra, while only the $\omega_{pe}\cos\theta$ modes contribute to $F_{\parallel}$. This dynamical friction results from the pitch angle scattering in the quasilinear response at the first gyro-resonance which appears like a friction when projected on the parrallel axis. The origin and physical mechanism of the friction term is discussed in sec II and Appendix A. Its relation to the overall solution is clarified in section V, with the derivation of the flow pattern in momentum space which characterizes the steady state. The terminology for labeling the kinetic coefficients is, unfortunately, ambiguous, owing to the variety of forms in which the

Fokker-Planck equation can be written. This leads to confusion over what does and does not constitute a friction.

The effect of the enhanced plasma wave spectrum on the self-consistent particle distribution function in Fig 1, can be understood in the following way. First for comparison consider the bulk electrons with $p_{||} < p_r$. For these, the collisional dynamic friction exceeds the electric field acceleration, hence an individual (test) electron would tend to slow down. In order to have a steady state, this deceleration must be balanced by an outward velocity space diffusion flux as is produced by a negative slope in the distribution function. This picture remains qualitatively correct out to the runaway momentum, p_r. Beyond the runaway momentum, the electric field dominates the collisional dynamic friction and an individual electron tends to be accelerated. In the collisional theory there is nothing to balance this tendency and electron runaway occurs. There is no steady state. With the waves present, it is still true that $eE > \bar{F}_{||}$ for some distance beyond p_r. The only way to maintain an equilibrium is then to balance the electric field acceleration by a back diffusion flux. This is precisely where the ω_{pe} modes come into play, maintaining the tail with a small but finite <u>positive</u> slope. This positive slope persists up to a sufficiently large momentum, that the dynamic friction from the $\omega_{pe}\cos\theta$ modes exceeds eE and cuts off the distrubution function.

It is easy to see that this state can be reached by the evolution of an initial (non-stationary) distribution with a flat tail. First particles accelerating through the runaway region pile up at the cutoff point. A positive slope then develops there (the possibility of this happening was discussed previously[24]). The ω_{pe} modes are then excited and flatten $f_{||}$ by the backward diffusion of particles, until the small residual slope of the steady state is achieved at marginal stability.

These are the results obtained by examining the distribution function at a fixed radius. However because the dynamic friction is produced by the convectively unstable $\omega_{pe}\cos\theta$ modes, we would expect that the distribution function would develop a radial structure. This is indeed the case, as is shown in fig 3 which is a plot of $f_{||}$ in $p_{||}$, r space. The parallel distribution function is flat in the shaded region and zero outside.

To complete the picture, it is necessary to determine the perpendicular momentum space structure of the distribution function. Actually $f_{||}(p_{||})$ can be found [6] without knowing this, but then the origin of the dynamic friction and the precise nature of the steady state are unclear. In particular, the balance of friction and diffusion just described only applies globally in the consideration of $f_{||}(p_{||})$. When the full distribution function in the $p_{\perp}$, $p_{||}$ plane is considered locally, such a balance does not occur, and a momentum space flow results.

To calculate the full $f(p_{||}, p_{\perp})$, it is necesary to identify five separate asymptotic regions for the kinetic equation of the tail electrons. This is done in sec III. We continue the scheme of Kruskal and Bernstein, numbering the tail regions V - X, so that we match to region IV of the classical solution. In sec IV, the procedure for obtaining f asymptotically is described and carried out explicitly to determine $T_{\perp}$.

Finally, to clarify the nature of the steady state, we revert to a Lagrangian description and calculate the electron flow lines in momentum space in Sec V. The flow lines close on themselves to form vortices as shown in Fig 4 .

II. LINEAR STABILITY ANALYSIS

We outline here the stability properties of the electrostatic waves which resonate with the runaway electrons. To be consistent with the energies obtained by the runaways, it will be necessary to obtain relativistically correct growth rates. This we do by identifying a simple transformation rule to convert the usual dielectric function into a relativistic one.

The transformation is obtained by writing down the linearized Vlasov equation for the one particle distribution function $\tilde{f}(\underline{p},\underline{r},t)$ in relativistic form,[25]

$$\frac{\partial \tilde{f}}{\partial t} + \frac{1}{m\gamma}\,\underline{p} \cdot \frac{\partial}{\partial \underline{r}}\,\tilde{f} - \frac{1}{\gamma}\,\underline{p} \times \underline{\Omega}_0 \cdot \frac{\partial}{\partial \underline{p}}\,\tilde{f} - q\,\frac{\partial \tilde{\phi}}{\partial \underline{r}} \cdot \frac{\partial f_0}{\partial \underline{p}} = 0 \qquad (10)$$

where $\underline{\Omega}_0 = q\underline{B}_0/mc$, q is the signed charge, m is the non-relativistic mass, B_0 is the applied magnetic field, c is the speed of light, $\bar{p}$ the momentum, $\tilde{\phi}$, $\tilde{f}$, f_0 are the perturbed potential, distribution function and the steady state distribution function respectively and $\gamma^2 = 1 + p^2/m^2c^2$. Equation (10) can be obtained from the non-relativistic Vlasov equation by

$$\bar{V} \rightarrow \bar{p}/m\gamma \qquad (11)$$

$$\Omega_0 \rightarrow \Omega_0/\gamma \qquad (12)$$

$$\partial/\partial V_\perp \rightarrow m\partial/\partial p_\perp \qquad (13)$$

$$\partial/\partial V_\parallel \rightarrow m\partial/\partial p_\parallel \qquad (14)$$

$$\partial/\partial \phi \rightarrow \partial/\partial \phi \qquad (15)$$

$$\int d^3V \rightarrow \int d^3p \qquad (16)$$

where ϕ in (15) is the azimuthal angle.

It is easy to see that the procedure of obtaining the electrostatic dielectric function commutes with the operations (11) to (16), so that they may be applied directly to the usual non-relativistic dielectric function.[21] The real and imaginary parts are thus given by

$$\varepsilon_r(\omega,\underline{k}) = 1 + \sum_{s,n} (\omega_{ps}^2/k^2) \int d^3p \, [J_n^2(k_\perp p_\perp/m\Omega_s)\mathcal{L}_s^{(n)} f_s/(\omega - k_\parallel p_\parallel/m_s\gamma - n\Omega_s/\gamma)] \quad (17)$$

$$\varepsilon_i(\omega,k) = \sum_{s,n} \int d^3p \, (\omega_{ps}^2/k^2) \, J_n^2 \, (k_\perp p_\perp/m\Omega_s)$$
$$\times \, \delta(\omega - k_\parallel p_\parallel/m\gamma - n\Omega_s/\gamma)\mathcal{L}_s^{(n)} \, f \quad (18)$$

where the sums are over species and harmonics of Ω_s, J_n is the Bessel function, and $\mathcal{L}_s^{(n)} = m_s k_\parallel \partial/\partial p_\parallel + (n\Omega_s m_s/p_\perp)\partial/\partial p_\perp$.

The relevant waves have very high phase velocities, $\omega/k_\parallel >> v_e$, so that thermal corrections to the dispersion are negligible. The density of tail electrons is assumed to be sufficiently small, $n_T/n << 1$, that they will not affect the frequency of oscillation but only the growth rate. In this limit, Eq. (17) reduces to,

$$\varepsilon_r = 1 - \omega_{pi}^2/\omega^2 - \omega_{pe}^2/\omega^2 \; k_\parallel{}^2/k^2 - \omega_{pe}^2/(\omega^2 - \Omega_e^2) \; k_\perp{}^2/k^2 \quad (19)$$

when $\omega^2 << \Omega_e^2$, the real part of the frequency is given by

$$\omega^2 \simeq \omega_{pi}^2 \; (1 + m_i/m_e \; k_\parallel{}^2/k^2) \quad (20)$$

and finally, for $k_\parallel{}^2/k^2 >> m_e/m_i$,

$$\omega \simeq \omega_{pe} \, k_\parallel/k \quad (21)$$

which is the limit we utilize. Unstable lower hybrid waves with $k_\parallel{}^2/k^2 < m_e/m_i$ can be excited at high plasma densities when the runaway tail is very long. however, in such cases, the total runaway number is extremely small and their effects are minor.

The waves considered can be destabilized in two different ways. For modes driven by the $n = 0$ or Landau resonance, $\omega = k_{\parallel} p_{\parallel}/m\gamma$, the growth rate, in the absence of collisional damping, is

$$\omega_i/\omega = \pi(p_{\parallel}/\gamma)^2 \partial f_{\parallel}/\partial p_{\parallel} \,(\, p_{\parallel} = m\gamma\omega/k_{\parallel}\,). \tag{22}$$

Note that the growth rate is maximized at the largest frequency of oscillation or when $k_\perp \simeq 0$. Since the radial group velocity vanishes as $k_\perp \to 0$, we expect an absolute instability with $\omega \simeq \omega_{pe}$, whenever $f_{\parallel}$ has a positive slope. We refer to such modes as "ω_{pe} modes".

For the gyroresonance driven modes, at $\omega - k_{\parallel} p_{\parallel}/m\gamma - n\Omega_e/\gamma = 0$, we take the limit $p_\perp{}^2 << p_{\parallel}{}^2$, $k_\perp p_\perp/m\,\Omega_e << 1$, and $\gamma\omega << \Omega_e$, which can be verified a posteriori. The $n = \pm 1$ resonances are then dominant and we have

$$\omega_i/\omega = \sum_{n=\pm 1} \pi/4 \;\; \omega_{pe}^2/\Omega_e^2 \;\; k_\perp^2/k^2 \;\; m\gamma\partial/\partial p_{\parallel}\, T_\perp \; f_{\parallel}\,(-n\Omega_e m/k_{\parallel})$$

$$- \pi/4 \; n \; k_\perp^2/k^2 \;\; \omega_{pe}^2/\Omega_e \;\; m\gamma/k_{\parallel} \;\; f_{\parallel}\,(-n\Omega_e/k_{\parallel}) \tag{23}$$

where $\gamma^2 \simeq 1 + p_{\parallel}^2/m^2c^2$, $T_\perp$ and $f_{\parallel}$ are defined in Eq (5) and (6), and the second term in (23) results from an integration by parts. The parallel derivative term will turn out to be small, so we neglect it for the moment (it has an additional destabilizing influence for the modes we consider). Assuming negligible Landau damping for the mode considered, instability will occur if $f_{\parallel}(m\Omega_e/k_{\parallel}) > f_{\parallel}(-m\Omega_e/k_{\parallel})$. With the tremendous anisotropy in the parallel distribution function, this condition is generally satisfied, and (23) becomes

$$\omega_i/\omega_{pe} = \pi/4 \;\; \omega_{pe}^2/\Omega_e^2 \;\; k_\perp^2/k^2 \;\; m\gamma\Omega_e/k \;\; f_{\parallel}\,(m\Omega_e/k_{\parallel}). \tag{24}$$

These waves have large radial group velocities $v_{g_\perp} \sim v_e$, hence the convection time across the plasma column, $L/v_{g_\perp}$, is always short compared to the growth time ω_i^{-1}. Provided the coherent reflections from the edge are small, the instability is convective with a growth factor of

$$\lambda_{\underline{k}} \equiv \omega_i L/v_{g_\perp} = \pi/4 \;\; L \;\; \omega_{pe}^2/\Omega_e^2 \;\; k_\perp/k_{\|} \;\; \gamma m\Omega_e f_{\|} (m\Omega_e/k_{\|}) \tag{25}$$

The growth factor is large when $k_\perp/k_{\|} > 1$. The maximum $k_\perp$ is determined by the minimum phase velocity at which Landau damping is negligible, i.e. the runaway velocity. Thus, $k_\perp \simeq \omega_{pe}/v_R$ and $\lambda_{\underline{k}}$ increases with decreasing $k_{\|}$. Since $k_{\|} \simeq m\Omega_e/p_{\|}$, for constant $f_{\|}$, the growth rate increases with momentum. The dominant convective modes thus have $\omega = \omega_{pe} k_{\|}/k_\perp < \omega_{pe}$ and we refer to these as the "$\omega_{pe}\cos\theta$" modes. The resulting enhanced wave spectrum, for distributions of the runaway type, are summarized in fig 2.

To clarify the mechanisms by which these instabilities are produced, and, more important, to facilitate the discussion of their effect on the distribution function, we briefly examine the quasilinear response at the two resonances. Using the conservation of energy and momentum between the resonant particles and the unstable waves, one can obtain the particle diffusion paths [23]. The details can be found in Appendix B. For the $n = 0$ interaction ($\omega = \omega_{pe}$ spectrum), the well known result is that

$$p_\perp = \text{constant} \tag{26}$$

Thus, unstable waves at the Landau resonance diffuse a test particle along $p_\perp$ = constant trajectories to lower and higher values of $p_{\|}$ with equal probability (see Fig 5). However, with a local positive slope in the distribution function, there is a net scattering of particles to lower energies thus tending to wipe out the positive slope and provide a source of energy to amplify the waves. For the n = -1 gyroresonance driven waves (at $\omega = \omega_{pe} k_{\parallel} / k$), the diffusion paths are significantly different. In the limit of $k_{\perp} \gg k_{\parallel}$, the diffusion paths are given by

$$(p_{\parallel} - m\omega_{pe}/k_{\perp})^2 + p_{\perp}^2 = \text{constant}, \quad (27.)$$

that is the particles diffuse along circles centered at the wave phase velocity. Again a test particle gets scattered with equal probability in either direction along the diffusion path, as shown in Fig 5 . Since this scattering decreases the particle's total energy, the wave is amplified (provided a negligible number of particles exist at the n = +1 resonance). This accounts for the last term in Eq (23). Finally, the tendency to remove gradients along the diffusion path accounts for the first term in Eq (23).

III. Kinetic Equations for the Tail Electrons and the Unstable Waves

In this section we will derive the limiting forms of the wave and particle kinetic equations appropriate to the calculation of the runaway tail. The dominant scattering terms in the particle kinetic equation are those due to collisions and to the $n = 0,-1$ quasilinear diffusion. These terms dominate in different parts of momentum space and their ordering defines the five asymptotic regions of the tail distribution function.

To evaluate the kinetic coefficients in the particle equation, we need the spectral energy distribution of the waves. For the $\omega_{pe}\cos\theta$ modes, the spectral density can be obtained directly by integration of the wave kinetic equation, since the modes are convectively unstable. While this is clearly a hypothesis, data does tend to confirm it. Very low plasma density discharges undergo relaxed oscillations[26] with the characteristics expected from the $\omega_{pe}\cos\theta$ mode when it is absolutely unstable[27,28]. This regime disappears abruptly as the plasma density is raised, suggesting the transition to a convective instability.[29]

The ω_{pe} instability, however, is absolute with a large growth rate, and it is necessary to find its saturated state. Specifically, we take the saturated state of the ω_{pe} modes to be determined by marginal stability, with the growth balanced by some damping mechanism (e.g. collisions). This criterion specifies the slope of the parallel distribution function. The diffusion coefficient, $D_{\parallel}$, needed to maintain this known steady state is found from the particle (parallel) kinetic equation, and $D_{\parallel}$, in turn, determines the spectral density. This marginal stability analysis (including the smooth matching to the rest of the distribution function) is described in the present section. The diffusion coefficient so obtained is then used with the full kinetic equation to find the complete distribution function in Sec IV.

The particle kinetic equation, including collisional, wave and particle discreteness effects is written[30]

$$\frac{\partial f}{\partial t} + \underline{F}_0 \cdot \frac{\partial}{\partial \underline{p}} f = \left.\frac{\partial f}{\partial t}\right|_C + \left.\frac{\partial f}{\partial t}\right|_{QL} - \frac{\partial}{\partial \underline{p}} \cdot \underline{J}, \tag{28}$$

where $\underline{F}_0$ denotes the zero order forces. The effects of spatial diffusion are of order $\rho_e^2/a^2 \ll 1$ compared to velocity space diffusion and have been ignored. The first term on the right hand side is given by Eq. (1) (or Equation (2) at high energies), the second term contains the quasilinear terms (wave-particle, wave-wave, nonlinear Landau damping) and $\underline{J}$ is the current due to particle discreteness (Cherenkov emission of waves). The validity of the non-relativistic form of Eq (28) has been established for both the stable and weakly unstable plasma regimes.[31] For the situation we consider, where a well developed unstable spectrum is present, the term due to discreteness is negligible. Furthermore, the wave-particle terms in the quasilinear operator are dominant, with the n = 0 ($\omega = k_{\|}p_{\|}/m\gamma$) and the n = -1 ($\omega = k_{\|}p_{\|}/m\gamma + \Omega_e/\gamma$) resonances being the most important since $k_\perp^2\rho_e^2 \ll 1$. Thus, in the steady state, the kinetic equation for the tail electrons reduces to

$$eE \frac{\partial f}{\partial p_{\|}} = C_c(f) + C_0(f) + C_{-1}(f), \tag{29}$$

where E is the applied electric field, the C's denote the collisional operators and the subscripts have the obvious meanings.

The term due to collisions is given in Eq. (2). Expanding for $p_\perp \ll p_{\|}$, $p_{\|} > p_R$, it is

$$C_c(f) \simeq \nu_{ei} p_e \left(\frac{\gamma}{p_{\|}} \frac{1}{p_\perp} \frac{\partial}{\partial p_\perp} p_\perp \frac{\partial}{\partial p_\perp} f + \frac{\partial}{\partial p_{\|}} \frac{\gamma^2}{p_{\|}^3} f + \frac{\partial}{\partial p_{\|}} \frac{p_e^2 \gamma^3}{p_{\|}^3} \frac{\partial}{\partial p_{\|}} f \right) \tag{30}$$

The quasilinear operators are obtained by applying the transformation (11) - (16) to the non relativistic form[30] giving

$$C_0 + C_{-1} = \sum_{n=0,-1} 16\pi^2 e^2/m^2 \int_{k_\parallel > 0} d^3k \, (\omega_{\underline{k}} \partial\varepsilon/\partial\omega_{\underline{k}})^{-1} \, \varepsilon_{\underline{k}}/k^2$$

$$\times \mathcal{L}^{(n)} \, J_n^2 \, (k_\perp p_\perp/m\Omega_e) \, \delta \, (\omega_k - k_\parallel \, p_\parallel \,/m\gamma - n\Omega_e/\gamma) \mathcal{L}^{(n)}, \qquad (31)$$

where ε_k is the field plus particle energy density,

$$\gamma^2 \simeq 1 + p_\parallel^2 \,/m^2c^2$$

and

$$\mathcal{L}^{(n)} = mk_\parallel \, \partial/\partial p_\parallel + (n\Omega_e m^2/p_\perp) \, \partial/\partial p_\perp .$$

The restriction $k_\parallel > 0$ on the integral reflects the positive sign of the phase velocity of the unstable waves. Frequencies are taken positive in eq (31), with negative frequencies accounting for the factor of two which has been included. Expanding the Bessel function for $k_\perp^2 \rho_e^2 \ll 1$, and noting $\omega_{\underline{k}} \partial\varepsilon/\partial\omega_{\underline{k}} = 2$ for the plasma waves, these become

$$C_0(f) = 8\pi^2 e^2 \int_{k_\parallel > 0} d^3k \varepsilon_{\underline{k}} k_\parallel^2/k^2 \partial/\partial p_\parallel \, \delta(\omega_{pe} k_\parallel/k - k_\parallel p_\parallel/m\gamma)(\partial/\partial p_\parallel) f \qquad (32)$$

$$C_{-1}(f) = 8\pi^2 e^2 \int_{k_\parallel > 0} d^3k \varepsilon_{\underline{k}} \, k_\perp^2/k^2 \, [(k_\parallel \,/m\Omega_e) \, \partial/\partial p_\parallel - (1/p_\perp)\partial/\partial p_\perp \,]$$

$$\times \, (p_\perp^2/4) \, \delta(\omega_{pe} k_\parallel \,/k - k_\parallel \, p_\parallel \,/m\gamma + \Omega_e/\gamma)$$

$$\times \, ((k_\parallel \,/m\Omega_e)\partial/\partial p_\parallel - (1/p_\perp)\partial/\partial p_\perp) f. \qquad (33)$$

While the full spectrum appears in each of these operators, the dominant contributions to eq (32) and eq (33) come from the ω_{pe} and the $\omega_{pe}\cos\theta$ modes respectively.

Writing the operators in eq (31) in terms of the total wave energy is a helpful simplification of the equations. In this description, the energy in the non-resonant particles is included with the waves

Equation (30) describes the resonant distribution function, the non-resonant distribution function is unnecessary and all the quasilinear conservation theorems are satisfied (Appendix B).

We now consider the marginal stability problem to determine $\varepsilon_{\underline{k}}$ for the ω_{pe} modes. This utilizes the equation for the parallel distribution function, obtained from Equation (29) by the operation $\int 2\pi p_\perp dp_\perp$. There results

$$eE\partial f_{\parallel}/\partial p_{\parallel} = (\partial/\partial p_{\parallel})F_{\parallel T}f + (\partial/\partial p_{\parallel})D_{\parallel T}(\partial/\partial p_{\parallel})f, \quad (34)$$

where $F_{\parallel T} = F_C + F + \chi dT_\perp/dp_{\parallel}$, and $D_{\parallel T} = D_{c\parallel} + D_0 + \chi T_\perp$ with

$$F_C = \nu_{ei}p_e^3\gamma^2/p_{\parallel}^2 \quad (35)$$

$$D_{C_{\parallel}} = \nu_{ei}p_e^5\gamma^3/p_{\parallel}^3 \quad (36)$$

$$D_0 = 8\pi^2e^2\int d^3k\varepsilon_{\underline{k}}\ k_{\parallel}^2/k^2\ \delta(\omega_{pe}k_{\parallel}/k - k_{\parallel}p_{\parallel}/m\gamma) \quad (37)$$

$$F = 8\pi^2e^2\int d^3k\varepsilon_{\underline{k}}\ k_\perp^2/k^2\ k_{\parallel}/2m\Omega_e\ \delta(\omega_{pe}k_{\parallel}/k - k_{\parallel}p_{\parallel}/m\gamma + \Omega_e/\gamma) \quad (38)$$

$$\chi = 8\pi^2e^2\int d^3k\varepsilon_{\underline{k}}\ k_\perp^2/k^2\ k_{\parallel}^2/2m\Omega_e^2\ \delta(\omega_{pe}k_{\parallel}/k - k_{\parallel}p_{\parallel}/m\gamma + \Omega_e/\gamma) \quad (39)$$

Upon doing the $k_{\parallel}$ integrals in (38) and (39), we find, for $\gamma\omega_{pe}k_{\parallel}/k \ll \Omega_e$, that

$$\chi = mF/p_{\parallel} \quad (40)$$

Equation (34) can be integrated once, using the boundary condition corresponding to the condition that there be no flux of particle across the surface $p_{||} = p_R$, that is $\partial f_{||}/\partial p_{||} = 0$ at $p_{||} = p_R$, gives

$$D_{||T} \partial f_{||}/\partial p_{||} = (eE - F_{||T}) f_{||} . \tag{41}$$

Just beyond the runaway momentum, where the $\omega_{pe}\cos\theta$ modes are stable, $eE > F_T$ and Eq (41) implies that $f_{||}$ has a positive slope. This is to be compared with the slope at marginal stability where collisional damping balances the growth rate of the ω_{pe} modes,

$$\partial f_{||}/\partial p_{||} \Big|_{MS} = 1/\pi \quad \nu_{ei}/\omega_{pe} \; \gamma^2/p_{||}^2 \tag{42}$$

For some distance beyond p_R (where $F_T \simeq F_C$ is only slightly less than eE) the slope obtained from Eq (41) will be less than that from Eq (42). In this region, V, the ω_{pe} modes are stable and (41) determines $f_{||}$. At the point $p_{||} = p_0$, the two slopes are equal and for $p_{||} > p_0$, Eq (42) determines $f_{||}$. This match between the stable and marginally stable regions of $f_{||}$ is a smooth one.

Using the slope given by Eq (42) in Eq (41) gives the diffusion coefficient from the ω_{pe} modes,

$$D_0(p_{||}) = - D_{C||} + \pi f_C \, \omega_{pe}/\nu_{ei} \; p_{||}^2/\gamma^2 \; (eE - F_C), \tag{43}$$

where we have used $f_{||} \simeq f_C \equiv f_{||}(p_R)$, since the slope is so small. Putting $D_0 = 0$ in (43) also determines p_0 which, since $\omega_{pe}/\nu_{ei} >> 1$, is close to p_R. Evidently, there is a region of very rapid change, a boundary layer, near p_0 where D_0 rises from zero to its asymptotic value

$$D_0 \simeq \pi f_C \, \omega_{pe}/\nu_{ei} \; p_{||}^2/\gamma^2 \; eE \tag{44}$$

The boundary layer is denoted as region VI. The region where Eq (44) applies is VII.

At large momentum, $p_{\parallel} > p_1$, where the unstable $\omega_{pe}\cos\theta$ modes produce significant friction, the slope in $f_{\parallel}$ again approaches zero, signifying the end of region VII. Here $F \gg F_C$ and the analog of Equation (43) is

$$D_0(p_{\parallel}) = -\chi T_{\perp} + \pi f_C \; \omega_{pe}/\nu_{ei} \; p_{\parallel}^2/\gamma^2 \quad (eE - F - \chi T_{\perp}'). \tag{45}$$

Putting $D_0 = 0$ in (45) gives p_2, the boundary to region VII. As before this is accompanied by a boundary layer, region VIII, bringing us to regions IX and X, where the $\omega_{pe}\cos\theta$ modes are dominant. The equation

$$eE = F + \chi T_{\perp}' \tag{46}$$

defines the line $p_{\parallel} = p_C$, the boundary between regions IX and X, where $\partial f_{\parallel}/\partial p_{\parallel} = 0$. For $p_{\parallel} > p_C$, the slope is negative. The diffusion coefficient is shown schematically in Fig. 6. In the Kruskal-Bernstein solution, region IV extends out to $p_{\parallel} \sim p_R(1+(E/E_r)^{1/3})$. We have replaced their region IV by our regions V, VI, and a small part of region VII-a. Our region V corresponds to Kruskal-Bernstein's region IV, when $p \sim p_R < p_0$. Their region V $(p_{\parallel} \gg p_R(1+(E/E_r)^{1/3})$ has been replaced with our regions VII-a - X.

To summarize, we write out the leading order kinetic equations in the different regions. Referring to Eq (34), (37) and (38), these are:

Region V $(p_R \le p_{\parallel} \le p_0)$

$$eE\partial f/\partial p_{\parallel} = C_C(f) \tag{47}$$

Region VIIa $(p_0 \le p_{\parallel} \le p_1)$

$$eE\partial f/\partial p_{\parallel} = C_C(f) + \partial/\partial p_{\parallel} D_0 \; \partial/\partial p_{\parallel} f, \tag{48}$$

Region VIIb $(p_1 \le p_{\parallel} \le p_2)$

$$\begin{aligned} eE\partial f/\partial p_{\parallel} = {} & \partial/\partial p_{\parallel} (D_0 + \tfrac{1}{2} p_{\perp}^2 F/p_{\parallel})\, \partial/\partial p_{\parallel} f \\ & - 1/p_{\perp}\, \partial/\partial p_{\perp} \tfrac{1}{2} p_{\perp}^2 F\, \partial/\partial p_{\parallel} f - \partial/\partial p_{\parallel} \tfrac{1}{2} p_{\perp}^2 F/p_{\perp}\, \partial/\partial p_{\perp} f \\ & + \tfrac{1}{2} p_{\parallel} F/p_{\perp}\, \partial/\partial p_{\perp}\, p_{\perp}\, \partial/\partial p_{\perp} f, \end{aligned} \tag{49}$$

Region IX, X $(p_2 \le p_\parallel)$

$$eE\partial f/\partial p_\parallel = \partial/\partial p_\parallel \frac{1}{2} p_\perp^2 F/p_\parallel \partial/\partial p_\parallel f - 1/p_\perp \partial/\partial p_\perp \frac{1}{2} p_\perp^2 F \partial/\partial p_\parallel f$$
$$- \partial/\partial p_\parallel \frac{1}{2} p_\perp^2 F 1/p_\perp \partial/\partial p_\perp f$$
$$+ \frac{1}{2} p_\parallel F 1/p_\perp \partial/\partial p_\perp p_\perp \partial/\partial p_\perp f, \tag{50}$$

where D_0 is given by Eq (44) in region VII. Equations (48) and (49) also apply in the boundary layers, regions VI and VIII respectively, except that one must use the more exact expressions (43) and (45) for D_0.

The saturated level of the ω_{pe} fluctuations follows from Eq (37) using Eq (44). We find

$$\varepsilon^P_{k_\parallel} = \int d^2k_\perp \varepsilon^P_{\underline{k}} = 1/8\pi\ m^2/e^2\ \omega_p^4/k_\parallel^3\ E/E_r\ n_T/n \tag{51}$$

To verify that this level is consistent with the assumptions of quasilinear theory, we evaluate the autocorrelation time, $\tau_{AC} \simeq (\Delta k_\parallel |V_{g_\parallel} - V_{ph}|)^{-1} \sim (k_\parallel V_{ph})^{-1} \sim \omega_{pe}^{-1}$, and the trapping time $\tau_{tr} \sim (e^2/mk_\parallel^2\ \Delta k_\parallel\ \varepsilon_{k_\parallel})^{-1}$. The ratio

$$(\tau_{AC}/\tau_{tr}) = 1/(2\pi)^2\ E/E_r\ n_T/n \ll 1, \tag{52}$$

is always small, as required.

The convective, $\omega_{pe}\cos\theta$ modes, are described by the wave transport equation

$$\underline{V}_g \cdot \nabla \varepsilon_{\underline{k}} - 2\ \omega_i\ \varepsilon_{\underline{k}} = P_{\underline{k}} \tag{53}$$

where $\underline{V}_g$ is the group velocity, ω_i is the growth rate as given in Eq (24), and $P_{\underline{k}}$ is the emission due to particle discreteness.[32] Equation (53) describes the total energy in the mode at $\omega = \omega_{pe} k_\parallel /k$ and can be thought of as the integral over the band of frequencies centered on $\omega = \omega_{pe} k_\parallel /k$. This equation has recently been discussed in some detail[33,34] the latter paper emphasizing its limitations. The case we treat, with a steady state plasma and neglecting plasma gradients, is straightforward and the meaning of eq. (52) is unambiguous.

The emission resulting from discreteness is easily obtained by the test particle method. In the limit of weak damping or growth, the emission concentrates in a narrow line. Integrating over frequency then gives the net emission into the mode which is,

$$P_{\underline{k}} = m^2/16\pi^2 \quad \omega_{pe}^4 \quad (k_{||}/k)^4 \quad f_{||}(m\omega_{\underline{k}}/k_{||}), \tag{54}$$

where we have included only the Cherenkov ($\omega_{\underline{k}} = k_{||} v_{||}$) emission term since (with $k_\perp^2 \rho_e^2 << 1$) it is larger than the emission at the gyroresonances. This, i.e. eq. (53) description does not have any divergences in a finite system. In an infinite system $\mathcal{E}_{\underline{k}}$ would diverge as marginal stability is approached from the stable side, since the absorption vanishes.

IV. THE SOLUTION OF THE KINETIC EQUATIONS

In general, our procedure is to develop an expansion for each region of the particle kinetic equation and then match these together asymptotically. If the detailed solution within the boundary layers (Regions VI and VIII) is not needed, they can be replaced by jump conditions on the derivatives of f and this substantially simplifies the matching procedure. In this way, we obtain f in terms of known quantities and the unknown friction coefficient, F, for the $\omega_{pe}\cos\theta$ modes. The last step is to calculate F, making the solution self-consistent.

When the ω_{pe} modes are stabilized by collisional damping, as we assume here, the expansion can be formally cast in terms of the small parameter E/E_r (since D_0 is a complicated function of E/E_r) - just as in the classical runaway theory [1]. While it is tempting to do this, generality is lost in the process and such a calculation could not be readily modified to include alternative saturation mechanisms. We prefer instead to keep the expansion parameter implicit, carrying out the solution to lowest non-trivial order in each region and then matching. Since the solution in the largest region, VII, is nearly constant in $p_{||}$ and expandable in series form, on does not have the problem of calculating large exponents. The meticulous accuracy required in the classical runaway problem is not needed here.

The point of departure from the classical solution is in region IV very close to p_R, and the region labeled V by Kruskal and Bernstein is eliminated. Furthermore, we treat region IV, a boundary layer, different from Kruskal and Bernstein. This is an important point which, however, belongs with the classical solution. We discuss it here only to the extent required to match the tail and classical solutions together. The runaway rate is adequately determined by the distribution function at the end of region III and not significantly altered by this match.

An outline of our procedure is as follows. The coefficient of the first parallel derivative, $eE - F_c$, vanishes at p_R. For this reason, the second parallel derivative, although small, must be retained, making the kinetic equation elliptic in region IV. This means that boundary data is required for a unique specification and hence the solutions in both region III and region IV must be known. Specifically, the necessary conditions are

$$f(p_R^-, p_\perp),\ f(p_0^-, p_\perp),\ f(p_\parallel, \infty) = 0,\ \partial f/\partial p_\perp(p_\parallel, 0) = 0.$$

The function $f(p_R^-, p_\perp)$, in the space where regions III and IV overlap is known from the classical solution in III. But with $f(p_0^-, p_\perp)$ unknown until the entire problem is solved, the solution in IV will contain one undetermined constant (function of $p_\perp$). Classically, region IV extends to $p_\parallel \approx p_R(1+(E/E_r)^{1/3}) \gg p_0$, we have thus labeled $p_R \leq p_\parallel \leq p_0$ as V.

Region V terminates at p_0 with the onset of the ω_{pe} modes and the appearance of the coefficient D_0. This occurs very close to p_R, see eq.(43), in fact $p_0 - p_R \lesssim (E/E_r)^{1/3} p_R$ which indicates a negligible change in f from p_R to p_0.

Region VI, the boundary layer were D_0 changes rapidly, is replaced here by simple jump conditions. These are obtained by integration of the kinetic eq. (29), across the layer from $p_\parallel = p_0^-$ to $p_\parallel = p_0^+$. This gives, to an accuracy of order $(p_0^+ - p_0^-)/p_0 \ll 1$,

$$f(p_0^+, p_\perp) = f(p_0^-, p_\perp), \tag{55}$$

$$[D_0(p_0^+) + D_{c_\parallel}(p_0^+)]\ \partial f(p_0^+, p_\perp)/\partial p_\parallel = D_{c_\parallel}(p_0^-)\ \partial f(p_0^-, p_\perp)/\partial p_\parallel, \tag{56}$$

where the effects of the $n = -1$ terms were neglected since they don't come into play until $p_\parallel > p_1$.

Application of eq (55) and (56) brings us to region VII. In region VII the kinetic equation is again elliptic and hence we require, $f(p_0^+, p_\perp)$,

$f(p_2^-,p_\perp)$, $\partial f(p_\parallel,0)/\partial p_\perp = 0$, $f(p_\parallel,\infty) = 0$, for a unique specification. Thus before completing the solution in region VII, we have to determine $f(p_2^-,p_\perp)$, which is, again, unknown until the entire solution is found.

The boundary layer, VIII, is also replaced by jump conditions,

$$f(p_2^+,p_\perp) = f(p_2^-,p_\perp), \tag{57}$$

$$D_\parallel (p_2^+)\ \partial f(p_2^+,p_\perp)/\partial p_\parallel = [D_0(p_2^-) + D_\parallel (p_2^-)]\ \partial f(p_2^-,p_\perp)/\partial p_\parallel , \tag{58}$$

which connect into region IX. The kinetic equation, for $p_\parallel > p_2^+$, is parabolic (only the n = -1 quasilinear terms) so the appropriate boundary conditions are $f(p_2^+,p_\perp)$. This changeover to a parabolic equation permits the completion of the solution, since the jump conditions (56) and (58) are now sufficient to determine the functions $f(p_0^-,p_\perp)$ and $f(p_2^-,p_\perp)$.

We now turn to the evaluation of f region by region. We first give the calculation formally for the whole distribution function and afterwards carry out explicitly the determination of $T_\perp(p_\parallel)$.

<u>Region VII</u> $(p_0^+ \leq p_\parallel \leq p_2^-)$

The largest term in eq. (48,49) is the $D_0(p_\parallel)$ term. Seeking an expansion for the distribution function in inverse powers of D_0, then generates the following sequence,

$$\partial/\partial p_\parallel\ D_0\ \partial/\partial p_\parallel f^{(0)} = 0 \tag{59}$$

$$\partial/\partial p_\parallel\ D_0\ \partial/\partial p_\parallel\ f^{(1)} = -1/p_\perp\ \partial/\partial p_\perp\ p_\perp (D_{c_\perp} + \tfrac{1}{2} p_\parallel F)\ \partial/\partial p_\perp\ f^{(0)} \tag{60}$$

where $D_{c_\perp} = \nu_{ei} p_e^3 \gamma/p_\parallel$, F is defined in eq. (38), and terms in (60) involving parallel derivatives, of order D_0^{-2} or $p_\perp^2/p_\parallel^2$, have been discarded. The boundary conditions to be used with eq (59) and (60) are

$$f^{(0)}(p_0^+,p_\perp) = f(p_0^-,p_\perp) \quad (61)$$

$$f^{(1)}(p_0^+,p_\perp) = 0 \quad (62)$$

$$f^{(0)}(p_2^-,p_\perp) = f(p_2^-,p_\perp) \quad (63)$$

$$f^{(1)}(p_2^-,p_\perp) = 0 \quad (64)$$

Integrating Eq. 59,60 we get

$$f^{(0)}(p_{||},p_\perp) = f(p_0^+,p_\perp) + g_1(p_\perp) \int_{p_0^+}^{p_{||}} dp/D_0(p) \quad (65)$$

$$f^{(1)}(p_{||},p_\perp) = \int_{p_0^+}^{p_{||}} dp'/D_0(p')\{ g_2(p_\perp) + eE \int_{p_0^+}^{p'} dp''\partial f^{(0)}/\partial p'' - \int_{p_0^-}^{p'} dp'' 1/p_\perp \, \partial/\partial p_\perp \; p_\perp(D_{c_\perp} + p''F)\partial f^{(0)}/\partial p_\perp \} \quad (66)$$

and

$$g_1(p\ ) = \{f(p_2^-,p_\perp) - f(p_0^+,p_\perp)\}/ \int_{p_0^+}^{p_2^-} dp/D_0(p) \quad (67)$$

$$g_2(p\ ) = \{ \int_{p_0^+}^{p_2^-} dp'/D_0(p') \; [-eE \int_{p_0^-}^{p'} dp'' \; \partial f^{(0)}/\partial p'' + \int_{p_2^+}^{p'} dp'' \; 1/p_\perp \, \partial/\partial p_\perp \; p_\perp(D_{c_\perp} + p''F) \; \partial f^{(0)}/\partial p_\perp \;]\} \; 1/ \int_{p_0^+}^{p_2} dp/D_0(p) \quad (68)$$

The matching at p_0 and p_2 will be used to determine the unknown functions.

Region IX $(p_2^+ \leq p_{||} \leq p_c)$

To solve eq (50), we exploit the disparity of scales in $p_\perp$ and $p_{||}$, writing f in the factorized form

$$f(p_{||},p_\perp) = f_{||}(p_{||})f_\perp(p_{||},p_\perp) \quad (69)$$

where $f_{||}$ is given by eq. (5) and $f_\perp$ satisfies the normaliztion condition

$$2\pi \int_0^\infty dp_\perp p_\perp f_\perp = 1$$

Since $p_\perp \ll p_{||}$, the dominant term on the right hand side of equation (50) is the perpendicular diffusion term. This creates a rapid spreading of f in the perpendicular direction, but does not effect $f_{||}$, which is a slowly varying function of $p_{||}$. The equation for $f_\perp$ thus becomes

$$eE\ \partial f_\perp/\partial p_\parallel = \frac{1}{2}\, p_\parallel F\ \ 1/p_\perp\ \partial/\partial p_\perp\ p_\perp\ \partial/\partial p_\perp f_\perp \tag{70}$$

In effect, eq (70) is the fast scale part of eq (50). Application of $\int dp_\perp p_\perp$ to eq (50) annihilates the fast operator, leaving equation (41) for the slow variations. Combining the solutions to equations (41) and (70) gives the distribution function in region IX,

$$f(p_\parallel, p_\perp) = f_c/2\pi m T_\perp(p_\parallel)\quad \exp(-p_\perp^2/2mT_\perp(p_\parallel)), \tag{71}$$

$$T_\perp(p_\parallel) = T_\perp(p_2^+) + \int_{p_2^+}^{p_\parallel} dp\quad pF/meE \tag{72}$$

<u>Region X</u> $(p_\parallel > p_c)$

The self-consistent evaluation of $f_\parallel$ in this region is extremely awkward, involving a determination of the $p_\parallel$, $p_\perp$ and r dependences of f together with $F(r,p_\parallel)$; we do not have the benefit, as in the other regions, of a constant $f_\parallel$ to lowest order. We therefore restrict the discussion to a qualitative description of the cutoff.

To this end, we write equation (50) in the form

$$eE/p_\parallel\ \partial f/\partial p_\parallel = \mathcal{L}\ p_\perp^2 F/2\ \mathcal{L}\ f \tag{73}$$

where $\mathcal{L} = 1/p_\parallel\ \partial/\partial p_\parallel - 1/p_\perp\ \partial/\partial p_\perp$ is a pitch angle scattering operator. When $F > eE$, $p_\perp^2 \ll p_\parallel^2$, the leading order solution to (73) has $\mathcal{L} f = 0$ so that f is constant along the diffusion paths. With the boundary data given on $p_\parallel = p_c$ as $f(p_{\perp 0}) = f_c/2\pi m T_{\perp c}\ \ \exp(-p_{\perp 0}^2/2mT_{\perp c})$ and the diffusion paths, $p_\parallel^2 + p_\perp^2 = p_{\perp 0}^2 + p_c^2$, this gives, for $p_\parallel > p_c$,

$$f(p_\perp, p_\parallel) = f_c/2\pi m T_{\perp c}\ \ \exp(-p_\perp^2/2mT_{\perp c} - (p_\parallel^2 - p_c^2)/2mT_{\perp c}). \tag{74}$$

Integrating over $p_\perp$ results in

$$f_{\parallel}(p_{\parallel}) = f_c \exp(-(p_{\parallel}^2 - p_c^2)/2mT_{\perp c})$$

demonstrating a rapid exponential decay.

Calculation of $T_\perp(p_\parallel)$, for $p_R < p_\parallel < p_c$

Since f is approximately Gaussian in the perpendicular direction, we use the form $f = f_\parallel / 2\pi mT \exp(-p_\perp^2/2mT_\perp)$ so that, by taking moments in the preceding formalism, the problem reduces to a series of ordinary differential equations for $T_\perp(p_\parallel)$.

. The jump condition at p_2, region VIII results by taking the perpendicular energy moment of equation (58), giving

$$D_0 \partial/\partial p_\parallel T_\perp f_\parallel)_- + 2mF/p_\parallel \partial/\partial p_\parallel T_\perp^2 f_\parallel)_- = 2mF/p_\parallel \partial/\partial p_\parallel T_\perp^2 f_\parallel)_+ \quad (75)$$

Using the continuity of $f_\parallel$ and $T_\perp$, which follows from eq. (57), and the $f_\parallel$ equation ( ), this becomes

$$(D_0 + 4mF/p_{\parallel}\ T_\perp)\ \partial T_\perp^-/\partial p_{\parallel} = 4mF/p_{\parallel}\ T_\perp\ \partial T_\perp^+/\partial p_{\parallel}\ , \tag{76}$$

which is the desired jump condition on the derivatives of $T_\perp$.

Using Eq. 72 for $\partial T_\perp^+/\partial p_{\parallel}$ and Eq. 65, 66 for $\partial T_\perp^-/\partial p_{\parallel}$ after some algebra, we find

$$T_\perp(p_2) \cong T_\perp(p_0^+)[1 + 4F^2/eE \int_{p_0^+}^{p_2} dp/D_0(p)] \tag{77}$$

We should now use the jump condition at p_0 to determine $T_\perp(p_0^+)$, but because our Region V is so small, we shall assume that $T_\perp(p_0^+) \cong T_\perp(p_R)$ with negligibly small error. Using the value of $D_0(p_{\parallel})$ in Eq. 44, we find

$$T_\perp(p_2) \simeq T_\perp(p_R)\ [1 + \frac{4}{\pi}(F/eE)^2\ n/n_T\ \ \nu_{ei}/\omega_{pe}\ \ p_e/p_2\ (p_2-p_0)/p_0$$
$$\times\ p_0/p_e\ p_2/p_e(p_e/mc)^2\ ] \tag{78}$$

Using $B_0 = 40k$; $n \simeq 4 \times 10^{13}$, E = 0.01 volts/cm, $T_e \simeq 0.8$ keV, we find that the two terms are of equal order,

$$T_\perp(p_2) \simeq T_\perp(p_R)[1 + 2.4/(1 + \sqrt{5}\ )^2\ ] \tag{79}$$

where we used eq (?) for $T_\perp(p_R)$. The heating as is expected is quite small in region VII (see Fig 7).

Evaluation of the Cutoff Momentum, p_c

We require the spectral energy density of the $\omega_{pe}\cos\theta$ modes. Since $f_{\parallel} \simeq f_c$ in the region of interest, $p_R \le p_{\parallel} \le p_c$, this can be obtained by a direct integration of eq. (53). We carry this out asymptotically for large growth factor, $\lambda_k << 1$, (see eq. (25), which is the appropriate limit for finding the cutoff.

We thus consider a cylinder of radius a (Fig 8) and look for the Green's function solution to

$$(v_g \cdot \nabla - 2\omega_i)\ G(\underline{r},\underline{r}') = \delta(\underline{r} - \underline{r}') \tag{80}$$

where r, r' are the coordinates of the obsrevation and source points respectively. The only waves which contribute to the spectral energy density at r are those which when emmitted at r' propogate through the observation point at r. That means we can transform into a coordinate system where one of the axes is parallel to the line joining (r, r') and the other coordinate is orthogonal to it (see fig. 8), hence eq. (80) can be written

$$(v_g\ \partial/x_{\parallel} - 2\omega_i)\ G(\underline{x}, \underline{x}') = \delta(x_{\parallel} - x_{\parallel}')\delta(x_{\perp} - x_{\perp}') \tag{81}$$

Now decomposing G into $G(\bar{x},\bar{x}') = g(x_{\parallel})\ \delta(x_{\perp} - x_{\perp}')$ and substituting this into eq. (81), integrating over $x_{\perp}'$ and solving for the simple one dimensional Green's function, we get

$$G(\underline{x},\underline{x}', v_{g_{\perp}}) = 1/v_g\ \exp p(x,x')\ H(x_{\parallel} - x_{\parallel}')\delta(x - x_{\perp}') \tag{82}$$

where $p(x,x') = 2\omega_i/v_g \int_x^{x'} ds$ is the distance between (x, x') and H, δ denote the usual Heaviside unit step and Dirac delta functions respectively. The spectral energy density is then obtained from

$$\varepsilon_k = \int d^2x' \, P_k(\underline{x}')G(\underline{x},\underline{x}',v_{g\perp}) \tag{83}$$

Since we are treating a homogeneous plasma, the emission function is independent of the spatial location. The integral in $\mathcal{E}$ (83) is just an integral over the Green's function. To evaluate this, we transform into a polar coordinate system where θ, θ', ϕ denote the angles of the observation point, source point and the group velocity (Fig. 8). In that case

we have $x_{||} = r\cos\theta$, $x'_{||} = r'\cos(\phi - \theta')$, $x_\perp = r\sin\theta$, and $x'_\perp = r'\sin(\phi - \theta')$.

In addition, using the law of cosines, we have

$$p(x,x') = 2\omega_i/v_g \, (r^2 + r'^2 \mp 2rr'\cos\theta')^{1/2} \tag{84}$$

$$\int_0^{2\pi} d\phi \, \mathcal{E}_k = P_k/v_g \int_0^{2\pi} d\phi \int_0^a r'dr' \int_0^{2\pi} d\theta' e^p \, H(r\cos\theta - r'\cos(\phi - \theta'))$$
$$\times \, \delta(r\sin\theta - r'\sin(\phi - \theta')) \tag{85}$$

Performing the θ' integral first,

$$\mathcal{E}_{k_\perp,k_{||}} = P_k/v_g \int_0^{2\pi} d\phi \int_0^a r'dr' \quad e^p/r'\cos(\phi - \theta')\Big|_{\theta'=\theta_0} \tag{86}$$

where θ_0 is the solution to the $\sin(\phi - \theta_0') = r\sin\theta/r'$. Note that the integral in $\mathcal{E}$ (86) is maximized with $\theta' \simeq \pi$ which requires that $\phi \simeq 0$. The maximum contribution to the spectral energy density comes from the waves which propogate through the axis of the plasma as shown by the dashed conic region in Fig 8.

To do the asymptotic evaluation, we solve $\sin(\phi - \theta_0) = r/r' \sin\phi$ near $\theta_0 \simeq \pi$ and $\phi \simeq 0$. We define $\theta_0 = \pi + \delta$ and find that $\phi \simeq \delta/(1 + r/r')$. Substituting this into eq (86), retaining the dominant terms for $\delta^2 << 1$, and extending the integration limits on δ gives

$$\varepsilon_{k_\perp, k_{||}} = P_k/v_g \int_0^a dr'r' \int_{-\infty}^{\infty} d\delta \, (1/(r+r')) \exp[2\omega_i/v_g \, (r+r') \times (1 - \tfrac{1}{2}\delta^2 rr'^3/(r+r')^4)] \tag{87}$$

The remaining integrals can be carried out asymptotically with the dominant contribution in the r' integral coming from r' = a. This yields (for $r \neq 0$),

$$\varepsilon_{k_\perp, k_{||}} = P_{k_\perp, k_{||}}/2\omega_i \sqrt{\pi(a+r)/r\lambda_{\underline{k}}} \, \exp(2\lambda_{\underline{k}} (r+a)/r) \tag{88}$$

where $\lambda_{\underline{k}}$ is given by equation (25), with L = r.

The friction coefficient can now be evaluated with eq (88), the integral again being susceptible to asymptotic methods on account of the exponent. We find that the integrand maximizes at the minimum allowable $k_\perp$, which here is set by the condition that Landau damping be absent, $k_\perp \sim m\omega_{pe}/p_R$. This puts the phase velocity of the dominant modes at the runaway point. We find

$$F(p_{||}, r) = 1/\sqrt{2}\,\pi^2 e^2/\lambda_e^2 \, (n/n_T \; \lambda_e/\gamma r \; (p_R/p_{||})^3)^{3/2} (\Omega_e/\omega_{pe})^5 \times \exp(\pi n_T/n \, (\omega_{pe}/\Omega_e)^2 \, r/\lambda_e \; \gamma p_{||}/p_R)) \tag{89}$$

where $\lambda_e = v_e/\omega_{pe}$ is the Debye length.

The equation for the cutoff, equation (46), using equation (72) for $T_\perp'$ becomes

$$eE = F + F^2/eE = (1 + \sqrt{5})F/2 \tag{90}$$

Remarkably, the ratio of the electric force to the dynamic friction at the cutoff is given by the golden mean! Although this equation is transcendental the unknown appears in a large exponent, and the desired root can be found approximately to a very high accuracy. The details are given in Appendix C In the relativistic limit, $\gamma \simeq p_{||}/mc$, which is the most useful one in practice.

$$\frac{p_c^2}{p_e^2} \simeq \frac{1}{\pi}\frac{n}{n_T}\frac{\Omega_e^2}{\omega_{pe}^2}\frac{\lambda_e}{r}\frac{p_r mc}{p_e^2}(1+\epsilon_1)\ln\zeta_1 , \tag{91}$$

where

$\epsilon_1 = [3/(\ln\zeta_1 - 3)]\ \ln.[\ln\zeta_1/(\ln 27\xi_1)], \zeta_1 = \xi_1(\ln 27\ \xi_1)^3$, and $\xi_1 = (2^{3/2}/\pi(1 + 5^{1/2})(p_r/p_e)^{1/2}(\Omega_e/\omega_{pe})[eE/(e^2/\lambda_e^2)][(\lambda/r)(n/n_T)(mc/p_e)]^{3/2}$;

The nonrelativistic limit $\gamma \to 1$ is obtained by deleting the mc/p term in Eq (91) replacing 27 by $(9/2)^{9/2}$ and replacing 3/2 in the last term of ξ_1 by 3. Note that in eq (91), $p_c^2 \sim 1/r$. This radial dependence of the cutoff momentum arises because of the convective nature of the instability. To see this, refer to fig. 3 and recall that $p_{||} \simeq m\Omega_e/k_{||}$. Consider a fixed radius r_0 and suppose that at some wave number $k_{||}^o$ (momentum $p_{||}^o$), the distribution function has cut off. For a slightly smaller $k_{||}$, there are no particles at the resonant momentum and radius r_0 so that the growth must start at a smaller radius r where $f_{||}$ is not yet cut off. In fact, solving eq (90) for r instead of $p_{||}$, gives the cutoff radius

$$\frac{r_c}{\lambda_e} = \frac{1}{\pi}\ \frac{n}{n_T}\ \left(\frac{\Omega_e}{\omega_{pe}}\ \frac{mc}{p_{||}}\right)^2 \frac{P_R}{Mc}\ \frac{(1+\epsilon)\ln\zeta}{\sqrt{1+(mc/p_{||})^2}} \tag{92}$$

where

$\epsilon = [3/2/(\ln\zeta - 3/2)]\ \ln\ (\ln\zeta/\ln\ 1.84\xi)$, $\zeta = \xi(\ln\ 1.84\xi)^{3/2}$, $\xi = [2(2\pi)^{1/2}/(1 + 5^{1/2})]$ $(eE/(e^2/\lambda_e^2))(\omega_{pe}/\Omega_e)^2\ (p_e/p_r)\ (m\Omega_e/p_e k_{||})^3$ and $1.84 \approx (3/2)^{3/2}$.

The net result for $f_{||}$ as a function of r and $p_{||}$ is shown in Fig 3 .

V. MOMENTUM SPACE FLOW PATHS

In order to better clarify the nature of the solution just obtained, we compute the flow associated with the steady state distribution function. This is effectively a transformation to a Lagrangian description from the Eulerian one which was more convenient for the calculation of f. Note that the steady state kinetic equation can be written as the divergence of a current (in momentum space) or, with angular symmetry

$$\partial J_{\parallel}/\partial p_{\parallel} + 1/p_{\perp}\ \partial/\partial p_{\perp}\ p_{\perp} J_{\perp} = 0, \tag{93}$$

where $\underline{J}$ contains the collisional, n =0,−1 quasilinear and electric field fluxes or accelerations. Equation (93) is identically satisfied $\underline{J} = \overline{\nabla}\times\overline{\psi}$, with $\overline{\psi} = (\psi_{\parallel}, \psi_{\phi}, \psi_{\perp})$. The ϕ symmetry makes only one component $\psi_{\phi} \equiv \psi$ necessary, so that

$$J_{\perp} = -\ 1/p_{\perp}\ \partial/\partial p_{\perp}\ (p_{\perp}\psi), \tag{94}$$

$$J_{\parallel} = 1/p_{\perp}\ \partial/\partial p_{\parallel}\ (p_{\perp}\psi). \tag{95}$$

Taking $J_{\parallel}$ times eq. (94) and subtracting $J_{\perp}$ times eq. (95) gives

$$J_{\parallel}\ \partial/\partial p_{\parallel}\ (p_{\perp}\psi) + J_{\perp}\ \partial/\partial p_{\perp}\ (p_{\perp}\psi) = 0 \tag{96}$$

a quasilinear partial differential equation [35], whose solution is given by

$$dp_{\parallel}/ds = J_{\parallel}, \tag{97}$$

$$dp_{\perp}/ds = J_{\perp}, \tag{98}$$

$$d/ds\ (p_{\perp}\psi) = 0. \tag{99}$$

The characteristics as given by eq. (97) and (98) are the flow lines we seek. Having obtained a solution with the Eulerian description, $J_{\parallel}$ and $J_{\perp}$ are known, and the flow lines can be obtained by direct integration.

In regions VII, where the D_0 term dominates in the parallel flow, we have

$$dp_{\parallel}/ds = - D_0 \ f/T_\perp \ dT_\perp/dp_\perp \ \ (-1 + p_\perp^2/2mT_\perp) \tag{100}$$

$$dp_\perp/ds = \nu_{ei} p_e^3 \ \gamma/p_{\parallel} \ \ p_\perp/2mT_\perp \ f. \tag{101}$$

Thus for *a* perpendicular momentum, $p_\perp^2 < 2mT_\perp$, the flow is toward higher parallel momentum, while at higher perpendicular momentum the flow is reversed, as shown in Fig. 4 returning to the bulk. Since $dp_{\parallel}/dp_\perp \gg 1$, the lines are generally flat, nearly parallel to the $p_{\parallel}$ axis.

In region IX, where the electric field and pitch angle scattering from the $\omega_{pe}\cos\theta$ modes are dominant, the flow lines are

$$dp_{\parallel}/ds = eEf \ [1 - p_\perp^2/2mT_\perp \ F/eE \ (1 - F/eE) \ - (p_\perp^2/2mT_\perp \ F/eE)^2] \tag{102}$$

$$dp_\perp/ds = p_{\parallel} \, p_\perp/2mT_\perp \ fF \ \ [\, 1 - F/eE \ + F/eE \ \ p_\perp^2/2mT_\perp] \tag{103}$$

These show generally the same behavior as in region VII. The difference here is that for $p_\perp^2 \sim 2mT_\perp$, $dp_\perp/dp_{\parallel} >> 1$ and the flow lines curve very rapidly towards the vertical $p_\perp$ axis; most of the electrons turn around in this region.

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APPENDIX A: The Quasilinear Friction Force

We discuss systems described by the Fokker-Planck equation, restricting consideration to thermal and weakly turbulent situations. The standard form of this equation[36] is

$$\partial f/\partial t = - \partial/\partial v_i \; 1/m \; F_i f + \frac{1}{2} \partial^2/\partial v_i \partial v_j \; D_{ij} f, \quad \text{(A1)}$$

where

$$F_i/m \equiv \langle \Delta v_i/\Delta t \rangle, \quad \text{(A2)}$$

$$D_{ij} \equiv \langle \Delta v_i \Delta v_j/\Delta t \rangle. \quad \text{(A3)}$$

In taking the momentum moment of eq. (A1), the second term on the right annihilates. The coefficient F_i is clearly interpretable as a force.

Equation (A1) can also be written

$$\partial f/\partial t = - \partial/\partial v_i \; 1/m F_i' f + \frac{1}{2} \partial/\partial v_i \; D_{ij} \; \partial/\partial v_j \; f, \quad \text{(A4)}$$

where

$$F_i' = F_i - m/2 \; \partial D_{ij}/\partial v_j.$$

Now it happens for the special case of collisional Coulomb interactions,[37] that the relation

$$F_i^c/m = \partial/\partial v_j \; D_{ij}^c, \quad \text{(A5)}$$

holds. Thus for this case

$$\partial f/\partial t = - \partial/\partial v_i \; 1/2m \; F_i^c f + \partial/\partial v_i \; \frac{1}{2} D_{ij}^c \; \partial/\partial v_j \; f, \quad \text{(A6)}$$

and excepting the factor of 1/2, the coefficients are the same whether one uses eq. (A 1) or (A 4). The coefficent of the first term in (A 4), F_i', is often referred to as the force of dynamical friction.[38] This terminology can be misleading since the second term in (A4) also alters the momentum, thus effecting a force. For example in Quasilinear therory, $F_i^{QL} = m/2 \; \partial D_{ij}^{QL}/\partial v_j$, so that $F_i' = 0$, and one would say that there are no friction forces in quasilinear theory. While this is certainly true in the convention of eq. (A4), it <u>suggests</u> an absence of forces, which is not true. Clearly the waves can contain momentum and the extraction of it from the particle will result in a force.

The coefficients (A 2) and (A 3) can be computed directly[39] for an arbitrary (small) level of electric field fluctuations. The test particle self-fields (which are not in general related to the ambient field fluctuations) contribute to F_i which can be written

$$F_i(\underline{v}) = eE_i^s(\underline{v}) + m/2 \; \partial D_{ij}/\partial v_j . \qquad \text{(A7)}$$

Therefore, $-eE_i^s = F_i'$, and it is the self-fields that are neglected in quasi-linear theory. The force coefficient, eq. (A 7), is still non-zero in general. In quasilinear theory, upon integrating over one of the coordinate variables produces in certain situations a reduced equation which has the form of eq. (A 4).

APPENDIX B

(a) DIFFUSION PATHS USING CONSERVATION OF ENERGY AND MOMENTUM

We shall first use a simple physical argument to find when a resonant particle is moved out of resonance by quasilinear scattering and thus provides a source of energy for the waves.[23] We define

$$n_k = 1/8\pi^2 \quad \partial\varepsilon_r/\partial\omega_r \; E_k^2$$

as the density of waves in the neighborhood of wave number $\underline{k}$. Then the conservation of energy and parallel momentum between Δn_k waves having $\underline{k}$ values between $(k, k + \Delta k)$, resonating with N particles having velocities between $(\underline{v}, \underline{v} + \Delta\underline{v})$ leads to,

$$mN \, (v_{\parallel} \Delta v_{\parallel} + v_{\perp}\Delta v_{\perp}) + \omega_r \Delta n_k = 0 \tag{B1}$$

$$mN\Delta v_{\parallel} + k_{\parallel} \Delta n_k = 0 \tag{B2}$$

where $k_{\parallel}$ is determined by the wave particle resonance condition. The perpendicular momentum need not be conserved since the applied magnetic field can absorb momentum. Solving (B2) for Δn_k and substituting in (B1) leads to,

$$(v_{\parallel} - \omega_r/k_{\parallel}) \, \Delta v_{\parallel} + v_{\perp}\Delta v_{\perp} = 0 \tag{B3}$$

We study these diffusion paths for two specific resonances.

Consider first the Landau interaction at n = 0 which requires that $\omega_r = k_{\parallel} v_{\parallel}$ then we see that the diffusion paths are

$$v_{\perp} = \text{constant} \tag{B4}$$

That is, the particle is scattered along constant perpendicular energy paths. The preferred direction being specified by the local slope in the distribution function.

By combining the resonance condition for the $n \neq 0$ wave particle interaction, $\omega_r - k_{\parallel} v_{\parallel} - n\Omega = 0$, together with the definition of the wave phase velocity for the particular waves of interest, we can write $\omega_r / k_{\parallel}$ in eq. (B-3) in terms of $v_{\parallel}$. This is easy to do in the case of magnetized plasma waves, $\omega_r = \omega_{pe} k_{\parallel} / k$ when $k_{\perp} \gg k_{\parallel}$, and leads to,

$$\frac{1}{2}(v_{\parallel} - \omega_{pe}/k_{\perp})^2 + v_{\perp}^2/2 = \text{constant} \qquad \text{(B5)}$$

These are circles centered at the wave phase velocity.

Once again the preferred direction will be given by the local slopes in the distribution function (as seen by the diffusing particle).

-

A more satisfying way to derive these results would be to start from the quasilinear kinetic equation and construct an H theorem. The kinetic equation describing the quasilinear evolution of the resonant electron distribution function is given by,[23]

$$\partial f/\partial t = 8\pi^2 e^2/m^2 \sum_n \int d^3k \; E_k^2/k^2 \; \mathcal{L}^{(n)} J_n^2(k_{\perp} v_{\perp}/\Omega) \, \delta(\omega_k - k_{\parallel} v_{\parallel} - n\Omega) \mathcal{L}^{(n)} f \qquad \text{(B6)}$$

where $\mathcal{L}^{(n)} = k_{\parallel} \partial/\partial v_{\parallel} + n\Omega/v_{\perp} \; \partial/\partial v_{\perp}$, J_n is the Bessel function, E_k^2 is the electric field energy density, and the delta function insures that we onlypick out the resonant distribution function. Define $H = \int d^3v \; f \ln f$, then using (B5)

$$dH/dt = -8\pi^2 e^2/m^2 \sum_n \int d^3v \int d^3k \; E_k^2 J_n^2(k_\perp v_\perp/\Omega)/k^2 \delta(\omega - k_{||}v_{||} - n\Omega)(\mathcal{L}^{(n)} f)^2/f \quad \text{(B-7)}$$

This implies that the marginally stable asymptotic states of f are given by the zeroes of H. This occurs in two ways: if E_k^2 vanishes (trivial case since there are no waves present) or if

$$(\mathcal{L}^{(n)} f)^2 = 0 \qquad \text{(B8)}$$

with $E_k^2 \neq 0$.

Equation (B8) is a simple first order partial differential equation it can be integrated by the method of characteristics, giving

$$dv_{||}/d\bar{s} = 1 \qquad \text{(B9)}$$

$$dv_\perp/ds = n\Omega/v_\perp k_{||} \qquad \text{(B10)}$$

$$df/d\bar{s} = 0 \qquad \text{(B11)}$$

In addition $k_{||}$ is specified by the delta function selection in eq.(B10). Equation (B11) implies that f is constant on the diffusion paths. Integrating Eq. (B9-10) reproduces (B4-5). Note that all of these analyses are based upon the assumption that each of the gyroresonances can be treated without any interferences from all the other gyroresonances. A wave at phase velocity $(\omega_r/k_{||})$ can suffer Landau growth (damping) at that phase velocity, gyroresonance growth (n = -1) and gyroresonance damping (n = +1) and similarly for all the other gyroresonances. In the case of the runaway electron tail, the distribution is so anisotropic that the gyroresonance damping is negligible and Landau damping is also negligible since the distribution function is flat. One final note, when $\omega_r \ll \Omega_e$, then the diffusion paths are virtually identical to constant energy surfaces and there is very little free energy available to drive the instabilities.

(b) CONSERVATION THEOREMS IN QUASILINEAR THEORY

Finally, we briefly turn our attention to the separation of the distribution function into resonant ($v \simeq \omega/k$) and non resonant ($v >> \omega/k$) parts and the various conservation of energy and momentum theorems between the waves and particles. We shall treat only the simple one dimensional model since the results generalize quite easilty to the three dimensional case. The quasilinear kinetic equations[23,30] in a one dimensional electric plasma are,

$$\partial f(v,t)/\partial t = \partial/\partial u \quad D\, \partial/\partial u\, f(v,t) \tag{B12}$$

$$\partial E_k/\partial t = 2\,\omega_i\, E_k^2 \tag{B13}$$

$$D = 8\pi(e/m)^2 \int dk \quad E_k^2/i(kv - \omega) \tag{B14}$$

$$\varepsilon = 1 - (\omega_{pe}/k)^2 \, ⨍ \, 1/(v - \omega_r/k) \quad \partial f/\partial v - i\pi(\omega_{pe}/k)^2 \, \partial f/\partial v \Big|_{v=\omega_r/k} = 0 \tag{B15}$$

where $f(v,t)$ is the background distribution function, ω_i is the growth rate, E_k^2 is the electrostatic electric field energy density, $\varepsilon = 0$ $\varepsilon = 0$ characterizes the particular dispersion relation that we wish to study and gives both the frequency of oscillation ω_r and the growth (damping) rate ω_i. We take the principle part in the integral in eq. (B15) which is the same as integrating only over the non resonant distribution function. It is well known that eq. (B11 - B15) conserve particles, momentum and energy when the total distribution function (resonant plus non resonant piece) is considered. Since it is somewhat cumbersome to continuously treat the distribution function consisting of a resonant and a non resonant piece, we shall instead consider a modified set of kinetic equations,

$$\partial f^R(v,t)/\partial t \;=\; \partial/\partial v \;\; D_0^R \;\; \partial f^R(v,t)/\partial v \tag{B16}$$

$$\partial \varepsilon/\partial t = 2\omega_i \varepsilon_k \tag{B17}$$

$$D_0^R \;=\; 8\pi^2(e/m)^2 \int dk \;\; 1/v \; \varepsilon_k/(\partial(\omega_r \varepsilon_r)/\partial\omega_r) \;\; \delta[k - \omega_r/v] \tag{B18}$$

$$\varepsilon_k = \;\; \partial/\partial\omega_r(\omega_r\varepsilon_r) \; E_k^2 \tag{B19}$$

where the R on the distribution function and diffusion coefficient signifies that this is the resonant piece, and ε_k is the total wave energy density and consists of the electric field energy density plus the kinetic energy of the nonresonant particles, ε_r is the real part of the plasma permittivity function (B15). The total wave energy density ε_k is obtained from

$$\partial/\partial t \int E_k^2 \, dk \;+\; \partial/\partial t \;\int nmv^2 f^{NR} \;\;\equiv\;\; \partial/\partial t \;\int dk \; E_k^2 \; \partial(\omega_r\varepsilon_r)/\partial\omega_r \tag{B20}$$

and using the kinetic equation for the non resonant (NR) distribution function and electric field energy density. It is a simple matter to now show that the kinetic equations in (B16) and (B19) conserve particles, momentum and energy. In proving momentum conservation, the following result will be useful

$$\partial/\partial t \int nmvf^{NR} \;=\; 2 \int dk \; k \; E_k^2 \; \omega_i \;\; \partial\varepsilon_r/\partial\omega_r \tag{B21}$$

and in proving energy conservation it will be necessary to make use of

$$\partial\omega_r\varepsilon_r/\partial\omega_r \;=\; -\;\omega_r/k \;(\omega_{pe}/k)^2 \int \partial f/\partial v \;\; dv/(v - \omega_r/k)^2 \tag{B22}$$

$$\omega_i = -\;\varepsilon_i(\omega_r)/\partial\varepsilon_r(\omega_r)/\partial\omega_r \tag{B23}$$

where ε_i is the imaginary part of the plasma permittivity function in (B15)

The advantage of the above set of equations is that one no longer has to solve for the nonresonant distribution function as long as the total wave energy density is used in the diffusion coefficient. In addition, the waves now carry momentum, because the mechanical oscillation of the non resonant electrons has been included in the description of the waves. The electrostatic field itself does not carry any momentum.

APPENDIX : C

We outline a method of getting very accurate approximate solutions to transcendental equations of the form,

$$A = e^{Bx}/x^{\ell} \tag{C1}$$

A,B are constants and ℓ can be any power. In the limit where $Bx >> 1$, we look for solutions with $x >> x_0$ where x_0 is the point at which eq. (C-1) exhibits a minimum ($x_0 = \ell/B$). It is now convenient to define $y = x/x_0$ look for solutions with $y >> 1$. Taking the logarithms of eq. (C-1), we obtain,

$$y - \ln y - \frac{1}{\ell} \ln A(\ell/B)^{\ell} = 0 \tag{C2}$$

For $y >> 1$ this can be solved by iteration,

$$y^{(0)} = \frac{1}{\ell} \ln A(\ell/B)^{\ell} \tag{C3}$$

$$y^{(1)} = \ln y^{(0)} \tag{C4}$$

$$y^{(2)} = \ln y^{(1)} \tag{C5}$$

where $y^{(0)} >> y^{(1)} >> y^{(2)}$, which is kept up until $y^{(n)} > 1$, where it must be stopped. The remainder term then determines the error in the asymptotic series. Since the sequence generated consists of compounded logarithms, the terms decrease very rapidly. For our case of interest, the first two terms suffice to produce

$$y = x/(\ell/B) = \frac{1}{\ell} \ln((A/B^{\ell})(\ln(A(\ell/B)^{\ell})^{\ell})) \tag{C6}$$

This is a good approximation for the exponential term of eq. (C1) However it is sometimes necessary to improve upon the expansion in compounded logarithms.

This is done by performing a Newton-Raphson iteration using eq. (C6) as the initial guess. This leads to

$$x = (1 + \varepsilon)x_A \tag{C7}$$

where x_A is given in eq. (C-6) and

$$\varepsilon = (\ell/\ln\zeta - \ell)\ln(\ln\zeta/\ln A(\ell/B)^{\ell}) \tag{C8}$$

$$\zeta = (A/B^{\ell})\ [\ln A(\ell/B)^{\ell}]^{\ell} \tag{C9}$$

Figure Captions

Fig. 1

The parallel distribution function $f_{\parallel}(p_{\parallel}) = 2\pi\int p_{\perp}dp_{\perp}f$ as a function of $p_{\parallel}$. The drift velocity of the bulk is indicated by p_D and the runaway momentum by p_R. The classical (collisional) distribution function is shown dashed for $p_{\parallel} > p_R$. The positive slope due to the ω_{pe} modes is shown highly exaggerated for $p_R \leq p_{\parallel} \leq p_C$. The dynamic friction due to the $\omega_{pe}\cos\theta$ modes becomes effective for $p_{\parallel} \geq p_1 \cong \Omega_e p_R/\omega_{pe}$ and cuts the distribution function off at p_C. The $n = \pm 1$ gyroresonance interactions are shown. The non-relativistic picture can be obtained simply by letting $\gamma = 1$.

Fig. 2

The plasma wave spectrum consists of ω_{pe} modes $k_{\parallel} \simeq 0$ and $\omega_{pe}\cos\theta$ modes $k_{\perp} \gg k_{\parallel}$. The maximum k is limited by Landau damping $(\omega/k_{\parallel} \geq v_R)$. The X denotes the position of the maximally unstable waves for a finite length tail $(p_{\parallel} < p_C)$.

Fig. 3

The structure of the high energy tail in momentum and position space. The distribution function is equal to the classical one in the shaded region and is zero outside.

Fig. 4

Contours of the acceleration field stream function. The fact the lines close upon themselves is indicative of a steady state. In the dashed region between $p_0 < p_{\parallel} < p_{01}$, the flow lines have not been computed exactly.

Fig. 5

The quasilinear diffusion paths for the $n = 0$ (Landau) resonance and the $n = -1$ gyroresonance. In the Landau case, the diffusion paths are $p_{\perp}$ = constant while in the gyroresonance case they are circles centered at the wave phase velocity.

Fig. 6

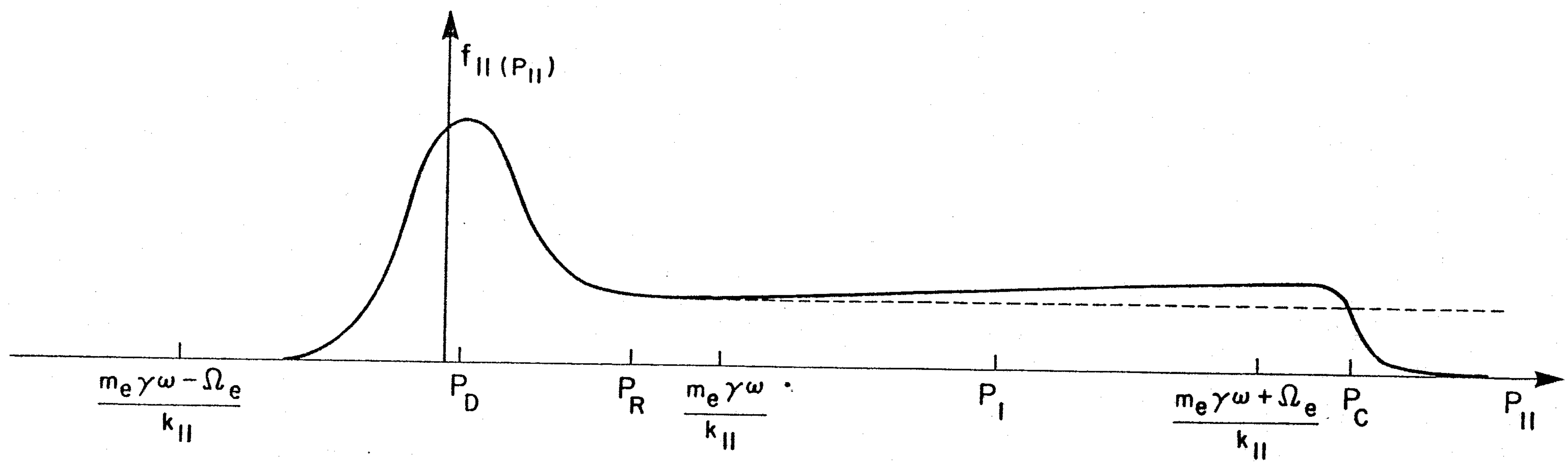
The diffusion coefficient due to the absolutely unstable ω_{pe} modes. Region IV of Kruskal-Bernstein extends approximately to $p_{\parallel} \sim p_R(1+(E/E_R)^{1/3}) >> p_0$ where $D_0(p_0) = 0$. Hence we have relabeled their region IV by our V, VI and part of VIIa. Their region V is replaced by our regions VII - X. Regions VI, VIII where D_0 varies rapidly are replaced by jump conditions.

Fig. 7

The perpendicular temperature as a function of momentum in the various regions. The heating in region VII is greatly exaggerated.

Fig. 8

Transformation of coordinates for the evaluation of the Green's Function from (r,θ) to coordinates centered on a line joining $(\bar{r}, \bar{r}')$. Waves propogating through the conical region (shaded) produce the dominant contribution to the fluctuation level at $(r,\theta = 0)$. The plasma radius is denoted by a.



$f_{II}(P_{II})$
$\frac{m_e \gamma \omega - \Omega_e}{k_{II}}$
P_D
P_R
$\frac{m_e \gamma \omega}{k_{II}}$
P_I
$\frac{m_e \gamma \omega + \Omega_e}{k_{II}}$
P_C
P_{II}

Fig 1

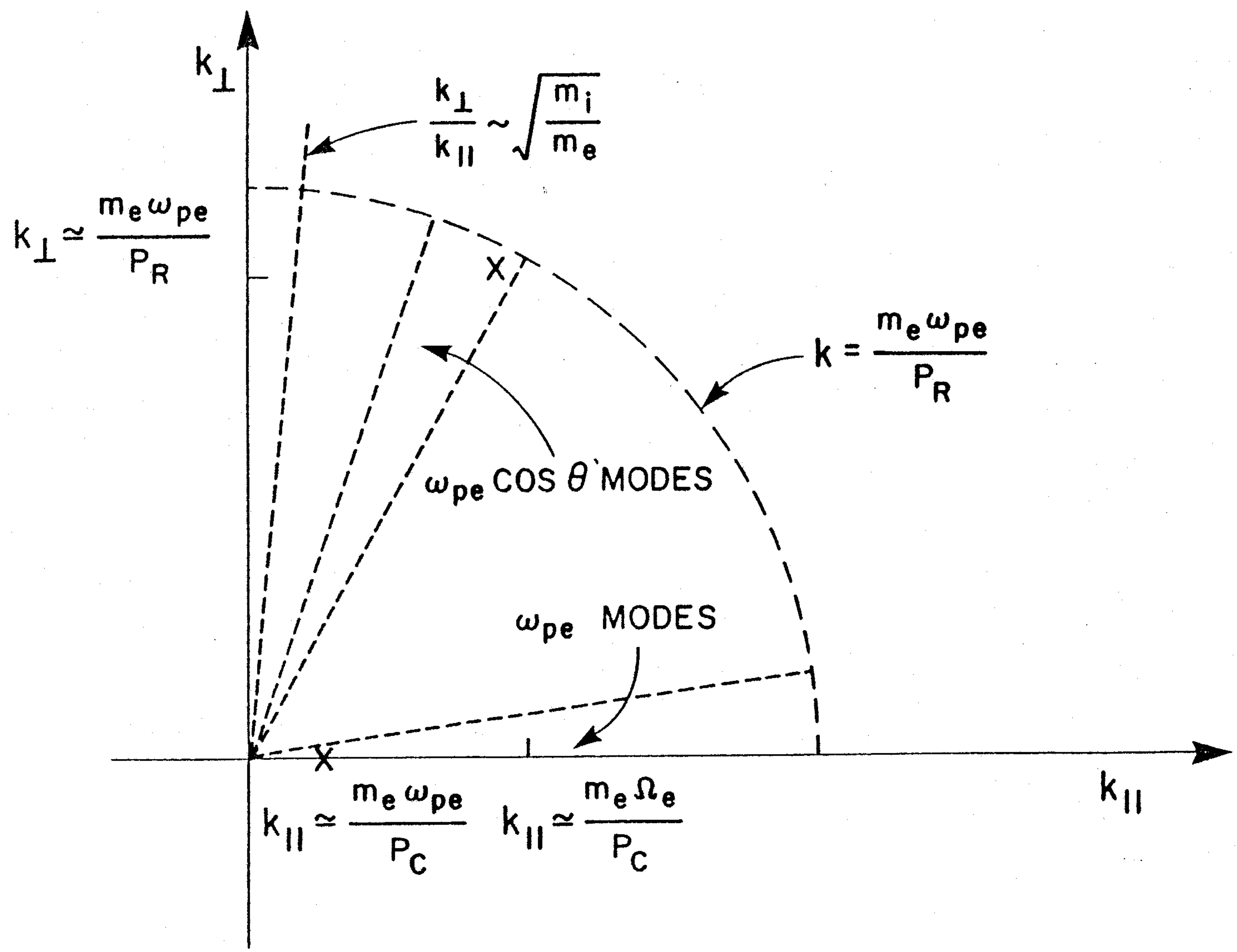
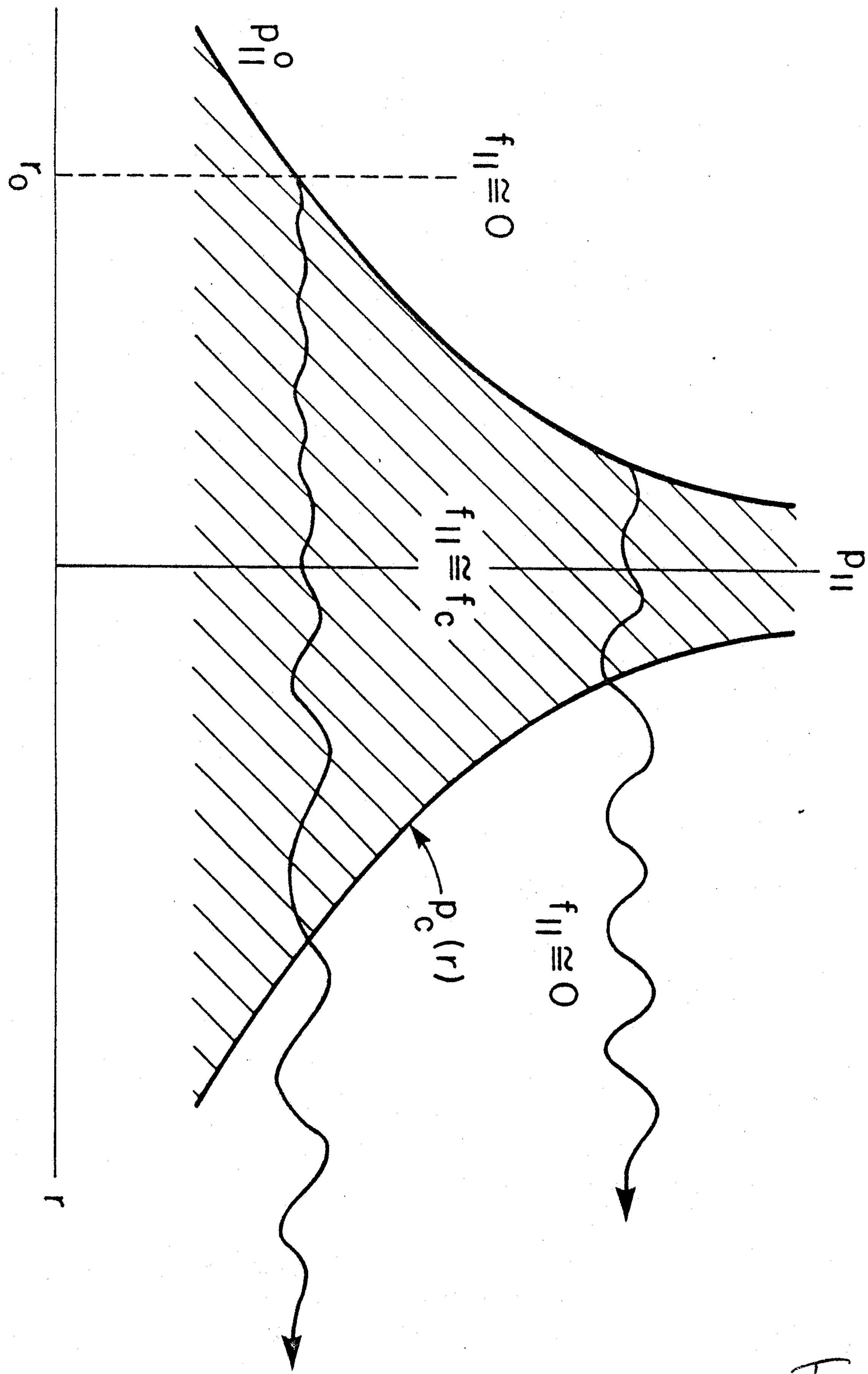

$k_\perp$
$\frac{k_\perp}{k_\parallel} \sim \sqrt{\frac{m_i}{m_e}}$
$k_\perp \simeq \frac{m_e \omega_{pe}}{P_R}$
$k = \frac{m_e \omega_{pe}}{P_R}$
ω_{pe} COS θ' MODES
ω_{pe} MODES
$k_\parallel \simeq \frac{m_e \omega_{pe}}{P_C}$
$k_\parallel \simeq \frac{m_e \Omega_e}{P_C}$
$k_\parallel$

Fig 2



$p_{\parallel}^{o}$
$f_{\parallel} \cong 0$
r_o
$f_{\parallel} \cong f_c$
$p_{\parallel}$
$p_c(r)$
$f_{\parallel} \cong 0$
r

Fig 3

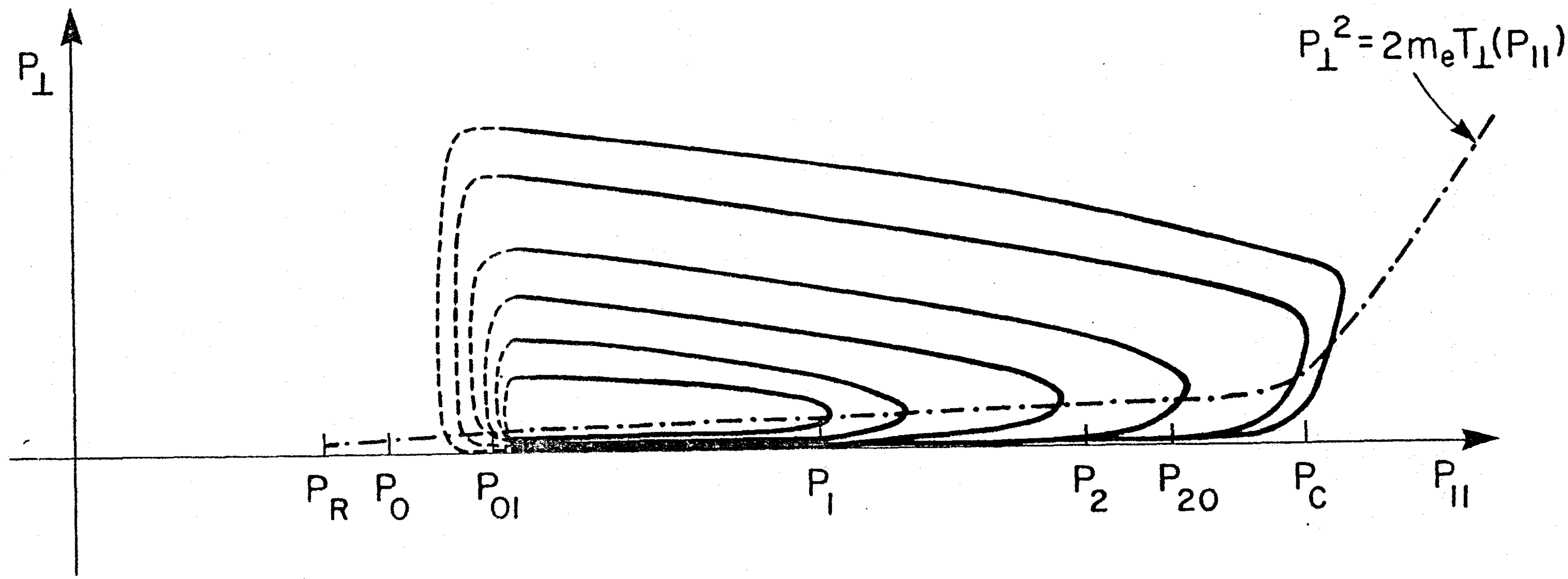


FIG 4

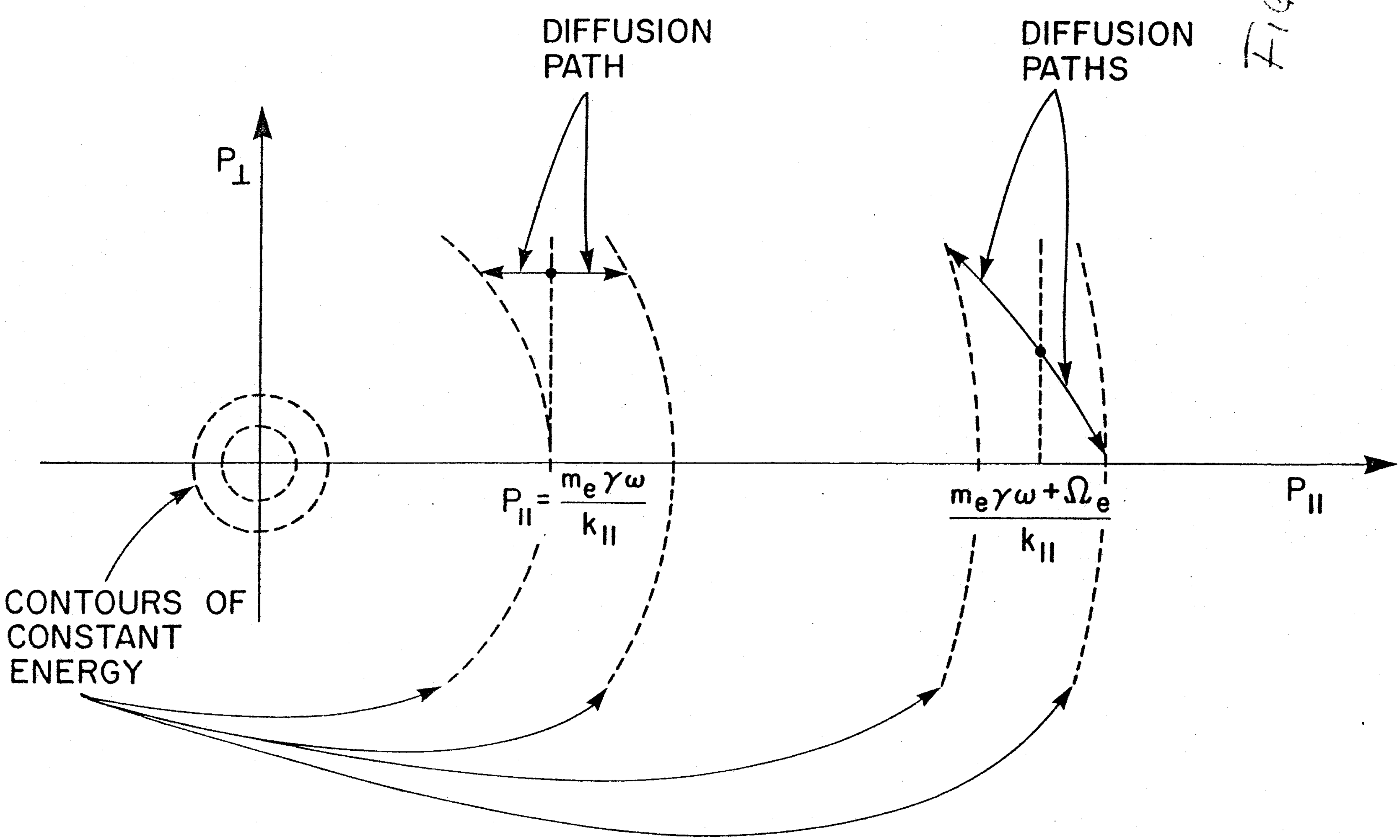

DIFFUSION PATH
DIFFUSION PATHS
$P_\perp$
$P_{\parallel} = \frac{m_e \gamma \omega}{k_{\parallel}}$
$\frac{m_e \gamma \omega + \Omega_e}{k_{\parallel}}$
$P_{\parallel}$
CONTOURS OF CONSTANT ENERGY

FIG 5

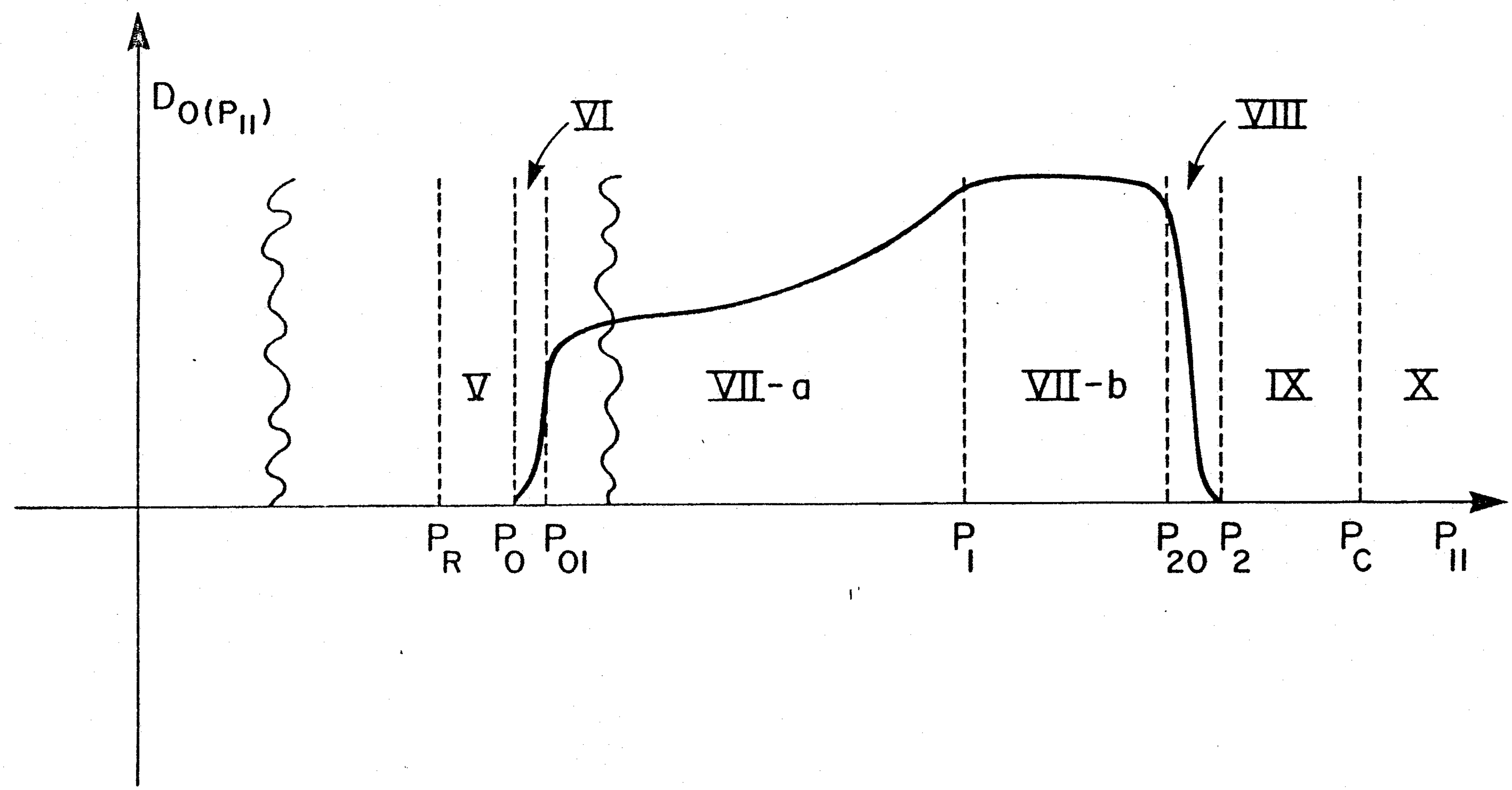


FIG 6

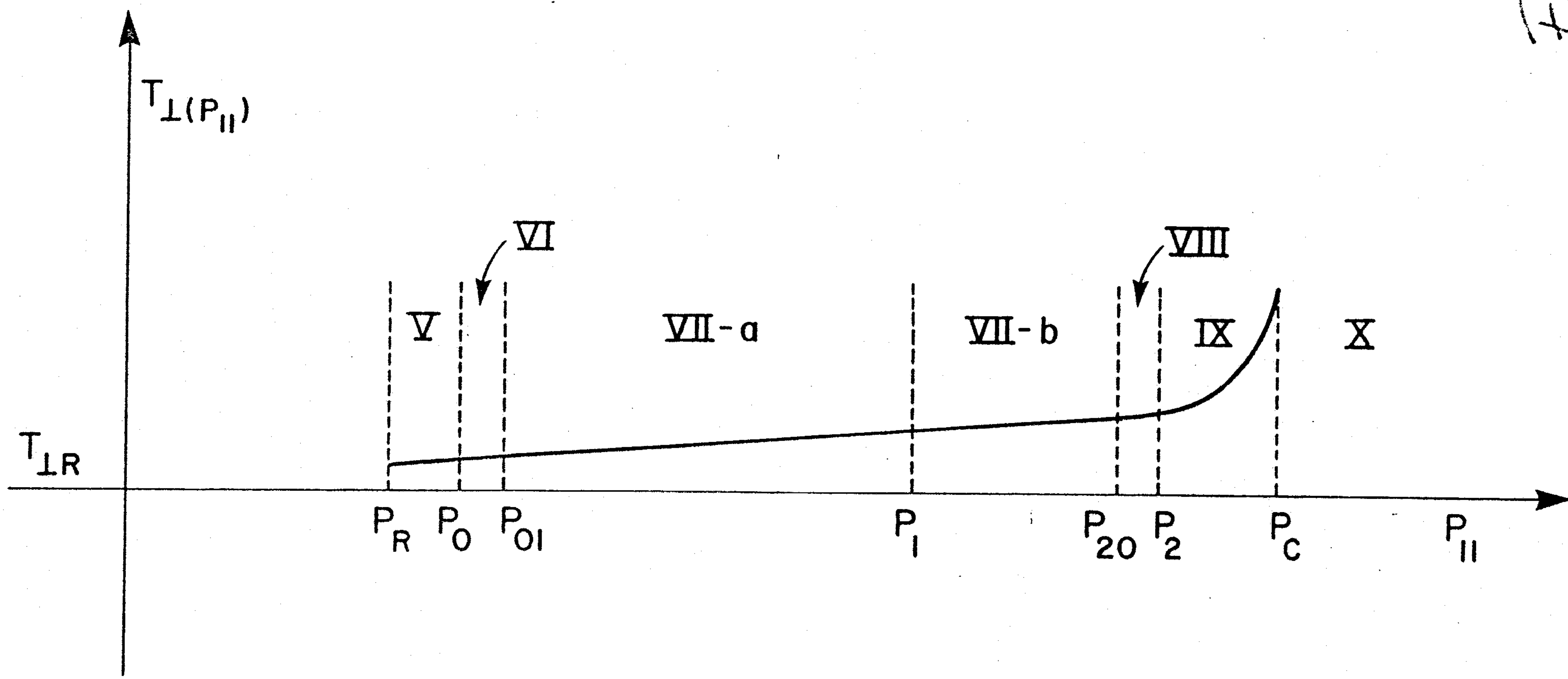


FIG 7

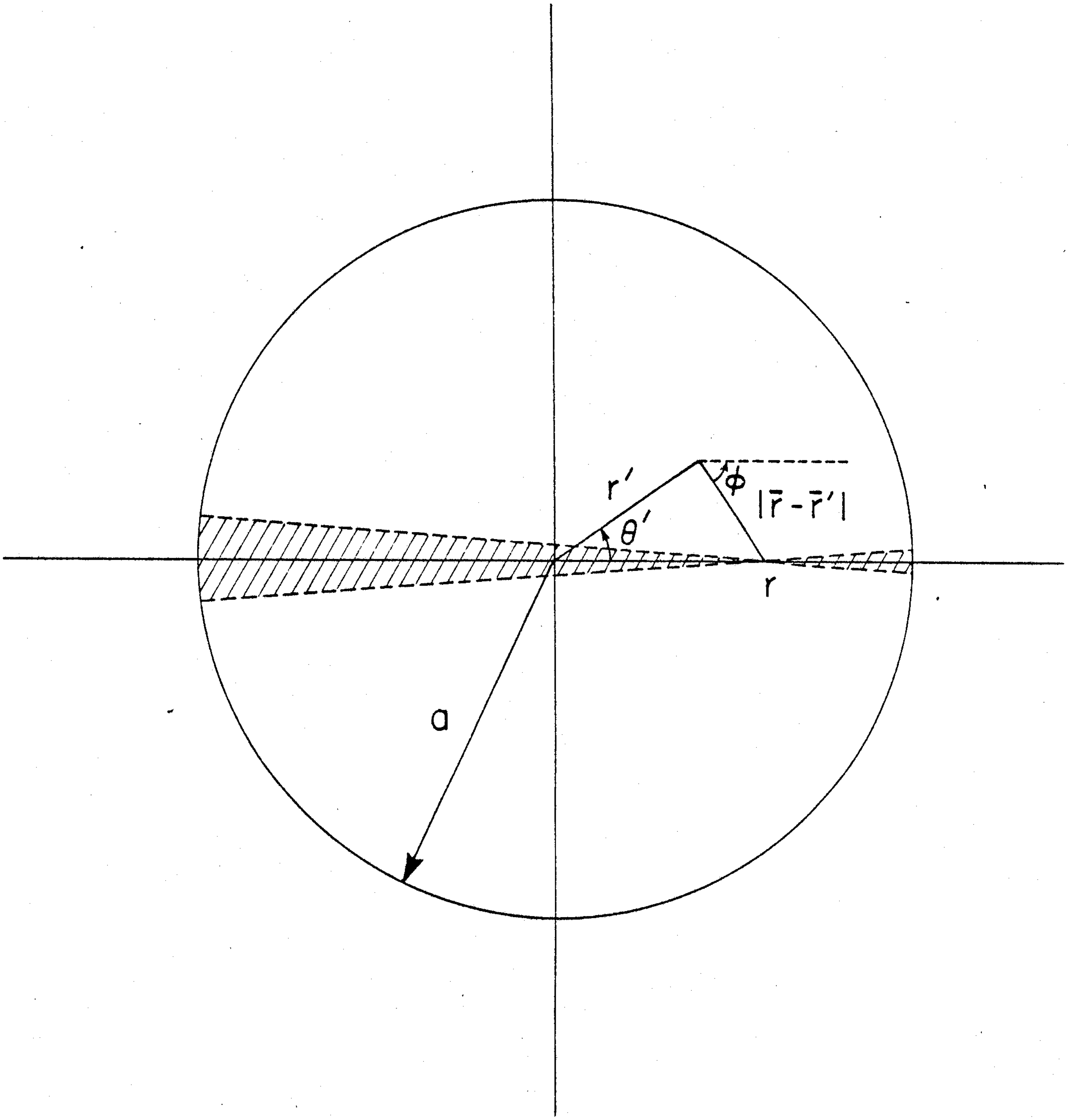


FIG 8