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STINESPRING'S THEOREM FOR MAPS ON HILBERT C^*-MODULES

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Abstract. We strengthen Mohammad B. Asadi's analogue of Stinespring's theorem for certain maps on Hilbert C^*-modules. We also show that any two minimal Stinespring representations are unitarily equivalent. We illustrate the main theorem with an example.

1. Introduction

Stinespring's representation theorem is a fundamental theorem in the theory of completely positive maps. It is a structure theorem for completely positive maps from a C^*-algebra into the C^*-algebra of bounded operators on a Hilbert space. This theorem provides a representation for completely positive maps, showing that they are simple modifications of $*$-homomorphisms (see [9] for details). One may consider it as a natural generalization of the well-known Gelfand-Naimark-Segal thoerem for states on C^*-algebras (see [2, Theorem 4.5.2, page 278] for details). Recently, a theorem which looks like Stinespring's theorem was presented by Mohammad B. Asadi in [1] for a class of unital maps on Hilbert C^*-modules. Here we strengthen this result by removing a technical condition of Asadi's theorem [1]. We also remove the assumption of unitality on maps under consideration. Further we prove uniqueness up to unitary equivalence for minimal representations, which is an important ingredient of structure theorems like GNS theorem and Stinespring's theorem. Now the result looks even more like Stinespring's theorem.

1.1. **Notations and Earlier Results.** We denote Hilbert spaces by H, H_1, H_2 etc and the corresponding inner product and the induced norm by $\langle ., . \rangle$ and $\| \cdot \|$ respectively. Our inner products are conjugate linear in the first variable and linear in the second variable. The space of bounded linear operators from H_1 to H_2 is denoted by $\mathcal{B}(H_1, H_2)$ and if $H_1 = H_2 = H$, then $B(H_1, H_2) = \mathcal{B}(H)$. The C^*-algebra of all $n \times n$ matrices with entries from a C^*-algebra $\mathcal{A}$ is denoted by $\mathcal{M}_n(\mathcal{A})$. If L is a subset of a Hilbert space, then $[L] := \overline{\text{span}}(L)$.

Now we consider maps on Hilbert C^*-modules. Let E be a Hilbert C^*-module over a C^*-algebra $\mathcal{A}$ (see [4] for details of Hilbert C^*-modules). Let $\phi : \mathcal{A} \to \mathcal{B}(H)$ be linear. Then ϕ is said to be a morphism if it is a $*$ homomorphism and nondegenerate (i.e., $\overline{\phi(\mathcal{A})H} = H$). We remind the reader that $\mathcal{B}(H_1, H_2)$ is

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a Hilbert $\mathcal{B}(H_1)$-module with respect to the inner product $\langle T, S\rangle = T^*S$. A map $\Phi : E \to \mathcal{B}(H_1, H_2)$ is said to be a

(1) ϕ**-map** if $\langle \Phi(x), \Phi(y)\rangle = \phi(\langle x, y\rangle)$ for all $x, y \in E$;
(2) ϕ**-morphism** if Φ is a ϕ-map and ϕ is a morphism;
(3) ϕ**-representation** if Φ is a ϕ-morphism and ϕ is a representation.

Note that a ϕ-morphism Φ is linear and satisfies $\Phi(xa) = \Phi(x)\phi(a)$ for every $x \in E$ and $a \in \mathcal{A}$. Several module versions of Stinespring theorem can be found in the literature. Typically they are structure theorems for completely positive maps in more general contexts ([3, 5, 6]). The result we are going to consider here are for ϕ-maps. M. Skeide has informed us that ϕ-morphisms are also known as ϕ-isometries in the literature (See [8] for further references). He has also remarked that as in the case of Stinespring's theorem the result below can be generalized further using the language of Hilbert modules.

Theorem 1.2. *(Mohammad B. Asadi [1]). If E is a Hilbert C^*-module over a unital C^*-algebra $\mathcal{A}$, $\phi : \mathcal{A} \to \mathcal{B}(H_1)$ is a completely positive map with $\phi(1) = 1$ and $\Phi : E \to \mathcal{B}(H_1, H_2)$ is a ϕ-map with the additional property $\Phi(x_0)\Phi(x_0)^* = I_{H_2}$ for some $x_0 \in E$, where H_1, H_2 are Hilbert spaces, then there exist Hilbert spaces K_1, K_2, isometries $V : H_1 \to K_1$, $W : H_2 \to K_2$, a $*$-homomorphism $\rho : \mathcal{A} \to \mathcal{B}(K_1)$ and a ρ-representation $\Psi : E \to \mathcal{B}(K_1, K_2)$ such that*

$$\phi(a) = V^*\rho(a)V, \quad \Phi(x) = W^*\Psi(x)V \quad \textit{for all } x \in E,\ a \in \mathcal{A}.$$

The proof of this Theorem as given in [1] is erroneous as the sesquilinear form defined there on $E \otimes H_2$ is not positive definite. This can be fixed by interchanging the indices i, j in the definition of this form. However such a modification yields a 'non-minimal' representation. Moreover, the technical condition to have $\Phi(x_0)\Phi(x_0)^* = I_{H_2}$ for some $x_0 \in E$ is completely unnecessary.

2. Main Results

In this Section we strengthen Asadi's theorem for a ϕ-map Φ and discuss the minimality of the representations.

Theorem 2.1. *Let $\mathcal{A}$ be a unital C^*-algebra and $\phi : \mathcal{A} \to \mathcal{B}(H_1)$ be a completely positive map. Let E be a Hilbert $\mathcal{A}$-module and $\Phi : E \to \mathcal{B}(H_1, H_2)$ be a ϕ-map. Then there exists a pair of triples (ρ, V, K_1) and (Ψ, W, K_2), where*

(1) K_1 and K_2 are Hilbert spaces;
(2) $\rho : \mathcal{A} \to \mathcal{B}(K_1)$ is a unital $*$-homomorphism and $\Psi : E \to \mathcal{B}(K_1, K_2)$ is a ρ-morphism;
(3) $V : H_1 \to K_1$ and $W : H_2 \to K_2$ are bounded linear operators;
(4) $\phi(a) = V^*\rho(a)V$, for all $a \in \mathcal{A}$ and $\Phi(x) = W^*\Psi(x)V$, for all $x \in E$.

Proof. We prove the theorem in two steps. **Step I:** Existence of ρ, V and K_1: This is the content of Stinespring's theorem [7, Theorem 4.1, page 43]. In fact we can choose a minimal Stinespring representation (ρ, V, K_1) for ϕ. That is $K_1 = [\rho(\mathcal{A})VH_1]$. **Step II:** Construction of Ψ, W and K_2: Let $K_2 := [\Phi(E)H_1]$. For $x \in E$, define $\Psi(x) : K_1 \to K_2$ by

$$\Psi(x)\Big(\sum_{j=1}^{n} \rho(a_j)Vh_j\Big) := \sum_{j=1}^{n} \Phi(xa_j)h_j, \quad a_j \in \mathcal{A},\ h_j \in H_1,\ j = 1, \dots, n,\ n \geq 1.$$

Since

$$\begin{aligned}||\Psi(x)\big(\sum_{j=1}^{n}\rho(a_j)Vh_j\big)||^2 &= \sum_{i,j=1}^{n}\langle h_j, V^*\rho(a_j^*\langle x,x\rangle a_i)Vh_i\rangle \\ &\leq \|\rho(\langle x,x\rangle)\|\,\|\sum_{j=1}^{n}\rho(a_j)Vh_j\|^2 \\ &\leq \|x\|^2\,\|\sum_{j=1}^{n}\rho(a_j)Vh_j\|^2,\end{aligned}$$

$\Psi(x)$ is well defined and bounded. Hence it can be extended to whole of K_1. This gives the required Ψ. To prove that Ψ is a ρ-morphism, let $x \in E, a_j \in \mathcal{A}, h_j \in H_1,\ j = 1, 2, \ldots, n,\ n \geq 1$. Then

$$\begin{aligned}\langle \Psi(x)^*\Psi(y)\big(\sum_{j=1}^{n}\rho(a_j)Vh_j\big), \sum_{i=1}^{n}\rho(a_i)Vh_i\rangle &= \sum_{i,j=1}^{n}\langle \phi(\langle xa_i, ya_j\rangle)h_j, h_i\rangle \\ &= \langle \rho(\langle x,y\rangle)\big(\sum_{j=1}^{n}\rho(a_j)Vh_j\big), \sum_{i=1}^{n}\rho(a_i)Vh_i\rangle.\end{aligned}$$

Thus $\Psi(x)^*\Psi(y) = \rho(\langle x,y\rangle)$ on the dense set $\mathrm{span}(\rho(A)VH_1)$ and hence they are equal on K_1. Note that $K_2 \subseteq H_2$. Let $W := P_{K_2}$, the orthogonal projection onto K_2. Then $W^* : K_2 \to H_2$ is the inclusion map. Hence $WW^* = I_{K_2}$. That is W is a co-isometry. Now for $x \in E$ and $h \in H_1$, we have $W^*\Psi(x)Vh = \Psi(x)Vh = \Psi(x)(\rho(1)Vh) = \Phi(x)h$. □

Definition 2.2. *Let ϕ and Φ be as in Theorem 2.1. We say that a pair of triples $\big((\rho, V, K_1), (\Psi, W, K_2)\big)$ is a **Stinespring representation** for (ϕ, Φ) if the conditions (1)-(3) of Theorem 2.1 are satisfied. Such a representation is said to be **minimal** if*

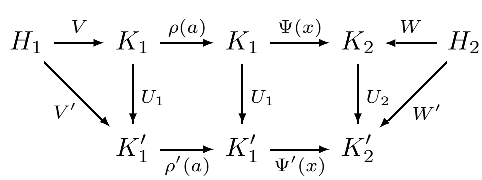
(a) $K_1 = [\rho(A)VH_1]$ and (b) $K_2 = [\Psi(E)VH_1]$.

Remark 2.3. Let ϕ and Φ be as in Theorem 2.1. The pair $\big((\rho, V, K_1), (\Psi, W, K_2)\big)$ obtained in the proof of Theorem 2.1, is a minimal representation for (ϕ, Φ).

Theorem 2.4. *Let ϕ and Φ be as in Theorem 2.1. Assume that $\big((\rho, V, K_1), (\Psi, W, K_2)\big)$ and $\big((\rho', V', K_1'), (\Psi', W', K_2')\big)$ are minimal representations for (ϕ, Φ). Then there exists unitary operators $U_1 : K_1 \to K_1'$ and $U_2 : K_2 \to K_2'$ such that*

1. *$U_1V = V'$, $U_1\rho(a) = \rho'(a)U_1$, for all $a \in \mathcal{A}$ and*
2. *$U_2W = W'$, $U_2\Psi(x) = \Psi'(x)U_1$, for all $x \in E$.*

That is, the following diagram commutes, for $a \in \mathcal{A}$ and $x \in E$:



Proof. Define $U_1 : \mathrm{span}(\rho(\mathcal{A})VH_1) \to \mathrm{span}(\rho'(\mathcal{A})V'H_1)$ by

$$U_1\big(\sum_{j=1}^{n}\rho(a_j)Vh_j\big) := \sum_{j=1}^{n}\rho'(a_j)V'h_j, \quad a_j \in \mathcal{A},\ h_j \in H_1, j = 1, \ldots, n,\ n \geq 1,$$

which can be seen to be an onto isometry and the unitary extension of this is the required map $U_1 : K_1 \to K_2$ ([7, Theorem 4.2, page 46]).

Now define $U_2 : \mathrm{span}(\Psi(E)VH_1) \to \mathrm{span}(\Psi'(E)V'H_1)$ by

$$U_2(\sum_{j=1}^{n} \Psi(x_j)Vh_j) := \sum_{j=1}^{n} \Psi'(x_j)V'h_j, \quad x_j \in E, h_j \in H_1, j = 1, 2, \ldots, n,\, n \geq 1.$$

Consider

$$\begin{aligned}\|\sum_{j=1}^{n} \Psi'(x_j)V'h_j\|^2 &= \sum_{i,j=1}^{n} \langle h_j, V'^* \rho'(\langle x_j, x_i\rangle)V'h_i\rangle \\ &= \sum_{i,j=1}^{n} \langle h_j, V^* \rho(\langle x_j, x_i\rangle)Vh_i\rangle \\ &= \|\sum_{j=1}^{n} \Psi(x_j)Vh_j\|^2.\end{aligned}$$

Thus U_2 is well defined and an isometry and can be extended to whole of K_2, call the extension U_2 itself, and being onto it is a unitary.

Since $\big((\rho, V, K_1), (\Psi, W, K_2)\big)$ and $\big((\rho', V', K_1'), (\Psi', W', K_2')\big)$ are representations for (ϕ, Φ), it follows that $\Phi(x) = W^*\Psi(x)V = W'^*\Psi'(x)V' = W'^*U_2\Psi(x)V$ and hence $(W^* - W'^*U_2)\Psi(x)V = 0$. Since $[\Psi(E)VH_1] = K_2$, it follows that $U_2W = W'$. As Ψ is a ρ-morphism and Ψ' is a ρ'-morphism, it can be shown that

$$U_2\Psi(x)\big(\sum_{j=1}^{n} \rho(a_j)Vh_j\big) = \Psi'(x)U_1\big(\sum_{j=1}^{n} \rho(a_j)Vh_j\big),$$

for all $x \in E$, $a_j \in \mathcal{A}, h_j \in H_1, 1 \leq j \leq n,\ n \geq 1$, concluding $U_2\Psi(x) = \Psi'(x)U_1$. □

Remark 2.5. Let $\big((\rho, V, K_1), (\Psi, W, K_2)\big)$ be a Stinespring representation for (ϕ, Φ). If ϕ is unital, then V is an isometry. If the representation is minimal, then W is a co-isometry by the proof of Theorem 2.1 and (2) of Theorem 2.4.

Example 2.6. Let $\mathcal{A} = \mathcal{M}_2(\mathbb{C}),\ H_1 = \mathbb{C}^2, H_2 = \mathbb{C}^8$ and $E = \mathcal{A} \oplus \mathcal{A}$. Let $D = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$. Define $\phi : \mathcal{A} \to \mathcal{B}(H_1)$ by $\phi(A) = D \circ A$, for all $A \in \mathcal{A}$, here $\circ$ denote the Schur product. As D is positive, ϕ is a completely positive map (see [7, Theorem 3.7, page 31] for details). Let $D_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$ and $D_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} \end{pmatrix}$. Define $\Phi : E \to \mathcal{B}(H_1, H_2)$ by

$$\Phi\big(A_1 \oplus A_2\big) = \begin{pmatrix} \frac{\sqrt{3}}{\sqrt{2}}A_1D_1 & \frac{\sqrt{3}}{\sqrt{2}}A_2D_1 & \frac{1}{\sqrt{2}}A_1D_2 & \frac{1}{\sqrt{2}}A_2D_2 \end{pmatrix}^{tr},\ A_1, A_2 \in \mathcal{A}.$$

It can be verified that Φ is a ϕ-map.

Let $K_1 = \mathbb{C}^4$ and $K_2 = H_2$. In this case $\rho : \mathcal{A} \to \mathcal{B}(K_1),\ V : H_1 \to K_1$ and $\Psi : E \to \mathcal{B}(K_1, K_2)$ are given by

$$V = \begin{pmatrix} \frac{\sqrt{3}}{\sqrt{2}}D_1 \\ \frac{1}{\sqrt{2}}D_2 \end{pmatrix}, \rho(A) = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}, \Psi\big(A_1 \oplus A_2\big) = \begin{pmatrix} A_1 & A_2 & 0 & 0 \\ 0 & 0 & A_1 & A_2 \end{pmatrix}^{tr}.$$

$A, A_1, A_2 \in \mathcal{A}$. Clearly Ψ is a ρ-morphism and $\Phi(A_1 \oplus A_1) = W^*\Psi(A_1 \oplus A_2)V$, where $W = I_{H_2}$. This example illustrates Theorem (2.1). Note that in this example there does not exists an $x_0 \in E$ with the property that $\Phi(x_0)\Phi(x_0)^* = I_{H_2}$, which is an assumption in Theorem 1.2.

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