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MATHEMATICAL MODELS

Stochastic analysis of a controlled queue with heterogeneous servers and constant retrial rate[1]

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Abstract—In this paper we analyze a controlled retrial queue with several exponential heterogeneous servers in which the time between two successive repeated attempts is independent of the number of customers applying for the service. The customers upon arrival are queued in the orbit or enters service area according to the control policy. This system is analyzed as controlled quasi-birth-and-death (QBD) process. It is showed that the optimal control policy is of threshold and monotone type. We give the explicit formula for the approximation to the optimal threshold levels and propose value iteration algorithm for the exact calculation of the levels. The steady-state analysis is performed using matrix-geometric approach. The main performance characteristics are calculated for the system under optimal threshold policy (OTP) and compared with the same characteristics for the model under scheduling threshold policy (STP) and other heuristic policies, e.g. the usage of the Fastest Free Server (FFS) or Random Server Selection (RSS).

1. INTRODUCTION

Different types of multi-server retrial queueing systems under a variety of modifications for single and multiple server cases have found applications in local area networks and communication protocols. For some concrete examples we refer to [3, 5, 21]. The multi-server retrial queues have been extensively studied for homogeneous servers (equal service intensities) [2, 4, 7] and for heterogeneous servers (different service intensities) [13, 20]. The queues with heterogeneous servers are extensively used for modeling real systems, e.g. the models with servers that have different types of processors as a consequence of system upgrade, nodes in telecommunication networks with links of different capacities, nodes in wireless systems serving different mobile users, are assumed to involve heterogeneous servers.

The system with heterogeneous servers are mostly investigated with respect to the heuristic service policies, e.g. when the servers must be activated according to the Fastest Free Server (FFS) or Random Service Selection (RSS) policies. The motivation for considering the controlled queues comes from the fact that many of the studied systems with heterogeneous servers do not incorporate facilities (controllers) that allow to pursue different control policies, possibly based upon state dependent decisions. But if the servers are heterogeneous then the controller may considerably improve the system's performance in comparison with mentioned above heuristic policies, e.g. by reducing the sojourn time (or the number of customers in the system): it is better to wait until the faster server will be idle then to occupy the slower one. It was shown in [18, 19] that classical queueing systems under optimal control policy are superior in performance to the homogeneous ones with the same total service time.

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The queueing systems that assumed to be dynamically controlled may be treated in the framework of Markov decision models [14, 15, 16]. Controlled classical queues with heterogeneous servers were studied in [17, 18]. It was shown that the dynamic programming value function has specific monotonicity properties that imply a threshold structure of the optimal control policy which minimizes the mean number of customers in the system. It means that for each server there exists some threshold level that specifies the number of customers in the queue when this server must be activated. Analogous results as a conjecture were formulated in [6] for the queues with classical retrial policy and phase-type interarrival and service time distributions. But the given proof for the monotonicity of increments in exponential case is not complete since it is necessary to prove some additional convexity properties for the value function similar to the inequalities shown for the classical queue in [9].

The analysis of controlled queue with constant retrial rate includes the following contributions:

(a) We prove the convex property of the value function that confirms the threshold structure of the optimal control policy.

(b) Analysis of the boundary to the areas with different optimal threshold levels we obtain the approximation formula for the explicit calculation of threshold levels.

(c) We model the system by means of a quasi-birth-and-death (QBD) process with policy dependent infinitesimal matrix and demonstrate the application of the general theory of matrix-geometric solutions [12] for deriving the steady-state distribution and stability conditions for different control policies.

(d) We calculate the main performance characteristics for the system with different control policies and study the influence of these policies on the quality of service. It is shown that the heterogeneous system may be superior in performance to the homogeneous one with the same total service time.

The paper is organized as follows. In Section 2, we describe the mathematical model and specify the optimization problem. In Section 3, we investigate the structural properties of the optimal control policy which appears to be of threshold and monotone type. In Section 4, we apply the results from the theory of QBD processes to the steady-state analysis of the system under optimal control policy. The steady-state distributions for the system under other control policies are investigated in Section 5. In Section 6, we derive the performance measures and in Section 7, we present some illustrative numerical results for the system under different control policies.

In further sections we will use the notations e, e_j and I for the suitably dimensioned column-vector consisting of 1's, column vector with 1 in the j-th (beginning from 0-th) position and 0 elsewhere, and an identity matrix, respectively.

2. DESCRIPTION OF THE MATHEMATICAL MODEL

Consider the queueing model $M/M/c$ in which primary customers arrive according to a Poisson stream with rate λ, c heterogeneous exponential servers with rates $\mu_1 > \mu_2 > \cdots > \mu_c$, constant retrial rate $\gamma > 0$ and the number of places in the retrial orbit $2 \leq K \leq \infty$. According to the control policy an arriving customer can join the orbit or have direct access to the accessible idle servers. The arrival process, service times and retrial times are assumed to be mutually independent.

Let $Q(t)$ is the number of customers in the retrial orbit at time t, $D_j(t),\ 1 \leq j \leq c$ describe the states of the servers at this time,

$$D_j(t) = \begin{cases} 0, & \text{if the } j\text{-th server is idle at time } t \text{ and} \\ 1, & \text{if the } j\text{-th server is busy.} \end{cases}$$

The observed process

$$\{X(t)\}_{t\geq 0} = \{Q(t), D_1(t), \ldots, D_c(t)\}_{t\geq 0} \tag{1}$$

is a continuous-time Markov process with state space defined as

$$E = \{x = (q, d_1, \ldots, d_c); 0 \leq q \leq K,\, d_j = \{0,1\},\, 1 \leq j \leq c\} \equiv \mathbb{N} \times \{0,1\}^c,$$

where q and d_j, $1 \leq j \leq c$ denote, respectively, the number of customers on the orbit and states of the servers and $q(x)$, $d_j(x)$ specify the components of the state x. Let $J_0(x)$ and $J_1(x)$ be the sets of labels assigned to idle and busy servers, respectively, i.e.

$$J_0(x) = \{j : d_j(x) = 0\},\ J_1(x) = \{j : d_j(x) = 1\}.$$

The control decision in this model are the choice of assigning or not assigning a server whenever primary arrival or retrial customer tries to enter a service area. Define by $U = \{0, 1, \ldots, c\}$ a set of admissible controls and by $U(x) \subseteq U$

$$U(x) = \begin{cases} J_0(x) \cup \{0\}, & \text{for } x \text{ if } q(x) < K, \\ J_0(x), & \text{for } x \text{ if } q(x) = K < \infty. \end{cases}$$

a set of admissible controls in state x. Thus each time t^- just before a transition upon arrival (primary or retrial) when the system state is $X(t^-) = x$ one of available in this state control actions $u \in U(x)$ must be chosen, where $u = 0$ has the meaning "send the job to the orbit"and $u = k \geq 1$ has the meaning "switch on server k".

A deterministic control policy is then meant by any function $f : E \to U$ subject to the constraint $f(x) \in U(x)$ for any $x \in E$. When f is adopted as a policy, $u = f(x)$ is applied whenever the system is in state x.

The components $a_{xy}(u)$ of the infinitesimal generator of the Markov process $\{X(t)\}_{t\geq 0}$ depend on the control action u and satisfies

$$a_{xy}(u) = \begin{cases} \lambda, & \text{for } y = x + e_u,\ u \in U(x) \\ \mu_j, & \text{for } y = x - e_j,\ j \in J_1(x), \\ \gamma, & \text{for } y = x - e_0 + e_u,\ u \in U(x - e_0) \\ 0, & \text{otherwise} \end{cases}$$

and $a_x(u) = -a_{xx}(u) = \sum_{y \neq x} a_{xy}(u)$.

For the model under consideration we look for some deterministic control policy f^* which minimizes the long-run average sojourn time. This is equivalent to minimizing the long-run average number of customers in the system

$$g(f) = \lim_{T\to\infty} \frac{1}{T} \mathbb{E}_x^f \left[\int_0^T (Q(t) + D_1(t) + D_2(t))dt \right], \tag{2}$$

where $\mathbb{E}_x^f$ denotes the expectation with respect to the probability measure induced by the policy f on the process $\{X(t)\}$ with initial state x. The policy f^* is said to be optimal if

$$g = g(f^*) \leq g(f),\ x \in E$$

for any other admissible stationary deterministic policy f.

3. ON OPTIMAL CONTROL POLICY

3.1. Optimality of always using faster server

Our objective is to prove that the faster server upon arrival of a primary or retrial customer must be switched on whenever it is idle. To prove it we need the following property of the dynamic programming operator B

$$Bv(x) = \min_u \left[\frac{1}{a_x(u)} \left(x^t e + \sum_{y \neq x} a_{xy}(u) v(y) - g \right) \right], \; x \in E \tag{3}$$

mapping any positive function $v : E \to \mathbb{R}_+$ into a positive function.

Lemma 1. *Assume the function $v : E \to \mathbb{R}_+$ has the following properties*

1. $v(x) \le v(x + e_i)$, $i \in J_0(x)$,
2. $v(x) \le v(x + e_0)$,
3. $v(x + e_j) \le v(x + e_i)$, $i, j \in J_0(x)$, $\mu_j \ge \mu_i$,
4. $v(x + e_1) \le v(x + e_0)$.

Then the operator B defined by (3) preserves these properties for the function v.

Proof. To prove the properties of lemma 1 we rewrite the equation (3) in the form

$$Bv(x) = \frac{1}{\lambda + \gamma + \sum_{j=1}^{c} \mu_j} \Bigg(x^t e + \lambda \min_{u \in U(x)} v(x + e_u) + \mathbf{1}_{\{q(x)>0\}} \gamma \min_{u \in U(x - e_0)} v(x - e_0 + e_u) \tag{4}$$
$$+ \mathbf{1}_{\{q(x)=0\}} \gamma v(x) + \sum_{j \in J_1(x)} \mu_j v(x - e_j) + \sum_{j \in J_0(x)} \mu_j v(x) - g \Bigg).$$

without loss of generality assuming that the rate $\lambda + \gamma + \sum_{j=1}^{c} \mu_j = 1$ For the inequality 1 taking the last equation for the states $x + e_i$ and x we get

$$\begin{aligned}
&Bv(x + e_i) - Bv(x) \\
&= 1 + \lambda [\min_{u \in U(x+e_i)} \{v(x + e_i + e_u)\} - \min_{u \in U(x)} \{v(x + e_u)\}] \\
&+ \mathbf{1}_{\{q(x)>0\}} \gamma [\min_{u \in U(x+e_i-e_0)} \{v(x + e_i - e_0 + e_u)\} - \min_{u \in U(x-e_0)} \{v(x - e_0 + e_u)\}] \\
&+ \mathbf{1}_{\{q(x)=0\}} \gamma [v(x + e_i) - v(x)] \\
&+ \sum_{l \in J_1(x+e_i)} \mu_l [v(x + e_i - e_l) - v(x - e_l)] + \sum_{l \in J_0(x+e_i)} \mu_l [v(x + e_i) - v(x)] \ge 0.
\end{aligned}$$

It can be easily seen that all terms in the right hand side of the equation are nonnegative due to the property of the function $v(x)$: $v(x) \le v(x + e_i)$. Indeed we have

$$\min_{u \in U(x+e_i)} \{v(x + e_i + e_u)\} \ge \min_{u \in U(x+e_i)} \{v(x + e_u)\} \ge \min_{u \in U(x)} \{v(x + e_u)\}$$
$$\min_{u \in U(x+e_i-e_0)} \{v(x + e_i - e_0 + e_u)\} \ge \min_{u \in U(x+e_i-e_0)} \{v(x + e_u - e_0)\} \ge \min_{u \in U(x-e_0)} \{v(x + e_u - e_0)\}$$

where the inequalities follow from the monotonicity assumption of the function v and from the relations $U(x + e_i) \subset U(x)$ and $U(x + e_i - e_0) \subset U(x - e_0)$ together with the fact that the minimum does not increase upon expanding the minimization set.

The inequality 2 can be shown similarly. Indeed,

$$\begin{aligned}
&Bv(x+e_0) - Bv(x) \\
&= 1 + \lambda[\min_{u\in U(x+e_0)}\{v(x+e_0+e_u)\} - \min_{u\in U(x)}\{v(x+e_u)\}] \\
&+ \mathbf{1}_{\{q(x)>0\}}\gamma[\min_{u\in U(x)}\{v(x+e_u)\} - \min_{u\in U(x-e_0)}\{v(x-e_0+e_u)\}] \\
&+ \mathbf{1}_{\{q(x)=0\}}\gamma[\min_{u\in U(x)}\{v(x+e_u)\} - v(x)] \\
&+ \sum_{l\in J_1(x)}\mu_l[v(x+e_0-e_l) - v(x-e_l)] + \sum_{l\in J_0(x)}\mu_l[v(x+e_0)-v(x)] \geq 0,
\end{aligned}$$

due to the monotonicity properties of the function v.

For the inequality 3 we take the equation (4) for the states $x+e_i$ and $x+e_j$

$$\begin{aligned}
&Bv(x+e_i) - Bv(x+e_j) \\
&= \lambda[\min_{u\in U(x+e_i)}\{v(x+e_i+e_u)\} - \min_{u\in U(x+e_j)}\{v(x+e_j+e_u)\}] \\
&+ \mathbf{1}_{\{q(x)>0\}}\gamma[\min_{u\in U(x+e_i-e_0)}\{v(x+e_i-e_0+e_u)\} - \min_{u\in U(x+e_j-e_0)}\{v(x+e_j-e_0+e_u)\}] \\
&+ \mathbf{1}_{\{q(x)=0\}}\gamma[v(x+e_i)-v(x+e_j)] + \mu_j[v(x+e_i)-v(x)] - \mu_i[v(x+e_j)-v(x)] \\
&+ \sum_{l\in J_1(x+e_i+e_j)}\mu_l[v(x+e_i-e_l) - v(x+e_j-e_l)] + \sum_{l\in J_0(x+e_i+e_j)}\mu_l[v(x+e_i)-v(x+e_j)] \geq 0.
\end{aligned}$$

The last expression is nonnegative if $\mu_j \geq \mu_i$ since

$$\begin{aligned}
&\min_{u\in U(x+e_i)}\{v(x+e_i+e_u)\} = \min_{u\in U(x+e_i+e_j)}\{\min\{v(x+e_i+e_j), v(x+e_i+e_u)\}\} \geq \\
&\min_{u\in U(x+e_i+e_j)}\{\min\{v(x+e_i+e_j), v(x+e_j+e_u)\}\} = \min_{u\in U(x+e_j)}\{v(x+e_j+e_u)\}
\end{aligned}$$

by virtue of inequality 3 for the function v. The same is true for the term with coefficient γ if we take the inequality 3 for the $x-e_0$.

At last for the inequality 4 we have

$$\begin{aligned}
&Bv(x+e_0) - Bv(x+e_1) \\
&= \lambda[\min_{u\in U(x+e_0)}\{v(x+e_0+e_u)\} - \min_{u\in U(x+e_1)}\{v(x+e_1+e_u)\}] \\
&+ \mathbf{1}_{\{q(x)>0\}}\gamma[\min_{u\in U(x)}\{v(x+e_u)\} - \min_{u\in U(x+e_1-e_0)}\{v(x+e_1-e_0+e_u)\}] \\
&+ \mathbf{1}_{\{q(x)=0\}}\gamma[\min_{u\in U(x)} v(x+e_u) - v(x+e_1)] + \mu_1[v(x+e_0)-v(x)] \\
&+ \sum_{l\in J_1(x+e_1)}\mu_l[v(x+e_0-e_l) - v(x+e_1-e_l)] + \sum_{l\in J_0(x+e_1)}\mu_l[v(x+e_0)-v(x+e_1)] \geq 0.
\end{aligned}$$

by virtue of the inequalities 3 and 4.

Theorem 1. *The optimal policy has the following property: Whenever upon arrival (primary or retrial) the fast server is available, this server should serve a customer.*

Proof. As in Lin and Kumar [10] the proof combines the value iteration algorithm with lemma 1. The value function $v(x)$ satisfies

$$Bv(x) = v(x),\ x \in E.$$

Let

$$v_{n+1}(x) = \min_u \left[\frac{1}{a_x(u)} \left(x^t e + \sum_y a_{xy}(u) v_n(y) - g_n \right) \right] = Bv_n(x),\ x \in E.$$

In spite of the operator B not being a contractive operator for the long-run average problem, if we put the starting iteration $v_0(x) = 0,\ x \in E$, then the function v_n is a monotonously increasing function so that

$$v(x) = \lim_{n\to\infty} B^n v_0(x),\ x \in E.$$

Note that the initial choice of v_0 satisfies the assumption of lemma 1. Hence all v_n as well as the limit have this property, therefore it holds also for the value function $v^*(x)$.

The structure of the equation (4) allows to express the optimal policy $f(x)$ in each state $x \in E$ through the value function $v^*(x)$ in the form

$$f(x) = \arg\min_{u\in U(x)} \{v(x + e_u)\},\ x \in E. \tag{5}$$

Therefore the the assumption of lemma 1 for the value function implies that if it is necessary to switch on some server it must be the fastest available one.

3.2. Optimality of threshold policy for $c = 2$

We have shown that the first server must be activated whenever the customer arrives and it is free. Note that the values $v(x + e_2)$ and $v(x + e_0)$ for $d_1(x) = 1$ can not be compared in the same way as before, this assertion does not allow us to determine whether a second server must be activated. But we can show that the increments $v(x + e_0) - v(x + e_2)$ are monotone with respect to the queue length.

Lemma 2. *Assume the positive function $v : E \to \mathbb{R}_+$ has the following property*

3. $v(x + e_0) - v(x + e_2) \le v(x + 2e_0) - v(x + e_0 + e_2)$,
4. $v(x + e_0) - v(x) \le v(x + e_2 + e_0) - v(x + e_2)$,
5. $2v(x + e_0) \le v(x) + v(x + 2e_0)$.

Then the operator B defined by (3) preserves these properties for the function v.

Proof. The inequality 3 of the function v is known as superconvexity, the inequality 2 - as supermodularity and the inequality 3 - as convexity. Note, that the summing 3 and 4 implies 5.

We have to show that the operator B preserves the mentioned properties. To prove the inequality 3 we substitute the equation (4) in corresponding point, then we have

$$\begin{aligned}
&Bv(x + e_0) - Bv(x + e_2) - Bv(x + 2e_0) + Bv(x + e_0 + e_2) \\
&= \lambda[\min_{u\in U(x+e_0)} \{v(x + e_0 + e_u)\} - \min_{u\in U(x+e_2)} \{v(x + e_2 + e_u)\} \\
&- \min_{u\in U(x+2e_0)} \{v(x + 2e_0 + e_u)\} + \min_{u\in U(x+e_0+e_2)} \{v(x + e_0 + e_2 + e_u)\}] \\
&+ \mathbf{1}_{\{q(x)>0\}} \gamma[\min_{u\in U(x)} \{v(x + e_u)\} - \min_{u\in U(x+e_2-e_0)} \{v(x - e_0 + e_2 + e_u)\}
\end{aligned}$$

$$
\begin{aligned}
&- \min_{u\in U(x+e_0)}\{v(x+e_0+e_u)\} + \min_{u\in U(x+e_2)}\{v(x+e_2+e_u)\}] \\
&+ \mathbf{1}_{\{q(x)=0\}}\gamma[\min_{u\in U(x)}\{v(x+e_u)\} - v(x+e_2) \\
&- \min_{u\in U(x+e_0)}\{v(x+e_0+e_u)\} + \min_{u\in U(x+e_2)}\{v(x+e_2+e_u)\}] \\
&+ \mu_1[v(x+e_0-e_1) - v(x+e_2-e_1) - v(x+2e_0-e_1) + v(x+e_0+e_2-e_1)] \\
&+ \mu_2[v(x+e_0) - v(x) - v(x+2e_0) + v(x+e_0)] \le 0
\end{aligned}
$$

The last two terms with the coefficients μ_1 and μ_2 are nonpositive with respect to the inequalities 3 and 5 of lemma 2. To prove the assertion that the term with coefficient λ is nonpositive we distinguish two possible cases

1. $f(x+e_2) = f(x+2e_0) = 0$. In this case we have

$$
\begin{aligned}
&\min_{u\in U(x+e_0)}\{v(x+e_0+e_u)\} - v(x+e_2+e_0) - v(x+3e_0) + \min_{u\in U(x+e_0+e_2)}\{v(x+e_0+e_2+e_u)\} \\
&\le v(x+2e_0) - v(x+e_2+e_0) - v(x+3e_0) + v(x+2e_0+e_2) \le 0
\end{aligned}
$$

according to the assumption of lemma 2 in the state $x+e_0$.

2. $f(x+e_2) = 0$ and $f(x+2e_0) = 2$. In this case we get

$$
\begin{aligned}
&\min_{u\in U(x+e_0)}\{v(x+e_0+e_u)\} - v(x+e_2+e_0) - v(x+2e_0+e_2) + \min_{u\in U(x+e_0+e_2)}\{v(x+e_0+e_2+e_u)\} \\
&\le v(x+e_0+e_2) - v(x+e_2+e_0) - v(x+2e_0+e_2) + v(x+e_0+e_2+e_0) = 0
\end{aligned}
$$

For the term with coefficient $\mathbf{1}_{\{q(x)>0\}}\gamma$ the nonpositivity follows from the previous cases by replacing x with $x-e_0$. Inequality of the term with coefficient $\mathbf{1}_{\{q(x)=0\}}\gamma$ can be shown similarly by considering two cases $f(x+e_0)=0$ and $f(x+e_0)=2$.

For the inequality 4 we have

$$
\begin{aligned}
&Bv(x+e_0) - Bv(x) - Bv(x+e_2+e_0) + Bv(x+e_2) \\
&= \lambda[\min_{u\in U(x+e_0)}\{v(x+e_0+e_u)\} - \min_{u\in U(x)}\{v(x+e_u)\} \\
&- \min_{u\in U(x+e_2+e_0)}\{v(x+e_2+e_0+e_u)\} + \min_{u\in U(x+e_2)}\{v(x+e_2+e_u)\}] \\
&+ \mathbf{1}_{\{q(x)>0\}}\gamma[\min_{u\in U(x)}\{v(x+e_u)\} - \min_{u\in U(x-e_0)}\{v(x-e_0+e_u)\} \\
&- \min_{u\in U(x+e_2)}\{v(x+e_2+e_u)\} + \min_{u\in U(x+e_2-e_0)}\{v(x+e_2-e_0+e_u)\}] \\
&+ \mathbf{1}_{\{q(x)=0\}}\gamma[\min_{u\in U(x)}\{v(x+e_u)\} - v(x) \\
&- \min_{u\in U(x+e_2)}\{v(x+e_2+e_u)\} + v(x+e_2)] \\
&+ \mu_1[v(x+e_0-e_1) - v(x-e_1) - v(x+e_2+e_0-e_1) + v(x+e_2-e_1)] \\
&+ \mu_2[v(x+e_0) - v(x) - v(x+e_0) + v(x)] \le 0
\end{aligned}
$$

The last two terms with coefficients μ_1 and μ_2 are nonpositive with respect to the inequality 3. For the term with λ we have two subcases, namely

1. $f(x) = f(x+e_2+e_0) = 0$.

$$
\begin{aligned}
&\min_{u\in U(x+e_0)}\{v(x+e_0+e_u)\} - v(x+e_0) - v(x+2e_0+e_2) + \min_{u\in U(x+e_2)}\{v(x+e_2+e_u)\} \\
&\le v(x+2e_0) - v(x+e_0) - v(x+2e_0+e_2) + v(x+e_2+e_u) \le 0
\end{aligned}
$$

according to the assumption of lemma 2 in the state $x + e_0$.

2. $f(x) = 2$ and $f(x + e_2 + e_0) = 0$.

$$\min_{u\in U(x+e_0)}\{v(x + e_0 + e_u)\} - v(x + e_2) - v(x + 2e_0 + e_2) + \min_{u\in U(x+e_2)}\{v(x + e_2 + e_u)\}$$
$$\leq 2v(x + e_0 + e_2) - v(x + e_2) - v(x + 2e_0 + e_2) \leq 0$$

using the inequality 5.

The term with the coefficient $\mathbf{1}_{\{q(x)>0\}}\gamma$ is nonpositive by applying the inequality 3 to the point $x - e_0$. At last the term with the coefficient $\mathbf{1}_{\{q(x)=0\}}\gamma$ is nonnegative due to the relation

$$\min_{u\in U(x)}\{v(x + e_u)\} - v(x) - \min_{u\in U(x+e_2)}\{v(x + e_2 + e_u)\} + v(x + e_2)$$
$$\leq v(x + e_0) - v(x) - v(x + e_2 + e_0) + v(x + e_2) \leq 0.$$

Theorem 2. *The optimal policy for the retrial queueing system $M/M/2$ with heterogeneous servers and constant retrial rate is of threshold and monotone type, i.e. the fastest idle server must be switched on whenever a primary or retrial customer arrives and another one must be switched on if and only if the fastest server is busy and the orbit length reaches the threshold level $q \geq q_2^*$.*

Proof. The assumption of lemma 2 holds for the value function v according to the monotone convergence property of the operator B, i.e.

$$v(x + e_0) - v(x + e_2) \leq v(x + 2e_0) - v(x + e_0 + e_2),\ x = (q, 1, 0),\ q \geq 0.$$

This inequality shows that the advantage to send the customer to the orbit and leave the second server idle decreases when the orbit size increases. According to the definition of the optimal control $f(x)$, $x \in E$ in (5) the following relations hold

$$f(x + e_0) = 0 \Rightarrow f(x) = 0$$
$$f(x) = 2 \Rightarrow f(x + e_0) = 2.$$

Hence there exists a level of the orbit size q_2^* when upon arrival the second server must be activated. And if in some state x the optimal control is to allocate a customer to the orbit then this control is optimal for all the states y.

3.3. Calculation of threshold levels

The analytical representation of the threshold level is quite complicated, but by means of the value iteration algorithm it can be done numerically. If future arrivals are not taken into account (scheduling problem), i.e. when the objective is to minimize the sojourn time for an individual customer, the explicit solution for the threshold level exists

Theorem 3 (Scheduling threshold policy). *If $\lambda = 0$ then there exists a threshold level*

$$q_2^* = \left\lfloor \frac{\gamma}{\mu_1 + \gamma}\left(\frac{\mu_1}{\mu_2} - 1\right)\right\rfloor, \tag{6}$$

such that if $q \geq q_2^*$ in state $x = (q, 1, 0)$ then upon retrial arrival it is optimal to dispatch a customer to the slower server, if $q > q_2^*$ then the slower server must be idle.

Proof. Consider the system with customers without future arrival, i.e. $\lambda = 0$. For this system it is obviously that the mean number of customers $g = 0$. Therefore the relative function v satisfies the recurrent equations for $q \geq 1$ which after some transformations have the form

$$\begin{aligned} v(0,1,1) &= \frac{\mu_1+\mu_2}{\mu_1\mu_2},\ v(0,0,1) = \frac{1}{\mu_2}, \\ v(q-1,1,0) &= \frac{q}{\mu_1} + v(q-1,0,0) = \frac{q}{\mu_1} + \frac{(q-1)q}{2}\left(\frac{1}{\gamma}+\frac{1}{\mu_1}\right), \\ v(q-1,1,1) &= \frac{1}{\mu_1+\mu_2}(q+1+\mu_1 v(q-1,0,1)+\mu_2 v(q-1,1,0)) \\ &= \frac{1}{\mu_2} + v(q-1,1,0). \end{aligned} \tag{7}$$

Since $q = q_2^*$ is an optimal orbit size to switch on the second server, it is necessary to require the following condition: The value function in the state $x = (q-1,1,1)$, in which the controller begins to use the second server with respect to the optimal policy must be less then the value function in the state, in which the controller does not use the second server, i.e.

$$v(q-1,1,1) < v(q,1,0).$$

Substituting the corresponding expressions (7) into the last inequality we get

$$\frac{1}{\mu_2} + \frac{q}{\mu_1} + \frac{(q-1)q}{2}\left(\frac{1}{\gamma}+\frac{1}{\mu_1}\right) < \frac{q+1}{\mu_1} + \frac{q(q+1)}{2}\left(\frac{1}{\gamma}+\frac{1}{\mu_1}\right).$$

Thus the second server must be switched on if

$$q > \left\lfloor \frac{\gamma}{\mu_1+\gamma}\left(\frac{\mu_1}{\mu_2}-1\right)\right\rfloor.$$

Further we obtain an approximation to the optimal threshold level q_2^* in general case with arrivals. This approximation can be obtained by analysis of the asymptote to the boundary between the areas where the optimal threshold value is q_2^* and q_2^*+1. Note that the term $\frac{\mu_1}{\mu_2}-1$ represents the threshold level for the scheduling problem in case of classical queue, see e.g. [1] and can be interpreted as the average number of customers that can be served by the first server while the second one is busy. The relation (6) has additional term $\frac{\gamma}{\mu_1+\gamma}$ that denotes the probability of the retrial arrival in the state with busy faster server. Now we can provide the following assertion.

Assertion 1. *The approximation to the threshold level of the retrial system with arrivals is of the form*

$$q_2^* \approx \tilde{q}_2^* = \left\lfloor \frac{\gamma}{\mu_1+\gamma}\tilde{q}_{2c}^* \right\rfloor, \tag{8}$$

$$\tilde{q}_{2c}^* = \left\lfloor \frac{\mu_1-\lambda+\sqrt{(\mu_1-\lambda)^2+4\mu_2\lambda}}{2\mu_2} - 1\right\rfloor, \tag{9}$$

where $\tilde{q}_{2c}^*$ is an approximation to the threshold level of the ordinary queue.

Proof. Consider an ordinary queueing system without retrials. To calculate the value q_{2c}^* we display the optimal thresholds as a function of μ_1/λ and μ_2/λ and get semi-infinite areas with boundaries that can be approximated by the lines

$$\lambda/\mu_2 = \frac{\lambda/\mu_1}{q_2^*+1} - \frac{q_2^*}{(q_2^*+1)^2}.$$

The latter expression implies the relation (9).

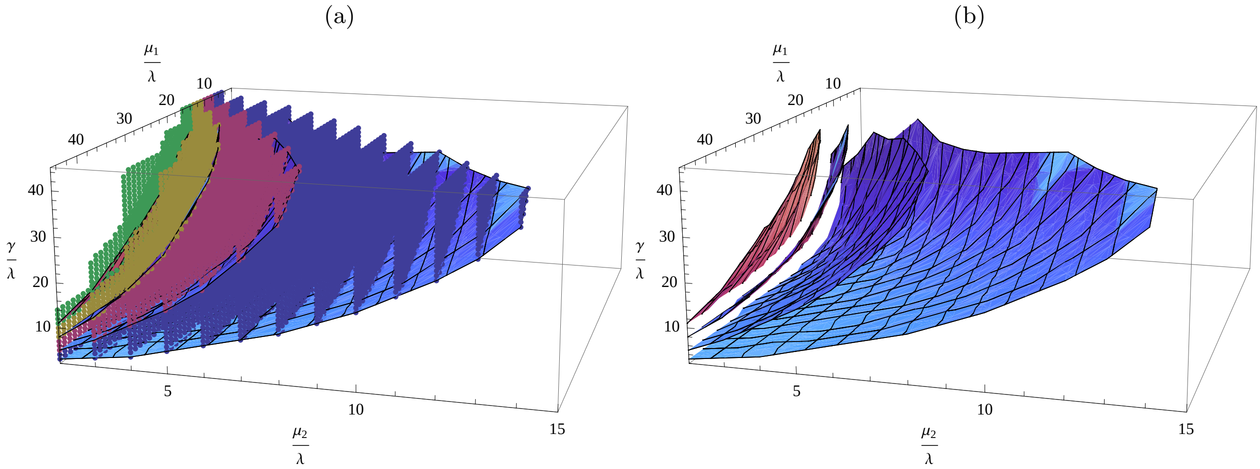

Figure 1. Areas with optimal threshold level $q_2^* = 1, 2, 3, 4$ and corresponding boundary surfaces

This relation coincides with the result obtained in [11] for the analogous model with probabilistic decision rule. The figure 1 shows that the obtained asymptotic surfaces quite good approximate the actual boundaries. Numerical results show that the difference between performance values g for the approximation and optimal threshold level never exceeds 1.5 percent.

Assertion 2. *In heavy traffic case, $\lambda \approx \mu_1 + \mu_2$, we can assume the suboptimality of the Fastest Free Server (FFS) discipline.*

Proof. It is also confirmed by the approximation formula (8). Indeed, taking into account that $\mu_1 \geq \mu_2$ we get,

$$\left\lfloor \frac{\mu_1 - (\mu_1 + \mu_2) + \sqrt{(\mu_1 - (\mu_1 + \mu_2))^2 + 4\mu_2(\mu_1 + \mu_2)}}{2\mu_1} - 1 \right\rfloor \leq \left\lfloor \sqrt{\left(\frac{\mu_2}{2\mu_1}\right)^2 + \frac{\mu_2}{\mu_1}\left(1 + \frac{\mu_2}{\mu_1}\right)} - 1 \right\rfloor < 1.$$

Assertion 3. *In light traffic case, $\lambda < \mu_2$, we can assume the suboptimality of the scheduling threshold policy (STP).*

Proof. Under the given condition we get the following upper and low bounds for the approximation (9)

$$\left\lfloor \frac{\mu_1 - \lambda + \sqrt{(\mu_1 - \lambda)^2 + 4\mu_2\lambda}}{2\mu_2} - 1 \right\rfloor \leq \left\lfloor \frac{\mu_1}{2\mu_2} - \frac{\lambda}{2\mu_2} + \sqrt{\left(\frac{1}{2}\left(\frac{\mu_1}{\mu_2} - \frac{\lambda}{\mu_2}\right) + \sqrt{\frac{\lambda}{\mu_2}}\right)^2} - 1 \right\rfloor$$
$$= \left\lfloor \frac{\mu_1}{\mu_2} - \frac{\lambda}{\mu_2} + \sqrt{\frac{\lambda}{\mu_2}} - 1 \right\rfloor \leq \left\lfloor \frac{\mu_1}{\mu_2} \right\rfloor,$$
$$\left\lfloor \frac{\mu_1 - \lambda + \sqrt{(\mu_1 - \lambda)^2 + 4\mu_2\lambda}}{2\mu_2} - 1 \right\rfloor \geq \left\lfloor \frac{\mu_1}{2\mu_2} - \frac{\lambda}{2\mu_2} + \sqrt{\frac{1}{4}\left(\frac{\mu_1}{\mu_2} - \frac{\lambda}{\mu_2}\right)^2} - 1 \right\rfloor$$
$$= \left\lfloor \frac{\mu_1}{\mu_2} - \frac{\lambda}{\mu_2} - 1 \right\rfloor \geq \left\lfloor \frac{\mu_1}{\mu_2} - 2 \right\rfloor.$$

Numerical investigation of the systems with more than two servers leads to the following conjecture about the approximation to the threshold levels.

Assertion 4 (Approximation to the threshold levels). *Using the asymptote to the boundary between the area where the optimal threshold for the j-th server is q_j^* and the area with optimal threshold q_j^*+1, the following approximations to the optimal threshold levels can be calculated by*

$$q_j^* \approx \tilde{q}_j^* = \left\lfloor \frac{\gamma}{\gamma + \sum_{k=1}^{j-1}\mu_j} \left(\frac{1}{2\mu_j} \left[\sum_{k=1}^{j-1}\mu_k - \lambda + \sqrt{\left(\sum_{k=1}^{j-1}\mu_k - \lambda\right)^2 + 4(j-1)\mu_j\lambda} \right] - (j-1) \right) \right\rfloor. \quad (10)$$

In case of more than two servers the exact values of optimal thresholds can be calculated numerically. The system with infinite retrial group will be approximated by the system with finite but enough large orbit capacity. We have used a C++ program implementing a *policy iteration algorithm*:

Algorithm 1.
Step 1. Selection of initial control policy, e.g. FFS

$$f_0(x) = \arg\min_{u \in J_0(x)} \left\{ \frac{1}{\mu_u} \right\}, \; x \in E.$$

Step 2. For the given control policy $f_i(x)$ solve the system of equations iteratively

$$\begin{aligned} & v_0^i(x) = 0, \; x \in E, \quad g_n^i = \lambda v_n^i(e_{f^i(0)}) \\ & v_{n+1}^i(x) = \frac{1}{\lambda + \sum_{j \in J_1(x)}\mu_j + \mathbf{1}_{\{q(x)>0\}}\gamma} \Big(x^t e + \lambda v_n^i(x + e_{f_i(0)}) + \sum_{j \in J_1(x)} \mu_j v_n^i(x - e_j) \\ & \qquad + \mathbf{1}_{\{q(x)>0\}} \gamma v_n^i(x - e_0 + e_{f_i(x-e_0)}) - g_n^i \Big), \; x \in E. \end{aligned}$$

Step 3. For the given solution of previous system of equations $v^i(x)$ find the improved policy by

$$f_{i+1}(x) = \arg\min_{u \in U(x)} \{v^i(x + e_u)\}, \; x \in E$$

Step 4. Repeat Step 1 and Step 2 until the policies for two neighbor iterations will be equal for all states, i.e.

$$f_i(x) = f_{i+1}(x) = f^*(x) = \arg\min_{u \in U(x)} \{v^*(x + e_u)\}, \; x \in E.$$

Step 5. Calculate the optimal threshold level q_2^* by

$$q_2^* = q + 1 \text{ if } f^*(q-1,1,0) = 0 \text{ and } f^*(q,1,0) = 2.$$

4. STEADY-STATE DISTRIBUTION OF THE SYSTEM UNDER OPTIMAL POLICY

In this section we derive the equilibrium state distribution under the optimal threshold policy (OTP). The derivation works via a standard matrix-geometric approach [12], taking into account the special structure of the boundary states where not all servers are active. To distinguish the system under OTP from other control policies which will be discussed later we will use the upper index $'(1)'$ for the concerned values. Let q_2^* be the threshold level for activation of the second server. As it was mentioned above they can be found numerically (e.g. using the policy iteration algorithm [8]).

Consider a Markov process $\{X(t)\}_{t\geq 0}$ defined by (1) with a state space E. This process is a QBD process with block - three-diagonal infinitesimal matrix. Note that the blocks have different sizes depending on the queue length.

4.1. Finite retrial group

First consider the system with finite retrial group, i.e. $K < \infty$. In this case the states are partitioned as follows:

- block 0 includes the single state: $(0, 0, 0)$;
- block 1 includes the states: $(0, 0, 1)$, $(0, 1, 0)$, $(1, 0, 0)$;
- block 2 includes the states: $(0, 1, 1)$, $(1, 0, 1)$, $(1, 1, 0)$, $(2, 0, 0)$;
- blocks i, $3 \le i \le K$ include the states:
  $(i-2, 1, 1)$, $(i-1, 0, 1)$, $(i-1, 1, 0)$, $(i, 0, 0)$;
- block $K+1$ includes the states:
  $(K-1, 1, 1)$, $(K, 0, 1)$, $(K, 1, 0)$;
- block $K+2$ includes the single state: $(K, 1, 1)$;

Denote by $\Lambda^{(1)}$ the infinitesimal matrix of dimension $4(K+1)$ for the system under OTP,

$$\Lambda^{(1)} = \tag{11}$$

$$\begin{pmatrix}
-\lambda & A_1^{(1)} & 0 & 0 & 0 & 0 & \dots & 0 \\
D_0^{(1)} & -(C_1^{(1)} - B_1^{(1)}) & A_2^{(1)} & 0 & 0 & 0 & \dots & 0 \\
0 & D_1^{(1)} & -(C_2^{(1)} - B_2^{(1)}) & A_3^{(1)} & 0 & 0 & \dots & 0 \\
0 & 0 & D_2^{(1)} & -(C_2^{(1)} - B_2^{(1)}) & A_3^{(1)} & 0 & \dots & 0 \\
\ddots & \ddots & \ddots & \ddots & \ddots & & \dots & 0 \\
0 & \dots & 0 & D_2^{(1)} & -(C_3^{(1)} - B_3^{(1)}) & A_4^{(1)} & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \dots & 0 & 0 & D_3^{(1)} & -(C_3^{(1)} - B_3^{(1)}) & A_5^{(1)} & 0 \\
0 & \dots & 0 & 0 & 0 & D_3^{(1)} & -(C_4^{(1)} - B_4^{(1)}) & A_6^{(1)} \\
0 & \dots & 0 & 0 & 0 & 0 & D_4^{(1)} & -M
\end{pmatrix},$$

where

$$\lambda + A_1^{(1)}e = D_0^{(1)}e - (C_1^{(1)} - B_1^{(1)})e + A_2^{(1)}e = D_1^{(1)}e - (C_2^{(1)} - B_2^{(1)})e + A_3^{(1)}e =$$
$$D_2^{(1)}e - (C_2^{(1)} - B_2^{(1)})e + A_3^{(1)}e = (D_2^{(1)} - (C_3^{(1)} - B_3^{(1)}) + A_4^{(1)})e = D_3^{(1)}e - (C_3^{(1)} - B_3^{(1)})e + A_5^{(1)}e =$$
$$D_3^{(1)}e - (C_4^{(1)} - B_4^{(1)})e + A_6^{(1)}e = D_4^{(1)}e - M = 0,$$

$M = \mu_1 + \mu_2$. Matrices $A_i^{(1)}$ and $B_i^{(1)}$ represent primary and retrial arrivals, respectively, depending on whether the queue length are above or below threshold level:

$$A_1^{(1)} = \begin{pmatrix} 0 & \lambda & 0 \end{pmatrix},\ A_2^{(1)} = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & \lambda & 0 \end{pmatrix},\ A_3^{(1)} = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & \lambda & 0 \end{pmatrix},\ A_4^{(1)} = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 \end{pmatrix},\ A_5^{(1)} = \begin{pmatrix} \lambda & 0 & 0 \\ \lambda & 0 & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & \lambda \end{pmatrix},$$

$$A_6^{(1)} = \begin{pmatrix} \lambda \\ \lambda \\ \lambda \end{pmatrix},\ B_1^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \gamma & 0 \end{pmatrix},\ B_2^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma & 0 \end{pmatrix},\ B_3^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \gamma & 0 & 0 & 0 \\ \gamma & 0 & 0 & 0 \\ 0 & 0 & \gamma & 0 \end{pmatrix},\ B_4^{(1)} = \begin{pmatrix} 0 & 0 & 0 \\ \gamma & 0 & 0 \\ \gamma & 0 & 0 \end{pmatrix}.$$

Matrices $C_i^{(1)}$ represent cases when the system stays at certain states:

$$C_1^{(1)} = \begin{pmatrix} \lambda + \mu_2 & 0 & 0 \\ 0 & \lambda + \mu_1 & 0 \\ 0 & 0 & \lambda + \gamma \end{pmatrix},\ C_2^{(1)} = \begin{pmatrix} \lambda + M & 0 & 0 & 0 \\ 0 & \lambda + \mu_2 + \gamma & 0 & 0 \\ 0 & 0 & \lambda + \mu_1 & 0 \\ 0 & 0 & 0 & \lambda + \gamma \end{pmatrix},$$

$$C_3^{(1)} = \begin{pmatrix} \lambda + M & 0 & 0 & 0 \\ 0 & \lambda + \mu_2 + \gamma & 0 & 0 \\ 0 & 0 & \lambda + \mu_1 + \gamma & 0 \\ 0 & 0 & 0 & \lambda + \gamma \end{pmatrix},\ C_4^{(1)} = \begin{pmatrix} \lambda + M & 0 & 0 \\ 0 & \lambda + \mu_2 + \gamma & 0 \\ 0 & 0 & \lambda + \mu_1 + \gamma \end{pmatrix}.$$

Matrices $D_i^{(1)}$ represent departures with elements depending on active servers:

$$D_0^{(1)} = \begin{pmatrix} \mu_2 \\ \mu_1 \\ 0 \end{pmatrix}, D_1^{(1)} = \begin{pmatrix} \mu_1 & \mu_2 & 0 \\ 0 & 0 & \mu_2 \\ 0 & 0 & \mu_1 \\ 0 & 0 & 0 \end{pmatrix}, D_2^{(1)} = \begin{pmatrix} 0 & \mu_1 & \mu_2 & 0 \\ 0 & 0 & 0 & \mu_2 \\ 0 & 0 & 0 & \mu_1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, D_3^{(1)} = \begin{pmatrix} 0 & \mu_1 & \mu_2 & 0 \\ 0 & 0 & 0 & \mu_2 \\ 0 & 0 & 0 & \mu_1 \end{pmatrix}, D_4^{(1)} = \begin{pmatrix} 0 & \mu_1 & \mu_2 \end{pmatrix}.$$

Denote by $\pi^{(1)} = (\pi_0^{(1)}, \pi_1^{(1)}, \dots)$ the row-vector of the steady-state probabilities,

$$\pi^{(1)} = \{\pi_x^{(1)} = \pi_{(q,d_1,d_2)}^{(1)} : \ x \in E\} = \lim_{t\to\infty} \mathbb{P}\{X(t) = x\},$$

by $\{\pi_k^{(1)} : \ k \geq 0\}$ — the subvectors that specify the states with k jobs in the system.

Obviously the row-vector $\pi^{(1)}$ of the steady-state probabilities of the system under optimal policy satisfies the equations

$$\pi^{(1)}\Lambda = 0, \ \pi^{(1)}e = 1. \tag{12}$$

The probabilities for the system states can be represented in the form of recursive relation with some matrices $M_k^{(1)}$,

$$\pi_k^{(1)} = \pi_{k+1}^{(1)} M_k^{(1)}, \quad k = 0, 1, \dots, K+1,$$

namely

$$\begin{aligned}
\pi_0^{(1)} &= \pi_1^{(1)} D_0^{(1)} \frac{1}{\lambda} = \pi_1^{(1)} M_0^{(1)}; \\
\pi_1^{(1)} &= \pi_2^{(1)} D_1^{(1)} (C_1^{(1)} - B_1^{(1)} - D_0^{(1)} \frac{1}{\lambda} A_1^{(1)})^{-1} = \pi_2^{(1)} M_1^{(1)}; \\
\pi_2^{(1)} &= \pi_3^{(1)} D_2^{(1)} (C_2^{(1)} - B_2^{(1)} - M_1^{(1)} A_2^{(1)})^{-1} = \pi_3^{(1)} M_2^{(1)}; \\
\pi_3^{(1)} &= \pi_4^{(1)} D_2^{(1)} (C_2^{(1)} - B_2^{(1)} - M_2^{(1)} A_3^{(1)})^{-1} = \pi_4^{(1)} M_3^{(1)}; \\
\pi_i^{(1)} &= \pi_{i+1}^{(1)} D_2^{(1)} (C_2^{(1)} - B_2^{(1)} - M_{i-1}^{(1)} A_3^{(1)})^{-1} = \pi_{i+1}^{(1)} M_i^{(1)}, \qquad 4 \leq i \leq q_2^*; \\
\pi_i^{(1)} &= \pi_{i+1}^{(1)} D_2^{(1)} (C_3^{(1)} - B_3^{(1)} - M_{i-1}^{(1)} A_4^{(1)})^{-1} = \pi_{i+1}^{(1)} M_i^{(1)}, \qquad q_2^* + 1 \leq i \leq K-1; \\
\pi_K^{(1)} &= \pi_{K+1}^{(1)} D_3^{(1)} (C_3^{(1)} - B_3^{(1)} - M_{K-1}^{(1)} A_4^{(1)})^{-1} = \pi_{K+1}^{(1)} M_K^{(1)}; \\
\pi_{K+1}^{(1)} &= \pi_{K+2}^{(1)} D_4^{(1)} (C_4^{(1)} - B_4^{(1)} - M_K^{(1)} A_5^{(1)})^{-1} = \pi_{K+2}^{(1)} M_{K+1}^{(1)},
\end{aligned} \tag{13}$$

where all inverses are well defined because the matrices involved are clearly non-singular using their probabilistic interpretation as transition matrices of a transient chain.

To calculate the values $\pi_k^{(1)}$ it is necessary:

Algorithm 2.

Step 1. Evaluate the matrices $M_k^{(1)}$, $k = 0, \dots, K+1$ according to relation (13).

Step 2 Evaluate the value $\pi_{K+2}^{(1)} = \pi_{(K,1,1)}^{(1)}$ from the normalization condition

$$1 = \sum_{k=0}^{K+2} \pi_k^{(1)} e = \pi_{K+2}^{(1)} + \pi_{K+2}^{(1)} \sum_{i=0}^{K+1} \prod_{j=K+1-i}^{K+1} M_j^{(1)} e = \pi_{K+2}^{(1)} \left[1 + \sum_{i=0}^{K+1} \prod_{j=K+1-i}^{K+1} M_j^{(1)} e \right]. \tag{14}$$

Step 3. Substitute $\pi_{K+2}^{(1)}$ into

$$\pi_i^{(1)} = \pi_{K+2}^{(1)} \prod_{j=K+1-i}^{K+1} M_j^{(1)}, \qquad i = 0, \dots, K+1.$$

Note that this method works efficiently as long as $K < \infty$ is not too large. But for large K the matrix geometric solution corresponding to $K = \infty$ is a good approximation.

4.2. Infinite retrial group

In case of infinite retrial group $K = \infty$ the infinitesimal matrix $\Lambda^{(1)}$ has infinite size and is obtained from the matrix (11) by removing the last three rows.

We consider first the matrix-geometric part of the equations, above the threshold level q_2^*:

$$\pi^{(1)}_{q_2^*+j} A_4^{(1)} + \pi^{(1)}_{q_2^*+j+2} D_2^{(1)} = \pi^{(1)}_{q_2^*+j+1}(C_3^{(1)} - B_3^{(1)}),\ j \geq 0.$$

Conjecturing the matrix-geometric form

$$\pi^{(1)}_{q_2^*+j} = \pi^{(1)}_{q_2^*}(R^{(1)})^j$$

where $R^{(1)}$ is the minimal non-negative solution to the matrix equation

$$(R^{(1)})^2 D_2^{(1)} - R^{(1)}(C_3^{(1)} - B_3^{(1)}) + A_4^{(1)} = 0, \tag{15}$$

which is typically solved numerically using the following iteration procedure:

$$\begin{aligned} &R^{(1)}(0) = 0, \\ &R^{(1)}(n+1) = A_4^{(1)}(C_3^{(1)} - B_3^{(1)})^{-1} + (R^{(1)})^2(n) D_2^{(1)}(C_3^{(1)} - B_3^{(1)})^{-1}, \end{aligned} \tag{16}$$

where the iteration halts when entries in $R^{(1)}(n+1)$ and $R^{(1)}(n)$ differ in absolute value by less that a given small constant.

The general theory [12] states that the necessary and sufficient condition for stability is

$$p^{(1)} D_2^{(1)} e > p^{(1)} A_4^{(1)} e,$$

where $p^{(1)} = (p_0^{(1)}, p_1^{(1)}, p_2^{(1)}, p_3^{(1)})$ is a stationary probability vector given by $p^{(1)}(A_4^{(1)} - (C_3^{(1)} - B_3^{(1)}) + D_2^{(1)}) = 0$, $p^{(1)}e = 1$.

Theorem 4. *For the system under optimal policy, the stationary vector $p^{(1)}$ of $A_4^{(1)} - (C_3^{(1)} - B_3^{(1)}) + D_2^{(1)}$ is given by*

$$\begin{aligned} p_0^{(1)} &= \frac{(\lambda+\gamma)^2(\lambda+\mu_2+\gamma)}{(\lambda+\mu_1+\gamma)((\lambda+\gamma)(\lambda+2\mu_1+\gamma)+\mu_2 M)}, \\ p_1^{(1)} &= \frac{\mu_1}{\lambda+\mu_2+\gamma} p_0^{(1)}, \\ p_2^{(1)} &= \frac{\mu_2(\lambda+M+\gamma)}{(\lambda+\gamma)(\lambda+\mu_2+\gamma)} p_0^{(1)}, \\ p_3^{(1)} &= \frac{\mu_1\mu_2(2(\lambda+\gamma)+M)}{(\lambda+\gamma)^2(\lambda+\mu_2+\gamma)} p_0^{(1)}. \end{aligned}$$

The system is stable if and only if the load factor $\rho^{(1)}$ satisfies

$$\rho^{(1)} = \frac{\lambda(\lambda+\gamma)^2(\lambda+\mu_2+\gamma)}{M\gamma(\lambda+\gamma)^2 + \gamma\mu_1\mu_2(3(\lambda+\gamma)+\mu_1) + \mu_2^2\gamma(\lambda+\mu_1+\gamma)} < 1. \tag{17}$$

Proof. By elementary calculations.

If the system is stable then all eigenvalues of R must be less than 1 and the geometric sequence $\sum_{j=0}^{\infty}(R^{(1)})^j = (I - R^{(1)})^{-1}$ converges.

Equations for the boundary states below the threshold level are still to be solved, namely:

$$\begin{aligned}
&\pi_0^{(1)} = \pi_1^{(1)} D_0^{(1)} \frac{1}{\lambda} = \pi_1^{(1)} M_0^{(1)};\\
&\pi_1^{(1)} = \pi_2^{(1)} D_1^{(1)} (C_1^{(1)} - B_1^{(1)} - D_0^{(1)} \frac{1}{\lambda} A_1^{(1)})^{-1} = \pi_2^{(1)} M_1^{(1)};\\
&\pi_2^{(1)} = \pi_3^{(1)} D_2^{(1)} (C_2^{(1)} - B_2^{(1)} - M_1^{(1)} A_2^{(1)})^{-1} = \pi_3^{(1)} M_2^{(1)};\\
&\pi_3^{(1)} = \pi_4^{(1)} D_2^{(1)} (C_2^{(1)} - B_2^{(1)} - M_2^{(1)} A_3^{(1)})^{-1} = \pi_4^{(1)} M_3^{(1)};\\
&\pi_i^{(1)} = \pi_{i+1}^{(1)} D_2^{(1)} (C_2^{(1)} - B_2^{(1)} - M_{i-1}^{(1)} A_3^{(1)})^{-1} = \pi_{i+1}^{(1)} M_i^{(1)}, \qquad 4 \le i \le q_2^* - 1;\\
&\pi_{q_2^*}^{(1)} = \pi_{q_2^*}^{(1)} (M_{q_2^*-1}^{(1)} A_3^{(1)} + R^{(1)} D_2^{(1)})(C_2^{(1)} - B_2^{(1)})^{-1}.
\end{aligned} \tag{18}$$

The following algorithm is used to in the calculations:

Algorithm 3.

Step 1. Solve (15) for matrix $R^{(1)}$, using iterations (16).

Step 2. Evaluate the Matrices $M_j^{(1)}$ for $j = 0, \dots, q_2^* - 1$.

Step 3. Evaluate the value $\pi_{q_2^*}^{(1)}$ by solving the normalisation condition

$$\begin{aligned}
1 = \sum_{k=0}^{\infty} \pi_k^{(1)} e &= \pi_{q_2^*}^{(1)} \sum_{i=0}^{q_2^*-1} \prod_{j=q_2^*-1-i}^{q_2^*-1} M_j^{(1)} e + \pi_{q_2^*}^{(1)} \sum_{j=0}^{\infty} (R^{(1)})^j e \\
&= \pi_{q_2^*}^{(1)} \Big[\sum_{i=0}^{q_2^*-1} \prod_{j=q_2^*-1-i}^{q_2^*-1} M_j^{(1)} e + (I - R^{(1)})^{-1} e \Big].
\end{aligned} \tag{19}$$

with the equation

$$\pi_{q_2^*}^{(1)} (M_{q_2^*-1}^{(1)} A_3^{(1)} - (C_2^{(1)} - B_2^{(1)}) + R^{(1)} D_2^{(1)}) = 0.$$

Step 4. Substitute $\pi_{q_2^*}^{(1)}$ in

$$\pi_i^{(1)} = \pi_{q_2^*}^{(1)} \prod_{j=q_2^*-1-i}^{q_2^*-1} M_j^{(1)} \tag{20}$$

for the values $i = 0, \dots, q_2^* - 1$ and calculate

$$\pi_{q_2^*+j}^{(1)} = \pi_{q_2^*}^{(1)} (R^{(1)})^j \tag{21}$$

for $j > 0$.

5. STEADY-STATE DISTRIBUTIONS OF THE SYSTEM UNDER HEURISTIC POLICIES

To measure the advantages of optimal threshold policy (OTP) we consider four alternative models, namely

- Scheduling threshold policy (STP)
- Fastest free server selection (FFS)

– Random server selection (RSS)
– System with homogeneous servers (HS) and the same total service time.

The models STP, FFS, RSS and HS will be denoted by index $m = \{2, 3, 4, 5\}$, respectively. The formulas for the calculation of the steady-state distribution of the system under the STP are exactly the same as for the optimal policy with only one exception that the threshold level may differ from the optimal one. For the homogeneous system to calculate the performance characteristics we use the steady-state distribution obtained in [19].

In the next subsections calculation procedures are treated for systems under the FFS and RSS policies.

5.1. Finite retrial group

In STP case the fastest server must be busy whenever the arrival occurs whereas the slower server can be switched on only if upon arrival of a primary or retrial customer the orbit has length defined by (6). The policy FSS means that the fastest free server must be occupied upon an arrival of a primary or retrial customer. Under the policy RSS arrivals choose any free server with equal probability.

Since the system under FFS and RSS control policies is described by the same Markov process $\{X(t)\}_{t\geq 0}$ with the state space E as under the optimal control policy, we have a similar states partitioning.

Analogously as in previous sections we can write down the three-diagonal infinitesimal block matrices $\Lambda^{(m)}$, $m = \{2, 3, 4\}$ of dimension $4(K+1)$. It is obvious that for STP matrix has the same form as for the optimal policy. For policies FFS and RSS policies in case $K < \infty$ the infinitesimal matrices are of the form,

$$\Lambda^{(m)} = \begin{pmatrix} -\lambda & A_1^{(m)} & 0 & 0 & 0 & \dots & 0 \\ D_0^{(m)} & -(C_1^{(m)} - B_1^{(m)}) & A_2^{(m)} & 0 & 0 & \dots & 0 \\ 0 & D_1^{(m)} & -(C_3^{(m)} - B_3^{(m)}) & A_4^{(m)} & 0 & \dots & 0 \\ 0 & 0 & D_2^{(m)} & -(C_3^{(m)} - B_3^{(m)}) & A_4^{(m)} & \dots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & & 0 \\ 0 & \dots & 0 & D_3^{(m)} & -(C_3^{(m)} - B_3^{(m)}) & A_5^{(m)} & 0 \\ 0 & \dots & 0 & 0 & D_3^{(m)} & -(C_4^{(m)} - B_4^{(m)}) & A_6^{(m)} \\ 0 & \dots & 0 & 0 & 0 & D_4^{(m)} & M \end{pmatrix}$$

where $A_1^{(3)} = A_1^{(1)}$, $A_2^{(3)} = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 \end{pmatrix}$, $A_i^{(3)} = A_i^{(1)}$, $i = 4, 5, 6$, $B_1^{(3)} = B_1^{(1)}$, $B_3^{(3)} = B_3^{(1)}$, $B_4^{(3)} = B_4^{(4)} = B_4^{(1)}$, $C_1^{(3)} = C_1^{(4)} = C_1^{(1)}$, $C_i^{(3)} = C_i^{(4)} = C_i^{(1)}$, $i = 3, 4$, $D_i^{(3)} = D_i^{(4)} = D_i^{(1)}$, $0 \leq i \leq 4$,

$A_1^{(4)} = \begin{pmatrix} \frac{\lambda}{2} & \frac{\lambda}{2} & 0 \end{pmatrix}$, $A_2^{(4)} = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ 0 & \frac{\lambda}{2} & \frac{\lambda}{2} & 0 \end{pmatrix}$, $A_4^{(4)} = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ 0 & \frac{\lambda}{2} & \frac{\lambda}{2} & 0 \end{pmatrix}$, $A_5^{(4)} = \begin{pmatrix} \lambda & 0 & 0 \\ \lambda & 0 & 0 \\ \lambda & 0 & 0 \\ 0 & \frac{\lambda}{2} & \frac{\lambda}{2} \end{pmatrix}$,

$A_6^{(4)} = A_6^{(1)}$, $B_1^{(4)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\gamma}{2} & \frac{\gamma}{2} & 0 \end{pmatrix}$, $B_3^{(4)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \gamma & 0 & 0 & 0 \\ \gamma & 0 & 0 & 0 \\ 0 & \frac{\gamma}{2} & \frac{\gamma}{2} & 0 \end{pmatrix}$

The steady-state probability vector $\pi^{(m)}$, $m = \{2, 3, 4\}$ satisfies the system

$$\pi^{(m)} \Lambda^{(m)} = 0, \; \pi^{(m)} e = 1.$$

To solve the system for FFS and RSS policies we represent the equations in the form

$$
\begin{aligned}
\pi_0^{(m)} &= \pi_1^{(m)} D_0^{(m)} \frac{1}{\lambda} = \pi_1^{(m)} M_0^{(m)};\\
\pi_1^{(m)} &= \pi_2^{(m)} D_1^{(m)} (C_1^{(m)} - B_1^{(m)} - D_0^{(m)} \frac{1}{\lambda} A_1^{(m)})^{-1} = \pi_2^{(m)} M_1^{(m)};\\
\pi_2^{(m)} &= \pi_3^{(m)} D_2^{(m)} (C_3^{(m)} - B_3^{(m)} - M_1^{(m)} A_2^{(m)})^{-1} = \pi_3^{(m)} M_2^{(m)};\\
\pi_i^{(m)} &= \pi_{i+1}^{(m)} D_2^{(m)} (C_3^{(m)} - B_3^{(m)} - M_{i-1}^{(m)} A_4^{(m)})^{-1} = \pi_{i+1}^{(m)} M_i^{(m)}, \qquad 3 \le i \le K-1;\\
\pi_K^{(m)} &= \pi_{K+1}^{(m)} D_3^{(m)} (C_3^{(m)} - B_3^{(m)} - M_{K-1}^{(m)} A_4^{(m)})^{-1} = \pi_{K+1}^{(m)} M_K^{(m)};\\
\pi_{K+1}^{(m)} &= \pi_{K+2}^{(m)} D_4^{(m)} (C_4^{(m)} - B_4^{(m)} - M_K^{(m)} A_5^{(m)})^{-1} = \pi_{K+2}^{(m)} M_{K+1}^{(m)}, \qquad m = \{3,4\}.
\end{aligned} \tag{22}
$$

To calculate the values $\pi_k^{(m)}, \;\; m = \{3,4\}$ it is necessary:

Algorithm 4.
Step 1. Evaluate matrices $M_k^{(m)}$, $k = 0, \ldots, K+1$ by relation (22).
Step 2 Evaluate the value $\pi_{K+2}^{(m)}$ from the normalization condition

$$1 = \sum_{k=0}^{K+2} \pi_k^{(m)} e = \pi_{K+2}^{(m)} + \pi_{K+2}^{(m)} \sum_{i=0}^{K+1} \prod_{j=K+1-i}^{K+1} M_j^{(m)} e = \pi_{K+2}^{(m)} \left[1 + \sum_{i=0}^{K+1} \prod_{j=K+1-i}^{K+1} M_j^{(m)} e \right]. \tag{23}$$

Step 3. Substitute $\pi_{K+2}^{(m)}$ into

$$\pi_i^{(m)} = \pi_{K+2}^{(m)} \prod_{j=K+1-i}^{K+1} M_j^{(m)}.$$

for $i = 0, \ldots, K+1$.

5.2. Infinite retrial group

In case of infinite retrial group, $K = \infty$, the infinitesimal matrices $\Lambda^{(m)}$, $m = \{2,3,4\}$ are obtained from the above matrices by removing the last three rows. In this case we conjecture the matrix-geometric form

$$\pi_{2+j}^{(m)} = \pi_2^{(m)} (R^{(m)})^j, \; j \ge 0$$

where the matrix $R^{(m)}$ is to be found by solving the following quadratic equation

$$(R^{(m)})^2 D_2^{(m)} - R^{(m)} (C_3^{(m)} - B_3^{(m)}) + A_4^{(m)} = 0. \tag{24}$$

As before the necessary and sufficient condition for stability is $p^{(m)} D_2^{(m)} e > p^{(m)} A_4^{(m)} e$, where $p^{(m)}$ is a stationary probability vector given by $p^{(m)}(A_4^{(m)} - (C_3^{(m)} - B_3^{(m)}) + D_2^{(m)}) = 0$, $p^{(m)} e = 1$.

Theorem 5. *For the system under STP and FFS control policies, the stationary probability vector* $p^{(m)}$ *of* $A_4^{(m)} - (C_3^{(m)} - B_3^{(m)}) + D_2^{(m)}$, $m = \{2,3\}$ *is given by formulas 4. The system is stable if and only if the load factor* $\rho^{(m)}$, *defined by (17) satisfies*

$$\rho^{(m)} < 1, \, m = \{2,3\}.$$

Theorem 6. *For the system under RSS control policy, the stationary probability vector* $p^{(4)}$ *of* $A_4^{(4)} - (C_3^{(4)} - B_3^{(4)}) + D_2^{(4)}$ *is given by*

$$\begin{aligned} p_0^{(4)} &= \frac{(\lambda+\gamma)^2}{(\lambda+\gamma)(\lambda+M+\gamma)+2\mu_1\mu_2}, \\ p_1^{(4)} &= \frac{\mu_1}{\lambda+\gamma} p_0^{(4)}, \\ p_2^{(4)} &= \frac{\mu_2}{\lambda+\gamma} p_0^{(4)}, \\ p_3^{(4)} &= \frac{2\mu_1\mu_2}{(\lambda+\gamma)^2} p_0^{(4)}. \end{aligned}$$

The system is stable if and only if the load factor $\rho^{(4)}$ *satisfies*

$$\rho^{(4)} = \frac{\lambda(\lambda+\gamma)^2}{M\gamma(\lambda+\gamma)+2\gamma\mu_1\mu_2} < 1. \tag{25}$$

Proof. By elementary calculations.

Probabilities $\pi_0^{(m)}$ and $\pi_1^{(m)}$ satisfy relations

$$\begin{aligned} \pi_0^{(m)} &= \pi_1^{(m)} D_0^{(m)} \frac{1}{\lambda} = \pi_1^{(m)} M_0^{(m)}, \\ \pi_1^{(m)} &= \pi_2^{(m)} D_1^{(m)} (C_1^{(m)} - B_1^{(m)} - D_0^{(m)} \frac{1}{\lambda} A_1^{(m)})^{-1} = \pi_2^{(m)} M_1^{(m)}. \end{aligned} \tag{26}$$

In case $m = \{3, 4\}$ the following algorithm is used for the calculations:

Algorithm 5.
Step 1. Solve equations (24) for matrix $R^{(m)}$, using iterations starting from $R^{(m)}(0) = 0$.
Step 2. Evaluate matrices $M_0^{(m)}$ and $M_1^{(m)}$.
Step 3. Evaluate value $\pi_2^{(m)}$ by solving the normalizing condition

$$\begin{aligned} 1 &= \sum_{k=0}^{\infty} \pi_k^{(m)} e = \pi_2^{(m)} M_1^{(m)} M_0^{(m)} e + \pi_2^{(m)} M_1^{(m)} e + \pi_2^{(m)} \sum_{j=0}^{\infty} (R^{(m)})^j e \\ &= \pi_2^{(m)} \Big[M_1^{(m)} M_0^{(m)} e + M_1^{(m)} e + (I - R^{(m)})^{-1} e \Big]. \end{aligned} \tag{27}$$

with the equation

$$\pi_2^{(m)} (M_1^{(m)} A_2^{(m)} - (C_3^{(m)} - B_3^{(m)}) + R^{(m)} D_2^{(m)}) = 0.$$

Step 4. Substitute $\pi_2^{(m)}$ into

$$\pi_0^{(m)} = \pi_2^{(m)} M_1^{(m)} M_0^{(m)}, \; \pi_1^{(m)} = \pi_2^{(m)} M_1^{(m)} \tag{28}$$

and calculate

$$\pi_{2+j}^{(m)} = \pi_2^{(m)} (R^{(m)})^j, \qquad j > 0. \tag{29}$$

6. SYSTEM PERFORMANCE CHARACTERISTICS

By means of steady-state probabilities the main performance measures in case $K = \infty$ can be derived as follows.

Corollary.
Utilization of the system

$$\overline{U}^{(m)} = 1 - \pi_0^{(m)} \tag{30}$$

Utilization of the first and second server

$$\overline{U}_1^{(m)} = \pi_1^{(m)} e_1 + \left[\sum_{i=2}^{q_2^*-1} \pi_i^{(m)} + \pi_{q_2^*}^{(m)} (I - R^{(m)})^{-1}\right] (e_0 + e_2),\ m = \{1,2\} \tag{31}$$
$$\overline{U}_2^{(m)} = \pi_1^{(m)} e_0 + \left[\sum_{i=2}^{q_2^*-1} \pi_i^{(m)} + \pi_{q_2^*}^{(m)} (I - R^{(m)})^{-1}\right] (e_0 + e_1),\ m = \{1,2\}$$
$$\overline{U}_1^{(m)} = \pi_1^{(m)} e_1 + \pi_2^{(m)} (I - R^{(m)})^{-1} (e_0 + e_2),\ m = \{3,4\}$$
$$\overline{U}_2^{(m)} = \pi_1^{(m)} e_0 + \pi_2^{(m)} (I - R^{(m)})^{-1} (e_0 + e_1),\ m = \{3,4\}$$

Mean number of busy servers

$$\overline{C}^{(m)} = \pi_1^{(m)} (e_0 + e_1) + \left[\sum_{i=2}^{q_2^*-1} \pi_i^{(m)} + \pi_{q_2^*}^{(m)} (I - R^{(m)})^{-1}\right] (2e_0 + e_1 + e_2) = \overline{U}_1^{(m)} + \overline{U}_2^{(m)},\ m = \{1,2\} \tag{32}$$
$$\overline{C}^{(m)} = \pi_1^{(m)} (e_0 + e_1) + \pi_2^{(m)} (I - R^{(m)})^{-1} (2e_0 + e_1 + e_2) = \overline{U}_1^{(m)} + \overline{U}_2^{(m)},\ m = \{3,4\}$$

Mean number of customers waiting in the queue

$$\overline{Q}^{(m)} = \pi_1^{(m)} e_2 + \left[\sum_{i=2}^{q_2^*-1} i\pi_i^{(m)} + \pi_{q_2^*}^{(m)} (R^{(m)} (I - R^{(m)})^{-1} + q_2^* I)(I - R^{(m)})^{-1}\right] e \tag{33}$$
$$- \left[\sum_{i=2}^{q_2^*-1} \pi_i^{(m)} + \pi_{q_2^*}^{(m)} (I - R^{(m)})^{-1}\right] (2e_0 + e_1 + e_2),\ m = \{1,2\}$$
$$\overline{Q}^{(m)} = \pi_1^{(m)} e_2 + \pi_2^{(m)} [(R^{(m)} (I - R^{(m)})^{-1} + 2I)(I - R^{(m)})^{-1} e - (I - R^{(m)})^{-1} (2e_0 + e_1 + e_2)],\ m = \{3,4\}$$

Mean number of customers in the system

$$\overline{N}^{(m)} = \sum_{i=1}^{q_2^*-1} i\pi_i^{(m)} e + \pi_{q_2^*}^{(m)} (R^{(m)} (I - R^{(m)})^{-1} + q_2^* I)(I - R^{(m)})^{-1} e = \overline{C}^{(m)} + \overline{Q}^{(m)} \tag{34}$$

Mean waiting and sojourn times

$$\overline{W}^{(m)} = \frac{\overline{Q}^{(m)}}{\lambda}, \quad \overline{T}^{(m)} = \frac{\overline{N}^{(m)}}{\lambda} \tag{35}$$

Overall rate of retrials

$$\tilde{\gamma}_1^{(m)} = \gamma(1 - \pi_0^{(m)} - \pi_1^{(m)}(e_0 + e_1) - \pi_2^{(m)} e_0) \tag{36}$$

Rate of successful retrials

$$\tilde{\gamma}_2^{(m)} = \tilde{\gamma}_1^{(m)} - \gamma \left(\sum_{i=2}^{q_2^*-1} \pi_i^{(m)} e_2 + \left[\sum_{i=3}^{q_2^*-1} \pi_i^{(m)} + \pi_{q_2^*}^{(m)} (I - R)^{-1} \right] e_0 \right), \; m = \{1, 2\} \tag{37}$$

$$\tilde{\gamma}_2^{(m)} = \tilde{\gamma}_1^{(m)} - \gamma \pi_2^{(m)} (I - R)^{-1} e_0, \; m = \{3, 4\}$$

Fraction of successful retrials $\frac{\tilde{\gamma}_2^{(m)}}{\tilde{\gamma}_1^{(m)}}$

Blocking probability

$$P_{blocking}^{(m)} = \pi_1^{(m)} e_1 + \sum_{i=2}^{q_2^*-1} \pi_i^{(m)} e_2 + \left[\sum_{i=2}^{q_2^*-1} \pi_i^{(m)} + \pi_{q_2^*}^{(m)} (I - R^{(m)})^{-1} \right] e_0, \; m = \{1, 2\} \tag{38}$$

$$P_{blocking}^{(m)} = \pi_2^{(m)} (I - R^{(m)})^{-1} e_0, \; m = \{3, 4\}$$

Probability of immediate access

$$P_{access}^{(m)} = \pi_0^{(m)} + \pi_1^{(m)}(e_0 + e_2) + \sum_{i=2}^{q_2^*-1} \pi_i^{(m)} (e_1 + e_3) + \pi_{q_2^*}^{(m)} (I - R)^{-1} (e_1 + e_2 + e_3), \; m = \{1, 2\} \tag{39}$$

$$P_{access}^{(m)} = \pi_0^{(m)} + \pi_1^{(m)} e + \pi_2^{(m)} (I - R^{(m)})^{-1} (e_1 + e_2 + e_3), \; m = \{3, 4\}$$

Mean busy period

$$T_{busy}^{(m)} = \frac{1}{\lambda} \left(\frac{1}{\pi_0^{(m)}} - 1 \right). \tag{40}$$

Mean time in a regenerative cycle during which the orbiting customer is blocked

$$\overline{T}_{blocking}^{(m)} = P_{blocking}^{(m)} \left(\frac{1}{\lambda} + T_{busy}^{(m)} \right) = \frac{P_{blocking}^{(m)}}{\lambda \pi_0^{(m)}} \tag{41}$$

7. NUMERICAL RESULTS AND COMPARISON ANALYSIS

Consider the system $M/M/2$ with primary arrival rate λ, retrial rate γ and service rates μ_1 and μ_2. By means of *Mathematica* package the following procedures were developed :

– calculation of steady-state probabilities under optimal and empirical service disciplines, by using formulas (18–21) and (26)–(29),

– calculation of the performance characteristics under different service disciplines, by using formulas (30–41).

Next we present some numerical results to show the effect of parameters on the performance characteristics. In the tables below performance measures for the system under different selection policies $m = \{1, 2, 3, 4\}$ and for the homogeneous system $m = \{5\}$ are given for varying values of primary arrival, retrial and service rates, respectively

$\lambda, \mu_1, \mu_2, \gamma$	$\overline{U}^{(1)}$	$\overline{U}^{(2)}$	$\overline{U}^{(3)}$	$\overline{U}^{(4)}$	$\overline{U}^{(5)}$	$\overline{N}^{(1)}$	$\overline{N}^{(2)}$	$\overline{N}^{(3)}$	$\overline{N}^{(4)}$	$\overline{N}^{(5)}$
0.1,2.2,0.3,8.5	0.0459	0.0459	0.0552	0.1648	0.0769	0.0482	0.0482	0.0575	0.1717	0.0802
0.1,2.2,0.3,1.5	0.0485	0.0484	0.0554	0.1650	0.0771	0.0510	0.0510	0.0577	0.1722	0.0803
0.1,1.3,1.2,8.5	0.0744	0.0744	0.0744	0.0771	0.0769	0.0775	0.0775	0.0775	0.0803	0.0802
0.1,1.3,1.2,1.5	0.0745	0.0745	0.0745	0.0772	0.0771	0.0777	0.0777	0.0777	0.0805	0.0803
1.3,2.2,0.3,8.5	0.6775	0.6706	0.6706	0.8526	0.7034	1.6820	1.8538	1.8520	2.0310	1.5904
1.3,2.2,0.3,1.5	0.8692	0.8799	0.8814	0.9364	0.8043	4.4750	4.6931	4.4960	5.5240	2.8376
1.3,1.3,1.2,8.5	0.6992	0.6992	0.6992	0.7037	0.7034	1.5799	1.5799	1.5800	1.5900	1.5904
1.3,1.3,1.2,1.5	0.7997	0.7997	0.7997	0.8040	0.8043	2.7940	2.7940	2.7940	2.8250	2.8376

Table 1. Utilization and mean number of customers in the system

$\lambda, \mu_1, \mu_2, \gamma$	$\overline{W}^{(1)}$	$\overline{W}^{(2)}$	$\overline{W}^{(3)}$	$\overline{W}^{(4)}$	$\overline{W}^{(5)}$	$\overline{T}^{(1)}$	$\overline{T}^{(2)}$	$\overline{T}^{(3)}$	$\overline{T}^{(4)}$	$\overline{T}^{(5)}$
0.1,2.2,0.3,8.5	0.0275	0.0275	0.0012	0.0036	0.0017	0.4821	0.4821	0.5754	1.7173	0.8017
0.1,2.2,0.3,1.5	0.0546	0.0546	0.0025	0.0074	0.0034	0.5096	0.5096	0.5765	1.7218	0.8034
0.1,1.3,1.2,8.5	0.0016	0.0016	0.0016	0.0017	0.0017	0.7753	0.7753	0.7753	0.8028	0.8017
0.1,1.3,1.2,1.5	0.0033	0.0033	0.0033	0.0034	0.0034	0.7770	0.7770	0.7770	0.8046	0.8034
1.3,2.2,0.3,8.5	0.6437	0.9152	0.5253	0.5779	0.4234	1.2939	1.4260	1.4246	1.5623	1.2234
1.3,2.2,0.3,1.5	2.5848	2.8059	2.5317	3.2481	1.3828	3.4421	3.6101	3.4586	4.2493	2.1827
1.3,1.3,1.2,8.5	0.4202	0.4202	0.4202	0.4231	0.4234	1.2154	1.2154	1.2154	1.2233	1.2234
1.3,1.3,1.2,1.5	1.3545	1.3545	1.3545	1.3728	1.3828	2.1492	2.1492	2.1492	2.1727	2.1828

Table 2. Mean waiting time and mean sojourn time of a customer

The following figures represent the system utilization (Figure 2(a,b)), mean number of customers in the system (Figure 3(a,b)), mean waiting time (Figure 4(a,b)) and mean sojourn time of customers (Figure 5(a,b)) for service intensities $\mu_1 = 2.2$, $\mu_2 = 0.3$, retrial intensity $\gamma = 2.5$ for figures labeled by "a"and $\gamma = 18.5$ for figures labeled by "b"with varying primary arrival intensity $0.05 \leq \lambda \leq 1.7$. Analyzing the figures we have noticed the following.
1. When the primary arrival rate λ makes relative small contribution to the load factor of the system then the OPT and STP coincide that leads to the equal values of the performance measures. Otherwise, the policies as well as the performance measures are different.
2. When the primary arrival rate is quite large and the load factor tends to 1 ("heavy traffic") then difference between the policies can be neglected.
3. The curve of the mean waiting time for the FFS lies below other graphs, i.e. this policy minimizes the mean number of jobs in the orbit. This does not contradict with the optimality of the OTP since this policy optimizes the mean sojourn time, that is confirmed by the corresponding figure.
4. All curves are monotone except for the mean waiting time in case of the OTP and mean sojourn time in case of the RSS policy. In first case as λ increases the threshold levels of the OTP decreases that leads to the reduction of the waiting time in the queue. In second case the convexity of the curve is connected with the mean service time that contributes to the mean sojourn time. If the rate is very small a new arrival most likely sees two servers idle and under the RSS policy with equal probability occupies them. Upon increasing of the arrival rate the slower server will be made more busy and fastest server is used more intensively, that resulting in less service time.

(a) (b)

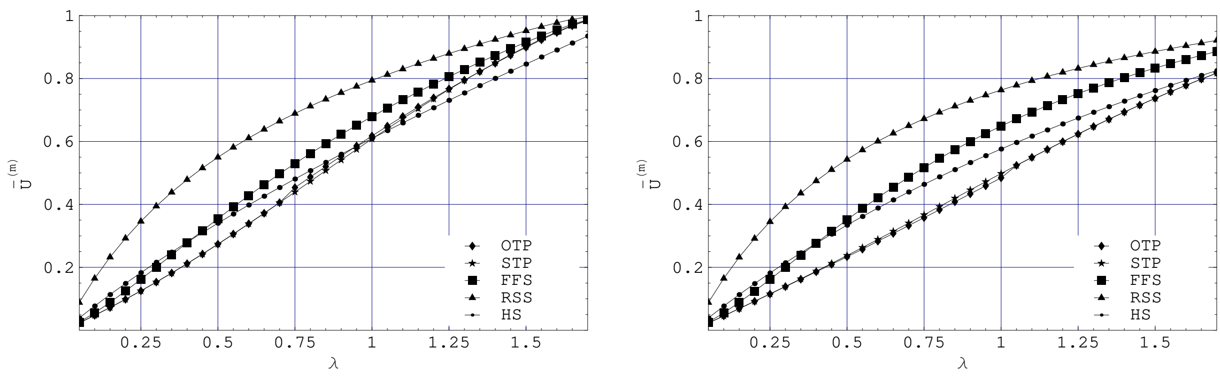

Figure 2. System utilization versus primary arrival and retrial rate (a) $\gamma = 2.5$ (b) $\gamma = 18.5$

(a) (b)

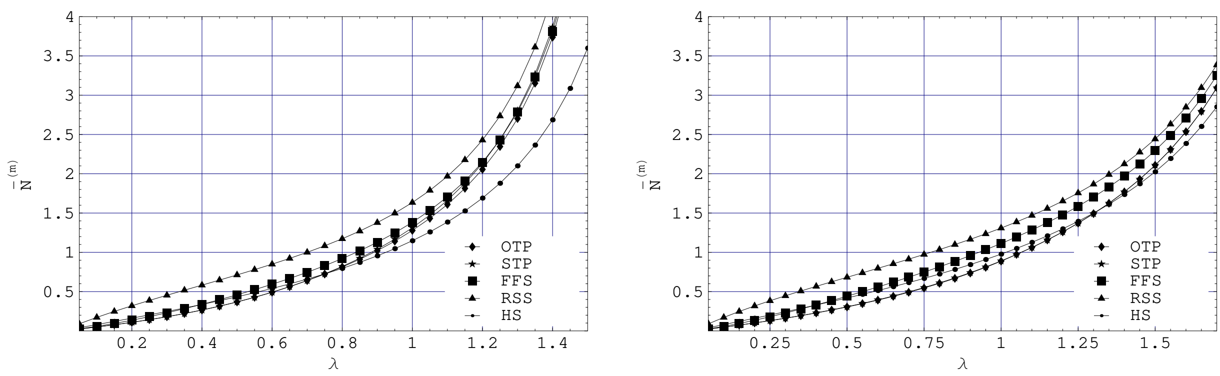

Figure 3. Mean number of customers versus primary arrival and retrial rate (a) $\gamma = 2.5$ (b) $\gamma = 18.5$

5. In comparison to the classical queues [19] where the system with heterogeneous servers under OTP is always superior in performance to the homogeneous one with the same total service time, in case of retrials the system with homogeneous servers has advantage over heterogeneous one only if $\mu_2 < \lambda$ for large μ_1/λ and small γ/λ or vice versa.

The next figures represent the system utilization (Figure 6(a)), mean number of customers in the system (Figure 6(b)), mean waiting time (Figure 7(a)) and mean sojourn time of a customer (Figure 7(b)) for service intensities $\mu_1 = 2.2,\ \mu_2 = 0.3$, primary arrival rate $\lambda = 0.5$ with varying retrial rate $0.05 \leq \gamma \leq 10$.

With respect to the given performance measures we observe the following.
1. For small retrial rate the threshold control policies prescribe to use the slower server when $q_2^* = 1$, that coincides with the heuristic policies (FFS, RSS). As the retrial rate increases the advantage of the threshold policies with respect to the mean number of customers in the system or mean sojourn time becomes more evident.
2. For large retrial rate the illustrated curves converges to their asymptotic values that resemble very close to the corresponding performance measures of the classical $M/M/2$ queue.
3. All curves monotone decrease except for the mean waiting time in case of threshold policies

(a) (b)

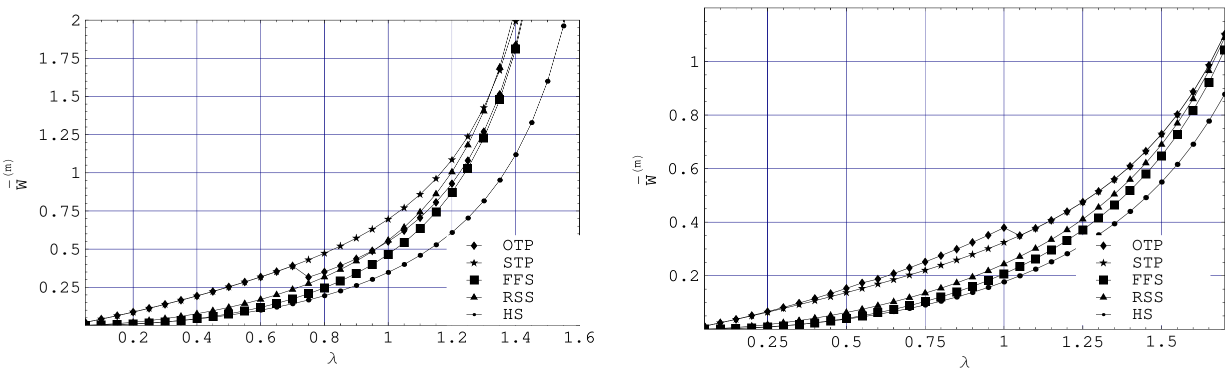

Figure 4. Mean waiting time versus primary arrival and retrial rate (a) $\gamma = 2.5$ (b) $\gamma = 18.5$

(a) (b)

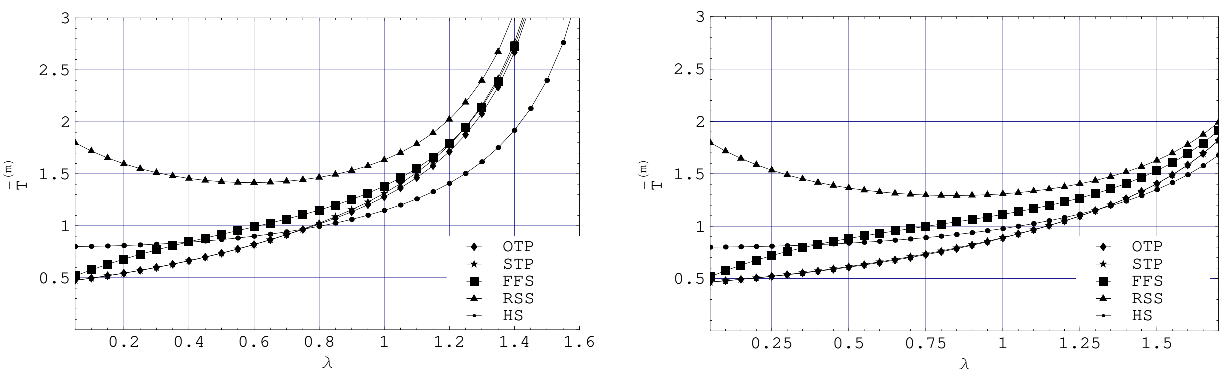

Figure 5. Mean sojourn time versus primary arrival and retrial rate (a) $\gamma = 2.5$ (b) $\gamma = 18.5$

(OPT, STP). It can be explained by the fact that as γ increases the threshold levels also increase that in turn leads to the significant increasing of the mean waiting time.

Figure 8 displays the influence of γ on $\overline{T}^{(m)}_{\text{blocking}}$, $m = \{1, 2, 3, 4\}$, graphs are labeled by (a,b,c,d), respectively.

1. Pictures show that the curves for all the models start by decreasing to a minimum and then converges to their asymptotic values. These results coincide with the corresponding outcomes obtained in [2] for the uncontrollable homogeneous retrial queues with direct access to the server facility.
2. The step structure of the curves in case of threshold policies (OTP, STP) we interpret also by changing of the threshold levels which implies the significant changing of the mean blocking time.
3. When the retrial rate is small the mean blocking time for threshold policies (OTP, STP) is quite small in comparison with the heuristic policies (FFS, RSS). But as γ increases, the level q_2^* increases as well, that leads to the larger mean blocking time.

(a) (b)

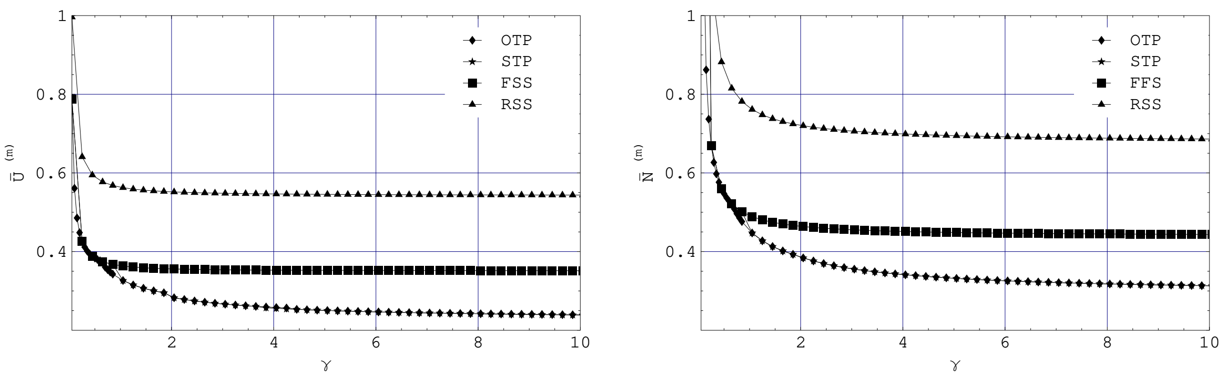

Figure 6. (a) System utilization and (b) mean number of customers in the system versus retrial rate

(a) (b)

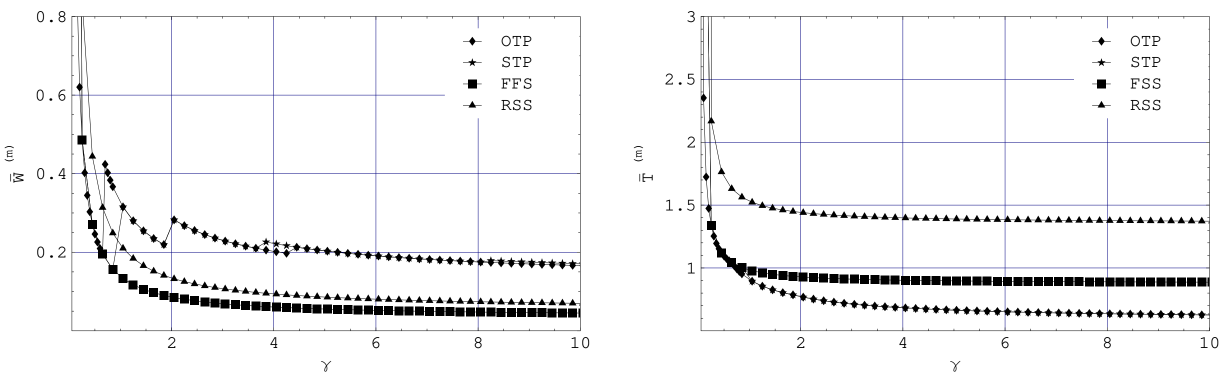

Figure 7. (a) Mean waiting time and (b) mean sojourn time of a customer versus retrial rate

8. CONCLUSION

The present paper considered a controllable queueing system with heterogeneous servers, constant retrial rate. It was proved that similarly to the classical queueing systems the optimal control policy of the retrial system is of threshold and monotone type, i.e. the fastest server must be used whenever the customer comes to service area, while the slower server has to be switched on only if the number of customers in orbit reaches some prespecified level. We have derived the formula for the calculation of the optimal threshold levels for scheduling problem and approximation for the general case and presented an efficient algorithm for the numerical calculation of these levels in general case. Examining some numerical results we conjecture the suboptimality of the STP in so called "light traffic case"when the arrival rate is smaller that the slowest server and FSS in "heavy traffic case"when the load factor tends to 1.

We showed that the controllable model with heterogeneous servers and given control policy may be treated as QBD process with block states and transition submatrices. Thus the general theory of matrix-geometric solutions was applicable for developing numerical methods.

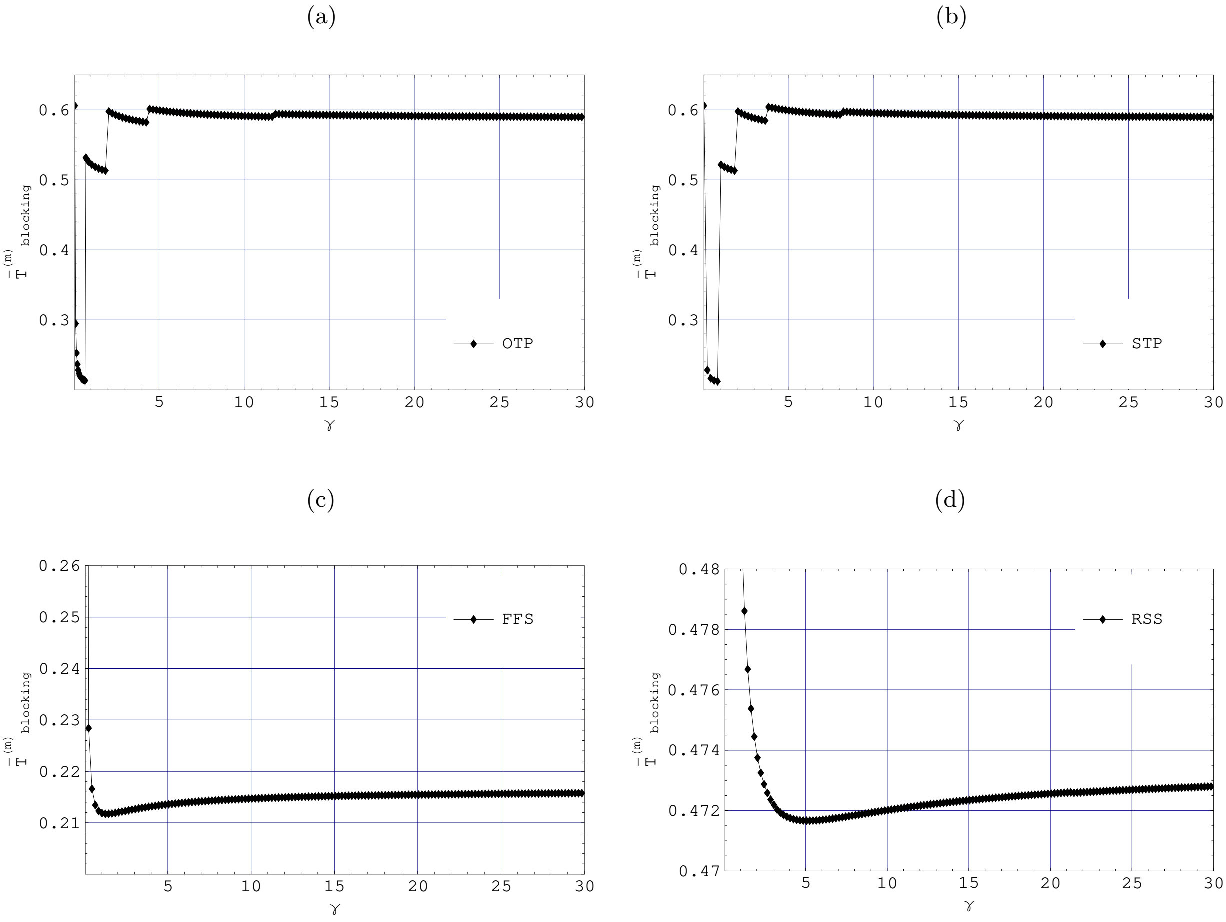

Figure 8. Mean blocking time in a regenerative cycle versus retrial rate

By using the structured properties of the optimal control policy we derived an efficient algorithm for the calculation of steady state probabilities and mean performance characteristics for the system under optimal and other heuristic control policies. Some performance measures were calculated and compared under different control policies. Numerical analysis showed that for the relative small load factor the threshold policies (OTP, STP) have a great advantage in mean number of customers in the system or mean sojourn time over heuristic policies (FFS, RSS) while the difference between threshold policies was negligible. As the load factor tends to 1 the advantages of the threshold policies decreased and the performance measures converged to similar values. Other performance characteristics under optimal policy may be worse as under heuristic policies because the optimal policy corresponds to the certain optimization criterion that minimizes the specified performance measure. Necessary and sufficient conditions for the stability of the systems were obtained. In case of constant retrial rate these conditions depended on all system parameters.

The presented methods can also be extended to some other models, e.g. with more than two servers or where interarrival and service times are dependent on a modulating Markov process, or with phase type times. These cases are not discussed because they would require very complicated notation to write down the algorithms, but the extensions are natural and possible in principle.

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