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Stochastic completeness and volume growth

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Abstract

Let M be a geodesically complete Riemannian manifold and $p \in C^{\infty}((0,\infty) \times M \times M)$ be its heat kernel. It has the following property $\int_M p(t,x,y)dy \leq 1$. The *heat kernel* has the following stochastic interpretation. For $x \in M$ and $U \subset M$ open, $\int_U p(t,x,y)dy$ is the probability that a random path emanating from x lies in U at time t. Thus if we have strict inequality $\int_M p(t,x,y)dy < 1$, then there is positive probability that the random path will reach infinity in finite time t. This motivates the following definition. A geodesically complete connected Riemannian manifold is called *stochastically complete* if $\int_M p(t,x,y)dy = 1$.

- Complete Riemannian manifolds with Ricci curvature bounded below are stochastically complete.
- For any $x \in M$ denote the closed ball of radius $r > 0$ about x by $B(x,r)$. Write $V(x,r) := \mathrm{vol}(B(x,r))$ and $S(x,r) := \mathrm{area}(\partial B(x,r))$. Grigor'yan's criterion, says that if

$$\int^{\infty} \frac{r}{\log V(x,r)} = \infty$$

  for some $x \in M$, then M is stochastically complete. This criteria can be applied if $V(x,r) \leq \exp(c \cdot r^2)$ for some $c > 0$ and all $r \geq r_0$.
- *Model manifolds* are $\mathbb{R}^n$ equipped with the metric $g = dr^2 + f^2(r) g_{\mathbb{S}^{n-1}}$ where $f\colon [0,\infty) \to \mathbb{R}$ is a smooth function such that $f(0) = 0$, $f'(0) = 1$ and $f(t) > 0$ for $t > 0$. Here $r = |x|$ is the distance from the origin $o \in \mathbb{R}^n$ and $g_{\mathbb{S}^{n-1}}$ is the standard metric of $\mathbb{S}^{n-1}$. Model manifolds are stochastically complete if and only if

$$\int^{\infty} \frac{V(o,r)}{S(o,r)} dr = \infty.$$

- Grigor'yan asked/conjectured if/that for a general geodesically complete manifold M the condition

$$\int^{\infty} \frac{V(x,r)}{S(x,r)} dr = \infty \tag{0.1}$$

  for some $x \in M$ is sufficient for stochastically completeness.

Our main result is the construction of counter-examples to this conjecture.

Theorem 0.1 (Bar-Bessa). *In any dimension $n \geq 2$ there exists a geodesically complete but stochastically incomplete connected Riemannian manifold M such that for some $x \in M$ the volume grow condition 0.1 holds.*

Remark 0.1. *Conversely, one may ask if on a general geodesically complete manifold M the condition*

$$\int^{\infty} \frac{V(x,r)}{S(x,r)} dr < \infty \tag{0.2}$$

for some $x \in M$ is sufficient for stochastically incompleteness. But this is false too. We construct a counter-example in the same spirit as in the Theorem 0.1.

Remark 0.2. *The construction is based on a result due to Pigola-Rigoli-Setti where it is shown that stochastic completeness is equivalent to a weak form of the Omori-Yau maximum principle.*