

# Stochastic delay foraging arena predator-prey system with Markov switching

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## Abstract

In this paper we introduce white noise, telegraph noise and time delay to the two-dimensional foraging arena population system describing the prey and predator abundance. The aim is to find out how the interactions between white noise, telegraph noise and time delay affect the dynamics of the population system. Firstly the existence of a global positive solution is verified. Then the long-time properties including the stochastically ultimate boundedness, extinction and some other asymptotic pathwise estimation of this population system are studied. Finally the main results are illustrated by two examples.

**Keywords:** stochastic predator-prey model, Brownian motion, Markov chain, time delay, ultimate boundedness

## 1 Introduction

The predator-prey models have been widely studied recently. A general system to describe the dynamics of prey and predator populations could be represented by

$$\frac{dx(t)}{dt} = \lambda_1(x(t))x(t) - \lambda_2(x(t), y(t))y(t) \quad (1.1a)$$

$$\frac{dy(t)}{dt} = \gamma\lambda_2(x(t), y(t))y(t) - \lambda_3(y(t))y(t), \quad (1.1b)$$

where  $x(t)$  and  $y(t)$  refer to the population densities of prey and predator at time  $t$ ,  $\lambda_1(x)$  is the per capita net prey growth in absence of predator,  $\lambda_2(x, y)$  is the density-dependent uptake response of consumers,  $\gamma$  is the trophic efficiency ranging from 0 to 1 and  $\lambda_3(y)$  is the consumers death rate. Especially,  $\lambda_1(x)$  takes the form of  $\lambda_1(x) = r$  (exponential growth) or  $\lambda_1(x) = r(1 - \frac{x}{K})$  (logical growth) [1], where  $r$  is the intrinsic growth rate and  $K$  is the carrying capacity. Moreover  $\lambda_2(x, y)$  is called the “functional response” in the prey equations (1.1a) and the “numerical response” in the consumers equation (1.1b) [2, 3]. It is initially assumed that  $\lambda_2(x, y) = \lambda_2(x)$ , i.e. the response function of prey depends on the prey density only. In this case, prey and predator individuals encounter each other randomly in space and time. For instance, the classic Lotka-Volterra type response is a direct linear function of prey density; Holling type II response gives  $\lambda_2(x) = u_1x/(u_2 + x)$ , with  $u_1$  a maximum uptake rate by the predator and  $u_2$  a prey half-saturation coefficient and Holling type III functional response is represented by  $\lambda_2(x) = u_1x^2/(u_2^2 + x^2)$ . This is then challenged by some ecologists with the fact that the functional response over ecological time scale depends on both the prey and predator abundance [4]. Then the consumer-density dependence of per capita uptake rate was introduced. The ratio-dependent functional response is pointed out by Arditi and Ginzburg [5, 6] to represent sharing of resources, behavioural interference between consumers to their mutual impairment, enhanced escape

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reactions by prey, sheltering in refuges with increasing predator density [7] or the foraging of predators in a patchy prey environment [8]. However, this formulation has been criticised mainly because the uptake rate trends to infinity as consumer abundance tends to zero [4]. Hence the concerned model fails to satisfy the continuity condition at origin. To alleviate this property, the foraging arena model is pointed out by [9, 10] with

$$\lambda_2(x, y) = u_1x/(w + u_3y) = sx/(\beta + y),$$

where  $\beta = w/u_3 =$  consumer density at half maximum per capita uptake rate and  $u_1/w = s/\beta =$  maximum per capita uptake rate by predator. Foraging arenas are common in aquatic systems. They are formed by a series of mechanisms such as the restrictions of the consumer distributions in response to the predation risk due to their own predators and the risk-sensitive foraging behaviour by their prey [10]. Especially, the classic Lotka-Volterra model and the Holling types assume that the individual prey and predator items are distributed in a spatially uniform way. While the foraging arena model considers the spatial and temporal restrictions in predator and prey activities. The foraging arena theory has been widely used in fisheries science to explain and model responses of harvested ecosystems. This is done mainly through the application of Ecosim which is the dynamic modelling part of an ecosystem modelling software suite called *Ecopath with Ecosim (EwE)*. Ecosim is built around foraging arena theory and is capable of fitting historical data on responses of multiple fish populations to harvesting and changes in primary production regimes [9, 10]. The two-dimensional foraging arena predator-prey model can be represented as follows:

$$\begin{aligned} d\bar{x}_1(t) &= \bar{x}_1(t) \left( a - b\bar{x}_1(t) - \frac{s\bar{x}_2(t)}{\beta + \bar{x}_2(t)} \right) dt, \\ d\bar{x}_2(t) &= \bar{x}_2(t) \left( \frac{h\bar{x}_1(t)}{\beta + \bar{x}_2(t)} - c - f\bar{x}_2(t) \right) dt, \end{aligned} \tag{1.2}$$

where  $\bar{x}_1(t)$  and  $\bar{x}_2(t)$  denote the population densities of prey and predator at time  $t$  and  $a, b, s, \beta, h, c$  and  $f$  are all positive constants. More precisely,  $a$  is the intrinsic growth rate of prey,  $c$  is the density-dependent mortality rate of consumers,  $h = \gamma s$ ,  $b$  and  $f$  are the quadratic mortality rates of prey and predator respectively. We set  $\bar{x}(t) = (\bar{x}_1(t), \bar{x}_2(t))^T$  as the solution to model (1.2) with the initial value  $\bar{x}_0 = (\bar{x}_1(0), \bar{x}_2(0))^T$ . In model (1.2), there are two non-negative trivial equilibrium points  $\bar{E}_0 = (0, 0)$  and  $\bar{E}_1 = (\frac{a}{b}, 0)$ . Also an unique interior equilibrium point  $\bar{E}^*(\bar{x}_1^*, \bar{x}_2^*)$  with the nullclines

$$\begin{aligned} (a - b\bar{x}_1^*)(\beta + \bar{x}_2^*) &= s\bar{x}_2^*, \\ (\beta + \bar{x}_2^*)(c + f\bar{x}_2^*) &= h\bar{x}_1^* \end{aligned}$$

exists and is globally asymptotically stable provided that  $a > \frac{b\beta c}{h}$  [11].

In fact, population systems are always subject to the complex variations. A natural response is to consider stochastic models. An extensive literature is concerned with the effects of environmental variability on the predator-prey populations [12–28]. Mao et al. [29] pointed out an important fact in the Lotka-Volterra models that the environmental noise can suppress a potential population explosion. In [13–15], the  $n$ -dimensional delay Lotka-Volterra models with different types of environmental noise were studied and the unique asymptotic behaviours of these SDE models are explored. Takeuchi et al.[30] discussed a surprising effect of a colour noise on a Lotka-Volterra model. The stochastic predator-prey systems with Holling II response are also well studied [16–19]. According to Liu et al. [18], the long-time behaviours were explored as well as its stationary distribution. Zhang et al. [31] also studied the stochastic Holling type II model with Markovian switching and jumps. Moreover, the more complicated ratio-dependent response is also explored by some authors. Ji et al.[20] established the conditions for species in a ratio-dependent population system to be either extinct or persistent. However, we are not aware of any literature addressing this issue for the foraging arena model. This is the motivation for us to study the stochastic versions of the foraging arena system. Obviously the intrinsic prey growth rate and the consumer death rate in system (1.2) are varied by some environmental factors such as climate fluctuations. Suppose that  $a$  and  $c$  are stochastically perturbed with

$$a \rightarrow a + \sigma_1 \dot{B}_1(t) \quad \text{and} \quad c \rightarrow c + \sigma_2 \dot{B}_2(t),$$

where  $B_1(t)$  and  $B_2(t)$  are two independent Brownian motions with the intensities represented by two positive constants  $\sigma_1$  and  $\sigma_2$ . As a result this perturbed system is given by

$$\begin{aligned} dx_1(t) &= x_1(t) \left( a - bx_1(t) - \frac{sx_2(t)}{\beta + x_2(t)} \right) dt + \sigma_1 x_1(t) dB_1(t), \\ dx_2(t) &= x_2(t) \left( \frac{hx_1(t)}{\beta + x_2(t)} - c - fx_2(t) \right) dt + \sigma_2 x_2(t) dB_2(t). \end{aligned} \quad (1.3)$$

The analytical results on model (1.3) can be found in [25]. Additionally, some time delays are inevitable in population interactions. For instance, gestation may result in time delays [32]. In this paper we incorporate time delay into the predator-prey system to examine how it would affect the population system. This leads to the following system:

$$\begin{aligned} dx_1(t) &= x_1(t) \left( a - bx_1(t) - \frac{sx_2(t)}{\beta + x_2(t)} \right) dt + \sigma_1 x_1(t) dB_1(t), \\ dx_2(t) &= x_2(t) \left( \frac{hx_1(t - \tau)}{\beta + x_2(t - \tau)} - c - fx_2(t) \right) dt + \sigma_2 x_2(t) dB_2(t), \end{aligned} \quad (1.4)$$

where  $\tau$  is the constant delay due to gestation. We also would like to investigate the effect of telegraph noise on the population dynamics. Telegraph noise can characterise the systems where the structures and parameters experience abrupt changes due to environmental disturbances and changing subsystem interconnections [22]. Recall that telegraph noise can be described as a switching between two or more regimes of environments [33, 34]. The regime switching can be modelled by a finite-state Markov chain [33]. We let  $r(t), t \geq 0$  be a right-continuous Markov chain on the probability space taking values in the state space  $\mathbb{S} = \{1, 2, \dots, N\}$  with the generator  $\Gamma = (\gamma_{uv})$  given by

$$\mathbb{P}\{r(t + \delta) = v | r(t) = u\} = \begin{cases} \gamma_{uv}\delta + o(\delta), & \text{if } u \neq v, \\ 1 + \gamma_{uu}\delta + o(\delta) & \text{if } u = v, \end{cases}$$

where  $\delta > 0$ . Here  $\gamma_{uv} \geq 0$  is the transition rate from  $u$  to  $v$  if  $u \neq v$  while  $\gamma_{uu} = -\sum_{v \neq u} \gamma_{uv}$ . Such process is called a continuous-time finite Markov chain. We suppose that all the Markov chains are finite and all states are stable. For such a Markov chain, almost every sample path is a right continuous step function with a finite number of sample jumps in any finite subinterval of  $\mathbb{R}_+$ . There is a sequence  $\{\eta_n\}_{n \geq 0}$  of finite-valued  $\mathcal{F}_t$ -stopping times such that  $0 = \eta_0 < \eta_1 < \dots < \eta_n \rightarrow \infty$  almost surely and

$$r(t) = \sum_{n=0}^{\infty} r(\eta_n) \mathbf{I}_{[\eta_n, \eta_{n+1})}(t).$$

The switching is memoryless and the waiting time for the next switch has an exponential distribution with parameter  $-\gamma_{ii}$ , given that  $r(\eta_n) = i$ . Namely

$$\mathbb{P}(\eta_{n+1} - \eta_n \geq T | r(\eta_n) = i) = e^{\gamma_{ii}T}, \quad \forall T \geq 0.$$

We also assume that the Markov chain  $r(\cdot)$  is independent of the Brownian motion  $B(\cdot)$  and is irreducible. Under this condition, the Markov chain has a unique stationary distribution  $\pi = (\pi_1, \pi_2, \dots, \pi_N) \in \mathbb{R}^{1 \times N}$  which can be defined by solving the following linear equation

$$\pi \Gamma = 0$$

subject to

$$\sum_{i=1}^N \pi_i = 1 \text{ and } \pi_i > 0 \text{ for all } i \in \mathbb{S}.$$

Now we will introduce the Markov switching into the SDE system (1.4). Suppose the Markov chain  $r(t)$  in the state space  $\mathbb{S} = \{1, 2, \dots, N\}$  controls the switching between the environmental regimes, the population system (1.4) then becomes

$$dx_1(t) = x_1(t) \left( a(r(t)) - b(r(t))x_1(t) - \frac{s(r(t))x_2(t)}{\beta(r(t)) + x_2(t)} \right) dt + \sigma_1(r(t))x_1(t) dB_1(t) \quad (1.5a)$$

$$dx_2(t) = x_2(t) \left( \frac{h(r(t))x_1(t-\tau)}{\beta(r(t)) + x_2(t-\tau)} - c(r(t)) - f(r(t))x_2(t) \right) dt + \sigma_2(r(t))x_2(t)dB_2(t), \quad (1.5b)$$

where the system parameters  $a(i), b(i), s(i), \beta(i), h(i), c(i), f(i), \sigma_1(i)$  and  $\sigma_2(i)$  are all positive constants for all  $i \in \mathbb{S}$ . We set  $x(t) = (x_1(t), x_2(t))^T$  as the solution to model (1.5) with the initial value  $x_0 = (x_1(0), x_2(0))^T$ . Throughout this paper, unless otherwise specified, we let  $(\Omega, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$  be a complete probability space with a filtration  $\{\mathcal{F}_t\}_{t \geq 0}$  satisfying the usual conditions. We also define  $\mathcal{F}_\infty = \sigma(\bigcup_{t \geq 0} \mathcal{F}_t)$ , i.e. the  $\sigma$ -algebra generated by  $\bigcup_{t \geq 0} \mathcal{F}_t$ . Let  $B(t) = (B_1(t), B_2(t))^T$  be a two-dimensional Brownian motion defined on this probability space. We denote by  $\mathbb{R}_+^2$  the positive cone in  $\mathbb{R}^2$ , that is  $\mathbb{R}_+^2 = \{x \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\}$ . We also set  $\inf \emptyset = \infty$ . If  $A$  is a vector or matrix, its transpose is denoted by  $A^T$ . If  $A$  is a matrix, its trace norm is  $|A| = \sqrt{\text{trace}(A^T A)}$  whilst its operator norm is  $\|A\| = \sup\{|Ax| : |x| = 1\}$ . If  $A$  is a symmetric matrix, its smallest and largest eigenvalues are denoted by  $\lambda_{\min}(A)$  and  $\lambda_{\max}(A)$ .

In this paper, we first prove the existence of a global positive solution of system (1.5). Next the asymptotic moment average of system (1.5) is explored. In section 4 and 5, we investigate the long-time behaviours of each species. Computer simulations based on the Euler-Maruyama scheme are performed to illustrate our theory.

## 2 Global Positive Solution

It is an essential property for a population system to have a unique positive solution. We found that the coefficients of model (1.5) do not obey the linear growth condition, though they are locally Lipschitz continuous. This suggests that the solution may exit from  $\mathbb{R}_+^2$  space at a finite time. The following theorem shows the existence and uniqueness of a positive global solution to model (1.5).

**Theorem 2.1.** *For any given initial value  $\{x(t) : -\tau \leq t \leq 0\} \in C([-\tau, 0]; \mathbb{R}_+^2)$ , there is a unique solution  $x(t)$  to equation (1.5) on  $t \geq -\tau$  and the solution will remain in  $\mathbb{R}_+^2$  with probability one, namely  $x(t) \in \mathbb{R}_+^2$  for all  $t \geq -\tau$  almost surely.*

*Proof.* Since the coefficients of equation (1.5) are locally Lipschitz continuous, for any given initial value  $\{x(t) : -\tau \leq t \leq 0\} \in C([-\tau, 0]; \mathbb{R}_+^2)$ , there is a unique maximal local solution  $x(t)$  on  $t \in [-\tau, \tau_e)$ , where  $\tau_e$  is the explosion time (exit time) from  $\mathbb{R}_+^2$ . To show that  $x(t) \in \mathbb{R}_+^2$  a.s. for all  $t \geq 0$ , we need to verify  $\tau_e = \infty$  a.s. Let  $k_0 > 0$  be sufficiently large for

$$\frac{1}{k_0} < \min_{-\tau \leq t \leq 0} |x(t)| \leq \max_{-\tau \leq t \leq 0} |x(t)| < k_0.$$

For each integer  $k \geq k_0$ , define the stopping time

$$\tau_k = \inf\{t \in [0, \tau_e) : x_i(t) \notin \left(\frac{1}{k}, k\right) \text{ for some } i = 1, 2\}.$$

Obviously,  $\tau_k$  is increasing as  $k \rightarrow \infty$ . Set  $\tau_\infty := \lim_{k \rightarrow \infty} \tau_k$  and whence  $\tau_\infty \leq \tau_e$  a.s. Hence to complete the proof, we need to show that

$$\tau_\infty = \infty \quad \text{a.s.} \quad (2.1)$$

Define a  $C^2$ -function  $V : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$  by  $V(x) = x_1 - \log x_1 + x_2 - \log x_2$ . From the Itô formula [22, 35],

$$\begin{aligned} dV(x(t), r(t)) &= LV(x(t), x(t-\tau), r(t))dt + \sigma_1(r(t))(x_1(t) - 1)dB_1(t) \\ &\quad + \sigma_2(r(t))(x_2(t) - 1)dB_2(t), \end{aligned} \quad (2.2)$$

where

$$\begin{aligned} LV(x, y, i) &= -a(i) + \frac{s(i)x_2}{\beta(i) + x_2} + c(i) + \frac{\sigma_1^2(i)}{2} + \frac{\sigma_2^2(i)}{2} + (a(i) + b(i))x_1 + (f(i) \\ &\quad - c(i))x_2 - b(i)x_1^2 - f(i)x_2^2 + \frac{h(i)x_2y_1}{\beta(i) + y_2} - \frac{h(i)y_1}{\beta(i) + y_2}, \end{aligned} \quad (2.3)$$

with  $x$  replaced by  $x(t)$ ,  $y$  replaced by  $x(t - \tau)$  and  $i$  replaced by  $r(t)$  in (2.2). The Young inequality then indicates that

$$\frac{h(i)x_2y_1}{\beta(i) + y_2} \leq \frac{h(i)x_2y_1}{\beta(i)} = \frac{h(i)y_1}{\beta(i)\sqrt{f(i)}}\sqrt{f(i)}x_2 \leq \frac{h^2(i)}{2\beta^2(i)f(i)}y_1^2 + \frac{f(i)}{2}x_2^2$$

It is then followed from (2.3) that

$$\begin{aligned} LV(x, y, i) &\leq -a(i) + s(i) + c(i) + \frac{\sigma_1^2(i)}{2} + \frac{\sigma_2^2(i)}{2} + (a(i) + b(i))x_1 + (f(i) - c(i))x_2 \\ &\quad - b(i)x_1^2 - \frac{f(i)}{2}x_2^2 + \frac{h^2(i)}{2\beta^2(i)f(i)}y_1^2 \end{aligned}$$

Hence there exist three positive constants  $K_1$ ,  $K_2$  and  $K_3$  for

$$LV(x, y, i) \leq K_1\left(1 + \frac{|x|}{2}\right) - K_2|x|^2 + K_3y_1^2. \quad (2.4)$$

Note that  $|x| \leq 2V(x)$ . Equation (2.2) is then followed from (2.4) that

$$\begin{aligned} dV(x(t), r(t)) &\leq [K_1(1 + V(x(t))) - K_2|x(t)|^2 + K_3x_1^2(t - \tau)]dt \\ &\quad + \sigma_1(r(t))(x_1(t) - 1)dB_1(t) + \sigma_2(r(t))(x_2(t) - 1)dB_2(t). \end{aligned}$$

For any  $k \geq k_0$  and  $t_1 \in [0, \tau]$ , we obtain

$$\mathbb{E}V(x(t_1 \wedge \tau_k)) \leq K_4 + K_1\mathbb{E}\int_0^{t_1 \wedge \tau_k} V(x(t))dt - K_2\mathbb{E}\int_0^{t_1 \wedge \tau_k} |x(t)|^2dt, \quad (2.5)$$

where

$$K_4 = V(x(0)) + K_1\tau + K_3\int_0^\tau x_1^2(t - \tau)dt < \infty.$$

We then obtain from (2.5) that

$$\mathbb{E}V(x(t_1 \wedge \tau_k)) \leq K_4 + K_1\int_0^{t_1} \mathbb{E}V(x(\tau_k \wedge t))dt.$$

This and the Gronwall inequality [35] imply that

$$\mathbb{E}V(x(t_1 \wedge \tau_k)) \leq K_4e^{\tau K_1} \quad \text{for } 0 \leq t_1 \leq \tau, k \geq k_0. \quad (2.6)$$

It then follows that

$$\mathbb{E}V(x(\tau_k \wedge \tau)) \leq K_4e^{\tau K_1} \quad \text{for } k \geq k_0.$$

We can hence show that  $\tau_\infty \geq \tau$  a.s. [29, 35, 36]. Additionally, letting  $k \rightarrow \infty$  in (2.6) gives

$$\mathbb{E}V(x(t_1)) \leq K_4e^{\tau K_1} \quad \text{for } 0 \leq t_1 \leq \tau.$$

By setting  $t_1 = \tau$  in (2.5) and then letting  $k \rightarrow \infty$ , we have

$$\mathbb{E}\int_0^\tau |x(t)|^2dt \leq \frac{1}{K_2}(K_4 + \tau K_1 K_4 e^{\tau K_1}) < \infty. \quad (2.7)$$

For  $t_2 \in (\tau, 2\tau]$ ,

$$\begin{aligned} \mathbb{E}V(x(t_2 \wedge \tau_k)) &\leq K_5 + K_1\mathbb{E}\int_0^{t_2 \wedge \tau_k} V(x(t))dt - K_2\mathbb{E}\int_0^{t_2 \wedge \tau_k} |x(t)|^2dt \\ &\quad + K_3\mathbb{E}\int_0^{t_2 \wedge \tau_k - \tau} x_1^2(t)dt \\ &\leq \tilde{K}_5 + K_1\mathbb{E}\int_0^{t_2 \wedge \tau_k} V(x(t))dt - K_2\mathbb{E}\int_0^{t_2 \wedge \tau_k} |x(t)|^2dt, \end{aligned} \quad (2.8)$$

where  $K_5 = V(x(0)) + 2K_1\tau + K_3 \int_0^\tau x_1^2(t - \tau)dt$  and  $\tilde{K}_5 = K_5 + \frac{1}{K_2}(K_4 + \tau K_1 K_4 e^{\tau K_1}) < \infty$  in view of (2.7). Similarly we obtain that  $\tau_\infty \geq 2\tau$  a.s. and

$$\mathbb{E}V(x(t_2)) \leq \tilde{K}_5 e^{2\tau K_1}.$$

By setting  $t_2 = 2\tau$  in (2.8) and then letting  $k \rightarrow \infty$  yields that

$$\mathbb{E} \int_0^{2\tau} |x(t)|^2 dt \leq \frac{1}{K_2} (\tilde{K}_5 + 2\tau K_1 \tilde{K}_5 e^{2\tau K_1}) < \infty.$$

Repeating this procedure, one can show  $\tau_\infty \geq m\tau$  with probability one for any integer  $m \geq 1$ . Therefore  $\tau_\infty = \infty$  a.s.  $\square$

### 3 Stochastically Ultimate Boundedness

After analysing the global positive solution to model (1.5), we now explore the conditions for system (1.5) to be stochastically ultimately bounded.

**Theorem 3.1.** *If*

$$h(i) \leq \beta(i)f(i) \text{ for all } i \in \mathbb{S}, \quad (3.1)$$

*then for any  $\theta > 0$ , there exists a positive constant  $K(\theta)$  such that for any initial value  $\{x(t) : -\tau \leq t \leq 0\} \in C([- \tau, 0]; \mathbb{R}_+^2)$ ,*

$$\limsup_{t \rightarrow \infty} \mathbb{E}|x(t)|^\theta \leq K(\theta).$$

*Proof.* Condition (3.1) yields that there exists a constant  $\tilde{\theta} > 1$  sufficiently large such that

$$\frac{e^\tau h^{\tilde{\theta}+1}(i)}{(\tilde{\theta} + 1)\beta^{\tilde{\theta}+1}(i)f^{\tilde{\theta}}(i)} < \hat{b} \quad \text{for all } i \in \mathbb{S}.$$

We first consider the case when  $\theta \geq \tilde{\theta}$ . It then follows that

$$\frac{e^\tau h^{\theta+1}(i)}{(\theta + 1)\beta^{\theta+1}(i)f^\theta(i)} < \hat{b} \quad \text{for all } i \in \mathbb{S}. \quad (3.2)$$

Applying the Itô formula to  $e^t(x_1^\theta(t) + x_2^\theta(t))$ ,

$$\begin{aligned} d(e^t(x_1^\theta(t) + x_2^\theta(t))) &= e^t \phi(x(t), x(t - \tau), r(t))dt + \theta \sigma_1(r(t))e^t x_1^\theta(t) dB_1(t) \\ &\quad + \theta \sigma_2(r(t))e^t x_2^\theta(t) dB_2(t), \end{aligned} \quad (3.3)$$

where

$$\begin{aligned} \phi(x, y, i) &= \left( a(i)\theta + \frac{1}{2}\theta(\theta - 1)\sigma_1^2(i) + 1 \right) x_1^\theta + \left( -c(i)\theta + \frac{1}{2}\theta(\theta - 1)\sigma_2^2(i) + 1 \right) x_2^\theta \\ &\quad - \frac{s(i)\theta x_1^\theta x_2}{\beta(i) + x_2} + \frac{h(i)\theta y_1 x_2^\theta}{\beta(i) + y_2} - b(i)\theta x_1^{\theta+1} - f(i)\theta x_2^{\theta+1}. \end{aligned}$$

Integrating on both sides of (3.3) and then taking expectation then yields that

$$\begin{aligned} \mathbb{E} \left[ e^{t \wedge \tau_k} (x_1^\theta(t \wedge \tau_k) + x_2^\theta(t \wedge \tau_k)) \right] &= x_1^\theta(0) + x_2^\theta(0) \\ &\quad + \int_0^{t \wedge \tau_k} e^u \phi(x(u), x(u - \tau), r(u)) du. \end{aligned}$$

From the Young inequality, for all  $i \in \mathbb{S}$

$$\frac{h(i)y_1x_2^\theta}{\beta(i) + y_2} \leq \frac{h(i)y_1}{\beta(i)f^{\frac{\theta}{\theta+1}}(i)} \cdot f^{\frac{\theta}{\theta+1}}(i)x_2^\theta \leq \frac{h^{\theta+1}(i)}{(\theta+1)\beta^{\theta+1}(i)f^\theta(i)}y_1^{\theta+1} + \frac{\theta f(i)}{\theta+1}x_2^{\theta+1}.$$

Hence

$$\phi(x, y, i) \leq \phi_1(x, i) + \phi_2(y, i) - b(i)\theta x_1^{\theta+1}$$

with

$$\begin{aligned} \phi_1(x, i) &= \left( a(i)\theta + \frac{1}{2}\theta(\theta-1)\sigma_1^2(i) + 1 \right) x_1^\theta + \left( -c(i)\theta + \frac{1}{2}\theta(\theta-1)\sigma_2^2(i) + 1 \right) x_2^\theta \\ &\quad - \frac{f(i)\theta}{\theta+1}x_2^{\theta+1} \end{aligned}$$

and

$$\phi_2(y, i) = \frac{\theta h^{\theta+1}(i)}{(\theta+1)\beta^{\theta+1}(i)f^\theta(i)}y_1^{\theta+1}.$$

Note that

$$\begin{aligned} &\int_0^{t \wedge \tau_k} e^u \phi_2(x_1(u-\tau), r(u)) du \\ &\leq \frac{\theta e^\tau}{\theta+1} \int_{-\tau}^{t \wedge \tau_k} \frac{h^{\theta+1}(r(u+\tau))e^u}{\beta^{\theta+1}(r(u+\tau))f^\theta(r(u+\tau))} x_1^{\theta+1}(u) du \\ &\leq \frac{\theta \check{h}^{\theta+1} e^\tau}{(\theta+1)\hat{\beta}^{\theta+1}\hat{f}^\theta} \int_{-\tau}^0 x_1^{\theta+1}(u) du + \frac{\theta e^\tau}{\theta+1} \int_0^{t \wedge \tau_k} \frac{h^{\theta+1}(r(u+\tau))e^u}{\beta^{\theta+1}(r(u+\tau))f^\theta(r(u+\tau))} x_1^{\theta+1}(u) du \end{aligned}$$

Hence

$$\begin{aligned} &\mathbb{E} \left[ e^{t \wedge \tau_k} (x_1^\theta(t \wedge \tau_k) + x_2^\theta(t \wedge \tau_k)) \right] \\ &\leq x_1^\theta(0) + x_2^\theta(0) + \int_0^{t \wedge \tau_k} e^u \left[ \phi_1(x(u), r(u)) + \phi_2(x(u-\tau), r(u)) - b(r(u))\theta x_1^{\theta+1} \right] du \\ &\leq x_1^\theta(0) + x_2^\theta(0) + \frac{\theta \check{h}^{\theta+1} e^\tau}{(\theta+1)\hat{\beta}^{\theta+1}\hat{f}^\theta} \int_{-\tau}^0 x_1^{\theta+1}(u) du + \int_0^{t \wedge \tau_k} e^u \left[ \phi_1(x(u), r(u)) \right. \\ &\quad \left. + \left( \frac{e^\tau h^{\theta+1}(r(u+\tau))}{(\theta+1)\beta^{\theta+1}(r(u+\tau))f^\theta(r(u+\tau))} - b(r(u)) \right) \theta x_1^{\theta+1}(u) \right] du \end{aligned}$$

This and (3.2) indicate that there is a positive constant  $K^*(\theta)$  such that

$$\begin{aligned} &\mathbb{E} \left[ e^{(t \wedge \tau_k)} (x_1^\theta(t \wedge \tau_k) + x_2^\theta(t \wedge \tau_k)) \right] \\ &\leq x_1^\theta(0) + x_2^\theta(0) + \frac{\theta \check{h}^{\theta+1} e^\tau}{(\theta+1)\hat{\beta}^{\theta+1}\hat{f}^\theta} \int_{-\tau}^0 x_1^{\theta+1}(u) du + K^*(\theta) \int_0^{t \wedge \tau_k} e^u du. \end{aligned}$$

Letting  $k \rightarrow \infty$  and then  $t \rightarrow \infty$  yields

$$\limsup_{t \rightarrow \infty} \mathbb{E}[x_1^\theta(t) + x_2^\theta(t)] \leq K^*(\theta). \quad (3.4)$$

On the other hand, we have

$$|x|^\theta \leq 2 \max(x_1^\theta, x_2^\theta), \text{ so } |x|^\theta \leq 2^{\theta/2} \max(x_1^\theta, x_2^\theta) \leq 2^{\theta/2}(x_1^\theta + x_2^\theta).$$

As a result,

$$\limsup_{t \rightarrow \infty} \mathbb{E}|x(t)|^\theta \leq 2^{\theta/2} \limsup_{t \rightarrow \infty} \mathbb{E}[x_1^\theta(t) + x_2^\theta(t)] \leq 2^{\theta/2} K^*(\theta) = K(\theta). \quad (3.5)$$

For  $0 < \theta < \tilde{\theta}$ , Hölder's inequality yields

$$\mathbb{E}|x(t)|^\theta \leq (\mathbb{E}|x(t)|^{\tilde{\theta}})^{\frac{\theta}{\tilde{\theta}}}.$$

Hence from (3.5)

$$\limsup_{t \rightarrow \infty} \mathbb{E}|x(t)|^\theta \leq \limsup_{t \rightarrow \infty} (\mathbb{E}|x(t)|^{\tilde{\theta}})^{\frac{\theta}{\tilde{\theta}}} \leq K(\theta).$$

□

According to Chebyshev's inequality [35], Theorem 3.1 reveals that for any constant  $\theta > 0$ , there exists a positive constant  $K(\theta)$  such that for any initial value  $\{x(t) : -\tau \leq t \leq 0\} \in C([-\tau, 0]; \mathbb{R}_+^2)$  and constants  $D_1, D_2 > 0$ , we have

$$\limsup_{t \rightarrow \infty} \mathbb{P}(x_1(t) \geq D_1) = \limsup_{t \rightarrow \infty} \mathbb{E}[\mathbf{I}_{x_1(t) \geq D_1}] \leq \limsup_{t \rightarrow \infty} \mathbb{E}\left[\frac{x_1^\theta(t)}{D_1^\theta} \mathbf{I}_{x_1(t) \geq D_1}\right] \leq \limsup_{t \rightarrow \infty} \frac{\mathbb{E}[x_1^\theta(t)]}{D_1^\theta} \leq \frac{K(\theta)}{D_1^\theta}$$

and similarly

$$\limsup_{t \rightarrow \infty} \mathbb{P}(x_2(t) \geq D_2) \leq \frac{K(\theta)}{D_2^\theta}$$

under condition 3.1, where  $\mathbf{I}$  is the indicator function. From the biological point of view, this implies that it is unlikely for either populations to become too large ultimately.

## 4 Extinction

In this section, we investigate the conditions for the system to be extinct.

**Theorem 4.1.** *For any initial value  $\{x(t) : -\tau \leq t \leq 0\} \in C([-\tau, 0]; \mathbb{R}_+^2)$ , if*

$$\lambda_1 := \sum_{i \in \mathbb{S}} \pi_i \left( a(i) - \frac{\sigma_1^2(i)}{2} \right) < 0, \quad (4.1)$$

*both  $x_1(t)$  and  $x_2(t)$  tend to zero exponentially as  $t \rightarrow \infty$  with probability one.*

*Proof.* Applying the Itô formula on  $\log x_1(t)$ , we have

$$\begin{aligned} d \log x_1(t) &= \left( a(r(t)) - b(r(t))x_1(t) - \frac{\sigma_1^2(r(t))}{2} - \frac{s(r(t))x_2(t)}{\beta(r(t)) + x_2(t)} \right) dt \\ &\quad + \sigma_1(r(t)) dB_1(t) \\ &\leq \left( a(r(t)) - \frac{\sigma_1^2(r(t))}{2} \right) dt + \sigma_1(r(t)) dB_1(t). \end{aligned} \quad (4.2)$$

Integrating from 0 to  $t$  and dividing by  $t$ , we get

$$\frac{1}{t} \log x_1(t) \leq \frac{1}{t} \log x_1(0) + \frac{1}{t} \int_0^t \left( a(r(u)) - \frac{\sigma_1^2(r(u))}{2} \right) du + \frac{\tilde{\sigma}_1 B_1(t)}{t}.$$

Letting  $t \rightarrow \infty$  and by the strong law of large numbers for martingales [35]

$$\lim_{t \rightarrow \infty} \frac{\tilde{\sigma}_1 B_1(t)}{t} = 0 \quad \text{a.s.}$$

Thus by the ergodic property of the Markov chain,

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log x_1(t) \leq \lambda_1 < 0 \quad \text{a.s.}$$



as required. It follows that

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t x_1(u) du = 0 \quad \text{a.s.} \quad (4.3)$$

Meanwhile

$$\begin{aligned} d \log x_2(t) &= \left( \frac{h(r(t))x_1(t-\tau)}{\beta(r(t)) + x_2(t-\tau)} - c(r(t)) - \frac{\sigma_2^2(r(t))}{2} - f(r(t))x_2(t) \right) dt \\ &\quad + \sigma_2(r(t))dB_2(t). \end{aligned} \quad (4.4)$$

It follows that

$$\frac{\log x_2(t)}{t} \leq \frac{1}{t} \left( \log x_2(0) + \frac{\check{h}}{\check{\beta}} \int_{-\tau}^0 x_1(u) du + \frac{\check{h}}{\check{\beta}} \int_0^t x_1(u) du \right) - \left( \hat{c} + \frac{\hat{\sigma}_2^2}{2} \right) + \frac{\check{\sigma}_2 B_2(t)}{t}.$$

Letting  $t \rightarrow \infty$  and recalling equation (4.3),

$$\limsup_{t \rightarrow \infty} \frac{\log x_2(t)}{t} \leq - \left( \hat{c} + \frac{\hat{\sigma}_2^2}{2} \right) < 0 \quad \text{a.s.}$$

□

## 5 Pathwise Estimation

In this section, we discuss the long-time properties of the solutions of system (1.5) pathwisely.

**Theorem 5.1.** *Assume that condition (3.1) holds. Then for any initial value  $\{x(t) : -\tau \leq t \leq 0\} \in C([- \tau, 0]; \mathbb{R}_+^2)$ ,*

$$\limsup_{t \rightarrow \infty} \frac{\log(x_1(t) + x_2(t))}{\log t} \leq 1 \quad \text{a.s.}$$

*Proof.* From the Young inequality,

$$\begin{aligned} &d[x_1(t) + x_2(t)] \\ &\leq \left[ a(r(t))x_1(t) + \frac{h(r(t))x_1(t-\tau)x_2(t)}{\beta(r(t)) + x_2(t-\tau)} \right] dt + \sigma_1(r(t))x_1(t)dB_1(t) \\ &\quad + \sigma_2(r(t))x_2(t)dB_2(t) \\ &\leq \left[ \check{a}x_1(t) + \frac{\check{h}^2}{2\check{\beta}^2}x_1^2(t-\tau) + \frac{1}{2}x_2^2(t) \right] dt + \sigma_1(r(t))x_1(t)dB_1(t) + \sigma_2(r(t))x_2(t)dB_2(t). \end{aligned}$$

Then we have

$$\begin{aligned} &\mathbb{E} \left[ \sup_{t \leq u \leq t+1} (x_1(u) + x_2(u)) \right] \leq \mathbb{E}[x_1(t) + x_2(t)] + \check{a} \int_t^{t+1} \mathbb{E}[x_1(v)] dv \\ &\quad + \frac{\check{h}^2}{2\check{\beta}^2} \int_t^{t+1} \mathbb{E}[x_1^2(u-\tau)] du + \frac{1}{2} \int_t^{t+1} \mathbb{E}[x_2^2(u)] du \\ &\quad + \mathbb{E} \left( \sup_{t \leq u \leq t+1} \int_t^u \sigma_1(r(v))x_1(v)dB_1(v) \right) + \mathbb{E} \left( \sup_{t \leq u \leq t+1} \int_t^u \sigma_2(r(v))x_2(v)dB_2(v) \right). \end{aligned} \quad (5.1)$$

By the Burkholder-Davis-Gundy inequality,

$$\begin{aligned} &\mathbb{E} \left( \sup_{t \leq u \leq t+1} \int_t^u \sigma_1(r(v))x_1(v)dB_1(v) \right) \leq 4\sqrt{2} \mathbb{E} \left( \int_t^{t+1} \check{\sigma}_1^2 x_1^2(v) dv \right)^{\frac{1}{2}} \\ &\leq \mathbb{E} \left( \sup_{t \leq u \leq t+1} x_1(u) \cdot 32\check{\sigma}_1^2 \int_t^{t+1} x_1(v) dv \right)^{\frac{1}{2}} \\ &\leq \mathbb{E} \left( \frac{1}{2} \sup_{t \leq u \leq t+1} x_1(u) + 16\check{\sigma}_1^2 \int_t^{t+1} x_1(v) dv \right) \\ &= \frac{1}{2} \mathbb{E} \left( \sup_{t \leq u \leq t+1} x_1(u) \right) + 16\check{\sigma}_1^2 \int_t^{t+1} \mathbb{E}[x_1(v)] dv. \end{aligned}$$

Similarly, we have

$$\mathbb{E}\left(\sup_{t \leq u \leq t+1} \int_t^u \sigma_2(r(s))x_2(s)dB_2(s)\right) \leq \frac{1}{2}\mathbb{E}\left(\sup_{t \leq u \leq t+1} x_2(u)\right) + 16\check{\sigma}_2^2 \int_t^{t+1} \mathbb{E}[x_2(v)]dv.$$

Hence (5.1) is then followed by

$$\begin{aligned} \mathbb{E}\left[\sup_{t \leq u \leq t+1} (x_1(u) + x_2(u))\right] &\leq 2\mathbb{E}[x_1(t) + x_2(t)] + 2\check{a} \int_t^{t+1} \mathbb{E}[x_1(v)]dv \\ &+ \frac{\check{h}^2}{\check{\beta}^2} \int_{t-\tau}^{t+1-\tau} \mathbb{E}[x_1^2(v)]dv + \int_t^{t+1} \mathbb{E}[x_2^2(v)]dv + 32(\check{\sigma}_1^2 \vee \check{\sigma}_2^2) \int_t^{t+1} \mathbb{E}[x_1(v) + x_2(v)]dv. \end{aligned}$$

Letting  $t \rightarrow \infty$  and making use of (3.4), we obtain

$$\limsup_{t \rightarrow \infty} \mathbb{E}\left[\sup_{t \leq u \leq t+1} (x_1(u) + x_2(u))\right] \leq 2\left(1 + \check{a} + 16(\check{\sigma}_1^2 \vee \check{\sigma}_2^2)\right)K^*(1) + \left(\frac{\check{h}^2}{\check{\beta}^2} \vee 1\right)K^*(2).$$

Hence there is a positive constant  $\tilde{K}$  such that

$$\mathbb{E}\left[\sup_{k_1 \leq u \leq k_1+1} (x_1(u) + x_2(u))\right] \leq \tilde{K} \text{ for } k_1 = 1, 2, \dots$$

Let  $\epsilon > 0$  be arbitrary. By the Chebychev inequality,

$$\mathbb{P}\left[\sup_{k_1 \leq u \leq k_1+1} (x_1(u) + x_2(u)) > k_1^{1+\epsilon}\right] \leq \frac{\mathbb{E}\left[\sup_{k_1 \leq u \leq k_1+1} (x_1(u) + x_2(u))\right]}{k_1^{1+\epsilon}} \leq \frac{\tilde{K}}{k_1^{1+\epsilon}}$$

for  $k_1 = 1, 2, \dots$ . By the Borel-Cantelli lemma, for almost all  $\omega \in \Omega$ ,

$$\sup_{k_1 \leq t \leq k_1+1} [x_1(t) + x_2(t)] \leq k_1^{1+\epsilon} \text{ holds for all but finitely many } k_1.$$

Hence there exists a  $\tilde{k}(\omega)$ , if  $k_1 \geq \tilde{k}$  and  $k_1 \leq t \leq k_1 + 1$ ,

$$\frac{\log [x_1(t) + x_2(t)]}{\log t} \leq \frac{\log \left[\sup_{k_1 \leq t \leq k_1+1} (x_1(t) + x_2(t))\right]}{\log t} \leq \frac{\log k_1^{1+\epsilon}}{\log t} \leq 1 + \epsilon \quad \text{a.s.}$$

Consequently,

$$\limsup_{t \rightarrow \infty} \frac{\log [x_1(t) + x_2(t)]}{\log t} \leq 1 + \epsilon \quad \text{a.s.}$$

Letting  $\epsilon \rightarrow 0$ , we obtain the required assertion.  $\square$

Theorem 5.1 shows that for arbitrary small  $\epsilon > 0$ , there is a positive random variable  $t_1 = t_1(\omega)$  such that with probability one,

$$\frac{\log [x_1(t) + x_2(t)]}{\log t} \leq 1 + \epsilon \text{ for all } t \geq t_1.$$

Hence we have

$$x_1(t) + x_2(t) \leq t^{1+\epsilon} \text{ for all } t \geq t_1.$$

It then follows that

$$x_1(t) + x_2(t) \leq \sup_{0 \leq t \leq t_1} [x_1(t) + x_2(t)] + t^{1+\epsilon} \text{ for all } t \geq 0.$$

This means that the total amount of prey and predator species will grow at most polynomially with order close to 1.

**Lemma 5.2.** *A one-dimensional Brownian motion  $\{W_t\}_{t \geq 0}$  has the property that for almost every  $\omega \in \Omega$ ,*

$$\lim_{t \rightarrow \infty} \frac{\min_{0 \leq u \leq t} W(u, \omega)}{t} = \lim_{t \rightarrow \infty} \frac{\max_{0 \leq u \leq t} W(u, \omega)}{t} = 0. \quad (5.2)$$

*Proof.* According to [35, p.16], for any  $n > 0$ , there exists a positive random variable  $\rho_n$  such that for almost every  $\omega \in \Omega$ ,

$$-(1+n)\sqrt{2t \log \log t} \leq W(t, \omega) \leq (1+n)\sqrt{2t \log \log t} \quad \text{for all } t \geq \rho_n(\omega).$$

It then follows that for almost every  $\omega \in \Omega$ ,

$$\begin{aligned} \min_{0 \leq u \leq \rho_n(\omega)} W(u, \omega) - (1+n)\sqrt{2t \log \log t} &\leq \min_{0 \leq u \leq t} W(u, \omega) \leq \max_{0 \leq u \leq t} W(u, \omega) \\ &\leq \max_{0 \leq u \leq \rho_n(\omega)} W(u, \omega) + (1+n)\sqrt{2t \log \log t}. \end{aligned}$$

This implies

$$0 \leq \lim_{t \rightarrow \infty} \frac{\min_{0 \leq u \leq t} W(u)}{t} \leq \lim_{t \rightarrow \infty} \frac{\max_{0 \leq u \leq t} W(u)}{t} \leq 0 \quad \text{a.s.}$$

This gives the required assertion (5.2) immediately.  $\square$

Based on Lemma 5.2, the following theorem shows some other asymptotic properties of the prey and predator populations.

**Theorem 5.3.** *For any initial value  $\{x(t) : -\tau \leq t \leq 0\} \in C([-\tau, 0]; \mathbb{R}_+^2)$ , if*

$$a(i) - \frac{\sigma_1^2(i)}{2} := q(i) > 0 \quad \text{for all } i \in \mathbb{S}, \quad (5.3)$$

then

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t x_1(u) du \leq \frac{\lambda_1}{\hat{b}} \quad \text{a.s.}$$

In particular, if also

- (i)  $\lambda_2 > 0$ , then  $\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t x_1(u) du \geq \frac{\lambda_2}{\hat{b}}$  almost surely;
- (ii)  $\frac{\check{h}}{\hat{b}\hat{\beta}} \lambda_1 - \lambda_3 < 0$ , then  $\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t x_1(u) du \geq \frac{\lambda_1}{\hat{b}}$  and  $x_2$  dies out exponentially almost surely,

where

$$\lambda_2 = \sum_{i \in \mathbb{S}} \pi_i \left( a(i) - s(i) - \frac{\sigma_1^2(i)}{2} \right) \quad \text{and} \quad \lambda_3 = \sum_{i \in \mathbb{S}} \pi_i \left( c(i) + \frac{\sigma_2^2(i)}{2} \right).$$

*Proof.* Applying Itô's formula on  $\frac{1}{x_1(t)}$  gives

$$d\left(\frac{1}{x_1(t)}\right) = \left(\frac{1}{x_1(t)} \left(\frac{s(r(t))x_2(t)}{\beta(r(t)) + x_2(t)} - a(r(t)) + \sigma_1^2(r(t))\right) + b(r(t))\right) dt - \frac{\sigma_1(r(t))}{x_1(t)} dB_1(t).$$

Hence by the variation-of-constants formula (see e.g. [35, p.98-99]),

$$\begin{aligned}
\frac{1}{x_1(t)} &= \exp \left( \int_0^t \left( \frac{1}{2} \sigma_1^2(r(u)) - a(r(u)) + \frac{s(r(u))x_2(u)}{\beta(r(u)) + x_2(u)} \right) du - \int_0^t \sigma_1(r(u)) dB_1(u) \right) \cdot \\
&\left[ \frac{1}{x_1(0)} + \int_0^t b(r(u)) \exp \left( \int_0^u \left( a(r(v)) - \frac{s(r(v))x_2(v)}{\beta(r(v)) + x_2(v)} - \frac{1}{2} \sigma_1^2(r(v)) \right) dv \right. \right. \\
&\left. \left. + \int_0^u \sigma_1(r(v)) dB_1(v) \right) du \right] \\
&= \exp \left( - \int_0^t \sigma_1(r(u)) dB_1(u) \right) \left[ \frac{1}{x_1(0)} \exp \left( \int_0^t \left( -a(r(u)) + \frac{1}{2} \sigma_1^2(r(u)) \right. \right. \right. \\
&\left. \left. + \frac{s(r(u))x_2(u)}{\beta(r(u)) + x_2(u)} \right) du \right) + \int_0^t b(r(u)) \exp \left( \int_u^t \left( -a(r(v)) + \frac{1}{2} \sigma_1^2(r(v)) \right. \right. \\
&\left. \left. + \frac{s(r(v))x_2(v)}{\beta(r(v)) + x_2(v)} \right) dv + \int_0^u \sigma_1(r(v)) dB_1(v) \right) du \right]. \tag{5.4}
\end{aligned}$$

On the one hand, (5.4) leads to

$$\begin{aligned}
\frac{1}{x_1(t)} &\leq \exp \left( - \int_0^t \sigma_1(r(u)) dB_1(u) \right) \left[ \frac{1}{x_1(0)} \exp \left( \int_0^t \left( -a(r(u)) + \frac{1}{2} \sigma_1^2(r(u)) \right. \right. \right. \\
&\left. \left. + \frac{s(r(u))x_2(u)}{\beta(r(u)) + x_2(u)} \right) du \right) + \check{b} \exp \left( \check{\sigma}_1 \max_{0 \leq u \leq t} B_1(u) + \int_0^t \frac{s(r(u))x_2(u)}{\beta(r(u)) + x_2(u)} du \right) \cdot \\
&\int_0^t \exp(-\hat{q}(t-u)) du \Big] \\
&\leq \exp \left( \check{\sigma}_1 \max_{0 \leq u \leq t} B_1(u) - \hat{\sigma}_1 B_1(t) + \int_0^t \frac{s(r(u))x_2(u)}{\beta(r(u)) + x_2(u)} du \right) \left[ \frac{1}{x_1(0)} \exp(-\hat{q}t) \right. \\
&\left. + \check{b} \int_0^t \exp(-\hat{q}(t-u)) du \right] \\
&= \exp \left( \check{\sigma}_1 \max_{0 \leq u \leq t} B_1(u) - \hat{\sigma}_1 B_1(t) + \int_0^t \frac{s(r(u))x_2(u)}{\beta(r(u)) + x_2(u)} du \right) \left[ \frac{1}{x_1(0)} \exp(-\hat{q}t) \right. \\
&\left. + \frac{\check{b}(1 - \exp(-\hat{q}t))}{\hat{q}} \right],
\end{aligned}$$

It then follows that

$$\frac{\log x_1(t)}{t} \geq -\frac{\log N_1(t)}{t} - \frac{\check{\sigma}_1 \max_{0 \leq u \leq t} B_1(u) - \hat{\sigma}_1 B_1(t)}{t} - \frac{1}{t} \int_0^t \frac{s(r(u))x_2(u)}{\beta(r(u)) + x_2(u)} du, \tag{5.5}$$

where

$$N_1(t) = \frac{1}{x_1(0)} \exp(-\hat{q}t) + \frac{\check{b}(1 - \exp(-\hat{q}t))}{\hat{q}}$$

and  $\sup_{0 \leq t < \infty} N_1(t) < \infty$  under condition (5.3). By (4.2) and (5.5),

$$\begin{aligned}
\frac{1}{t} \int_0^t x_1(u) du &\leq \frac{1}{\hat{b}t} \int_0^t \left( a(r(u)) - \frac{\sigma_1^2(r(u))}{2} \right) du - \frac{\log x_1(t)}{\hat{b}t} + \frac{\log x_1(0)}{\hat{b}t} \\
&\quad - \frac{1}{\hat{b}t} \int_0^t \frac{s(r(u))x_2(u)}{\beta(r(u)) + x_2(u)} du + \frac{\check{\sigma}_1}{\hat{b}t} B_1(t) \\
&\leq \frac{1}{\hat{b}t} \int_0^t \left( a(r(u)) - \frac{\sigma_1^2(r(u))}{2} \right) du + \frac{\log N_1(t)}{\hat{b}t} + \frac{\log x_1(0)}{\hat{b}t} + \frac{\check{\sigma}_1}{\hat{b}t} B_1(t) \\
&\quad + \frac{\check{\sigma}_1 \max_{0 \leq u \leq t} B_1(u) - \hat{\sigma}_1 B_1(t)}{\hat{b}t}.
\end{aligned}$$

As  $t \rightarrow \infty$  and from the strong law of large numbers for martingales [35] and Lemma 5.2,

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t x_1(u) du \leq \frac{\lambda_1}{\hat{b}} \quad \text{a.s.} \tag{5.6}$$

On the other hand, (5.4) yields

$$\begin{aligned} \frac{1}{x_1(t)} &\geq \exp(-\check{\sigma}_1 B_1(t)) \left[ \frac{1}{x_1(0)} \exp\left(\int_0^t \left(-a(r(u)) + \frac{1}{2}\sigma_1^2(r(u))\right) du\right) \right. \\ &\quad \left. + \hat{b} \exp\left(\hat{\sigma}_1 \min_{0 \leq u \leq t} B_1(u)\right) \int_0^t \exp\left(\int_u^t \left(-a(r(v)) + \frac{1}{2}\sigma_1^2(r(v))\right) dv\right) du \right] \\ &\geq \exp\left(\hat{\sigma}_1 \min_{0 \leq u \leq t} B_1(u) - \check{\sigma}_1 B_1(t)\right) \left[ \frac{1}{x_1(0)} \exp(-\check{q}t) + \frac{\hat{b}(1 - \exp(-\check{q}t))}{\check{q}} \right], \end{aligned}$$

It follows that

$$\frac{\log x_1(t)}{t} \leq -\frac{\hat{\sigma}_1 \min_{0 \leq u \leq t} B_1(u) - \hat{\sigma}_1 B_1(t)}{t} - \frac{\log N_2(t)}{t},$$

where

$$N_2(t) = \frac{1}{x_1(0)} \exp(-\check{q}t) + \frac{\hat{b}(1 - \exp(-\check{q}t))}{\check{q}}$$

and  $\sup_{0 \leq t < \infty} N_2(t) < \infty$  under condition (5.3). This leads to

$$\limsup_{t \rightarrow \infty} \frac{\log x_1(t)}{t} \leq 0 \tag{5.7}$$

(i) Equation (4.2) and (5.7) indicate

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t x_1(u) du \geq \lim_{t \rightarrow \infty} \frac{1}{\check{b}t} \int_0^t \left( a(r(u)) - \frac{\sigma_1^2(r(u))}{2} - s(r(u)) \right) du = \frac{\lambda_2}{\check{b}} > 0.$$

This and (5.6) yield

$$\frac{\lambda_2}{\check{b}} \leq \liminf_{t \rightarrow \infty} \int_0^t x_1(u) du \leq \limsup_{t \rightarrow \infty} \int_0^t x_1(u) du \leq \frac{\lambda_1}{\hat{b}} \quad \text{a.s.}$$

(ii) From equation (4.4),

$$d \log x_2(t) \leq \left( \frac{h(r(t))x_1(t - \tau)}{\beta(r(t))} - c(r(t)) - \frac{\sigma_2^2(r(t))}{2} \right) dt + \sigma_2(r(t)) dB_2(t).$$

Hence

$$\begin{aligned} \log x_2(t) &\leq \log x_2(0) + \frac{\check{h}}{\check{\beta}} \int_{-\tau}^0 x_1(u) du + \frac{\check{h}}{\hat{\beta}} \int_0^t x_1(u) du - \int_0^t \left( c(r(u)) + \frac{\sigma_2^2(r(u))}{2} \right) du \\ &\quad + \check{\sigma}_2 B_2(t). \end{aligned}$$

This and (5.6) yield

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{1}{t} \log(x_2(t)) &\leq \frac{\check{h}}{\check{\beta}} \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t x_1(u) du - \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \left( c(r(u)) + \frac{\sigma_2^2(r(u))}{2} \right) du \\ &\leq \frac{\check{h}\lambda_1}{\hat{b}\check{\beta}} - \lambda_3 < 0. \end{aligned}$$

Hence for arbitrary small  $\zeta > 0$ , there exists  $t_\zeta$  such that

$$\mathbb{P}(\Omega_\zeta) \geq 1 - \zeta \quad \text{where } \Omega_\zeta = \left\{ \omega : \frac{s(r(t))x_2(t, \omega)}{\check{b}(\beta(r(t)) + x_2(t, \omega))} \leq \zeta \text{ for } t \geq t_\zeta \right\}.$$

It then follows from (4.2), (5.7) and Lemma 5.2 that for any  $\omega \in \Omega_\zeta$ ,

$$\liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t x_1(u) du \geq \frac{\lambda_1}{\check{b}} - \zeta.$$

Letting  $\zeta \rightarrow 0$  and recalling (5.6) yields

$$\frac{\lambda_1}{\tilde{b}} \leq \liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t x_1(u) du \leq \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t x_1(u) du \leq \frac{\lambda_1}{\tilde{b}} \quad \text{a.s.}$$

□

Theorem 5.3(i) suggests that if condition (5.3) holds and  $\lambda_2 > 0$ , then the prey population is bounded in average ultimately. On the other hand, from Theorem 5.3(ii), suppose that condition (5.3) is satisfied and  $\frac{\tilde{h}}{b\beta} \lambda_1 - \lambda_3 < 0$ , then the prey species is persistent in average while the predator die out ultimately.

## 6 Numerical Simulations

It is worth pointing out that for some  $j \in \mathbb{S}$ , if the environmental noise is big enough, in the sense that  $a(j) - \frac{\sigma_1^2(j)}{2} < 0$ , then in the subsystem

$$\begin{aligned} dx_1(t) &= x_1(t) \left( a(j) - b(j)x_1(t) - \frac{s(j)x_2(t)}{\beta(j) + x_2(t)} \right) dt + \sigma_1(j)x_1(t)dB_1(t) \\ dx_2(t) &= x_2(t) \left( \frac{h(j)x_1(t-\tau)}{\beta(j) + x_2(t-\tau)} - c(j) - f(j)x_2(t) \right) dt + \sigma_2(j)x_2(t)dB_2(t), \end{aligned} \quad (6.1)$$

both prey and predator populations are extinct (Theorem 4.1). On the other hand, for some  $j \in \mathbb{S}$ , if  $a(j) - \frac{\sigma_1^2(j)}{2} > 0$  and  $\frac{h(j)}{b(j)\beta(j)}(a(j) - \frac{1}{2}\sigma_1^2(j)) - c(j) - \frac{1}{2}\sigma_2^2(j) < 0$ , we obtain from Theorem 5.3 (ii) that in the subsystem (6.1), the prey species is persistent:

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t x_1(u) du = \frac{2a(j) - \sigma_1^2(j)}{2b(j)} \quad \text{a.s.}$$

while the predators die out ultimately. In addition, Theorem 4.1 yields that if in some subsystems the preys are persistent and in some others the prey species are extinct, then due to the presence of the Markov switching, in the overall system both populations could be extinct if  $\lambda_1$  is negative. The following two examples indicate the impacts of Markov switching on the population dynamics. The Euler-Maruyama (EM) scheme is used for the computer simulations [37]. From Mao [38], the EM approximate solutions are convergent to the true solution of model (1.5) in probability. We shall assume that all the parameters are given in appropriate units [37].

**Example 6.1.** We assume that model (1.5) switches from one to the other according to the movement of the Markov chain  $r(t)$  in the state space  $\mathbb{S} = \{1, 2\}$  with the coefficients defined in Table 1. Given

Parameters	$a(i)$	$b(i)$	$s(i)$	$\beta(i)$	$h(i)$	$c(i)$	$f(i)$	$\sigma_1(i)$	$\sigma_2(i)$
$i = 1$	0.4	1	1	2.5	0.8	3	2	1.5	0.5
$i = 2$	1.5	1.5	0.8	2	0.64	2	0.5	0.8	1

Table 1: Parameters of SDE model (1.5).

the generator of the Markov chain  $r(t)$  as

$$\Gamma = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} \quad (6.2)$$

with the unique stationary distribution  $\pi = (\pi_1, \pi_2) = (\frac{2}{3}, \frac{1}{3})$ . Then  $\lambda_1 = -0.09 < 0$  a.s. Therefore by Theorem 4.1, for any initial value  $\{x(t) : -\tau \leq t \leq 0\} \in C([-\tau, 0]; \mathbb{R}_+^2)$ , both the prey and consumers of system (1.5) will tend to zero exponentially with probability one. The computer simulation in

Figure 1 supports this result clearly, illustrating the extinction of both species. We then compute  $a(1) - \frac{1}{2}\sigma_1^2(1) = -0.725 < 0$ , which shows that both species in the first subsystem die out ultimately (Theorem 4.1). Then we compute  $a(2) - \frac{1}{2}\sigma_1^2(2) = 1.18 > 0$  and  $\frac{h(2)}{b(2)\beta(2)}(a(2) - \frac{1}{2}\sigma_1^2(2)) - c(2) - \frac{1}{2}\sigma_2^2(2) = -2.248 < 0$ . Hence in the second subsystem the prey population is persistent:

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t x_1(u) du = 0.787 \quad \text{a.s.}$$

and the consumers tend to zero exponentially almost surely (Theorem 5.3 (ii)). However due to the presence of Markov switching, the overall behaviour of both populations remains extinctive ultimately.

**Example 6.2.** Assume that model (1.5) switches from one to the other according to the movement of the Markov chain  $r(t)$  in the state space  $\mathbb{S} = \{1, 2\}$  with the coefficients defined in Table 2. Let the

Parameters	$a(i)$	$b(i)$	$s(i)$	$\beta(i)$	$h(i)$	$c(i)$	$f(i)$	$\sigma_1(i)$	$\sigma_2(i)$
$i = 1$	1	0.5	0.5	1	1	1.1	1.5	0.9	0.1
$i = 2$	1.5	1.5	0.8	2	0.64	2	0.5	0.8	1

Table 2: Parameters of SDE model (1.5).

generator of the Markov chain  $r(t)$  be

$$\Gamma = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \quad (6.3)$$

with the unique stationary distribution  $\pi = (\pi_1, \pi_2) = (\frac{1}{3}, \frac{2}{3})$ . Then  $a(i) - \frac{\sigma_1^2(i)}{2} > 0$  for  $i \in \{1, 2\}$  and  $\frac{h}{b\beta}\lambda_1 - \lambda_3 = -0.065 < 0$ . Hence from Theorem 5.3(ii), the solution to model (1.5) has the property that

$$0.657 \leq \liminf_{t \rightarrow \infty} \frac{1}{t} \int_0^t x_1(u) du \leq \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t x_1(u) du \leq 1.97$$

and  $x_2(t)$  goes to zero almost surely. The computer simulation shown in Figure 2 supports these results clearly.

## 7 Summary

In this paper, we have introduced white noise, telegraph noise and time delay to the classical foraging arena predator-prey system (1.2). Theorem 4.1 suggested that a bigger amplitude of environmental noise may destabilize the system. The presence of time delay makes the system become stochastically ultimately bounded only under certain parametric restriction (Theorem 3.1). Based on this, we then showed that the total amount of prey and predator species will grow at most polynomially with order close to one (Theorem 5.1). The existence of Markov switching makes a difference to the population behaviours. Especially, if the prey is persistent in some subsystems and is extinct in some other subsystems, due to the presence of the Markov switching, both populations in the overall system could be extinct as long as  $\lambda_1$  defined in (4.1) is negative. Numerical simulations were carried out to substantiate the analytical results.

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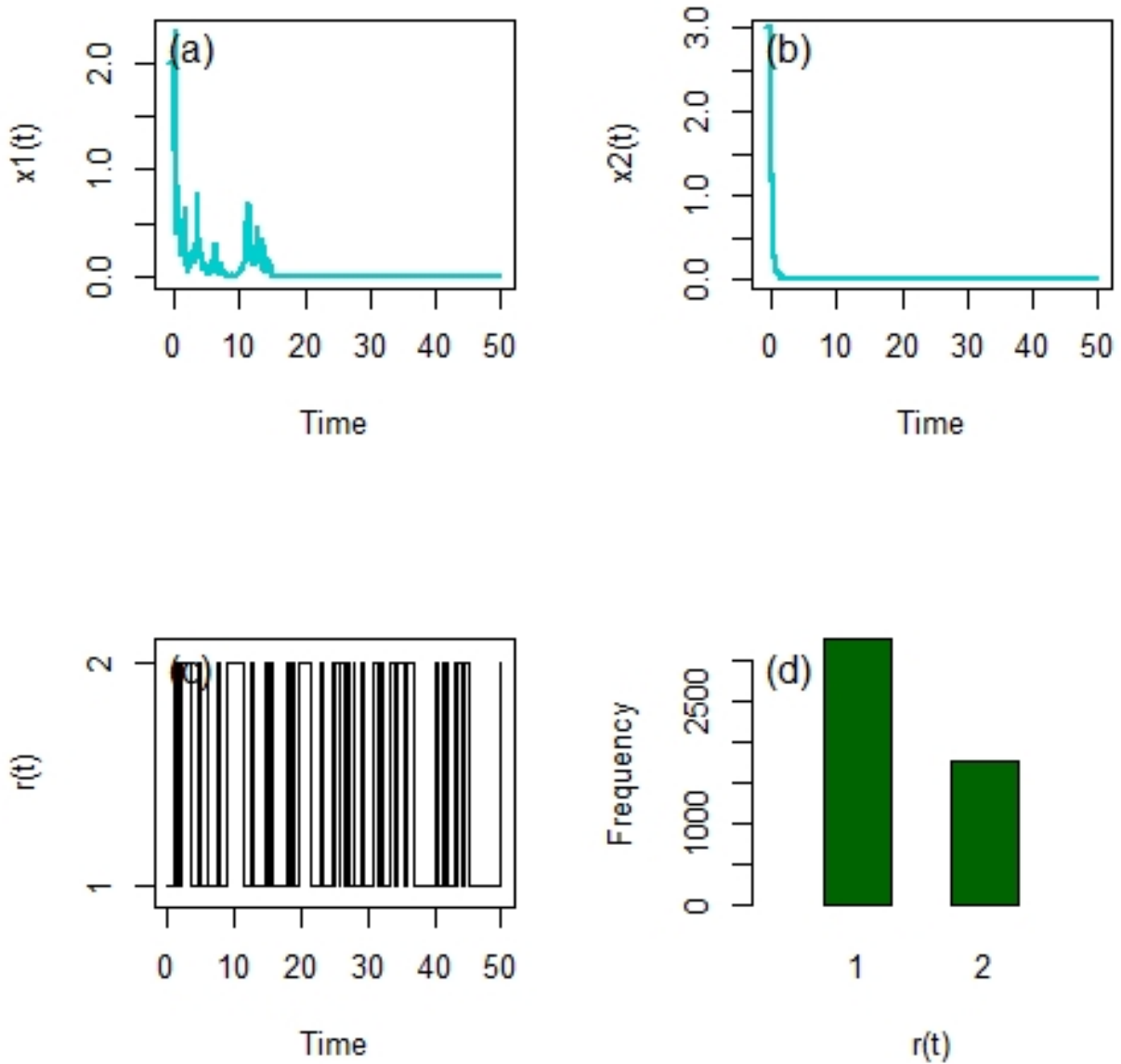


Figure 1: Computer simulations of the paths (a)  $x_1(t)$  and (b)  $x_2(t)$  of 5000 iterations of SDE model (1.5) using the EM scheme with stepsize  $\Delta = 0.01$  and initial value  $x(t) = (2, 3)^T$  for  $t \in [-1, 0]$  with the system parameters provided by Table 1 and the generator of the Markov chain  $r(t)$  given by (6.2). The trajectory and frequency of the Markov chain are shown in (c) and (d) respectively.



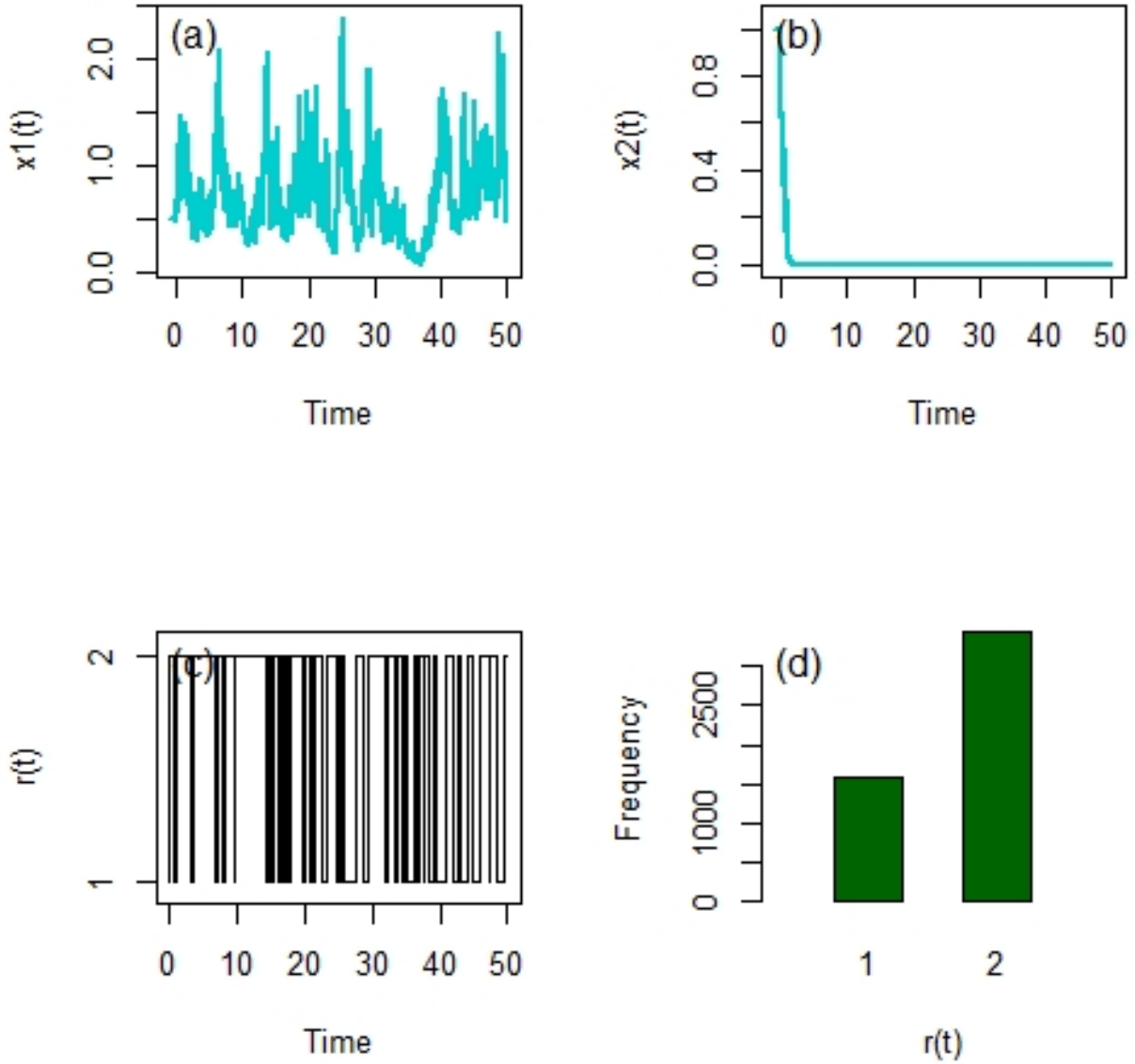


Figure 2: Computer simulations of the paths (a)  $x_1(t)$  and (b)  $x_2(t)$  of 5000 iterations of SDE model (1.5) using the EM scheme with stepsize  $\Delta = 0.01$  and initial value  $x(t) = (0.5, 1)^T$  for  $t \in [-1, 0]$  with the system parameters provided by Table 2 and the generator of the Markov chain  $r(t)$  given by (6.3). The trajectory and frequency of the Markov chain are given in (c) and (d) respectively.

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