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Stochastic Derivation of an Integral Equation for Probability Generating Functions

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Abstract Functional, integral and differential equations of transformed probability generating functions are generally recognized as powerful analytical tools for establishing characterizations of discrete probability distributions. The present paper establishes a characterization of the distribution of an important integral part model by incorporating an integral equation based on three fundamental transformed probability generating functions. Interpretations of such a characterization in analyzing and implementing information risk frequency reduction operations are also established.

1. Introduction

Stochastic multiplicative models incorporating two nonnegative random variables are generally recognized as very strong analytical tools of several fundamental areas of probability theory [3]. Moreover, such stochastic models have very useful applications in economics, management, operational research, risk theory, insurance, reliability theory, logistics, engineering, informatics, and other fundamental practical disciplines. If the two nonnegative random variables of a stochastic multiplicative model are independent, then the distribution function of this model belongs to the class of scale mixtures of distribution functions [6]. This class has substantially contributed to the investigation of unimodality, infinite divisibility, selfdecomposability, stability and other significant theoretical properties with particular practical importance of many fundamental classes of distribution functions [8]. The literature concentrating on the investigation of the theoretical properties and practical applications of stochastic multiplicative models incorporating two nonnegative and independent random variables is very broad

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[9]. Properties and practical applications in information risk management of a stochastic multiplicative model incorporating two nonnegative and independent random variables, one of which follows the standard uniform distribution and the other is the sum of two nonnegative and independent random variables, have been established by Artikis and Artikis [1].

The present paper is devoted to the establishment of properties and applications in information risk management of a discrete analogue of such a stochastic multiplicative model by making use of the stochastic model

$$M = [\Pi B],$$

where Π, B are independent random variables with Π taking values in the set $\mathbf{N} = \{1, 2, 3, \dots\}$, B following the standard uniform distribution and $[\Pi B]$ denoting the integral part of ΠB. If

$$P(\Pi = \pi) = p_\pi, \quad \pi = 1, 2, \dots$$

is the probability function of Π and $P_\Pi(z)$ is the probability generating function of Π, then

$$P(M = m) = \sum_{\pi=m+1}^{\infty} \frac{p_\pi}{\pi}, \quad m = 0, 1, 2, \dots$$

is the probability function of M and

$$P_M(z) = \frac{1}{1-z} \int_z^1 \frac{P_\Pi(\beta)}{\beta} \, d\beta$$

is the probability generating function of M [7].

The present paper concentrates on the implementation of two purposes. The first purpose is the establishment of a characterization of the distribution of the integral part model, based on the product of two independent random variables one of which follows the geometric distribution and the other follows the standard uniform distribution, by making use of an integral equation incorporating three fundamental transformed probability generating functions. The second purpose is the interpretation of such a characterization in the area of stochastic modelling of information risk frequency reduction operations.

2. Certain Transformations of Probability Generating Functions

The present section of the paper mainly concentrates on the consideration of certain known transformations of probability generating functions corresponding to discrete random variables with values in the set $\mathbf{N_0} = \{0, 1, 2, \dots\}$ and finite mean.

Let X be a discrete random variable with values in the set $\mathbf{N_0}$, probability generating function $P_X(z)$ and finite mean μ, then

$$P_U(z) = \frac{1}{\mu(z-1)} \log P_X(z), \tag{2.1}$$

is a probability generating function of a discrete random variable U with values in the set $\mathbf{N_0}$ and probability function having a unique mode at the point 0 if and only if, X is infinitely divisible [2]. The formula (2.1) is a transformation which maps the probability generating function $P_X(z)$ into the probability generating function $P_U(z)$. An integral equation incorporating the probability generating function $P_U(z)$ and the probability generating function of an integral part model, based on the product of two independent random variables one of which is distributed as the random variable $X+1$ and the other follows the standard uniform distribution, has been investigated by Artikis *et al* [2]. Moreover, the formula

$$P_X(z) = \exp[\mu(z-1)P_U(z)] \tag{2.2}$$

is a transformation which maps the probability generating function $P_U(z)$ into the probability generating function $P_X(z)$.

Let V be a discrete random variable with values in the set $\mathbf{N_0}$, probability generating function $P_V(z)$ and finite mean θ, then

$$P_J(z) = \frac{1-P_V(z)}{\theta(1-z)} \tag{2.3}$$

is the probability generating function of the renewal distribution corresponding to the distribution of the random variable V [4]. Formula (2.3) is a transformation which maps the probability generating function $P_V(z)$ into the renewal probability generating function $P_J(z)$.

The present paper makes use of the transformation in (2.2) and the transformation in (2.3) to establish a characterization of a discrete distribution.

3. Characterizing the Distribution of an Integral Part Model

The present section of the paper establishes a characterization of the integral part model based on the product of two independent random variables, one of which is geometrically distributed and the other follows the standard uniform distribution [2].

Theorem. *Let W, T and L be independent random variables with W following the standard uniform distribution and T, L distributed as the random variable S with probability generating function*

$$P_S(z) = \exp[\kappa(z-1)P_Y(z)], \quad \kappa > 0,$$

where $P_Y(z)$ is the probability generating function of a discrete random variable Y with values in the set $\mathbf{N_0}$, finite mean and probability function having a unique mode *at the point* 0. *The probability generating function of the random variable* Y *has the form*

$$P_Y(z) = \frac{p}{q(z-1)} \log\left(\frac{p}{1-qz}\right),$$

where

$$p = \frac{1}{\kappa+1}, \quad q = 1-p$$

if and only if the stochastic model

$$C = [(L+T+1)W],$$

denoting the integral part of

$$(L+T+1)W,$$

is equally distributed with the random variable

$$R$$

following the renewal distribution corresponding to the distribution of the random variable S.

Proof. Only the sufficient condition will be proved since the necessity condition can be proved by reversing the argument.

It is readily shown that the independence of the random variables

$$W, \ T, \ L$$

implies the independence of the random variables

$$L+T+1, \ W.$$

Hence, it is also readily shown that the probability generating function of the integral part model

$$C = [(L+T+1)W]$$

is given by

$$P_C(z) = \frac{1}{1-z}\int_z^1 \exp[2\kappa(w-1)P_Y(w)]\,dw. \tag{3.1}$$

Moreover, the probability generating function of the random variable R is given by

$$P_R(z) = \frac{1-\exp[\kappa(z-1)P_Y(z)]}{\kappa(1-z)}. \tag{3.2}$$

From (3.1), (3.2) and the assumption that the random variables

$$R, \ C = [(L+T+1)W]$$

are equal in distribution, we get the integral equation

$$\frac{1-\exp[\kappa(z-1)P_Y(z)]}{\kappa(1-z)} = \frac{1}{1-z}\int_z^1 \exp[2\kappa(w-1)P_Y(w)]\,dw. \tag{3.3}$$

If we multiply both sides of the integral equation in (3.3) by $\kappa(1-z)$ and then differentiate, we get the differential equation

$$\exp[\kappa(z-1)P_Y(z)]\frac{d}{dz}[\kappa(z-1)P_Y(z)] = \kappa\exp[2\kappa(z-1)P_Y(z)] \tag{3.4}$$

which satisfies the boundary condition

$$P_Y(1) = 1.$$

Since the probability generating function

$$P_S(z) = \exp[(\kappa(z-1)P_Y(z)]$$

has no real roots, then the differential equation in (3.4) can be written in the form

$$\exp[-\kappa(z-1)P_Y(z)]\frac{d}{dz}[-\kappa(z-1)P_Y(z)] = -\kappa \tag{3.5}$$

Integrating the differential equation in (3.5) with due regard to the above boundary condition we get that

$$\exp[-\kappa(z-1)P_Y(z)] = \kappa+1-\kappa z \tag{3.6}$$

From (3.6) it follows that

$$P_Y(z) = \frac{p}{q(z-1)}\log\left(\frac{p}{1-qz}\right) \tag{3.7}$$

with

$$p = \frac{1}{\kappa+1}$$

and

$$q = \frac{\kappa}{\kappa+1}.$$

It is of some theoretical importance to mention that the above probability generating function is derived from the transformation (2.1) if the random variable X follows the geometric distribution with probability generating function

$$P_X(z) = \frac{p}{1-qz}\ .$$

It is easily demonstrated that the random variable with probability generating function $P_Y(z)$ in (3.7) can be written in the form of the integral part model

$$Y = [DH],$$

where D, H are independent random variables with D following the geometric distribution with probability generating function

$$P_D(z) = \frac{pz}{1-qz}$$

and H following the uniform distribution with probability density function

$$f_H(h) = 1.$$

An interpretation of the integral part model

$$C = [(L + T + 1)W]$$

incorporated by the present paper for characterizing the distribution of the integral part model

$$Y = [DH],$$

in the area of risk frequency reduction operations is the following. We suppose that the random variable L denotes the frequency of a risk and the random variable $T + 1$ denotes the frequency of another risk. We also suppose that the risks are of the same type. Since the random variable W follows the standard uniform distribution, then the integral part model

$$C = [(L + T + 1)W]$$

can be interpreted as the total risk frequency after applying a risk frequency reduction operation to these risks. The general recognition of risk frequency reduction operations as extremely important, for proactive treatment of risks threatening information systems of modern complex organizations, substantially supports the applicability of the proposed interpretation of the integral part model incorporated by this paper in analyzing and implementing such information risk management operations [5]. □

4. Conclusions

The establishment of a characterization for the distribution of an integral part model, based on the product of two independent random variables, one of which follows the geometric distribution and the other follows the standard uniform distribution, is the theoretical contribution of the paper. The practical contribution of the paper consists of interpreting such a characterization in the area of stochastic modelling of information risk frequency reduction operations.

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