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Stochastic Differential Equations for the Deterioration Processes of Engineering Systems

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ABSTRACT: A critical part of Life-Cycle Analysis of engineering systems is the modeling of their deterioration over time. A system might be subject to different deterioration processes that might impair its ability to sustain the future levels of demand. The attainment of a given level of deterioration might also prompt maintenance operations that may disrupt its ability to provide a regular service. Recently proposed formulations model the time-varying reliability of a system by looking at the evolution of state-variables that define the characteristics of the system. These state-dependent formulations rely heavily on the chosen models for the evolution of the state-variables over time. However, most models available in literature rely on simplifying assumptions that disregard the true nature of the processes, either by discretizing the time domain or by assuming independence among different processes acting on the system at the same time. This paper proposes to use a system of Stochastic Differential Equations to model the evolution of the state variables over time. The proposed formulation captures the continuous nature of the processes and takes into account the possible interactions among them. In addition, results from stochastic calculus could be used to facilitate the simulation of the processes and to obtain closed-form solutions for the distribution of the state variables over time. Moreover, the proposed models can be calibrated based on periodical monitoring of the state variables, should that be performed via Non-Destructive Evaluation or Structural Health Monitoring. A procedure for calibration is introduced and a brief explanatory example is provided.

1. INTRODUCTION

Engineering systems are continuously subject to aging and deterioration phenomena that may impair their ability to sustain the levels of demand for which they were originally designed. To perform a proper Life-Cycle Analysis of these systems, there is a need for a comprehensive formulation that includes the effects of these phenomena and translates them into time-varying estimates of the reliability for the systems (Gardoni 2019).

Deterioration processes can in general be separated into two main classes (Kumar et al. 2009); gradual deterioration processes (such as fatigue and corrosion) affect the performance of the system in a continuous fashion, while shock deterioration processes (such as earthquake and floods) affect the performance of the system at specific, instantaneous moments over its life-cycle.

Recent works have started to incorporate both gradual and shock deterioration processes into the life-cycle analysis of the systems. However, multiple deterioration processes are often considered independently. For example Ciampoli and Ellingwood (2002) looked at the effects of both gradual and shock deteriorations for the performance of concrete structures in nuclear power plants, but they only considered the randomness coming from the shock deterioration process, superimposing the effects of shocks to effects of a deterministic, independent gradual deterioration process. Kumar et al. (2009) developed a proper reliability analysis framework for reinforced concrete bridges in which both capacity and demand vary over time, but they also assumed independence between gradual and shock deterioration.

However, there might be interactions among the different deterioration processes affecting the system: For example, the occurrence of an earthquake might increase the length of the cracks of the system, which might in turn increase their exposure to hostile environmental conditions, leading to faster gradual deterioration. The interaction between different deterioration processes has only recently been acknowledged in the literature. For example, Jia and Gardoni (2018a, b, c) modeled the interaction among different deterioration processes by formulating the time-varying reliability of the system in terms of the evolution of the state variables, i.e. the physical quantities that define the state of a system.

Having physically sound models for the evolution of the state variables is critical to obtain accurate predictions of the reliability of the system at future moments in time. However, the understanding of most deterioration processes is limited.

The quality of the models can be assessed based on data collected in the field. Structural Health Monitoring (SHM) or Non-Destructive Evaluation (NDE) procedures can be used for this purpose. Some recent formulations for Bayesian updating of deterioration models choose to disregard the continuous nature of the process in favor of approximate, more tractable discrete formulations (Straub 2009). Others, relying on the probability density evolution method (Li and Chen 2008, Fan et al. 2017), require assumptions on the form of the models which might limit their applications to cases in which such form is known a priori.

This paper builds upon the framework proposed in Jia and Gardoni (2018a) by proposing a specific formulation for the time-varying reliability of the systems that is based on modeling the evolution of the state variables using Stochastic Differential Equations (SDEs). The proposed procedure (i) captures the continuous nature of the deterioration processes by not requiring to discretize the time domain for the analysis, (ii) is able to incorporate both gradual and shock deterioration processes into a unified formulation, (iii) captures the interactions among different deterioration processes, and (iv) allows to obtain closed-form solutions for the distributions of the state variables over time making use of formulations from stochastic calculus. The formulation uses semimartingale driving noises, which form the largest class of processes for which Itô and Stratonovich integrals can be defined (Grigoriu 2003), making it suitable for the vast majority of behaviors that could be observed in practice. In addition, we propose a method for calibrating the newly developed models based on information coming from inspections and Structural Health Monitoring.

2. GENERAL FRAMEWORK

Aging and deterioration processes affecting engineering systems are, in general, a function of a set of external conditions, which could be separated into environmental conditions (such as atmospheric pressure, relative humidity and temperatures) and consequences of shock occurrences (such as earthquakes and floods) (Jia and Gardoni 2018). Following Jia & Gardoni (2018a), we can define the vector of external conditions $\mathbf{Z}(t)$ as

$$\mathbf{Z}(t) = [\mathbf{E}(t), \mathbf{S}(t)] \tag{1}$$

where $\mathbf{E}(t)$ is the vector of environmental conditions, and $\mathbf{S}(t)$ is the vector of measures of shocks. Both vectors are time-dependent as the external conditions might be changing over time as a consequence of different phenomena (e.g. seasonality and climate change). Let the vector $\mathbf{X}(t) = \left[X_1(t), \dots, X_j, \dots, X_d(t)\right]^T$ denote the state variables of the system at time t, i.e. a collection of variables on which the capacity and the demand of the system depend (e.g. material properties and geometry). The state variables evolve over time as a consequence of the deterioration processes.

In general, if we define $\{\mathbf{Z}(t)\}$ as the sequence of all of the external conditions from time 0 to t, and $\mathbf{X}_0$ as the initial state variables at time $t = 0$, we can write $\mathbf{X}(t)$ in the compact form

$$\mathbf{X}(t) = \mathbf{X}[t, \mathbf{X}_0, \{\mathbf{Z}(t)\}; \mathbf{\Theta}_\mathbf{X}] \tag{2}$$

where $\mathbf{\Theta}_\mathbf{X}$ is a vector of model parameters. Following Gardoni et al. (2002, 2003), we can model the capacity of the system $C(t)$ and the demand imposed by external conditions $D(t)$ as functions of $\mathbf{X}(t)$, as

$$C\left[t, \mathbf{X}(t)\right] = C\left[t, \mathbf{X}(t); \mathbf{\Theta}_C\right] \tag{3}$$

$$D\left[t, \mathbf{X}(t)\right] = D\left[t, \mathbf{X}(t), \mathbf{S}(t); \mathbf{\Theta}_D\right] \tag{4}$$

where $\mathbf{\Theta}_C$ and $\mathbf{\Theta}_D$ are vectors of parameters for the models of $C(t)$ and $D(t)$ respectively. It is possible to obtain the time-variant reliability of the system by defining the limit state function (Ditlevsen & Madsen 1996, Gardoni 2017) as

$$g\left[t, \mathbf{X}(t)\right] = C\left[t, \mathbf{X}(t)\right] - D\left[t, \mathbf{X}(t)\right] \tag{5}$$

and compute the probability of failure of the system as

$$P_f\left[t, \mathbf{X}(t)\right] = P\left\{g\left[t, \mathbf{X}(t)\right] \le 0\right\}. \tag{6}$$

At time t, the rate of state change due to the j-th deterioration process $\dot{\mathbf{X}}_j(t)$ can be modeled as a function of $\mathbf{X}(t)$ (state-dependency) as

$$\dot{\mathbf{X}}_j(t) = \dot{\mathbf{X}}_j[t, \mathbf{X}(t), \{\mathbf{Z}(t)\}; \mathbf{\Theta}_\mathbf{X}]. \tag{7}$$

Then, the total rate of change $\dot{\mathbf{X}}(t)$ due to p deterioration processes can be written as the sum of the rates associated to the individual processes

$$\dot{\mathbf{X}}(t) = \sum_{j=1}^{p} \dot{\mathbf{X}}_j[t, \mathbf{X}(t), \{\mathbf{Z}(t)\}; \mathbf{\Theta}_\mathbf{X}]. \tag{8}$$

The framework described has been summarized in the flow chart in Figure *1*. The rest of the paper will focus on a detailed formulation for the evolution of the state variables over time (Eq. 7).

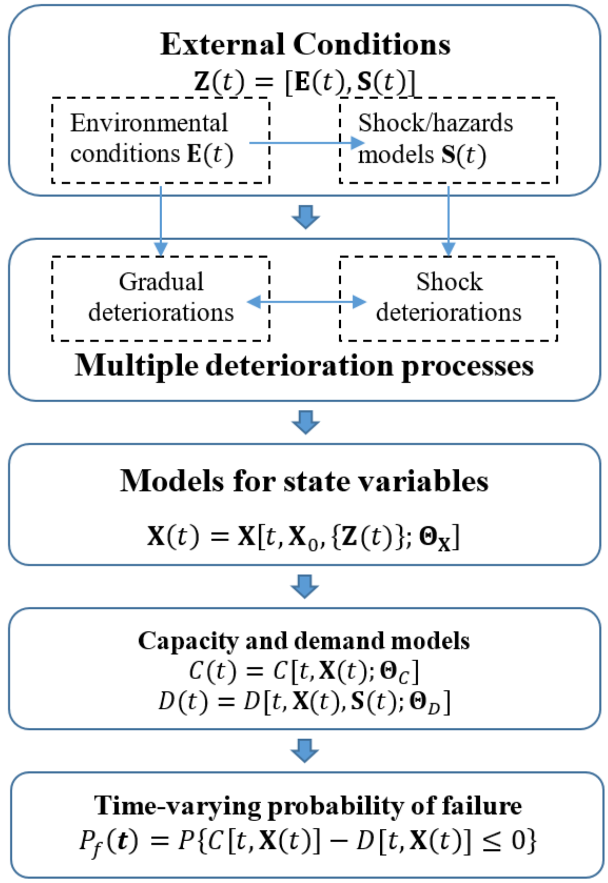

Figure 1: Flow chart of the proposed framework (adapted from Jia and Gardoni 2018a)

3. MODELING OF THE STATE VARIABLES OVER TIME

This section proposes a state-dependent formulation for the evolution of state variables using Stochastic Differential Equations (SDEs). The formulation follows the general form in Eq. 7 and can be plugged into the general framework presented in Section 2 to obtain the time-varying reliability of engineering systems. A typical SDE for the evolution of the $d \times 1$ vector of the state variables $\mathbf{X}(t|\mathbf{\Theta}_\mathbf{X})$ can be written as (adapted from Itô 1973)

$$\begin{aligned} \mathrm{d}\mathbf{X}(t; \mathbf{\Theta}_\mathbf{X}) = {} & \boldsymbol{\mu}\left[t, \mathbf{X}(t), \{\mathbf{Z}(t)\}; \mathbf{\Theta}_\boldsymbol{\mu}\right]\mathrm{d}t + \\ & \boldsymbol{\sigma}\left[t, \mathbf{X}(t), \{\mathbf{Z}(t)\}; \mathbf{\Theta}_\boldsymbol{\sigma}\right]\mathrm{d}\mathbf{S}(t; \mathbf{\Theta}_\mathbf{S}) \end{aligned} \tag{9}$$

where $\boldsymbol{\mu}[t, \mathbf{X}(t), \{\mathbf{Z}(t)\}; \mathbf{\Theta}_\boldsymbol{\mu}]$ is a $d \times 1$ vector of drift coefficients, quantifying the deterministic change of the quantity $\mathbf{X}(t; \mathbf{\Theta}_\mathbf{X})$ in the time interval $\mathrm{d}t$, $\boldsymbol{\sigma}[t,\mathbf{X}(t),\{\mathbf{Z}(t)\};\boldsymbol{\Theta}_{\boldsymbol{\sigma}}]$ is a $d\times d'$ matrix of diffusion coefficients, quantifying the randomness in the change of $\mathbf{X}(t;\boldsymbol{\Theta}_{\mathbf{X}})$ in the same time interval, $\boldsymbol{\Theta}_{\mathbf{X}}=[\boldsymbol{\Theta}_{\boldsymbol{\mu}},\boldsymbol{\Theta}_{\boldsymbol{\sigma}},\boldsymbol{\Theta}_{\mathbf{S}}]$ is the vector of model parameters and $\mathbf{S}(t)$ is a $d'\times 1$ driving noise vector of the stochastic process.

Different choices for the driving noise produce different results. In the most general case, the driving noise can be assumed to be of the semimartingale type. Defining the natural filtration $(\mathcal{F}_t)_{t\geq 0}$ of a stochastic process as the sigma-algebra generated by the collection of random variables up to time t, $\mathcal{F}_t=\sigma\{\mathbf{X}(s),s\leq t\}$ (Hajek 2015), a semimartingale noise $\mathbf{S}(t)$ can be decomposed as (Grigoriu 2013)

$$\mathbf{S}(t;\boldsymbol{\Theta}_{\mathbf{S}})=\mathbf{S}(0)+\mathbf{A}(t;\boldsymbol{\Theta}_{\mathbf{A}})+\mathbf{M}(t;\boldsymbol{\Theta}_{\mathbf{M}}) \quad (10)$$

where $\mathbf{S}(0)\in\mathcal{F}_0$, $\mathbf{A}$ is an $\mathcal{F}_t$-adapted process with samples of finite variation on each compact of $[0,\infty)$, $\mathbf{A}(0)=\mathbf{0}$, $\mathbf{M}$ is an $\mathcal{F}_t$-local martingale, $\mathbf{M}(0)=\mathbf{0}$ and $\boldsymbol{\Theta}_{\mathbf{S}}=[\boldsymbol{\Theta}_{\mathbf{A}},\boldsymbol{\Theta}_{\mathbf{M}}]$ is a vector of model parameters. The martingale component is in turn assumed to admit the representation

$$\mathbf{M}(t;\boldsymbol{\Theta}_{\mathbf{M}})=\mathbf{H}(t;\boldsymbol{\Theta}_{\mathbf{H}})\mathrm{d}\mathbf{B}(t)+\mathbf{K}(t;\boldsymbol{\Theta}_{\mathbf{K}})d\mathbf{C}(t;\boldsymbol{\Theta}_{\mathbf{C}}) \quad (11)$$

where the entries of the $d'\times d_b$ matrix $\mathbf{H}$ and of the $d'\times d_c$ matrix $\mathbf{K}$ are $\mathcal{F}_t$-adapted processes, the coordinates of $\mathbf{B}$ and $\mathbf{C}$ are independent Brownian motion and compensated Poisson processes, respectively, and $\boldsymbol{\Theta}_{\mathbf{M}}=[\boldsymbol{\Theta}_{\mathbf{H}},\boldsymbol{\Theta}_{\mathbf{K}},\boldsymbol{\Theta}_{\mathbf{C}}]$ is a vector of model parameters. Roughly speaking, $\mathbf{B}$ and $\mathbf{C}$ constitute the main components of the stochastic process solution of the SDE in Eq. 9, where $\mathbf{B}$ is responsible for the deterioration processes that are continuous in nature (gradual processes), and $\mathbf{C}$ is responsible for the deterioration processes that are described by jumps over time. The evolution of the single component of $\mathbf{X}(t;\boldsymbol{\Theta}_{\mathbf{X}})$, $X_j(t;\boldsymbol{\Theta}_{\mathbf{X}})$, is a function of $\mathbf{X}(t;\boldsymbol{\Theta}_{\mathbf{X}})$ itself, making the entire formulation state-dependent. Semimartingales form the largest class of processes for which Itô and Stratonovich integrals can be defined (Métivier 1982). In other words, the formulation in Eq. 9 is able to capture the largest class of behaviors for the evolution of the state variables over time.

3.1. Closed-form solutions for SDEs

In most practical applications, it can be assumed with minimal loss of generality that the driving noise component of Eq. 9 can be expressed as $\mathrm{d}\mathbf{S}(t)=\mathbf{H}(t)d\mathbf{B}(t)+\mathbf{K}(t)\mathrm{d}\mathbf{C}(t)$, that is, $\mathbf{S}$ is given by Eq. 10 with $\mathbf{S}(0)=0$ and $\mathbf{A}=0$ (model parameters were dropped to avoid heavy notation). Define $\widetilde{\mathbf{H}}=\boldsymbol{\sigma}[t,\mathbf{X}(t),\{\mathbf{Z}(t)\};\boldsymbol{\Theta}_{\boldsymbol{\sigma}}]\mathbf{H}(t)$ and $\widetilde{\mathbf{K}}=\boldsymbol{\sigma}[t,\mathbf{X}(t),\{\mathbf{Z}(t)\};\boldsymbol{\Theta}_{\boldsymbol{\sigma}}]\mathbf{K}(t)$. We can obtain a closed-form solution for the characteristic function of $\mathbf{X}(t)$ over time. Recall that the characteristic function of a random variable $\mathbf{X}(t)$ is defined as $\varphi(\mathbf{u};t)=E[\exp(i\mathbf{u}^T\mathbf{X}(t))]$, where $\mathbf{u}\in\mathbb{R}^d$ and $i=\sqrt{-1}$. With the above assumptions, the characteristic function of $\mathbf{X}(t)$ can be obtained as the solution of the following differential equation

$$\begin{aligned}\frac{\partial\varphi(\mathbf{u};t)}{\partial t}=&\\ &i\sum_{j=1}^{d}u_jE\left[e^{i\mathbf{u}^T\mathbf{X}(t)}\mu_i(\mathbf{X}(t),t)\right]\\ &-\frac{1}{2}\sum_{j,k}^{d}u_ju_kE\left[e^{i\mathbf{u}^T\mathbf{X}(t)}\left(\tilde{\mathbf{H}}(t)\tilde{\mathbf{H}}(t)^T\right)_{jk}\right]\\ &+\sum_{\alpha=1}^{d_c}\lambda_\alpha\left\{E\left[e^{i\mathbf{u}^T\mathbf{X}(t)}e^{i\sum_{j=1}^{d}\tilde{K}_{j\alpha}(t)\Delta C_\alpha(t)}\right]-\varphi(\mathbf{u};t)\right\}\end{aligned} \quad (12)$$

where the compound Poisson process component has been separated into its d_c terms with rate λ_α. As it will be shown in the example in Section 5, Eq. 12 becomes an ordinary partial differential equation if the drift and diffusion terms $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ can be written as polynomials of $\mathbf{X}(t)$. Under the restrictive assumption that the driving noise for the system is a Browian motion process, it is possible to obtain closed-form solutions for the probability density function (PDF) $f(\boldsymbol{x};t)$ of $\mathbf{X}(t|\boldsymbol{\Theta}_{\mathbf{X}})$. The results is known in literature as the Fokker-Planck equation and can be expressed as

$$\begin{aligned}\frac{\partial f(\mathbf{x};t)}{\partial t}=&-\sum_{j=1}^{d}\frac{\partial}{\partial x_j}\left[\mu_j(\mathbf{x},t)f(\mathbf{x};t)\right]\\ &+\frac{1}{2}\sum_{j,k=1}^{d}\frac{\partial^2}{\partial x_j\partial x_k}\left[\left(\boldsymbol{\sigma}(\mathbf{x},t)\boldsymbol{\sigma}(\mathbf{x},t)^T\right)_{jk}f(\mathbf{x};t)\right]\end{aligned} \quad (13)$$

with initial conditions $\mathbf{X}(0)=\mathbf{0}$.

Eqs. 12 and 13 can be used to obtain the distribution of the state variables over time to be inserted into the general framework in Section 2 and obtain the time-varying probability of failure for the system of interest.

4. CALIBRATION OF DETERIORATION MODELS BASED ON DATA FROM SHM/NDE

In general, the model parameters $\mathbf{\Theta_X}$ for the evolution of the state variables over time are not known a priori and need to be calibrated based on experimental results. Structural Health Monitoring and Non-Destructive Evaluation procedures can provide us with more or less sparse measurements for the values of $\mathbf{X}(t)$.

Assume that information is collected at N different moments in time $t_i^*, i = 1, \ldots, N$, providing N sets of M observations for the state variables at time t_i^* $\{\mathbf{x}_m \in \mathbb{R}^d : m = 1,2, \ldots, M\}$. The unknown probability distribution at time t_i^* can be modeled as a mixture of K different probability distributions from a parametric family (e.g. Gaussian)

$$f(\mathbf{x}; t_i^* \mid \mathbf{w}, \boldsymbol{\theta}_{1:K}) = \sum_{k=1}^{K} w_k f(\mathbf{x}; t_i^* \mid \theta_k) \qquad (14)$$

where $f(\cdot \mid \mathbf{w}, \boldsymbol{\theta}_{1:K})$ is the predicted probability density function, $\mathbf{w} = (w_1, \ldots, w_K)$ is the vector of mixture weight such that $\sum_{k=1}^{K} w_k = 1$ and $\boldsymbol{\theta}_{1:K} = (\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_K)$ is the vector of mixture component parameters. Both $\mathbf{w}$ and $\boldsymbol{\theta}_{1:K}$ need to be calibrated based on the collected data. This calibration can be achieved using Bayesian inference once the number K of probability distributions has been fixed a priori (Box and Tiao 2011). Tabandeh and Gardoni (2018) have recently proposed an alternative Dirichlet Process Mixture Model (DPMM) procedure that operates over an infinite dimensional parameter space and allows the number of K mixture distributions to grow indefinitely. Once a probability distribution has been estimated at each of the times t_i^*, the corresponding characteristic functions can be obtained as

$$\begin{aligned}\varphi^*(\mathbf{u}; t_i^*) &= E\left[e^{i\mathbf{u}^T \mathbf{X}(t_i^*)}\right] \\ &= \int_{-\infty}^{\infty} e^{i\mathbf{u}^T \mathbf{x}(t_i^*)} f(\mathbf{x}; t_i^* \mid \mathbf{w}, \boldsymbol{\theta}_{1:K}) \mathrm{d}\mathbf{x}.\end{aligned} \qquad (15)$$

Once we have obtained a characteristic function based on the observational data, the best estimate for the unknown parameters $\widehat{\mathbf{\Theta}}_\mathbf{X}$ in the models for the evolution of the state variables can be obtained by solving the following minimization problem

$$\begin{aligned}&\hat{\mathbf{\Theta}}_\mathbf{X} = \\ &\arg\min_{\mathbf{\Theta_X}} \sum_{i=1}^{N} \int_{\mathbb{R}^d} \left\| \varphi^*(\mathbf{u}; t_i^*) - \varphi(\mathbf{u}; t, \mathbf{\Theta_X}) \right\| \lambda(\mathrm{d}\mathbf{u})\end{aligned} \qquad (16)$$

where $\|\cdot\|$ is the complex norm and $\lambda(\mathrm{d}\boldsymbol{u})$ is an appropriate measure (Grigoriu 2000). By using the formulation in Eq. 16, we are minimizing the distance between the characteristic function obtained using the collected data and the characteristic function obtained as the solution of Eq. 12.

Section 5 will present a brief example of an application of the proposed procedure.

5. EXAMPLE

A stochastic deterioration process is assumed to be the solution of the following Stochastic Differential Equation

$$\mathrm{d}X(t) = \mu_1 X(t)\mathrm{d}t + \sigma_1 X(t)\mathrm{d}B(t) \qquad (17)$$

with $\mu_1 = 2.5$ and $\sigma_1 = 0.4$ (quantities are assumed to be dimensionless; in practice, units for the different quantities will be dependent on the problem being analyzed). The driving noise for the SDE, $B(t)$, is assumed to be a standard Brownian motion.

Figure 2 shows 50 realizations of the process in the time interval $[0,1]$. The process is known in literature as Geometric Brownian motion.

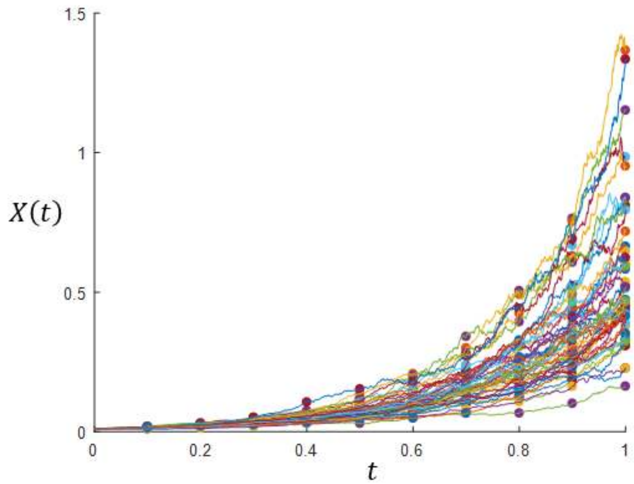

Figure 2: 50 realization of the geometric Brownian motion solution of Eq. 17

The Fokker-Planck equation (Eq. 13) for the SDE can be solved to obtain a closed-form solution of the time-varying probability density function of $X(t)$

$$f_X(x;\mu,\sigma,t) = \frac{1}{x\sigma_1\sqrt{2\pi t}} e^{-\frac{\left(\ln x - \ln x_0 - \left(\mu_1 - \frac{1}{2}\sigma^2\right)t\right)^2}{2\sigma_1^2 t}} \tag{18}$$

which corresponds to the PDF of a lognormal distribution with parameters $\lambda = \ln(x_0) + (\mu_1 - (1/2)\sigma^2)t$ and $\xi = \sigma_1\sqrt{t}$, where $x_0 = X(0)$. The closed-form distribution of the random variable $X(t)$ can then be plugged into the capacity and demand models for the system being analyzed to obtain the time-varying reliability of the system following the framework described in Section 2.

Assume now that the form and the values for the parameters in the model are unknown and need to be calibrated based on experimental observations. Assuming that 10 inspections are performed at regular intervals throughout the life-cycle of the system. The observations obtained from these inspections that are used to calibrate the unknown parameters in the model are shown in Figure 2 as filled dots. The SDE is assumed to have the general form in Eq. 9 where the semimartingale driving noise is assumed to be a standard Brownian motion ($\mathrm{d}S(t;\boldsymbol{\Theta}_S) = \mathrm{d}B(t)$). Dropping the compound Poisson process component of the driving noise is a reasonable assumption since there appear to be no jumps in the process by looking at the observations. With minimal loss of generality, we can assume the drift and diffusion terms to be expressed in polynomial form. In particular

$$\mu\left[t, X(t), \{\mathbf{Z}(t)\}; \boldsymbol{\Theta}_\mu\right] = \sum_{k=0}^{2} \mu_k X(t)^k \tag{19}$$

$$\sigma\left[t, X(t), \{\mathbf{Z}(t)\}; \boldsymbol{\Theta}_\sigma\right] = \sum_{k=0}^{1} \sigma_k X(t)^k \tag{20}$$

where the unknown parameters are $\boldsymbol{\Theta}_X = [\boldsymbol{\Theta}_\mu, \boldsymbol{\Theta}_\sigma] = [\mu_1, \mu_2, \mu_3, \sigma_1, \sigma_2]$.

In order to perform the minimization problem in Eq. 16, we need to solve the differential equation in Eq. 12 for the characteristic function of $X(t)$. Using Eq. 19 and Eq. 20 in Eq. 12 we obtain

$$\frac{\partial\varphi(u;t,\boldsymbol{\Theta}_X)}{\partial t} = iuE\left[e^{iuX(t)}\sum_{k=0}^{2}\mu_k X(t,\boldsymbol{\Theta}_X)^k\right] - \frac{1}{2}u^2 E\left\{e^{iuX(t)}\left[\sum_{k=0}^{1}\sigma_k X(t,\boldsymbol{\Theta}_X)^k\right]^2\right\}. \tag{21}$$

Eq. 21 can be simplified using the following property of the characteristic function (Grigoriu 2013)

$$E\left[X^k e^{iuX}\right] = (-i)^k \frac{\partial^k}{\partial u^k}\varphi(u). \tag{22}$$

Finally, the characteristic function of the state variable $X(t)$ can be obtained as the solution of the following partial differential equation

$$\begin{aligned} &\frac{\partial\varphi(u;t,\boldsymbol{\Theta}_X)}{\partial t} = \\ &\left[\frac{1}{2}u^2\sigma_1^2 - iu\mu_2\right]\frac{\partial^2\varphi(u;t,\boldsymbol{\Theta}_X)}{\partial u^2} \\ &+\left[u\mu_1 + iu^2\sigma_0\sigma_1\right]\frac{\partial\varphi(u;t,\boldsymbol{\Theta}_X)}{\partial u} \\ &+\left[\frac{1}{2}u^2\sigma_0^2 - iu\mu_0\right]\varphi(u;t,\boldsymbol{\Theta}_X). \end{aligned} \tag{23}$$

with boundary conditions coming from the properties of the characteristic function:

$$\varphi(u;t,\mathbf{\Theta}_X) = 1 \tag{24}$$

$$|\varphi(u;t,\mathbf{\Theta}_X)| \le 1 \tag{25}$$

$$\varphi(u;t,\mathbf{\Theta}_X) \xrightarrow{|u|\to\infty} 0 \tag{26}$$

$$\frac{\partial}{\partial u}\varphi(u;t,\mathbf{\Theta}_X) \xrightarrow{|u|\to\infty} 0 \tag{27}$$

After computing the empirical characteristic function at the times of inspections following the procedure described in Section 4, we perform the minimization problem in Eq. 16. The minimization reaches convergence and the minimizing values for the unknown parameters are shown in Table 1.

Table 1: Parameter values.

μ_0	μ_1	μ_2	σ_0	σ_1
0.073	*2.313*	*1.000*	*-0.031*	*0.444*

The calibrated values for $\mathbf{\Theta}_X$ are then used to simulate new realizations for the deterioration process. Figure 3 shows 20 realization of the newly calibrated process superimposed on the experimental observations used for calibration. The calibrated model is able to capture the original behavior of the process, as the realizations approximately follow the original path of the stochastic process.

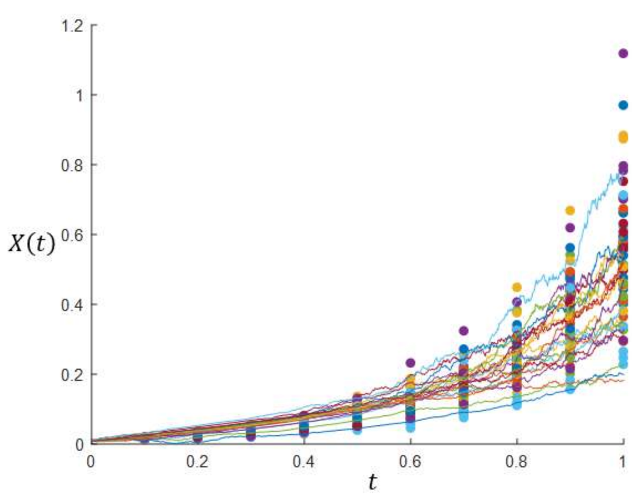

Figure 3: 20 realization of the stochastic process generated using the calibrated values for the unknown parameters, superimposed on the experimental observations

The results can be used to extrapolate the expected value for the state variable $X(t)$ at times following the latest inspection time. These values can then be used in conjunction with the framework for Life-Cycle Analysis presented in Section 2 to obtain the time-varying probability of failure for the system in the future.

6. CONCLUSIONS

A new formulation for the Life-Cycle Analysis of engineering systems was proposed. The formulation looks at the evolution of a set of variables that define the properties of the system (state variables) and models them using Stochastic Differential Equations. Efficient tools are available for the analysis of such stochastic processes, so that closed-form solutions can be obtained for the distribution of the state variables over time. Once capacity and demand are formulated in terms of the state variables, it is possible to obtain the time-varying probability of failure for the system.

Using Stochastic Differential Equations for deterioration processes is faithful to the true nature of the processes (which is continuous) with respect to more commonly used discrete formulations. In addition, the proposed framework is able to account for the interactions among multiple deterioration processes affecting the system and for the possible presence of shock deterioration processes that would cause jumps in the evolution of the state variables.

Finally, a procedure for the calibration of the processes based on a limited amount of experimental data was proposed. The procedure allows to incorporate into the framework the information coming from Structural Health Monitoring (SHM) and Non-Destructive Evaluation (NDE), in order to properly assess the reliability of the system at future times. A simple example was provided as an application of the proposed procedure for calibration.

Future work on this topic includes the investigation of the correlation between different processes acting on the system, as well as the development of new calibration tools that are able to provide confidence intervals for the estimated values for the unknown variables. One goal of this work is to develop a procedure that could be incorporated into a Bayesian framework, so that continuous updating of the models is possible as additional SHM or NDE data become available.

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