Stochastic dissipative PDE's and Gibbs measures

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Abstract

We study a class of dissipative nonlinear PDE's forced by a random force $\eta^\omega(t,x)$, with the space variable x varying in a bounded domain. The class contains the 2D Navier–Stokes equations (under periodic or Dirichlet boundary conditions), and the forces we consider are those common in statistical hydrodynamics: they are random fields smooth in x and stationary, short-correlated in time t. In this paper, we confine ourselves to "kick forces" of the form

$$\eta^\omega(t,x) = \sum_{k=-\infty}^{+\infty} \delta(t-kT)\eta_k(x),$$

where the η_k's are smooth bounded identically distributed random fields. The equation in question defines a Markov chain in an appropriately chosen phase space (a subset of a function space) that contains the zero function and is invariant for the (random) flow of the equation. Concerning this Markov chain, we prove the following main result (see Theorem 2.2): *The Markov chain has a unique invariant measure.* To prove this theorem, we present a construction assigning, to any invariant measure, a Gibbs measure for a 1D system with compact phase space and apply a version of Ruelle–Perron–Frobenius uniqueness theorem to the corresponding Gibbs system. We also discuss ergodic properties of the invariant measure and corresponding properties of the original randomly forced PDE.

Contents

0 Introduction

The paper deals with a class of randomly forced dissipative nonlinear PDE's. The class contains the 2D space-periodic Navier–Stokes equations and in the Introduction we confine ourselves to this important example:

$$\dot{u} - \delta\Delta u + (u\cdot\nabla)u + \nabla p = \eta^\omega(t,x), \quad \operatorname{div} u = 0, \tag{0.1}$$

where $x \in \mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$, $u = u(t,x)$ and $p = p(t,x)$. We assume that

$$\operatorname{div} \eta^\omega \equiv 0, \quad \langle \eta^\omega \rangle := \int_{\mathbb{T}^2} \eta^\omega dx \equiv 0$$

and study solutions $u(t,x)$ with zero mean value, i. e., $\langle u \rangle \equiv 0$. In the usual way [BV, CF, G, L], we exclude the pressure p from the equations, applying to (0.1) the projection to the linear space formed by divergence-free vector fields. Accordingly, we view (0.1) as a random dynamical system in a Sobolev phase space $\mathcal{H}^s, s \geq 0$, where

$$\mathcal{H}^s = \{u \in H^s(\mathbb{T}^2;\mathbb{R}^2) : \operatorname{div} u = 0\,,\ \langle u \rangle = 0\}.$$

Our concern is large-time asymptotic properties of this system.

It is traditional for statistical hydrodynamics to assume that the random force $\eta^\omega(t,x)$ in Equation (0.1) is smooth in x and is stationary short-correlated in time t. In mathematical literature, it is common to replace physically correct forces described above by random fields $\eta^\omega(t,x)$ that are smooth in x, while as a function of time t they are white noises, see [DZ, FlM, BKL, M, S1]. In this work, we take another mathematical model for the physically correct forces η, namely, a "kick force model" (see below). This model is sufficiently popular and suits our techniques the best. We believe that our approach also applies to equations with white noise forces in time[1] and plan to address these equations in a subsequent paper.

A kick force η corresponds to the situation when the system gets smooth random kicks with some time period T and evolves freely between the kicks. It means that the kth kick changes a solution $u(kT,x)$ to $u(kT+0,x) = u(kT,x)+\eta_k(x)$, while between the kicks, $u(t,x)$ satisfies Equation (0.1) with $\eta = 0$. This model is described by Equation (0.1) in which the force η is a δ-function of time t:

$$\eta^\omega(t,x) = \sum_{k\in\mathbb{Z}} \delta(t-kT)\eta_k(x). \tag{0.2}$$

Here the η_k's are independent identically distributed smooth random fields. To describe them, we expand η_k in the L_2-normalised trigonometric basis $\{e_j, j\in\mathbb{N}\}$ of the space $\mathcal{H}^s$:

$$\eta_k(x) = \sum_{j=1}^{\infty} b_j \xi_{jk} e_j(x). \tag{0.3}$$

In (0.3), ξ_{jk} $(j\in\mathbb{N}, k\in\mathbb{Z})$ are independent random variables uniformly distributed on the interval $[-1,1]$ and the real constants b_j satisfy the inequality

$$|b_j| \le C_r j^{-r} \quad \text{for all} \quad j,r\in\mathbb{N}. \tag{0.4}$$

(This assumption guarantees that the random fields are smooth.) The restrictions on distributions of ξ_{jk}'s and on the decay rate (0.4) can be weakened, see (2.7).

Due to the kick nature of the force, a solution $u^\omega(t,x)$ is completely described by its values $u_k(x)$ at the points $t=kT$, $k\in\mathbb{Z}$:

$$u_k(x) := u^\omega(kT+0,x), \quad k\in\mathbb{Z}.$$

Accordingly, from now on we treat (0.1) as a discrete-time random dynamical system in $\mathcal{H}^s$:

$$u_k = S(u_{k-1}) + \eta_k\,, \tag{0.5}$$

where the map $S:\mathcal{H}^s\to\mathcal{H}^s$ is a time-one shift along trajectories of Equation (0.1) with $\eta=0$.

[1] For a non-degenerate white noise force η, the set A in Theorems 0.1 and 0.2 below coincides with the whole space $\mathcal{H}^s$.

Due to (0.3) and (0.4), the random fields $\eta_k(x)$ are bounded in any C^l-norm, uniformly with respect to k and almost all ω. It follows that the solution of (0.5) with zero Cauchy data

$$u_0(x) = 0 \tag{0.6}$$

is also bounded in every $\mathcal{H}^l$-norm:

$$\|u_k\|_{\mathcal{H}^l} \le C'_l \quad \text{for} \quad k \ge 0 \quad \text{and almost all} \quad \omega. \tag{0.7}$$

Denoting by $A_k \subset \mathcal{H}^s$ the support of distribution of the random variable $u_k \in \mathcal{H}^s$, we conclude from (0.7) that the union $\cup_{k\ge 0} A_k$ is a precompact subset of $\mathcal{H}^s$. Its closure A is a compact set invariant for the random dynamical system (0.5), so the system defines a family of Markov chains in A. That is, given any Borel measure λ on A, Equation (0.5) admits a unique solution $(u_k(x), k \ge 0)$, which is a Markov chain in A such that the distribution of u_0 is λ (see [Re] and the main text). The measure λ is said to be *invariant* for (0.5) if the distributions of all random variables $u_k \in \mathcal{H}^s$ coincide with λ. The main result of this work is the following theorem:

Theorem 0.1. *There exists a finite integer $N \ge 0$ such that if $b_j \ne 0$ for $1 \le j \le N$, then the Markov chain defined by system* (0.5) *in A has a unique invariant measure λ. This measure is concentrated on smooth vector fields, i. e., $\lambda\big(\mathcal{H}^s \cap C^\infty(\mathbb{T}^2, \mathbb{R}^2)\big) = 1$, and its support in $\mathcal{H}^s$ is equal to A.*[2]

Existence of an invariant measure is an easy consequence of the usual Bogolyubov–Krylov argument (see [DZ]), whereas its *uniqueness* is a deep result (cf. [G, Section 6.1]).

The integer N depends on the viscosity $\delta > 0$ and on the $\mathcal{H}^s$-norm of the force η. In particular, the Markov chain in $A = A_\delta$ has a unique invariant measure for any $\delta > 0$ if all coefficients b_j are nonzero (and satisfy (0.4)).

The measure λ in Theorem 0.1 describes asymptotic behaviour in time for solutions of (0.5). That is, it describes the long-time behaviour of 2D fluid sloshed by random kicks.

Theorem 0.2. *Let $b_j \ne 0$ for $1 \le j \le N$, where N is defined in Theorem 0.1, and let $(u_k, k \ge 0)$ be any Markov chain in A satisfying* (0.5)*. Then the distributions of random variables $u_k \in \mathcal{H}^s$ weakly converge to λ.*

In particular, the solution of (0.5), (0.6) converges to λ in distribution as $k \to \infty$.

Since the random variables u_k are valued in the compact set $A \subset \mathcal{H}^s$, for any continuous function f on $\mathcal{H}^s$ we have the following convergence:

$$\mathbb{E} f(u_k) \to \int_A f(u)\, d\lambda(u) \quad \text{as} \quad k \to \infty.$$

In particular, choosing $s \ge 2$ and taking for f the function $f(u(\cdot)) = u^i(x)u^j(y)$ with some $1 \le i, j \le 2$ and $x, y \in \mathbb{T}^2$, we see that the 2-point correlation tensors $\mathbb{E}u^i(x)u^j(y)$ for a statistical solution of the 2D Navier–Stokes

[2]This means that A is the minimal closed subset of $\mathcal{H}^s$ which has full λ-measure .

equations (0.1) = (0.5) with zero initial condition (0.6) converge to corresponding correlation tensors for the measure λ.

The results of Theorems 0.1 and 0.2 were proved earlier for laminar flows (when the Reynolds number $\sim |\eta|\delta^{-1}$ is small), see [M], and for Equation (0.1) forced by a random force η that is non-smooth in x, see [FlM]. The above-mentioned works deal with the Navier–Stokes equations forced by a white noise force of the form

$$\eta(t,x) = \frac{d}{dt}\sum_{j=1}^{\infty} b_j \beta_j(t) e_j(x). \tag{0.8}$$

Here $\{e_j(x)\}$ is the trigonometric basis, β_j's are independent standard Brownian motions, and b_j's are real constants. The non-smoothness assumption imposed in [FlM] reeds as

$$cj^{-1/2} \le b_j \le Cj^{-3/8-\varepsilon} \quad \text{for all} \quad j \ge 1, \tag{0.9}$$

where c, C, and ε are positive constants. Due to the lower bound in (0.9), Equation (0.1) with right-hand side (0.8) defines, in a suitable low-smoothness Sobolev space $\mathcal{H}^s$, a Markov process that satisfies the strong Feller property. This allows one to get the uniqueness of an invariant measure as a corollary of a version of the Doob uniqueness theorem [DZ].

Another work related to Theorems 0.1 and 0.2 is the paper [S1] devoted to the Burgers equation. In [S1], uniqueness of an invariant measure is established without the smallness or non-smoothness assumption. However, the method applied there substantially employs the Cole–Hopf transformation, which integrates the Burgers equation. We also note that analysis of the resulting formulas uses some techniques developed to study Gibbs measures.

For a space-smooth random force η, the family of Markov processes defined by Equation (0.1) (or (0.5)) is not strongly Feller, see [DZ] and Remark 5.2. Therefore Doob's arguments do not apply to the problem (0.1)– (0.4). Our proof is based on other ideas sketched in Section 2.2. Very loosely, to prove the uniqueness we pass from Equation (0.5) to an equation for semi-trajectories $(u_k, k \le j)$, $j \in \mathbb{Z}$, and replace the latter by an equivalent system which is a direct sum of a random dynamical system in a space of sequences $(v_k \in \mathbb{R}^N, k \le 0)$ (N is the same as in Theorem 0.1) and of some trivial system. The new dynamical system turns out to be of the same type as those arising in studies of 1D Gibbs measures (see [Ru, S2, Bo]). To prove uniqueness of a 1D Gibbs measure, Ruelle proposed a Perron–Frobenius type theorem (see the same references). In Section 5, we use a version of his result to prove the uniqueness of an invariant measure. It is quite possible that some statements closely related to Ruelle's type theorem applied in this paper were known earlier. Still, since we failed to find an appropriate version in literature, a complete proof of the result we need is given in Section 4.

Our approach for proving the uniqueness applies to a large class of dissipative nonlinear systems described in Section 2.1 and do not use specifics of the Euler nonlinearity $(u \cdot \nabla)u$.

To complete the Introduction, we note that, in terms of the measures $\lambda = \lambda_\delta$, by the *turbulence problem* (for space-periodic 2D flow) is meant the problem of understanding limiting behaviour of the measure λ_δ as $\delta \to 0$. (Theorems 0.1 and 0.2 apply to Equation (0.1) with any $\delta > 0$ if in (0.3) all b_j's are positive.) In difference with what we said above regarding the equations with fixed $\delta > 0$, it is commonly believed that the limiting behaviour of the invariant measure depends essentially on properties of the Euler nonlinearity, cf. [BKL, G]. Corresponding results have no chance to be as general as the main results of this work (i. e., Theorems 2.2, 6.1, and 6.2 whose important particular cases are Theorems 0.1 and 0.2).

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Notation

Let $\mathbb{Z} = \mathbb{Z}_\infty$ be the set of all integers and, for $k \in \mathbb{Z}$, let $\mathbb{Z}_k$ be the set of integers that are no greater then k. For any set M, we denote by

$$\mathbf{M} = M^{\mathbb{Z}_0} = \prod_{l=-\infty}^{0} M, \quad \mathcal{M} = M^{\mathbb{Z}} = \prod_{l=-\infty}^{+\infty} M$$

the spaces of sequences $\boldsymbol{m} = (m_l, l \in \mathbb{Z}_0)$ and $\mathsf{n} = (n_l, l \in \mathbb{Z})$, respectively. For any $\mathsf{n} \in \mathcal{M}$ and $k \in \mathbb{Z}$, we write $\boldsymbol{n}^k = (n_l, l \in \mathbb{Z}_k)$ and regard the sequence $\boldsymbol{n}^k$ as an element of $\mathbf{M}$ identifying it with a shifted sequence that belongs to $\mathbf{M}$. As a rule, the superscripts of elements belonging to $\mathbf{M}$ or $\mathcal{M}$ will signify the "discrete time" while the subscripts will stand for the number of a component (e. g., $\boldsymbol{n}^k = (\dots, n^k_{-1}, n^k_0)$).

A set of sequences $\{\boldsymbol{m}^k \in \mathbf{M}, i < k < j\}$, $-\infty \le i < j \le \infty$, is said to be *compatible* if there exists a sequence $(m_l, l < j)$ such that $\boldsymbol{m}^k = (\dots, m_{k-1}, m_k)$ for $i < k < j$.

Let $\mathbf{X}$ be a Polish space, i. e., separable complete metric space. We shall use the following notation.

$[B]_{\mathbf{X}}$ is the closure in $\mathbf{X}$ of its subset B.
$B_{\mathbf{X}}(\boldsymbol{x}, r)$ is a ball in $\mathbf{X}$ of radius r centred at $\boldsymbol{x} \in \mathbf{X}$; we write $B_{\mathbf{X}}(r)$ if $\mathbf{X}$ has a selected point $\mathbf{0}$ and $\boldsymbol{x} = \mathbf{0}$.
$\mathcal{B}(\mathbf{X})$ is the σ-algebra of Borel subsets of $\mathbf{X}$.
$\mathcal{P}(\mathbf{X})$ is the set of probability measures on $(\mathbf{X}, \mathcal{B}(\mathbf{X}))$.
$\mathbf{C}(\mathbf{X})$ is the space of real-valued continuous functions on $\mathbf{X}$.
$\mathbf{C}_b(\mathbf{X})$ is the space of bounded functions $f \in \mathbf{C}(\mathbf{X})$. It is endowed with the norm

$$\|f\|_\infty := \sup_{\boldsymbol{x} \in \mathbf{X}} |f(\boldsymbol{x})|.$$

$L^1(\mathbf{X},\mu)$ is the space of Borel functions on $\mathbf{X}$ with finite norm

$$\|f\|_\mu := \int_{\mathbf{X}} \big|f(\boldsymbol{x})\big| \, d\mu(\boldsymbol{x}).$$

The integral of a function $f(\boldsymbol{x})$ over the space $\mathbf{X}$ with respect to a measure μ will sometimes be denoted by (μ, f):

$$(\mu, f) = \int_{\mathbf{X}} f(\boldsymbol{x}) \, d\mu(\boldsymbol{x}) = \int_{\mathbf{X}} f \, d\mu.$$

$\mathcal{D}(\xi)$ is the distribution of a random variable ξ.

1 Preliminaries

1.1 Invariant measures for a class of Markov chains

Let V be a separable Fréchet space, let $T : V \to V$ be a continuous mapping, and let $\{\eta_k, k \geq 1\}$ be a sequence of independent identically distributed (i.i.d.) V-valued random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Consider a family of homogeneous V-valued Markov chains (see [Re]) $\Theta^k = \Theta^k(v)$, $v \in V$, defined by the formulas

$$\Theta^0 = v, \tag{1.1}$$

$$\Theta^k = T(\Theta^{k-1}) + \eta_k, \tag{1.2}$$

where $k \geq 1$. Denote by $P(k, v, \Gamma)$ the corresponding transition function:

$$P(k, v, \Gamma) = \mathbb{P}\{\Theta^k(v) \in \Gamma\}, \quad v \in V, \quad \Gamma \in \mathcal{B}(V). \tag{1.3}$$

Note that if ν is the distribution of η_k, then

$$P(1, v, \Gamma) = \mathbb{P}\{T(v) + \eta_1 \in \Gamma\} = \nu\big(\Gamma - T(v)\big).$$

With the transition function (1.3) one associates the Markov operators

$$P_k f(v) = \int_V P(k, v, dz) f(z) : \mathbf{C}_b(V) \to \mathbf{C}_b(V), \tag{1.4}$$

$$P_k^* \mu(\Gamma) = \int_V P(k, v, \Gamma) d\mu(v) : \mathcal{P}(V) \to \mathcal{P}(V), \tag{1.5}$$

where $f \in \mathbf{C}_b(V)$ and $\mu \in \mathcal{P}(V)$. We shall write P and P^* instead of P_1 and P_1^*, respectively.

The fact that the image under P_k of a continuous bounded function belongs to $\mathbf{C}_b(V)$ is known as the Feller property. It follows from continuity of T. Indeed, if $v_n \in V$ converges to v, then

$$Pf(v_n) = \int_V P(1, v_n, dz) f(z) = \int_V f\big(z + T(v_n)\big) d\nu(z)$$

and therefore, by the dominated convergence theorem, $Pf(v_n) \to Pf(v)$ as $n \to \infty$.

We shall say that the transition function $P(k, v, \Gamma)$ and the Markov operators P_k and P_k^* *correspond to Equation* (1.2).

Definition 1.1. A random sequence $\{\Theta^k, k > k_0\}$, $k_0 \geq -\infty$, is called a solution of (1.2) if the following three conditions hold:

- $\{\Theta^k\}$ is a Markov chain;
- $\{\Theta^k\}$ satisfies (1.2) for $k > k_0 + 1$;
- the process is future-independent, i. e., Θ^k is independent of η_l, $l > k$.

In particular, any Markov chain $\Theta^k(u)$, $k \geq 0$, is a solution of (1.2).

Our goal is to study distributions of solutions for some equations of the form (1.2) that correspond to stochastic PDE's. Accordingly, our main interest here is not with the random variables Θ^k themselves, but rather with their distributions in V. For this reason, we do not distinguish equations of the form (1.2) with different probability spaces $(\Omega, \mathcal{F}, \mathbb{P})$ as soon as they give rise to the same transition function (1.3) (and hence to the same Markov operators (1.4) and (1.5)).

Denote by $\mathcal{D}(\xi)$ the distribution of a random variable ξ and by $\operatorname{supp} \mu$ the support of a measure μ (i. e., the smallest closed set of full measure). For Equation (1.2), the set of attainability A_k from zero by the time k is defined as

$$A_0 = \{0\}, \quad A_k = T(A_{k-1}) + \operatorname{supp} \mathcal{D}(\eta_k), \quad k \geq 1, \tag{1.6}$$

and the set of attainability from zero (in infinite time) has the form

$$A := \left[\bigcup_{k=0}^{\infty} A_k \right]_V . \tag{1.7}$$

Remark 1.2. The sets A_k and A can be interpreted in terms of the optimal control theory if we view (1.1), (1.2) as a controllable system, where η_k is the control chosen at the instance k. With this in mind, for given elements v and η_k, $k \geq 1$, we define the sequence (cf. (1.1), (1.2))

$$\Theta^0(v) = v, \quad \Theta^k(v; \eta_1, \dots, \eta_k) = T\big(\Theta^{k-1}(v; \eta_1, \dots, \eta_{k-1})\big) + \eta_k, \tag{1.8}$$

where $k \geq 1$. Obviously, A_k and A are the sets of attainability from zero (by finite or infinite time) for the system $\Theta^k(v; \eta_1, \dots, \eta_k)$ in the usual sense of the optimal control.

Let us denote by $\mathcal{P}(V, A)$ the set of Borel measures $\lambda \in \mathcal{P}(V)$ such that $\operatorname{supp} \lambda \subset A$ and fix an arbitrary $\lambda_0 \in \mathcal{P}(V, A)$. Let Θ^k be a Markov chain satisfying (1.2) for $k \geq 1$ such that $\mathcal{D}(\Theta^0) = \lambda_0$. Note that (1.3) is a transition function for Θ^k. Consider the Krylov–Bogolyubov averages of distributions of Θ^k:

$$\lambda_L = \frac{1}{L}\sum_{k=0}^{L-1}\mathcal{D}(\Theta^k) = \frac{1}{L}\sum_{k=0}^{L-1} P_k^*\lambda_0. \tag{1.9}$$

We recall that $\lambda \in \mathcal{P}(V)$ is called an *invariant measure* (for the family of Markov chains or for Equation (1.2)) if $P^*\mu = \mu$. The following assertion is a simple consequence of the Prokhorov and Krylov–Bogolyubov theorems (for instance, see [IW, Theorem I.2.6] and [DZ, Theorem 3.1.1]).

Proposition 1.3. *Assume that the set A is compact in V. Then the sequence $\{\lambda_L\}$ is tight in $\mathcal{P}(V)$ and, hence, has at least one limit point. Moreover, any limit point of $\{\lambda_L\}$ is an invariant measure for P^*, and its support is contained in A.*

Remark 1.4. By construction, the set A is invariant for any Markov chain Θ^k that satisfies (1.2) and whose initial distribution λ_0 is supported by A, i. e., $\Theta^k \in A$ almost surely. (For instance, if $\Theta^k = \Theta^k(v)$ is given by (1.1), (1.2) with $v \in A$, then A is invariant for Θ^k.) Redefining such a process Θ^k on a zero subset of Ω, we can assume that $\Theta^k \in A$ for all $\omega \in \Omega$ and regard Θ^k as a Markov chain with phase space A.

1.2 Stationary process corresponding to an invariant measure

Recall that we denote by $\nu \in \mathcal{P}(X)$ the distribution of η_k.

Proposition 1.5. *Assume that the set of attainability A is compact in V. Let $\lambda \in \mathcal{P}(V, A)$ be an invariant measure for P^*. Then there is a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$, a sequence of i.i.d. V-valued random variables $\tilde{\eta}_k$, $k \in \mathbb{Z}$, on $\widetilde{\Omega}$, and a stationary V-valued Markov chain $\mathsf{z} = (z_k, k \in \mathbb{Z})$ such that*

$$z_k = T(z_{k-1}) + \tilde{\eta}_k \quad \textit{for all} \quad \omega \in \widetilde{\Omega}, \quad k \in \mathbb{Z}. \tag{1.10}$$

Moreover, $\mathcal{D}(\tilde{\eta}_k) = \nu$ and $\mathcal{D}(z_k) = \lambda$ for any $k \in \mathbb{Z}$. Finally, the families $(z_l, l \le k)$ and $(\tilde{\eta}_l, l \ge k+1)$ are independent for all $k \in \mathbb{Z}$.

Proof. We apply standard arguments based on the Prokhorov and Skorokhod theorems (for instance, see [IW, Theorems I.2.6 and I.2.7]).

Define the space $\mathcal{V} = V^{\mathbb{Z}}$ endowed with the Tikhonov topology[3] and denote by $\{\xi_k, \zeta_k, k \in \mathbb{Z}\}$ an arbitrary family of independent V-valued random variables such that $\mathcal{D}(\xi_k) = \lambda$ and $\mathcal{D}(\zeta_k) = \nu$. Consider a sequence of V-valued Markov chains $\mathsf{x}^l = (x_k^l, k \in \mathbb{Z})$ defined as

$$x_k^l = \begin{cases} 0 & \text{for} \quad k \le -l-1, \\ \xi_l & \text{for} \quad k = -l, \\ T(x_{k-1}^l) + \zeta_k & \text{for} \quad k \ge -l+1. \end{cases} \tag{1.11}$$

[3]This means that a sequence $\boldsymbol{x}^j = (x_l^j, l \le 0)$ converges to $\boldsymbol{x} = (x_l, l \le 0)$ if and only if $x_l^j \to x_l$ as $j \to \infty$ for all $l \le 0$.

We claim that the sequence $\mathcal{D}(\mathsf{x}^l)$ is tight in $\mathcal{P}(\mathcal{V})$. Indeed, by construction, $\mathcal{D}(x_k^l) = \delta_0$ for $k \le -l-1$, where δ_0 is the Dirac measure concentrated at zero, and $\mathcal{D}(x_k^l) = \lambda$ for $k \ge -l$. Since the supports of $\mathcal{D}(x_k^l)$ are compact sets in V and an infinite product of compact sets is compact in the Tikhonov topology, we conclude that the measures $\boldsymbol{\lambda}_l = \mathcal{D}(\mathsf{x}^l)$, $l \ge 1$, form a tight family in $\mathcal{P}(\mathcal{V})$.

By the Prokhorov theorem, there is a sequence of integers $l_j \to +\infty$ and a measure $\boldsymbol{\lambda} \in \mathcal{P}(\mathcal{V})$ such that $\boldsymbol{\lambda}_{l_j} \to \boldsymbol{\lambda}$ in the weak topology of $\mathcal{P}(\mathcal{V})$. In view of the Skorokhod embedding theorem, there is a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$ and $\mathcal{V}$-valued random variables $\mathsf{z}^j = (z_k^j, k \in \mathbb{Z})$ and $\mathsf{z} = (z_k, k \in \mathbb{Z})$ such that

$$\mathcal{D}(\mathsf{z}^j) = \mathcal{D}(\mathsf{x}^{l_j}) = \boldsymbol{\lambda}_{l_j}, \quad \mathcal{D}(\mathsf{z}) = \boldsymbol{\lambda}, \tag{1.12}$$

$$\mathsf{z}^j \to \mathsf{z} \quad \text{as} \quad j \to \infty \quad \widetilde{\mathbb{P}}\text{-almost surely.} \tag{1.13}$$

We claim that z is the required Markov chain.

Indeed, the fact that $\mathcal{D}(z_k) = \lambda$ follows from the construction. Let us define the random variables

$$\tilde{\eta}_k = z_k - T(z_{k-1}), \quad \tilde{\eta}_k^l = z_k^l - T(z_{k-1}^l).$$

The first of this relations implies (1.10), and the second shows that $\mathcal{D}(\tilde{\eta}_k^l) = \mathcal{D}(\eta_k) = \nu$ for $k \ge 1-l$. Moreover, it follows from (1.13) and the continuity of T that, $\widetilde{\mathbb{P}}$-almost surely, $\tilde{\eta}_k^l \to \tilde{\eta}_k$ as $l \to \infty$ for any $k \in \mathbb{Z}$. Therefore, $\mathcal{D}(\tilde{\eta}_k) = \nu$ and

$$(z_k^l, \tilde{\eta}_k^l, k \in \mathbb{Z}) \to (z_k, \tilde{\eta}_k, k \in \mathbb{Z}) \quad \widetilde{\mathbb{P}}\text{-almost surely as} \quad l \to \infty,$$

whence we conclude that the corresponding distributions also converge. This implies the required assertions concerning the independence, which completes the proof of the proposition. $\square$

Since the underlying probability space is of no importance for the applications we deal with, in what follows, we shall drop the tildes and replace (1.10) by the original equation (1.2).

1.3 Gibbs system

In this section, we specify a class of Markov chains (1.1), (1.2) we are the most interested in.

Let X be a finite-dimensional Euclidean space with Lebesgue measure $d\alpha$, $\alpha \in X$, and let Y be a separable Fréchet space with a Borel measure[4] $d\ell_Y(\psi)$, $\psi \in Y$. Let us denote

$$\mathbf{X} = X^{\mathbb{Z}_0}, \quad \mathbf{Y} = Y^{\mathbb{Z}_0}.$$

We shall assume that $\mathbf{X}$ and $\mathbf{Y}$ are endowed with the Tikhonov topology. Let $(\varphi_k, k \ge 1)$ and $(\psi_k, k \ge 1)$ be two independent sequences of i.i.d. random variables with values in X and Y, respectively, such that, for any $k \ge 0$,

$$\mathcal{D}(\varphi_k) = D(\alpha)\, d\alpha, \quad \mathcal{D}(\psi_k) = d\ell_Y(\psi),$$

[4] In all applications the measure ℓ_Y will have a bounded support.

where $D(\alpha)$ is an integrable function on X. Let $T_0 : \mathbf{X}\times\mathbf{Y} \to X$ be a continuous mapping and let

$$T : \mathbf{X}\times\mathbf{Y} \to X \times Y, \quad T : \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{w} \end{pmatrix} \mapsto \begin{pmatrix} T_0(\boldsymbol{v}, \boldsymbol{w}) \\ 0 \end{pmatrix}.$$

Consider a family of Markov chains

$$\boldsymbol{\Upsilon}^k = \boldsymbol{\Upsilon}^k(\boldsymbol{U}) = \begin{pmatrix} \boldsymbol{\theta}^k(\boldsymbol{U}) \\ \boldsymbol{\zeta}^k(\boldsymbol{U}) \end{pmatrix} = \begin{pmatrix} \boldsymbol{\theta}^k \\ \boldsymbol{\zeta}^k \end{pmatrix}, \quad \boldsymbol{U} = \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{w} \end{pmatrix} \in \mathbf{X}\times\mathbf{Y},$$

with phase space $\mathbf{X}\times\mathbf{Y}$, defined by the formulas[5] (cf. (1.1), (1.2))

$$\boldsymbol{\Upsilon}^0 = \boldsymbol{U}, \tag{1.14}$$

$$\boldsymbol{\Upsilon}^k = \bigl(\boldsymbol{\Upsilon}^{k-1}, T\bigl(\boldsymbol{\Upsilon}^{k-1}\bigr) + \eta_k\bigr), \tag{1.15}$$

where $\eta_k = \binom{\varphi_k}{\psi_k}$ and (1.15) holds for any $k \ge 1$. Note that if (1.15) is viewed as an equation and a Markov chain $(\boldsymbol{\Upsilon}^k, k \in \mathbb{Z})$ is any its solution, then the sequences $\boldsymbol{\Upsilon}^k$ are compatible (see Notation) since for any integers k and $l \ge 1$ we have

$$\boldsymbol{\Upsilon}^{k+l} = \bigl(\boldsymbol{\Upsilon}^k, \Upsilon^{k+l}_{1-l}, \dots, \Upsilon^{k+l}_0\bigr). \tag{1.16}$$

For $l = 1$, relation (1.16) results from (1.15); the general case follows by induction.

As in Remark 1.2, with the family of Markov chains (1.14), (1.15) we associate the controllable system

$$\boldsymbol{\Upsilon}^k(\boldsymbol{U}; \sigma_1, \dots, \sigma_k) = \begin{pmatrix} \boldsymbol{\theta}^k(\boldsymbol{U}; \sigma_1, \dots, \sigma_k) \\ \boldsymbol{\zeta}^k(\boldsymbol{U}; \sigma_1, \dots, \sigma_k) \end{pmatrix}, \quad \sigma_k = \begin{pmatrix} \alpha_k \\ \beta_k \end{pmatrix} \in X \times Y, \quad k \ge 0,$$

given by the formulas

$$\boldsymbol{\Upsilon}^0(\boldsymbol{U}) = \boldsymbol{U}, \tag{1.17}$$

$$\boldsymbol{\Upsilon}^k(\boldsymbol{U}; \sigma_1, \dots, \sigma_k) = \bigl(\boldsymbol{\Upsilon}^{k-1}, T(\boldsymbol{\Upsilon}^{k-1}) + \sigma_k\bigr), \tag{1.18}$$

where $\boldsymbol{\Upsilon}^{k-1} = \boldsymbol{\Upsilon}^{k-1}(\boldsymbol{U}; \sigma_1, \dots, \sigma_{k-1})$.

Let us calculate the transition function $\mathfrak{P}(k, \boldsymbol{U}, \Gamma)$ for the family (1.14), (1.15). We begin with the case $k = 1$. For any set $I \subset \mathbb{R}$, we shall use capital Gothic letters with subscript I (for instance, $\mathfrak{B}_I$) to denote elements of the σ-algebra $\mathcal{B}\bigl((X \times Y)^{I\cap\mathbb{Z}_0}\bigr)$. Note that, for any vector $T' \in X \times Y$ of the form $T' = \binom{T_0'}{0}$ and any k, we have

$$\mathcal{D}(T' + \eta_k) = D(\alpha - T_0')\, d\alpha\, d\ell_Y.$$

[5] In accordance with our agreement, we identify the right-hand side in (1.15), which is an element of $(X \times Y)^{\mathbb{Z}_1}$, with the corresponding element in $\mathbf{X}\times\mathbf{Y}$.

Therefore, if a Borel set $\Gamma \in \mathcal{B}(\mathbf{X}\times\mathbf{Y})$ has the form $\Gamma = \mathfrak{B}_{(-\infty,-1]} \times \mathfrak{B}_{\{0\}}$, then

$$\mathfrak{P}(1,\boldsymbol{U},\Gamma) = \delta_{\boldsymbol{U}}(\mathfrak{B}_{(-\infty,-1]}) \int_{\mathfrak{B}_{\{0\}}} D\big(\alpha - T_0(\boldsymbol{U})\big)\, d\alpha\, d\ell(\psi). \tag{1.19}$$

We now turn to the case $k \geq 2$. Let $\Upsilon_0^k(\boldsymbol{U})$ be the zeroth component of the sequence $\boldsymbol{\Upsilon}^k(\boldsymbol{U})$, i. e., $\Upsilon_0^k(\boldsymbol{U}) = T\big(\boldsymbol{\Upsilon}^{k-1}(\boldsymbol{U})\big) + \eta_k$. By induction, the joint distribution of the random variables $\Upsilon_0^1(\boldsymbol{U}), \ldots, \Upsilon_0^k(\boldsymbol{U})$ in the direct product $(X\times Y)^{\{1,\ldots,k\}} = \prod_{l=1}^k X\times Y$ has the form

$$D_k^{\boldsymbol{U}} = \prod_{l=1}^{k} \big[D\big(\alpha_l - T_0(\boldsymbol{U};\sigma_1,\ldots,\sigma_{l-1})\big)\, d\ell(\sigma_l)\big], \tag{1.20}$$

where $d\ell = d\alpha\, d\ell_Y$ and $\sigma_l = \binom{\alpha_l}{\beta_l}$. Accordingly, for any set of the form

$$\Gamma = \mathfrak{B}_{(-\infty,-k]} \times \mathfrak{B}_{(-k,0]} \tag{1.21}$$

we have (cf. (1.19))

$$\mathfrak{P}(k,\boldsymbol{U},\Gamma) = \delta_{\boldsymbol{U}}(\mathfrak{B}_{(-\infty,-k]}) D_k^{\boldsymbol{U}}\big(\mathfrak{B}_{(-k,0]}\big). \tag{1.22}$$

Now let $\big\{\boldsymbol{\Upsilon}^k = (\Upsilon_l^k, l\in\mathbb{Z}_0)\big\}$ be a stationary Markov chain that is a solution of Equation (1.15) for $k\in\mathbb{Z}$ (see Definition 1.1). In this case, the sequence $\{\Upsilon_l = \Upsilon_0^l, l\in\mathbb{Z}\}$ formed of the zeroth components of $\boldsymbol{\Upsilon}^k$ is a stationary process in $X\times Y$. Since $\{\boldsymbol{\Upsilon}^k, k\geq 0\}$ is a compatible family, for any $n\in\mathbb{Z}$ we have

$$\boldsymbol{\Upsilon}^n = (\ldots,\Upsilon_{n-1},\Upsilon_n).$$

The conditional distribution of the k-vector $(\Upsilon_{n+1},\ldots,\Upsilon_{n+k})$ under the condition $\boldsymbol{\Upsilon}^n = \boldsymbol{U}\in\mathbf{X}\times\mathbf{Y}$ equals $D_k^{\boldsymbol{U}}$ (see (1.20)). Setting

$$\mathcal{H}_k(\boldsymbol{U};\sigma_1,\ldots,\sigma_k) := \log\prod_{l=1}^{k} D\big(\alpha_l - T_0(\boldsymbol{U};\sigma_1,\ldots,\sigma_{l-1})\big) \geq -\infty,$$

we obtain

$$\mathcal{D}(\Upsilon_{n+1},\ldots,\Upsilon_{n+k}|\,\boldsymbol{\Upsilon}^n = \boldsymbol{U}) = e^{\mathcal{H}_k(\boldsymbol{U};\sigma_{n+1},\ldots,\sigma_{n+k})} \prod_{j=n+1}^{n+k} d\ell(\sigma_j). \tag{1.23}$$

Clearly, the Hamiltonian $\mathcal{H}_k$ is stationary: it does not depend on n, but only on the vector $(\sigma_{n+1},\ldots,\sigma_{n+k})$ and the "past" $\boldsymbol{U}\in\mathbf{X}\times\mathbf{Y}$. This means that the random sequence $\{\Upsilon_n\}$ has a Gibbs distribution with the Hamiltonians $\{\mathcal{H}_k\}$ (for instance, see [S2, Bo]). Thus, any stationary Markov chain $\boldsymbol{\Upsilon}^m$, $m\in\mathbb{Z}$, that satisfies the above-mentioned conditions (in particular, (1.15) holds) defines a Gibbs measure for which the conditional distributions have the form (1.23). We conclude that the uniqueness of stationary solution for (1.15) follows from that of the Gibbs measure with conditional distributions (1.23). There are many results that guarantee uniqueness of a Gibbs measure for 1D systems (for instance, see [Do, Bo]). However, they do not apply to Gibbs systems associated with the Markov chains we are interested in because the corresponding Hamiltonians are equal to $-\infty$ on large parts of the phase space $(X \times Y)^{\{1,\dots,k\}} = \prod_{j=1}^{k} X \times Y$ (cf. assumption (2.4) in [Do]).

Remark 1.6. The results of this section remain true if the spaces V and $\mathbf{X} \times \mathbf{Y}$ are replaced by their closed subsets. In this case, we have to assume in addition that these subsets are invariant for the Markov chains introduced above.

2 Invariant measures for nonlinear dissipative semi-groups

2.1 Statement of the main results

Let H be a separable Hilbert space with norm $\|\cdot\|$ and let $S : H \to H$ be a (nonlinear) operator such that $S(0) = 0$ and the following conditions (A) – (C) hold:

(A) *For any $R > r > 0$ there are positive constants $C = C(R)$ and $a = a(R, r) < 1$ and an integer $n_0 = n_0(R, r) \geq 1$ such that*

$$\|S(u_1) - S(u_2)\| \leq C(R)\|u_1 - u_2\| \quad \textit{for all} \quad u_1, u_2 \in B_H(R), \tag{2.1}$$

$$\|S^n(u)\| \leq \max\{a\|u\|, r\} \quad \textit{for} \quad u \in B_H(R), \quad n \geq n_0. \tag{2.2}$$

For any compact set $K \subset H$, define a sequence of sets $\mathcal{A}_k(K) \subset H$ by the rule

$$\mathcal{A}_0(K) = \{0\}, \quad \mathcal{A}_k(K) = S\big(\mathcal{A}_{k-1}(K)\big) + K, \quad k \geq 1, \tag{2.3}$$

and denote

$$\mathcal{A}(K) = \left[\bigcup_{k=0}^{\infty} \mathcal{A}_k(K)\right]_H. \tag{2.4}$$

(B) The set $\mathcal{A}(K)$ is bounded in H for any compact set $K \subset H$.

Let $\{e_j\}$ be an orthonormal basis in H. For a given integer $N \geq 1$, denote by P_N and Q_N the orthogonal projections onto the closed subspaces H_N and[6] $H_N^\perp$ generated by the sets of vectors $\{e_1, \dots, e_N\}$ and $\{e_{N+1}, e_{N+2}, \dots\}$, respectively.

(C) *For any $R > 0$ there is a decreasing sequence $\gamma_N(R) > 0$ tending to zero as $N \to \infty$ such that*

$$\big\|\mathsf{Q}_N\big(S(u_1) - S(u_2)\big)\big\| \leq \gamma_N(R)\|u_1 - u_2\| \quad \textit{for all} \quad u_1, u_2 \in B_H(R). \tag{2.5}$$

[6]We denote by $E^\perp$ the orthogonal complement to a subspace E.

Let η_k, $k \geq 1$, be a sequence of independent H-valued random variables of the form

$$\eta_k = \sum_{j=1}^{\infty} b_j \xi_{jk} e_j, \tag{2.6}$$

where $b_j \geq 0$ are some constants such that

$$b := \left(\sum_{j=1}^{\infty} b_j^2 \right)^{1/2} < \infty, \tag{2.7}$$

and $\{\xi_{jk}\}$ is a family of independent real-valued random variables satisfying the following condition:

(D) *For any j, the random variables ξ_{jk}, $k \geq 1$, have the same distribution π_j such that $\pi_j(dr) = p_j(r)\,dr$, where the densities $p_j(r)$ are Lipschitz continuous and, moreover, $p_j(0) > 0$ and* $\operatorname{supp} p_j \subset [-1, 1]$.

Remark 2.1. We note straightaway that the densities $p_j(r)$ are allowed to be piecewise Lipschitz functions. In this case, all the results and their proofs remain the same, but some of the calculations become more cumbersome.

Define a family of Markov chains $\Theta^k = \Theta^k(u)$, $u \in H$, by the rule (cf. (1.1), (1.2))

$$\Theta^0 = u, \tag{2.8}$$

$$\Theta^k = S(\Theta^{k-1}) + \eta_k, \tag{2.9}$$

where $k \geq 1$, and denote by $P(k, u, \Gamma)$ the corresponding transition function (see (1.3)). Recall that Markov operators associated with $P(k, u, \Gamma)$ have the form (1.4) and (1.5).

Denote by ν the distribution of the i.i.d. random variables η_k and by A the set of attainability from zero for Equation (2.9) (see (1.7)). We recall that $\mathcal{P}(H, A)$ denotes the set of Borel measures on H whose support is contained in A. According to Remark 1.4, any Markov chain Θ^k that satisfies (2.9) and whose initial distribution is supported by A can be regarded as an A-valued Markov chain.

The theorem below is the main result of this paper.

Theorem 2.2. *Assume that conditions* (A) – (D) *hold. There is an integer $N = N(b) \geq 1$ such that if*

$$b_j > 0 \quad \textit{for} \quad j = 1, \ldots, N, \tag{2.10}$$

then P^* has a unique invariant measure $\lambda \in \mathcal{P}(H, A)$.

Remark 2.3. We shall apply Theorem 2.2 to stochastic PDE's of the form

$$\partial_t u + Lu + B(u) = \sum_{k=1}^{\infty} \delta(t - k)\eta_k, \tag{2.11}$$

where L is the generator of a parabolic semi-group, $B(u)$ is a nonlinear term, and η_k are i.i.d. random variables (see (2.6)). Denote by S_t the solving semi-group for Equation (2.11) with zero right-hand side. To apply Theorem 2.2, we set $S = S_1$ and check that conditions (A) – (C) are satisfied for S. Let us clarify informally what they mean.

Inequality (2.1) is nothing else but the condition of uniform Lipschitz continuity of S on any ball $B_H(R)$. Inequality (2.2) expresses the property of dissipativity of the semi-group S_t. Condition (B) means that the solution of (2.11) starting from zero is bounded in the phase space uniformly with respect to time and realisations of the random right-hand side. As a rule, this property holds for dissipative equations. Finally, in condition (C), $\{e_j\}$ is the complete set of eigenvectors of L, and inequality (2.5) follows from the fact that the corresponding eigenvalues tend to $+\infty$.

Ergodic properties of the invariant measure $\lambda \in \mathcal{P}(H, A)$ constructed in Theorem 2.2 are discussed in Section 6. The proof of Theorem 2.2, which is based on the constructions of Sections 3 and 4, is given in Section 5. Here we present the main steps of the proof.

2.2 Scheme of the proof of Theorem 2.2

2.2.1 Existence of an invariant measure and the corresponding stationary Markov chain

To prove the existence of an invariant measure $\lambda \in \mathcal{P}(H, A)$, note that $\operatorname{supp}\nu$ is a compact set (because it is a closed subset of the Hilbert cube defined by the sequence b_j) and therefore, by condition (B), the set of attainability $A = \mathcal{A}(\operatorname{supp}\nu)$ is bounded in H. It is easy to see that

$$A = S(A) + \operatorname{supp}\nu. \tag{2.12}$$

Now note that, in view of condition (C) and inequality (2.1) with $u_2 = 0$, there is a finite ε-net for the image under S of any bounded set in H. Hence, $S(A)$ is compact, and it follows from (2.12) that A is also compact in H. Proposition 1.3 now implies the required assertion.

It remains to check that the invariant measure is unique. The corresponding proof occupies Sections 3 – 5. Here we sketch it and develop some notations needed in the sequel.

Let $\lambda \in \mathcal{P}(H, A)$ be an invariant measure for P^*. By Proposition 1.5 (see also the remark after its proof), there is a stationary Markov chain $(u_k, k \in \mathbb{Z})$ and a family of independent random variables η_k such that $\mathcal{D}(u_k) = \lambda$, $\mathcal{D}(\eta_k) = \nu$, and

$$u_k = S(u_{k-1}) + \eta_k, \quad k \in \mathbb{Z}. \tag{2.13}$$

Introduce the linear space $\mathbf{H} = H^{\mathbb{Z}_0}$ endowed with the Tikhonov topology and consider a family of Markov chains $\mathbf{\Theta}^k = \mathbf{\Theta}^k(\boldsymbol{u})$ in $\mathbf{H}$ defined as

$$\mathbf{\Theta}^0 = \boldsymbol{u}, \tag{2.14}$$

$$\mathbf{\Theta}^k = \mathbf{S}\big(\mathbf{\Theta}^{k-1}\big) + \boldsymbol{\eta}_0^k, \tag{2.15}$$

where $k \geq 0$, $\boldsymbol{u} \in \mathbf{H}$, $\boldsymbol{\eta}_0^k = (\dots, 0, 0, \eta_k)$ and

$$\mathbf{S}(\boldsymbol{v}) = \big(\boldsymbol{v}, S(v_0)\big) = \big(\dots, v_{-2}, v_{-1}, v_0, S(v_0)\big), \quad \boldsymbol{v} = (v_l, l \leq 0) \in \mathbf{H}. \tag{2.16}$$

It is clear that the sequences[7] $\boldsymbol{u}^k := (u_l, l \in \mathbb{Z}_k) \in \mathbf{H}$ form a stationary compatible Markov chain satisfying Equation (2.15). (By compatibility we mean that almost all trajectories of the random process $\{\boldsymbol{u}^k, k \in \mathbb{Z}\}$ form compatible family of sequences.) Let us denote by $\mathbf{P}(k, \boldsymbol{u}, \Gamma)$, $\mathbf{P}_k$ and $\mathbf{P}_k^*$ the transition function and Markov semi-groups corresponding to (2.15) and by $\mathbf{A}$ the set of attainability from zero. What has been said implies that $\{\boldsymbol{u}^k\}$ defines an invariant measure $\boldsymbol{\lambda} \in \mathcal{P}(\mathbf{H}, \mathbf{A})$ for the semi-group $\mathbf{P}_k^*$. Since uniqueness of an invariant measure for $\mathbf{P}_k^*$ implies a similar property for the original semi-group P_k^*, it suffices to show that the invariant distribution for $\mathbf{P}_k^*$ is unique. However, this cannot be done directly because the noise $\boldsymbol{\eta}_0^k$ is effective only for the first N Fourier modes, whereas its projection to the (infinite-dimensional) subspace $H_N^\perp$ of codimension N may vanish. To overcome this difficulty and to prove that the distribution of a stationary solution $\{\boldsymbol{\Theta}^k, k \in \mathbb{Z}\}$ of (2.15) is uniquely defined, we perform an isomorphic transformation of $\{\boldsymbol{\Theta}^k\}$ that replaces a component of $\boldsymbol{\Theta}^k$ of codimension N (namely, its projection to the space $H_N^\perp$) by a random sequence (namely, by an appropriate component of the noise) whose "large-time behaviour" is known. The corresponding arguments are based on a Lyapunov–Schmidt type reduction.

2.2.2 Lyapunov–Schmidt type reduction

For any integer $N \geq 1$, we introduce the spaces $\mathbf{H}_N = (H_N)^{\mathbb{Z}_0}$ and $\mathbf{H}_N^\perp = (H_N^\perp)^{\mathbb{Z}_0}$ (where $H_N = \mathsf{P}_N H$ and $H_N^\perp = \mathsf{Q}_N H$) endowed with the Tikhonov topology. For the stationary sequences $(u_l, l \in \mathbb{Z})$ and $(\eta_l, l \in \mathbb{Z})$ defined above and for any integer k, we set

$$\boldsymbol{v}^k = (v_l, l \in \mathbb{Z}_k), \quad v_l = \mathsf{P}_N u_l; \quad \tilde{\boldsymbol{v}}^k = (\tilde{v}_l, l \in \mathbb{Z}_k), \quad \tilde{v}_l = \mathsf{Q}_N u_l, \tag{2.17}$$

and

$$\boldsymbol{\varphi}_0^k = (\dots, 0, 0, \varphi_k), \quad \boldsymbol{\psi}^k = (\dots, \psi_{k-1}, \psi_k), \quad \boldsymbol{\psi}_0^k = (\dots, 0, 0, \psi_k), \tag{2.18}$$

where $\varphi_k = \mathsf{P}_N \eta_k$ and $\psi_k = \mathsf{Q}_N \eta_k$. Let us denote by $\boldsymbol{\mathsf{P}}_N$ and $\boldsymbol{\mathsf{Q}}_N$ the projections

$$\begin{aligned} \boldsymbol{\mathsf{P}}_N &= \cdots \times \mathsf{P}_N \times \mathsf{P}_N : \mathbf{H} \to \mathbf{H}_N, & (u_l, l \leq 0) &\mapsto (\mathsf{P}_N u_l, l \leq 0), \\ \boldsymbol{\mathsf{Q}}_N &= \cdots \times \mathsf{Q}_N \times \mathsf{Q}_N : \mathbf{H} \to \mathbf{H}_N^\perp, & (u_l, l \leq 0) &\mapsto (\mathsf{Q}_N u_l, l \leq 0). \end{aligned}$$

Since the Markov chain $\{\boldsymbol{u}^k, k \in \mathbb{Z}\}$ satisfies Equation (2.15), applying P_N and Q_N to (2.15), we get the following system of two vector equations:

$$\boldsymbol{v}^j = \mathbf{P}_N \mathbf{S}(\boldsymbol{v}^{j-1}, \tilde{v}_{j-1}) + \boldsymbol{\varphi}_0^j, \tag{2.19}$$

$$\tilde{\boldsymbol{v}}^j = \mathbf{Q}_N \mathbf{S}(v_{j-1}, \tilde{\boldsymbol{v}}^{j-1}) + \boldsymbol{\psi}_0^j. \tag{2.20}$$

[7] The sequence $\boldsymbol{u}^k = (u_l, l \in \mathbb{Z}_k)$, $k \in \mathbb{Z}$, is regarded as an element of $\mathbf{H} = H^{\mathbb{Z}_0}$.

(We used the fact that $\mathbf{P}_N\mathbf{S}(\boldsymbol{u}^{j-1})$ does not depend on $\tilde{v}_l$ with $l \le j-2$ and similarly with $\mathbf{Q}_N\mathbf{S}(\boldsymbol{u}^{j-1})$.) The random sequences $\{\boldsymbol{v}^j\}$ and $\{\tilde{\boldsymbol{v}}^j\}$ are stationary and compatible since so is $\{\boldsymbol{u}^j\}$.

We now assume that $\boldsymbol{v}^r = (\dots, v_{r-1}, v_r)$ and $\boldsymbol{\psi}^r = (\dots, \psi_{r-1}, \psi_r)$ are bounded deterministic[8] sequences in H_N and $H_N^\perp$, respectively. It turns out that there exists a unique bounded sequence $(\dots, \tilde{v}_{r-1}, \tilde{v}_r)$ in $H_N^\perp$ such that $\tilde{\boldsymbol{v}}^j = (\tilde{v}_l, l \le j)$ satisfies Equation (2.20) for $j \le r$ with $\boldsymbol{v}^j = (v_l, l \le j)$ and $\boldsymbol{\psi}^j = (\dots, 0, \psi_j)$. Denote by

$$\mathcal{W}_0 : \mathbf{H}_N \times \mathbf{H}_N^\perp \to H_N^\perp \tag{2.21}$$

the operator defined by the relation

$$\tilde{v}_r = \mathcal{W}_0\big(\boldsymbol{v}^r, \boldsymbol{\psi}^r\big). \tag{2.22}$$

The operator $\mathcal{W}_0$ does not depend on r because Equation (2.19) is invariant with respect to shifts. A crucial property of $\mathcal{W}_0$ is that it forgets the past exponentially fast:

$$\left\| \frac{\partial \mathcal{W}_0}{\partial v_{r-j}} \right\| + \left\| \frac{\partial \mathcal{W}_0}{\partial \psi_{r-j}} \right\| \le C\, e^{-\varkappa j} \quad \text{for any} \quad j \ge 0, \tag{2.23}$$

where C and $\varkappa$ are positive constants.

We now return to the random equations (2.19) and (2.20). What has been said in the foregoing paragraph implies that we can solve Equation (2.20) with $j \le k$ for every $k \in \mathbb{Z}$ and express the random sequence $\tilde{\boldsymbol{v}}^k$ in terms of $\boldsymbol{v}^k$ and $\boldsymbol{\psi}^k$ using the operator $\mathcal{W}_0$. Substituting the result into (2.19), we conclude that the random sequence $\{\boldsymbol{v}^k, k \in \mathbb{Z}\}$ satisfies the equation

$$\boldsymbol{v}^k = \big(\boldsymbol{v}^{k-1}, T_0(\boldsymbol{v}^{k-1}, \boldsymbol{\psi}^{k-1})\big) + \boldsymbol{\varphi}_0^k, \tag{2.24}$$

where T_0 is an operator from $\mathbf{H}_N \times \mathbf{H}_N^\perp$ to H_N given by the formula

$$T_0(\boldsymbol{v}, \boldsymbol{\psi}) = \mathsf{P}_N S\big(v_0 + \mathcal{W}_0(\boldsymbol{v}, \boldsymbol{\psi})\big). \tag{2.25}$$

Consider now the family of Markov chains

$$\boldsymbol{\Upsilon}^k = \boldsymbol{\Upsilon}^k(\boldsymbol{U}) = \begin{pmatrix} \boldsymbol{\theta}^k \\ \boldsymbol{\zeta}^k \end{pmatrix}, \quad \boldsymbol{U} = \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{w} \end{pmatrix} \in \mathbf{H}_N \times \mathbf{H}_N^\perp,$$

with phase space $\mathbf{H}_N \times \mathbf{H}_N^\perp$, defined by (1.14) and (1.15), where the operator T_0 is given in (2.25). Let us denote by $\mathfrak{P}(k, \boldsymbol{U}, \Gamma)$, $\mathfrak{P}_k$, and $\mathfrak{P}_k^*$ the corresponding transition function and Markov semi-groups and by $\mathfrak{A}$ the set of attainability from zero (see (1.7)). We shall regard $\boldsymbol{\Upsilon}^k(\boldsymbol{U})$ as Markov chains in $\mathfrak{A}$ (see Remark 1.4). Assume that we can prove existence and uniqueness of an invariant

[8]In this paragraph, $\boldsymbol{v}^r$ and $\boldsymbol{\psi}^r$ stand for deterministic sequences that are not related to the random sequences defined in (2.17) and (2.18).

measure $\mu \in \mathcal{P}(\mathfrak{A})$ for $\mathfrak{P}_k^*$. By construction, the $(\mathbf{H}_N \times \mathbf{H}_N^\perp)$-valued random sequence

$$\boldsymbol{\Xi}^k = \begin{pmatrix} \boldsymbol{v}^k \\ \boldsymbol{\psi}^k \end{pmatrix} \tag{2.26}$$

(with $\boldsymbol{v}^k$ and $\boldsymbol{\psi}^k$ defined in (2.17) and (2.18), respectively) satisfies (1.15) for every $k \in \mathbb{Z}$. Moreover, it can be shown that $\boldsymbol{\Xi}^k$ is a stationary Markov chain and that its distribution $\Lambda = \mathcal{D}(\boldsymbol{\Xi}^k)$ is supported by $\mathfrak{A}$ (see Section 3.2), so that $\Lambda \in \mathcal{P}(\mathfrak{A})$ coincides with the unique invariant measure μ for $\mathfrak{P}_k^*$. It remains to note that $u_k = v_k + \mathcal{W}_0(\boldsymbol{v}^k, \boldsymbol{\psi}^k)$ and therefore $\mathcal{D}(z_k) = \lambda$ is also uniquely defined.

Thus, the problem of uniqueness of invariant distribution for the original Markov operator P_k^* is reduced to a similar question for $\mathfrak{P}_k^*$.

2.2.3 Uniqueness of invariant measure for $\mathfrak{P}_k^*$

The proof of the uniqueness is based on a version of the Ruelle–Perron–Frobenius (RPF) theorem presented in Section 4. Without going into details, let us explain the main idea. We deal with a family of Markov chains $\boldsymbol{\Upsilon}^k(\boldsymbol{U})$ in $\mathfrak{A}$. Using this fact and the dissipativity of the operator S (see (2.2)), it can be shown that $\boldsymbol{\Upsilon}^k(\boldsymbol{U})$ is irreducible (i.e., $\mathfrak{P}(k, \boldsymbol{U}, \mathfrak{O}) > 0$ for any $\boldsymbol{U} \in \mathfrak{A}$, an arbitrary open set $\mathfrak{O} \subset \mathfrak{A}$, and sufficiently large k). If the transition function were strong Feller, we could apply the Doob theorem (for instance, see [DZ, Theorem 4.2.1]) to prove uniqueness of invariant measure. However, the strong Feller property is not satisfied in the case under study (see [DZ, Sections 7.1 and 7.2] for necessary and sufficient conditions for the validity of the strong Feller property for infinite-dimensional systems), and to establish the uniqueness, we use a version of the RPF-theorem. Namely, we show that if the Markov family is "uniformly" irreducible and the operator $\mathfrak{P}_k$ possesses a very weak smoothing property, then the invariant measure is unique. Note that the proof of this fact substantially uses the exponential decay (2.23).

3 Lyapunov–Schmidt type reduction

3.1 Statement of the result

As in Section 2, we denote by H a separable Hilbert space with norm $\|\cdot\|$ and by S a nonlinear continuous operator in H. It is assumed that S satisfies conditions (B) and (C).

We recall that the spaces $\mathbf{H}_N = (H_N)^{\mathbb{Z}_0}$ and $\mathbf{H}_N^\perp = (H_N^\perp)^{\mathbb{Z}_0}$ are endowed with the Tikhonov topology. For any $R > 0$, define the bounded subsets

$$\mathbf{B}_N(R) = B_{H_N}(R)^{\mathbb{Z}_0} = \bigl\{\boldsymbol{v} = (v_l, l \le 0) \in \mathbf{H}_N : \|v_l\| \le R\bigr\},$$
$$\mathbf{B}_N^\perp(R) = B_{H_N^\perp}(R)^{\mathbb{Z}_0} = \bigl\{\boldsymbol{w} = (w_l, l \le 0) \in \mathbf{H}_N^\perp : \|w_l\| \le R\bigr\}.$$

For $\varkappa \ge 0$ and $\boldsymbol{u} \in \mathbf{H}$, we write

$$\|\boldsymbol{u}\|_\infty = \sup_{l \le 0} \|u_l\|, \quad \mathcal{M}(\varkappa)\boldsymbol{u} = (e^{\varkappa l} u_l, l \le 0).$$

It is easy to see that the Tikhonov topology on $\mathbf{B}_N(R)$ and $\mathbf{B}_N^\perp(R)$ coincides with the topology defined by the metric

$$d_\varkappa(\boldsymbol{u},\boldsymbol{v}) = \big\|\mathcal{M}(\varkappa)(\boldsymbol{u}-\boldsymbol{v})\big\|_\infty = \sup_{l\le 0} e^{\varkappa l}\|u_l - v_l\|, \tag{3.1}$$

where $\varkappa$ is an arbitrary positive number.

Consider the equation

$$u_k = S(u_{k-1}) + \eta_k, \quad k \le 0, \tag{3.2}$$

where $u_k, \eta_k \in H$. Application of Q_N to (3.2) results in

$$\tilde v_k = \mathsf{Q}_N S(v_{k-1} + \tilde v_{k-1}) + \psi_k, \quad k \le 0, \tag{3.3}$$

where $v_k = \mathsf{P}_N u_k$, $\tilde v_k = \mathsf{Q}_N u_k$, and $\psi_k = \mathsf{Q}_N \eta_k$. Let us abbreviate

$$B_{H_N}(R) \times B_{H_N^\perp}(b) = B_{R,b}, \quad \mathbf{B}_N(R)\times \mathbf{B}_N^\perp(b) = \mathbf{B}_{R,b}. \tag{3.4}$$

Theorem 3.1. *Assume that condition* (C) *holds. Let $R>0$, $b>0$, and γ, $0<\gamma<1$, be some constants and let an integer $N\ge 1$ be so large that $\gamma_N(\rho)\le\gamma$, where γ_N is the sequence in condition* (C) *and*

$$\rho := (R^2 + r^2)^{1/2}, \quad r := \frac{R\gamma + b}{1-\gamma}. \tag{3.5}$$

Then for any $\boldsymbol{v}\in\mathbf{B}_N(R)$ and $\boldsymbol{\psi}\in\mathbf{B}_N^\perp(b)$ Equation (3.3) *has a unique solution $\tilde{\boldsymbol{v}}\in\mathbf{B}_N^\perp(r)$. Moreover, for any $\varkappa$, $0\le\varkappa<-\ln\gamma$, the operator*

$$\mathcal{W} : \mathbf{B}_{R,b} \to \mathbf{B}_N^\perp(r), \quad (\boldsymbol{v},\boldsymbol{\psi}) \mapsto \tilde{\boldsymbol{v}}, \tag{3.6}$$

satisfies the inequality

$$\begin{aligned}\Big\|\mathcal{M}(\varkappa)\big(\mathcal{W}(\boldsymbol{v}^1,\boldsymbol{\psi}^1) - \mathcal{W}(\boldsymbol{v}^2,\boldsymbol{\psi}^2)\big)\Big\|_\infty &\le \\ \le (1-e^{\varkappa}\gamma)^{-1}\Big(e^{\varkappa}\gamma\big\|\mathcal{M}(\varkappa)(\boldsymbol{v}^1-\boldsymbol{v}^2)\big\|_\infty &+ \big\|\mathcal{M}(\varkappa)(\boldsymbol{\psi}^1-\boldsymbol{\psi}^2)\big\|_\infty\Big),\end{aligned} \tag{3.7}$$

where $\boldsymbol{v}^i\in\mathbf{B}_N(R)$, $\boldsymbol{\psi}^i\in\mathbf{B}_N^\perp(b)$, $i=1,2$. In particular, the operator $\mathcal{W}$ is continuous if the spaces entering (3.6) *are endowed with the Tikhonov topology.*

It follows from (3.3) that the operator $\mathcal{W}(\boldsymbol{v},\boldsymbol{\psi})$ is independent of the last component of $\boldsymbol{v}$.

Theorem 3.1 is a variant of the well-known result (originally due to [FP]) according to which the asymptotical dynamics of a nonlinear dissipative PDE is determined by the first N Fourier modes, where N is sufficiently large. Similar results are established by many authors for various purposes (for instance, to study attractors, inertial and integral manifolds, etc.). The proof of Theorem 3.1 is given in Appendix (see Section 8).

We now derive a simple corollary of Theorem 3.1. Let us write $\mathcal{W} = (\mathcal{W}_l, l \le 0)$, so that $\mathcal{W}_0(\boldsymbol{v}, \boldsymbol{\psi})$ is the zeroth (i. e., the last) component of $\mathcal{W}$. Till the end of this subsection, we shall abbreviate (v_l, ψ_l), $l \le 0$, to h_l. Denote by $\operatorname{Lip}_l \mathcal{W}_0$, $l \le 0$, the Lipschitz constant of $\mathcal{W}_0$ with respect to h_l that is uniform in all other arguments. In other words, $\operatorname{Lip}_l \mathcal{W}_0$ is the supremum over all

$$(\dots, h_{l-1}, h_{l+1}, \dots, h_0) \in \bigl(B_{R,b}\bigr)^{\mathbb{Z}_0 \setminus \{l\}}$$

of the Lipschitz constants of the functions

$$B_{R,b} \ni h \mapsto \mathcal{W}_0(\dots, h_{l-1}, h, h_{l+1}, \dots, h_0).$$

It follows from (3.7) that

$$\operatorname{Lip}_l \mathcal{W}_0 \le \bigl(1 - e^{\varkappa\gamma}\bigr)^{-1} e^{\varkappa l}. \tag{3.8}$$

This estimate shows that dependence of the function $\mathcal{W}_0$ on $h_l = (v_l, \psi_l)$ decays with l exponentially.

For later use, we make another obvious but helpful observation: if $\boldsymbol{v}^1 = (\boldsymbol{v}, v_1)$, $\boldsymbol{\psi}^1 = (\boldsymbol{\psi}, \psi_1)$, and $\tilde{\boldsymbol{v}} = \mathcal{W}(\boldsymbol{v}, \boldsymbol{\psi})$, then

$$\mathcal{W}(\boldsymbol{v}^1, \boldsymbol{\psi}^1) = \bigl(\tilde{\boldsymbol{v}}, \mathsf{Q}_N S(v_0 + \tilde{v}_0) + \psi_1\bigr). \tag{3.9}$$

Indeed, the right-hand side of (3.9) defines a sequence $\tilde{\boldsymbol{v}}' = (\tilde{v}_k', k \le 1)$ that satisfies Equation (3.2) for $k \le 1$ since $\tilde{\boldsymbol{v}}$ satisfies it for $k \le 0$.

3.2 Markov chain in the space $\boldsymbol{\mathfrak{H}}_N = \mathbf{H}_N \times \mathbf{H}_N^{\perp}$

We wish to define a family of Markov chains in $\boldsymbol{\mathfrak{H}}_N$ by the formulas

$$\boldsymbol{\Upsilon}^0 = \boldsymbol{U}, \qquad \boldsymbol{U} = \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{w} \end{pmatrix} \in \boldsymbol{\mathfrak{H}}_N, \tag{3.10}$$

$$\boldsymbol{\Upsilon}^k = \left(\boldsymbol{\Upsilon}^{k-1}, T\bigl(\boldsymbol{\Upsilon}^{k-1}\bigr) + \begin{pmatrix} \varphi_k \\ \psi_k \end{pmatrix}\right), \qquad T\begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{\psi} \end{pmatrix} = \begin{pmatrix} T_0(\boldsymbol{v}, \boldsymbol{\psi}) \\ 0 \end{pmatrix}, \tag{3.11}$$

where $\boldsymbol{\Upsilon}^k = \boldsymbol{\Upsilon}^k(\boldsymbol{U})$,

$$T_0(\boldsymbol{v}, \boldsymbol{\psi}) = \mathsf{P}_N S\bigl(v_0 + \mathcal{W}_0(\boldsymbol{v}, \boldsymbol{\psi})\bigr), \tag{3.12}$$

and v_0 is the last component of $\boldsymbol{v}$. However, the domain of definition of T_0 is only a part of the space $\boldsymbol{\mathfrak{H}}_N$, and therefore we have to choose carefully the corresponding phase space. To this end, find a constant $R > 0$ such that the set of attainability $A = \mathcal{A}(\operatorname{supp} \nu)$ for the original equation (2.9) is contained in $B_H(R)$ (see (2.4) and condition (B)). Let an integer $N \ge 1$ be so large that the conditions of Theorem 3.1 are satisfied. Denote by

$$\mathcal{W} = (\mathcal{W}_l, l \in \mathbb{Z}_0) : \mathbf{B}_{R,b} \to \mathbf{B}_N^{\perp}(r) \tag{3.13}$$

the operator constructed in Theorem 3.1. Obviously, we can define the kth element $\boldsymbol{\Upsilon}^k(\boldsymbol{U})$ if $\boldsymbol{\Upsilon}^{k-1}(\boldsymbol{U}) \in \mathbf{B}_{R,b}$. We claim that if $\boldsymbol{U} = \mathbf{0}$, where $\mathbf{0}$ is the element all of whose components are zero, then $\boldsymbol{\Upsilon}^k(\boldsymbol{U}) \in \mathbf{B}_{R,b}$ for all $k \geq 1$. Indeed, assume that the required inclusion is proved for $k \leq n-1$. Since $\|\psi_k\| \leq \|\eta_k\| \leq b$ by (2.6), (2.7) and condition (D), we have $\boldsymbol{\zeta}^k(\boldsymbol{U}) \in \mathbf{B}_N^\perp(b)$ for any $k \geq 0$. Let $\boldsymbol{\theta}^k(\boldsymbol{U}) = \big(\theta_l^k(\boldsymbol{U}), l \in \mathbb{Z}_0\big)$ be the first component of $\boldsymbol{\Upsilon}^k(\boldsymbol{U})$. Since, by the induction hypothesis,

$$\big(\theta_l^n(\boldsymbol{U}), l \leq -1\big) = \big(\theta_l^{n-1}, l \leq 0\big) = \boldsymbol{\theta}^{n-1}(\boldsymbol{U}) \in \mathbf{B}_N(R),$$

it suffices to show that $\theta_0^n(\boldsymbol{U}) \in B_{H_N}(R)$. However, it follows from the definition of $\boldsymbol{\theta}^k(\boldsymbol{U})$ that $\theta_0^n(\boldsymbol{U}) \in \mathsf{P}_N A_n$, where A_n is the set of attainability from zero by time n for Equation (2.9). It now remains to note that, according to the choice of R, we have $A_n \subset A \subset B_H(R)$ and hence $\mathsf{P}_N A_n \subset B_{H_N}(R)$.

Thus, if $\boldsymbol{U} = \mathbf{0}$, we can define $\boldsymbol{\Upsilon}^k(\boldsymbol{U})$ for any integer $k \geq 1$. Denote by $\mathfrak{A}$ the set of attainability from zero for Equation (3.11). Obvious argument based on the continuity of the operators entering the definition of $\boldsymbol{\Upsilon}^k(\boldsymbol{U})$ shows that formula (3.11), in which T_0 has the form (3.12), makes sense for any integer $k \geq 1$ and an arbitrary initial state $\boldsymbol{U} \in \mathfrak{A}$. We have thus obtained a family of Markov chains $\boldsymbol{\Upsilon}^k(\boldsymbol{U})$ with phase space $\mathfrak{A}$ which satisfy Equation (3.11).

An important observation is that Equation (3.11) with phase space $\mathfrak{A}$ is equivalent to (2.15) with phase space $\mathbf{A}$ in the sense that the sets of solutions for these two equations are in one-to-one correspondence. This result is stated below as Theorem 3.2.

Let $\varPhi\colon \mathbf{B}_{R,b} \to \mathbf{H}$ be a mapping that sends $\boldsymbol{U} = \big(U_l = \binom{v_l}{w_l}, l \leq 0\big)$ to

$$\varPhi(\boldsymbol{U}) = \boldsymbol{u} = \Big(u_l = \tbinom{v_l}{\tilde{v}_l} \in H_N \times H_N^\perp, l \leq 0\Big),$$

where

$$\tilde{\boldsymbol{v}} = \mathcal{W}(\boldsymbol{U}), \tag{3.14}$$

and let $\varPsi : \mathbf{H} \to \mathfrak{H}_N$ be a mapping that sends $\boldsymbol{u} = \big(\binom{v_l}{\tilde{v}_l}, l \leq 0\big) \in \mathbf{H}$ to $\varPsi(\boldsymbol{u}) = \boldsymbol{U} = \big(\binom{v_l}{w_l}, l \leq 0\big)$, where

$$w_l = \tilde{v}_l - \mathsf{Q}_N S(v_{l-1} + \tilde{v}_l), \quad l \leq 0. \tag{3.15}$$

It follows from Theorem 3.1 that $\varPhi$ is uniformly Lipschitz continuous. Moreover, inequality (2.1) implies that the restriction of $\varPsi$ to the set $\mathbf{B}_H(R) = B_H(R)^{\mathbb{Z}_0}$ is also uniformly Lipschitz continuous for any $R > 0$.

Theorem 3.2. *The operator $\varPhi$ defines a Lipschitz homeomorphism $\mathfrak{A} \to \mathbf{A}$ whose inverse is $\varPsi$. Moreover, a Markov chain $\big(\boldsymbol{\Upsilon}^k \in \mathfrak{A}, k > k_0 \geq -\infty\big)$ is a solution of* (3.11) *if and only if the chain $\big(\boldsymbol{u}_k = \varPsi(\boldsymbol{\Upsilon}^k) \in \mathbf{A}, k > k_0\big)$ is a solution of* (2.15)*.*

Proof. We first show that $\varPsi$ maps $\mathfrak{A}$ to $\mathbf{A}$. Denote by $\mathfrak{A}_k$ and $\mathbf{A}_k$ the sets of attainability from zero by the time k for Equations (3.11) and (2.15), respectively (see (1.6)). If $\boldsymbol{U} \in \mathfrak{A}_j$ for some $j \geq 1$, then there exists a trajectory

$(\boldsymbol{\Upsilon}^k, 0 \le k \le j)$ of (3.11), viewed as a controllable system, that is equal to zero for $k = 0$ and to $\boldsymbol{U}$ for $k = j$ (cf. Remark 1.2). It follows from the definition of $\mathcal{W}$ that the operator Φ sends $\boldsymbol{\Upsilon}^k$ to a trajectory of the controllable system (2.15) and that this trajectory is equal to zero for $k = 0$ and to $\Phi(\boldsymbol{U})$ for $k = j$. Hence, $\Phi(\boldsymbol{U}) \in \mathbf{A}_k$. By continuity, Φ maps $\mathfrak{A}$ to $\mathbf{A}$.

If $\boldsymbol{U} \in \mathfrak{A}$ and $\Phi(\boldsymbol{U}) = \boldsymbol{u} = \binom{\boldsymbol{v}}{\tilde{\boldsymbol{v}}}$, then $\tilde{\boldsymbol{v}}$ satisfies (3.3). Hence, (3.15) holds, and $\boldsymbol{U} = \Psi(\boldsymbol{u})$. Repeating the arguments applied above to Φ, we find that Ψ maps $\mathbf{A}_j$ to $\mathfrak{A}_j$ and, hence, $\mathbf{A}$ to $\mathfrak{A}$.

If $\boldsymbol{u} = \binom{\boldsymbol{v}}{\tilde{\boldsymbol{v}}} \in \mathbf{A}$ and $\Psi(\boldsymbol{u}) = \boldsymbol{U} = \binom{\boldsymbol{v}}{\boldsymbol{w}} \in \mathfrak{A}$, then $\boldsymbol{w}$ is defined by (3.15). Therefore $\tilde{\boldsymbol{v}}$ satisfies (3.3), and we have $\boldsymbol{u} = \Phi(\boldsymbol{U})$. Hence, $\Phi : \mathfrak{A} \to \mathbf{A}$ is a homeomorphism and $\Psi = \Phi^{-1}$.

Due to (3.9) and the definition of T (see (3.11)), the following diagram is commutative for any $\omega \in \Omega$:

$$\begin{array}{ccc} \boldsymbol{\Upsilon}^{k-1} & \xrightarrow{\Phi} & \boldsymbol{u}^{k-1} \\ \big\downarrow & & \big\downarrow \\ \boldsymbol{\Upsilon}^{k} & \xrightarrow{\Phi} & \boldsymbol{u}^{k} \end{array}$$

Here the left- and right-hand vertical arrows stand for the transformations in Equations (3.11) and (2.15), respectively. Since Φ is a homeomorphism, it defines a one-to-one correspondence between solutions of these two equations. $\square$

Corollary 3.3. *A Markov chain* $\big(\boldsymbol{\Upsilon}^k, k \in \mathbb{Z}\big)$ *is a stationary solution of* (3.11) *in* $\mathfrak{A}$ *if and only if* $\big(\boldsymbol{u}^k = \Phi(\boldsymbol{\Upsilon}^k) \in \mathbf{A}, k \in \mathbb{Z}\big)$ *is a stationary solution of* (2.15).

Corollary 3.4. *Equation* (3.11) *has a unique invariant measure supported by* $\mathfrak{A}$ *if and only if* (2.15) *possesses a unique invariant measure supported by* $\mathbf{A}$.

The second corollary follows from the first since Φ and Ψ transform identically distributed solutions of one equation to identically distributed solutions of the other.

By Theorem 3.2, the operator Φ transforms the family of Markov chains (3.10), (3.11) to the family (2.14), (2.15), where $\boldsymbol{u} = \Phi(\boldsymbol{U})$. Therefore the corresponding transition functions satisfy the relation

$$\mathfrak{P}(k, \boldsymbol{U}, \Gamma) = \mathbf{P}\big(k, \Phi(\boldsymbol{U}), \Phi(\Gamma)\big), \quad \boldsymbol{U} \in \mathfrak{A}, \quad \Gamma \in \mathcal{B}(\mathfrak{A}). \tag{3.16}$$

It follows that

$$\big(\mathbf{P}_k f\big) \circ \Phi = \mathfrak{P}_k(f \circ \Phi), \quad f \in \mathbf{C}_b(\mathbf{A}). \tag{3.17}$$

4 A version of the RPF-theorem

In this section, we prove a version of the RPF-theorem (with reservations concerning its novelty discussed in the Introduction). It provides a sufficient condition for the uniqueness of invariant measure for a Markov semi-group whose transition function is uniformly irreducible and possesses a smoothing property. This result will be used in Section 5 to prove Theorem 2.2.

4.1 Statement of the result

Let $\mathfrak{A}$ be a Polish space. A subset $\mathcal{R} \subset \mathbf{C}_b(\mathfrak{A})$ is called a *determining family for* $\mathcal{P}(\mathfrak{A})$ if for arbitrary measures $\mu_1, \mu_2 \in \mathcal{P}(\mathfrak{A})$ the condition

$$\int_{\mathfrak{A}} f(\boldsymbol{u})\, d\mu_1(\boldsymbol{u}) = \int_{\mathfrak{A}} f(\boldsymbol{u})\, d\mu_2(\boldsymbol{u}) \quad \text{for any} \quad f \in \mathcal{R}$$

implies that $\mu_1 = \mu_2$.

Let $\mathfrak{P}(k, \boldsymbol{u}, \Gamma)$, $\boldsymbol{u} \in \mathfrak{A}$, $\Gamma \in \mathcal{B}(\mathfrak{A})$, be a Feller transition function defined for nonnegative integers[9] k. Denote by

$$\mathfrak{P}_k : \mathbf{C}_b(\mathfrak{A}) \to \mathbf{C}_b(\mathfrak{A}), \quad \mathfrak{P}_k^* : \mathcal{P}(\mathfrak{A}) \to \mathcal{P}(\mathfrak{A}), \quad k \geq 0,$$

the Markov semi-groups associated with $\mathfrak{P}(k, \boldsymbol{u}, \Gamma)$.

For any function $f(\boldsymbol{u})$, denote by f^+ and f^- its positive and negative parts, respectively:

$$f^+ = \frac{1}{2}\big(f + |f|\big), \quad f^- = \frac{1}{2}\big(|f| - f\big).$$

We shall assume that the condition below is fulfilled.

(H) *There is a determining family $\mathcal{R}$ for $\mathcal{P}(\mathfrak{A})$ such that for any $f \in \mathcal{R}$ the function $f - c$ belongs to $\mathcal{R}$ for all $c \in \mathbb{R}$, and there is a constant $A = A_f > 1$ and an integer $k_0 = k_0(f) \geq 0$ such that the following property holds: if*

$$\sup_{\boldsymbol{u} \in \mathfrak{A}} f_k^+(\boldsymbol{u}) \geq \alpha \quad \textit{for all} \quad k \geq 0, \tag{4.1}$$

$$\sup_{\boldsymbol{u} \in \mathfrak{A}} f_k^-(\boldsymbol{u}) \geq \alpha \quad \textit{for all} \quad k \geq 0, \tag{4.2}$$

where $f_k^+ = \big(\mathfrak{P}_k f\big)^+$, $f_k^- = \big(\mathfrak{P}_k f\big)^-$, and $\alpha = \alpha(f) > 0$ is a constant not depending on k, then for any $k \geq k_0$ there is $l = l(k) > 0$ such that

$$\sup_{\boldsymbol{u} \in \mathfrak{A}} \big(\mathfrak{P}_l f_k^+\big)(\boldsymbol{u}) \leq A_f \inf_{\boldsymbol{u} \in \mathfrak{A}} \big(\mathfrak{P}_l f_k^+\big)(\boldsymbol{u}), \tag{4.3}$$

$$\sup_{\boldsymbol{u} \in \mathfrak{A}} \big(\mathfrak{P}_l f_k^-\big)(\boldsymbol{u}) \leq A_f \inf_{\boldsymbol{u} \in \mathfrak{A}} \big(\mathfrak{P}_l f_k^-\big)(\boldsymbol{u}). \tag{4.4}$$

Sufficient conditions guaranteeing the validity of (H) are given in Section 4.3; see conditions (H_1) and (H_2) there. The following theorem is the main result of this section.

Theorem 4.1. *Assume that condition* (H) *holds. Then the assertions below take place.*

(i) *Let $\mu \in \mathcal{P}(\mathfrak{A})$ be an invariant measure of $\mathfrak{P}_k^*$. Then, for any $f \in \mathcal{R}$,*

$$\mathfrak{P}_k f \to (\mu, f) \quad \textit{as} \quad t \to \infty \quad \textit{in} \quad L^1(\mathfrak{A}, \mu). \tag{4.5}$$

(ii) *The operator $\mathfrak{P}_k^*$ has at most one invariant measure $\mu \in \mathcal{P}(\mathfrak{A})$.*

[9] Theorem 4.1 established below remains true for continuous time. The statement of the corresponding result and its proof are literally the same as in the discrete case.

4.2 Proof of Theorem 4.1

Proof of (i). *Step 1.* Let $\mu \in \mathcal{P}(\mathfrak{A})$ be an invariant measure and let $f \in \mathcal{R}$. Without loss of generality, we can assume that

$$(\mu, f) = \int_{\mathfrak{A}} f(\boldsymbol{u})\, d\mu(\boldsymbol{u}) = 0. \tag{4.6}$$

The general case can be reduced to the former by the change $f \mapsto f - (\mu, f)$. Thus, we must prove that

$$\|\mathfrak{P}_k f\|_\mu = \int_{\mathfrak{A}} \big|\mathfrak{P}_k f\big|\, d\mu \to 0 \quad \text{as} \quad k \to \infty. \tag{4.7}$$

Note that $\mathfrak{P}_l^* \mu = \mu$ for any $l \geq 0$, and therefore

$$\begin{aligned}\|\mathfrak{P}_{k+l} f\|_\mu &= \int_{\mathfrak{A}} \big|\mathfrak{P}_l(\mathfrak{P}_k f)\big|\, d\mu \leq \int_{\mathfrak{A}} \mathfrak{P}_l\big|(\mathfrak{P}_k f)\big|\, d\mu \\ &= \int_{\mathfrak{A}} \big|(\mathfrak{P}_k f)\big|\, d(\mathfrak{P}_l^* \mu) = \int_{\mathfrak{A}} \big|(\mathfrak{P}_k f)\big|\, d\mu \\ &= \|\mathfrak{P}_k f\|_\mu.\end{aligned}$$

This means that $\|\mathfrak{P}_k f\|_\mu$ is a non-increasing sequence. Hence, the convergence (4.7) will be established if we show that for any $\varepsilon > 0$ there is $k = k_\varepsilon > 0$ such that

$$\|\mathfrak{P}_{k_\varepsilon} f\|_\mu \leq \varepsilon. \tag{4.8}$$

Step 2. We first assume that

$$\sup_{\boldsymbol{u} \in \mathfrak{A}} f_{k_s}^+(\boldsymbol{u}) \to 0 \quad \text{as} \quad s \to \infty,$$

where $\{k_s\}$ is a sequence of integers tending to $+\infty$. In this case

$$\int_{\mathfrak{A}} \big(\mathfrak{P}_{k_s} f\big)^+(\boldsymbol{u})\, d\mu(\boldsymbol{u}) = \int_{\mathfrak{A}} f_{k_s}^+(\boldsymbol{u})\, d\mu(\boldsymbol{u}) \to 0 \quad \text{as} \quad s \to \infty. \tag{4.9}$$

Moreover, it follows from (4.6) that

$$(\mu, f) = (\mathfrak{P}_k^* \mu, f) = (\mu, f_k) = 0, \quad k \geq 0,$$

where $f_k = \mathfrak{P}_k f$, and therefore

$$(\mu, f_k^+) = (\mu, f_k^-) \quad \text{for any} \quad k \geq 0. \tag{4.10}$$

Combining this with (4.9), we derive

$$(\mu, f_{k_s}^+) = (\mu, f_{k_s}^-) \to 0 \quad \text{as} \quad s \to \infty,$$

whence we conclude that (4.8) holds. A similar argument shows that if

$$\sup_{\boldsymbol{u} \in \mathfrak{A}} f_{k_s}^-(\boldsymbol{u}) \to 0 \quad \text{as} \quad s \to \infty$$

for a sequence $\{k_s\}$, then (4.8) is fulfilled.

Step 3. Thus, we can assume that (4.1) and (4.2) hold with a constant $\alpha > 0$. By condition (H), for any $k \ge k_0$ there is $l \ge 0$ such that (4.3) and (4.4) are satisfied. We claim that there is a sequence of integers $\{k_s, s \ge 1\}$ such that

$$\|\mathfrak{P}_{k_s} f\|_\mu \le a_f^s \|f\|_\mu, \quad s \ge 0, \tag{4.11}$$

where $a_f = 1 - A_f^{-1} < 1$.

The proof is by induction on s. Inequality (4.11) is obvious for $s = 0$. Assuming that (4.11) is established for $s \le r$, we now prove it for $s = r+1$. Set $k_{r+1} = k_r + l_r$, where $l_r \ge 0$ is the integer entering inequalities (4.3) and (4.4) with $k = k_r$. Note that, by (4.3) and (4.4), we have[10]

$$\int_{\mathfrak{A}} f_{k_r}^{\pm} d\mu = \int_{\mathfrak{A}} \mathfrak{P}_{l_r} f_{k_r}^{\pm} d\mu \le \sup_{\boldsymbol{u}\in\mathfrak{A}} \bigl(\mathfrak{P}_{l_r} f_{k_r}^{\pm}\bigr)(\boldsymbol{u}) \le A_f \inf_{\boldsymbol{u}\in\mathfrak{A}} \bigl(\mathfrak{P}_{l_r} f_{k_r}^{\pm}\bigr)(\boldsymbol{u})$$

whence

$$\mathfrak{P}_{l_r} f_{k_r}^{\pm}(\boldsymbol{u}) - A_f^{-1} \|f_{k_r}^{\pm}\|_\mu \ge 0 \quad \text{for} \quad \boldsymbol{u} \in \mathfrak{A}.$$

It follows that

$$\int_{\mathfrak{A}} \bigl|\mathfrak{P}_{l_r} f_{k_r}^{\pm} - A_f^{-1}\|f_{k_r}^{\pm}\|_\mu\bigr| \, d\mu = \int_{\mathfrak{A}} \bigl(\mathfrak{P}_{l_r} f_{k_r}^{\pm} - A_f^{-1}\|f_{k_r}^{\pm}\|_\mu\bigr) \, d\mu = a_f \|f_{k_r}^{\pm}\|_\mu.$$

We now estimate the expression $\|\mathfrak{P}_{k_{r+1}} f\|_\mu = \|\mathfrak{P}_{l_r} f_{k_r}\|_\mu$. In view of (4.10), we have

$$\|f_k^+\|_\mu = \|f_k^-\|_\mu \quad \text{for any} \quad k \ge 0, \tag{4.12}$$

and therefore

$$\begin{aligned}
\int_{\mathfrak{A}} \bigl|\mathfrak{P}_{l_r} f_{k_r}\bigr| d\mu &= \int_{\mathfrak{A}} \bigl|\mathfrak{P}_{l_r}(f_{k_r}^+ - f_{k_r}^-)\bigr| d\mu \\
&\le \int_{\mathfrak{A}} \bigl|\mathfrak{P}_{l_r} f_{k_r}^+ - A_f^{-1}\|f_{k_r}^+\|_\mu\bigr| \, d\mu + \int_{\mathfrak{A}} \bigl|\mathfrak{P}_{l_r} f_{k_r}^- - A_f^{-1}\|f_{k_r}^-\|_\mu\bigr| \, d\mu \\
&\le a_f\bigl(\|f_{k_r}^+\|_\mu + \|f_{k_r}^-\|_\mu\bigr) = a_f \|f_{k_r}\|_\mu.
\end{aligned}$$

Using the induction hypothesis, we derive

$$\int_{\mathfrak{A}} \bigl|\mathfrak{P}_{k_{r+1}} f\bigr| d\mu \le a_f \|\mathfrak{P}_{k_r} f\|_\mu \le a_f^{r+1} \|f\|_\mu,$$

which completes the proof of (4.11).

Inequality (4.8) is an obvious consequence of (4.11).

Proof of (ii). We first prove that any two invariant measures supported by $\mathfrak{A}$ are singular. To this end, we apply a well-known argument (for instance,

[10] Here and henceforth a formula involving the symbol $\pm$ is a brief writing for the two formulas corresponding to the upper and lower signs.

see [DZ, Proposition 3.2.5]). Let $\mu_1, \mu_2 \in \mathcal{P}(\mathfrak{A})$ be two different invariant measures. Since $\mathcal{R}$ is a determining family for $\mathcal{P}(\mathfrak{A})$, there is $f \in \mathcal{R}$ such that

$$(\mu_1, f) \neq (\mu_2, f). \tag{4.13}$$

By (i),

$$\mathfrak{P}_k f \to (\mu_i, f) \quad \text{as} \quad k \to \infty \quad \text{in} \quad L^1(\mathfrak{A}, \mu_i), \quad i = 1, 2.$$

Therefore, there is a sequence of integers $\{k_s\}$ tending to $+\infty$ such that

$$\mathfrak{P}_{k_s} f \to (\mu_i, f) \quad \text{as} \quad s \to \infty \quad \mu_i\text{-almost everywhere}, \quad i = 1, 2. \tag{4.14}$$

Denote by C_i, $i = 1, 2$, the set of points $\boldsymbol{u} \in \mathfrak{A}$ for which (4.14) takes place. We have $\mu_1(C_1) = \mu_2(C_2) = 1$ and, in view of (4.13), $C_1 \cap C_2 = \varnothing$. This means that μ_1 and μ_2 are singular.

We now assume that $\mu_1, \mu_2 \in \mathcal{P}(\mathfrak{A})$ are two different invariant measures for $\mathfrak{P}_k^*$. As is proved, they are singular. Consider the measure $\mu = (\mu_1 + \mu_2)/2$. It is clear that $\mu \in \mathcal{P}(\mathfrak{A})$ is an invariant measure and that μ and μ_1 are not singular. The contradiction obtained completes the proof of Theorem 4.1.

4.3 Sufficient conditions for application of Theorem 4.1

Assume that $\mathfrak{P}(k, \boldsymbol{u}, \Gamma)$, $\boldsymbol{u} \in \mathfrak{A}$, $\Gamma \in \mathcal{B}(\mathfrak{A})$, is a transition function satisfying the following conditions.

($\mathbf{H_1}$) *There is a determining family $\mathcal{R}_0$ for $\mathcal{P}(\mathfrak{A})$ such that if $f \in \mathcal{R}_0$, then the sequence $\mathfrak{P}_k f$, $k \geq k_0$, is uniformly equicontinuous, where k_0 is a nonnegative integer depending on f.*

($\mathbf{H_2}$) *For every $r > 0$ there are $\varepsilon > 0$ and $l \geq 1$ such that*

$$\mathfrak{P}(l, \boldsymbol{u}, B_{\mathfrak{A}}(\boldsymbol{a}, r)) \geq \varepsilon \quad \textit{for any} \quad \boldsymbol{u}, \boldsymbol{a} \in \mathfrak{A}. \tag{4.15}$$

Condition (H_1) can be called a "uniform Feller property". We impose it instead of the strong Feller property, which is common in arguments proving uniqueness of an invariant measure (see [DZ]), but which is not satisfied for the infinite-dimensional system we deal with (see Remark 5.2). Condition (H_2) is a slow-down version of the usual assumption that the measures $\mathfrak{P}(l, \boldsymbol{u}, \cdot)$ are absolutely continuous with respect to a reference measure on $\mathfrak{A}$ and the corresponding densities are positive uniformly in $\boldsymbol{u} \in \mathfrak{A}$ and $l \gg 1$. It can also be regarded as a condition of "uniform irreducibility" for the family of Markov chains in question.

Let $\mathcal{R}$ be the set of functions $f \in \mathbf{C}_b(\mathfrak{A})$ for which there is a constant $c \in \mathbb{R}$ such that $f - c \in \mathcal{R}_0$.

Theorem 4.2. *Let conditions* (H_1) *and* (H_2) *be satisfied. Then* (H) *is fulfilled for $\mathcal{R}$, and therefore assertions* (i) *and* (ii) *of Theorem 4.1 hold. Moreover,* $\operatorname{supp} \mu = \mathfrak{A}$. *Finally, if $\mathfrak{A}$ is a compact space, then the convergence in* (4.5) *takes place in the space* $\mathbf{C}_b(\mathfrak{A})$.

Proof. Let $f(\boldsymbol{u}) \in \mathcal{R}$ be an arbitrary function satisfying (4.1), where $f_k^{\pm} = (\mathfrak{P}_k f)^{\pm}$ and α is a positive constant. We must prove that (4.3) and (4.4) hold. We confine ourselves to the case of index $+$.

In view of condition (H_1), there is $r > 0$ and for any $k \geq k_0$ there is $\boldsymbol{u}_k \in \mathfrak{A}$ such that

$$\inf_{\boldsymbol{v} \in B_{\mathfrak{A}}(\boldsymbol{u}_k, r)} f_k^+(\boldsymbol{v}) \geq \sup_{\boldsymbol{v} \in \mathfrak{A}} f_k^+(\boldsymbol{v}) - \frac{\alpha}{2} \geq \frac{\alpha}{2} \quad \text{for} \quad k \geq k_0. \tag{4.16}$$

Let $\varepsilon > 0$ and $l \geq 1$ be the constants entering condition (H_2). In view of (4.15) and (4.16), we have

$$\begin{aligned} \big(\mathfrak{P}_l f_k^+\big)(\boldsymbol{u}) &= \int_{\mathfrak{A}} \mathfrak{P}(l, \boldsymbol{u}, d\boldsymbol{v}) f_k^+(\boldsymbol{v}) \geq \int_{B_{\mathfrak{A}}(\boldsymbol{u}_k, r)} \mathfrak{P}(l, \boldsymbol{u}, d\boldsymbol{v}) f_k^+(\boldsymbol{v}) \\ &\geq \mathfrak{P}\big(l, \boldsymbol{u}, B_{\mathfrak{A}}(\boldsymbol{u}_k, r)\big) \inf_{\boldsymbol{v} \in B_{\mathfrak{A}}(\boldsymbol{u}_k, r)} f_k^+(\boldsymbol{v}) \geq \frac{\alpha\varepsilon}{2}. \end{aligned} \tag{4.17}$$

On the other hand,

$$\big(\mathfrak{P}_l f_k^+\big)(\boldsymbol{u}) \leq \sup_{\boldsymbol{u} \in \mathfrak{A}} f_k^+(\boldsymbol{u}) \leq \sup_{\boldsymbol{u} \in \mathfrak{A}} \big|f_k(\boldsymbol{u})\big| \leq \sup_{\boldsymbol{u} \in \mathfrak{A}} \big|f(\boldsymbol{u})\big|. \tag{4.18}$$

Combining (4.17) and (4.18), we arrive at (4.3) with

$$A_f = 2(\alpha\varepsilon)^{-1} \sup_{\boldsymbol{u} \in \mathfrak{A}} \big|f(\boldsymbol{u})\big|.$$

Inequality (4.4) can be proved in a similar way.

We now assume that μ is an invariant measure for $\mathfrak{P}_k^*$ and show that $\operatorname{supp} \mu = \mathfrak{A}$. To this end, it suffices to check that

$$\mu\big(B_{\mathfrak{A}}(\boldsymbol{a}, r)\big) > 0 \quad \text{for any} \quad \boldsymbol{a} \in \mathfrak{A} \quad \text{and} \quad r > 0. \tag{4.19}$$

In view of the invariance of μ and inequality (4.15), we have

$$\mu\big(B_{\mathfrak{A}}(\boldsymbol{a}, r)\big) = \int_{\mathfrak{A}} \mathfrak{P}\big(l, \boldsymbol{u}, B_{\mathfrak{A}}(\boldsymbol{a}, r)\big)\, d\mu(\boldsymbol{u}) \geq \varepsilon,$$

which implies (4.19).

Finally, let $\mathfrak{A}$ be a compact space. We wish to show that

$$\mathfrak{P}_k f(\boldsymbol{u}) \to (\mu, f) \quad \text{as} \quad k \to \infty \quad \text{uniformly in} \quad \boldsymbol{u} \in \mathfrak{A}.$$

By condition (H_1) and the Arzelà–Ascoli theorem, there is a sequence of integers $k_j \to \infty$ such that $\mathfrak{P}_{k_j} f(\boldsymbol{u})$ converges uniformly to a function $g(\boldsymbol{u})$. In view of assertion (i) of Theorem 2.2, the function g must coincide with (μ, f) on the support of μ. Since $\operatorname{supp} \mu = \mathfrak{A}$, we have $g \equiv (\mu, f)$, and the whole sequence $\mathfrak{P}_k f$ converges to (μ, f). The proof of Theorem 4.2 is complete. □

Remark 4.3. If $\mathcal{R}$ is dense in $\mathbf{C}_b(\mathfrak{A})$, then the sequence $\mathfrak{P}_k f$ uniformly converges to (μ, f) for any $f \in \mathbf{C}_b(\mathfrak{A})$. Indeed, let a function $f_\varepsilon \in \mathcal{R}$ be such that $\|f - f_\varepsilon\|_\infty < \varepsilon$. We have

$$\big\|\mathfrak{P}_k f - (\nu, f)\big\|_\infty \leq \big\|\mathfrak{P}_k(f - f_\varepsilon)\big\|_\infty + \big\|\mathfrak{P}_k f_\varepsilon - (\nu, f_\varepsilon)\big\|_\infty + \big|(\mu, f - f_\varepsilon)\big|,$$

which implies the required assertion.

5 Proof of Theorem 2.2

5.1 Reduction to Theorem 4.1

Step 1. We recall that the original family of Markov chains $\Theta^k = \Theta^k(u)$ with phase space A is defined as

$$\Theta^0 = u, \tag{5.1}$$

$$\Theta^k = S(\Theta^{k-1}) + \eta_k, \tag{5.2}$$

where $k \geq 1$. As was shown in Section 2.2.1, there is an invariant measure $\lambda \in \mathcal{P}(A)$ for the semi-group P_k^* associated with (5.1), (5.2). Hence, we must establish the uniqueness.

We fix an arbitrary $R > 0$ such that $A \subset B_H(R)$, choose any constant γ, $0 < \gamma < 1$, and denote by N the smallest integer satisfying the condition (cf. (3.5))

$$\gamma_N(\rho) \leq \gamma, \quad \rho = \left(R^2 + r^2\right)^{1/2}, \quad r = \frac{R\gamma + b}{1 - \gamma}, \tag{5.3}$$

where γ_N is the sequence in condition (C). We claim that if condition (2.10) holds with the above choice of N, then the invariant measure is unique.

Step 2. By Proposition 1.5, an invariant measure $\lambda \in \mathcal{P}(H, A)$ for the family (5.1), (5.2) defines a stationary solution $(z_k, k \in \mathbb{Z})$ of (5.2), which gives rise to a stationary solution and, hence, to an invariant measure $\boldsymbol{\lambda} \in \mathcal{P}(\mathbf{H})$ for (2.15). We claim that $\operatorname{supp} \boldsymbol{\lambda} \subset \mathbf{A}$. Indeed, it suffices to show that if $\boldsymbol{u} \in \operatorname{supp} \boldsymbol{\lambda}$, then for any $\varepsilon > 0$ and an arbitrary integer $L \geq 0$ there is $\boldsymbol{u'} \in \mathbf{A}$ such that

$$\|u_l - u_l'\| \leq \varepsilon \quad \text{for} \quad -L \leq l \leq 0. \tag{5.4}$$

Fix arbitrary $\boldsymbol{u} \in \operatorname{supp} \boldsymbol{\lambda}$, $L \geq 0$, and $\varepsilon > 0$. It follows from the definition of the support of a measure that the event

$$z_l \in B_H(u_l, \varepsilon/2), \quad -L \leq l \leq 0,$$

has a positive probability. Since $\operatorname{supp} \mathcal{D}(z_l) \subset A$ and z_k satisfies Equation (5.2) for all $\omega \in \Omega$ and $k \in \mathbb{Z}$, there are realisations

$$\tilde{z}_l \in A \cap B_H(u_l, \varepsilon/2), \quad \tilde{\eta}_l \in \operatorname{supp} \nu, \quad -L \leq l \leq 0, \tag{5.5}$$

of the random variables z_l and η_l such that

$$\tilde{z}_l = S(\tilde{z}_{l-1}) + \tilde{\eta}_l, \quad 1 - L \leq l \leq 0. \tag{5.6}$$

Furthermore, since $\tilde{z}_{-L} \in A$, for any $\delta > 0$ there is an integer $j \geq 0$ and $u'_{-L} \in A_j$ such that $\|\tilde{z}_{-L} - u'_{-L}\| \leq \delta$. We now set

$$u_l' = S(u_{l-1}') + \tilde{\eta}_l, \quad 1 - L \leq l \leq 0. \tag{5.7}$$

It follows from (5.6), (5.7) and continuity of S (see (2.1)) that

$$\|u_l' - \tilde{z}_l\| \leq c(\delta) \quad \text{for} \quad -L \leq l \leq 0, \tag{5.8}$$

where $c(\delta) > 0$ goes to zero with δ. Comparing (5.5) and (5.8), we obtain the inequalities

$$\|u_l - u'_l\| \le \frac{\varepsilon}{2} + c(\delta), \quad -L \le l \le 0,$$

which imply (5.4) for sufficiently small $\delta > 0$.

It now remains to prove that the $(L+1)$-tuple $(u'_{-L}, \dots, u'_0)$ coincides with the last $L+1$ components of an element $\boldsymbol{u'} \in \mathbf{A}$, i.e., there are $u'_l \in H$, $l \le -1-L$, such that $(u'_l, l \in \mathbb{Z}_0) \in \mathbf{A}$. However, this assertion follows from the inclusion $u'_{-L} \in A_j$ and definition of A_j.

Thus, $\operatorname{supp} \boldsymbol{\lambda} \subset \mathbf{A}$, so that $\boldsymbol{\lambda} \in \mathcal{P}(\mathbf{H}, \mathbf{A})$. Clearly, different original invariant measures correspond to different invariant measures for (2.14), (2.15) since λ is the projections of $\boldsymbol{\lambda}$. Hence, it remains to check that the family of Markov chains $\boldsymbol{\Theta}^k(\boldsymbol{u})$ has a unique invariant measure $\boldsymbol{\lambda}$.

Step 3. Since (5.3) holds, Theorems 3.1 and 3.2 apply. Therefore, due to Corollary 3.4, it suffices to show that Equation (3.11) has a unique invariant measure μ supported by $\mathfrak{A}$. Then the measure $\boldsymbol{\lambda}$ is its image under the map Φ and is unique.

Step 4. By Theorem 4.1, to prove the uniqueness of an invariant measure $\mu \in \mathcal{P}(\mathfrak{A})$ for (3.11), it is sufficient to check that the transition function $\mathfrak{P}(k, \boldsymbol{U}, \Gamma)$ satisfies conditions (H_1) and (H_2).

5.2 Checking condition $(\mathbf{H_1})$

Recall that the space $\mathfrak{A}$ is endowed with the metric $d_\varkappa$ (see (3.1)) and that the topology defined on $\mathfrak{A}$ by $d_\varkappa$ coincides with the Tikhonov topology for any $\varkappa > 0$.

Proposition 5.1. *Assume that conditions* (A) – (D) *hold. Then the transition function* $\mathfrak{P}(k, \boldsymbol{U}, \Gamma)$ *satisfies condition* (H_1).

Proof. For a metric space X and an integer $k \ge 1$, we denote by X^k the direct product of k copies of X and endow it with the natural direct product metric.

Let $\mathcal{R}$ be the set of functions $f(\boldsymbol{U}) \in \mathbf{C}_b(\mathfrak{A})$ for which there is an integer $m \ge 0$ and a continuous function

$$F(v_{-m}, w_{-m}, \dots, v_0, w_0) \in \mathbf{C}_b\big((H_N \times H_N^\perp)^{m+1}\big) \tag{5.9}$$

such that

$$f(\boldsymbol{U}) = F(v_{-m}, \dots, w_0) \quad \text{for} \quad \boldsymbol{U} = \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{w} \end{pmatrix} = \begin{pmatrix} (v_l, l \le 0) \\ (w_l, l \le 0) \end{pmatrix} \in \mathfrak{A}. \tag{5.10}$$

Thus, $\mathcal{R}$ is the set of continuous functions on $\mathfrak{A}$ that depend on finitely many "coordinates." Clearly, $\mathcal{R}$ is invariant with respect to addition of a constant function and, moreover, it is a determining family for $\mathcal{P}(\mathfrak{A})$ because the Borel σ-algebra on $\mathfrak{A}$ is generated by the σ-algebras $\mathcal{B}_m(\mathfrak{A})$, $m \ge 0$, where, by definition, $\mathcal{B}_m(\mathfrak{A})$ consists of the sets of the form

$$\big\{\boldsymbol{U} \in \mathfrak{A} : (v_{-m}, w_{-m}, \dots, v_0, w_0) \in \Gamma\big\}, \quad \Gamma \in \mathcal{B}\big((H_N \times H_N^\perp)^{m+1}\big).$$

We now prove that, for any $f \in \mathcal{R}$, the family $\{\mathfrak{P}_k f, k \geq 0\}$ is uniformly equicontinuous. Since $\mathfrak{P}_k f \in \mathbf{C}_b(\mathfrak{A})$ for any $k \geq 0$ and $\mathfrak{A}$ is a compact space (in the Tikhonov topology), each of the functions $\mathfrak{P}_k f$ is uniformly continuous. Therefore it suffices to show that the family $\{\mathfrak{P}_k f, k \geq m+1\}$ is uniformly equicontinuous, where m is the integer in (5.9).

Denote by ν_N and $\nu_N^\perp$ the distributions of the random variables $\varphi_k = \mathsf{P}_N \eta_k$ and $\psi_k = \mathsf{Q}_N \eta_k$, respectively. It follows from condition (D) that $\nu_N \in \mathcal{P}(H_N)$ is absolutely continuous with respect to the Lebesgue measure on $H_N = \mathbb{R}^N$ and that the corresponding density has the form

$$D(\alpha) = \prod_{j=1}^N b_j^{-1} p_j(b_j^{-1}\alpha_j), \quad \alpha = (\alpha_1, \dots, \alpha_N) \in \mathbb{R}^N,$$

where $p_j(r)$ is the density of π_j (see condition (D)) and $b_j > 0$ are the constants in (2.4). Note that $D(\alpha)$ is Lipschitz continuous. It follows from (1.20) and (1.22) that if $f(\boldsymbol{U})$ is given by (5.10), then

$$\mathfrak{P}_k f(\boldsymbol{U}) = \int_{B_{R,b}^k} D_k(\boldsymbol{U}, \sigma_1, \dots, \sigma_k) F(\sigma_{k-m}, \dots, \sigma_k)\, d\ell(\sigma_1) \cdots d\ell(\sigma_k), \quad (5.11)$$

where $B_{R,b}^k := (B_{R,b})^k$, $B_{R,b} := B_{H_N}(R) \times B_{H_N^\perp}(b)$, $d\ell(\sigma)$ is the measure $d\alpha\, d\nu_N^\perp$ on $B_{R,b}$, and

$$D_k(\boldsymbol{U}; \sigma_1, \dots, \sigma_k) = \prod_{l=1}^k D\bigl(\alpha_l - T_0(\boldsymbol{U}, \sigma_1, \dots, \sigma_{l-1})\bigr). \quad (5.12)$$

Now note that the operator T_0 is defined on $\mathbf{B}_{R,b}$ and therefore formula (5.12) makes sense for any $\boldsymbol{U} \in \mathbf{B}_{R,b}$ and $\sigma_j \in B_{R,b}$, $1 \leq \sigma_j \leq k$.

We claim that $\mathfrak{P}_k f$ is a Lipschitz continuous function for any $k \geq m+1$ and that the corresponding Lipschitz constants are uniformly bounded. Indeed, for any Lipschitz continuous function $G(\boldsymbol{U})$ defined for $\boldsymbol{U} \in \mathbf{B}_{R,b}$, denote by $\operatorname{Lip} G$ and $\operatorname{Lip}_r G$, $r \leq 0$, its Lipschitz constants in $\boldsymbol{U}$ and $h_r = \binom{v_r}{w_r}$, respectively. (See the final paragraph of Section 3.1 for more detailed definition of $\operatorname{Lip}_r G$.) It follows from (5.12) that, for any integer $r \leq 0$,

$$\operatorname{Lip}_r D_k(\boldsymbol{U}; \sigma_1, \dots, \sigma_k) \leq L \sum_{j=1}^k D_{kj}(\boldsymbol{U}; \sigma_1, \dots, \sigma_k) \operatorname{Lip}_r T_0(\boldsymbol{U}, \sigma_1, \dots, \sigma_j),$$

where L is the Lipschitz constant for $D(\alpha)$ and

$$D_{kj}(\boldsymbol{U}; \sigma_1, \dots, \sigma_k) = \prod_{j \neq l=1}^k D\bigl(\alpha_l - T_0(\boldsymbol{U}, \sigma_1, \dots, \sigma_{l-1})\bigr).$$

Taking into account (2.1), (3.8), and (3.12), we conclude that

$$\operatorname{Lip}_r T_0(\boldsymbol{U}, \sigma_1, \dots, \sigma_j) \leq C_1 e^{\varkappa(r-j)},$$

where $C_1 > 0$ is a constant not depending on k, r, and $\boldsymbol{U}$. Since $d\ell = d\alpha\, d\nu_N^\perp$ and $D(\alpha)\, d\alpha$ and $d\nu_N^\perp$ are probability measures, we have

$$\int_{B_{R,b}^k} D_{kj}(\boldsymbol{U};\sigma_1,\dots,\sigma_k)\, d\ell(\sigma_1)\cdots d\ell(\sigma_k) \le C_2.$$

Therefore,

$$\int_{B_{R,b}^k} \operatorname{Lip}_r D_k(\boldsymbol{U};\sigma_1,\dots,\sigma_k)\, d\ell(\sigma_1)\cdots d\ell(\sigma_k) \le C_3 L\, e^{\varkappa r}, \quad r \le 0. \tag{5.13}$$

Using (5.11) and (5.13), we obtain

$$\begin{aligned}\operatorname{Lip}_r \mathfrak{P}_k f &\le \int_{B_{R,b}^k} \operatorname{Lip}_r\bigl(D_k(\boldsymbol{U},\sigma_1,\dots,\sigma_k)\bigr)F(\sigma_{k-m},\dots,\sigma_k)\, d\ell(\sigma_1)\cdots d\ell(\sigma_k)\\ &\le C_3 L\|F\|_\infty e^{\varkappa r} \le C_4 e^{\varkappa r}.\end{aligned}$$

Hence,

$$\operatorname{Lip} \mathfrak{P}_k f \le \sum_{r=-\infty}^{0} \operatorname{Lip}_r \mathfrak{P}_k f \le C_4 \sum_{r=-\infty}^{0} e^{\varkappa r} = C_4(1-e^{-\varkappa}).$$

Thus, the functions $\mathfrak{P}_k f$, $k \ge m+1$, are uniformly Lipschitz, and (H_1) follows. $\square$

Remark 5.2. Let us show that $\mathfrak{P}(k,\boldsymbol{U},\Gamma)$ is a continuous functions of $\boldsymbol{U} \in \mathfrak{A}$ if the Borel set $\Gamma \subset \mathfrak{A}$ depends on the last k coordinates, i. e., there is $\widetilde{\Gamma} \in \mathcal{B}\bigl((H_N \times H_N^\perp)^k\bigr)$ such that

$$\Gamma = \bigl\{\boldsymbol{V} = (V_l, l \le 0) \in \mathfrak{A} : (V_{1-k},\dots,V_0) \in \widetilde{\Gamma}\bigr\}.$$

Indeed, according to (5.11), we have

$$\mathfrak{P}(k,\boldsymbol{U},\Gamma) = \int_{\widetilde{\Gamma}} D_k(\boldsymbol{U},\sigma_1,\dots,\sigma_k)\, d\ell(\sigma_1)\cdots d\ell(\sigma_k),$$

and the required assertion follows from the continuity of the integrand as a function of $\boldsymbol{U}$ and the dominated convergence theorem.

It is not difficult to see, however, that the transition function $\mathfrak{P}(k,\boldsymbol{U},\Gamma)$ does not possess the strong Feller property. More exactly, assume that a Borel set $\Gamma \subset \mathfrak{A}$ has the form

$$\Gamma = \bigl\{\boldsymbol{V} = (V_l, l \le 0) \in \mathfrak{A} : (V_l, l \le -k) \in \widetilde{\Gamma}\bigr\},$$

where $\widetilde{\Gamma} \in \mathcal{B}\bigl((H_N \times H_N^\perp)^{\mathbb{Z}_{-k}}\bigr)$. In this case

$$\mathfrak{P}(k,\boldsymbol{U},\Gamma) = \mathbb{P}\bigl\{\boldsymbol{\Upsilon}^k(\boldsymbol{U}) \in \Gamma\bigr\} = \mathbb{P}\bigl\{\bigl(\boldsymbol{U},\Upsilon_{1-k}^k(\boldsymbol{U}),\dots,\Upsilon_0^k(\boldsymbol{U})\bigr) \in \Gamma\bigr\} = \chi_{\widetilde{\Gamma}}(\boldsymbol{U}),$$

where $\chi_{\widetilde{\Gamma}}$ is the characteristic function of the set $\widetilde{\Gamma}$. Hence, the function $\mathfrak{P}(k,\boldsymbol{U},\Gamma)$ is not continuous unless $\Gamma = \varnothing$ or $\Gamma = \mathfrak{A}$.

5.3 Checking condition $(\mathbf{H}_2)$

Recall that the space $\mathfrak{A}$ is endowed with metric $d_\varkappa$, $\varkappa > 0$ (see (3.1)).

Proposition 5.3. *Assume that conditions* (A) – (D) *hold. Then for any* $r > 0$ *there is an integer* $l > 0$ *and a constant* $\varepsilon > 0$ *such that*

$$\mathfrak{P}\big(l, \boldsymbol{U}, B_{\mathfrak{A}}(\boldsymbol{a}, r)\big) \geq \varepsilon \quad \textit{for any} \quad \boldsymbol{U}, \boldsymbol{a} \in \mathfrak{A}. \tag{5.14}$$

Proof. We first outline the main ideas. Since $\varPhi : \mathfrak{A} \to \mathbf{A}$ is a uniformly Lipschitz homeomorphism, it follows from (3.16) that Proposition 5.3 will be proved if we establish a similar assertion for $\mathbf{P}(k, \boldsymbol{u}, \Gamma)$: for any $r > 0$ there is an integer $l > 0$ and a constant $\varepsilon > 0$ such that

$$\mathbf{P}\big(l, \boldsymbol{u}, B_{\mathbf{A}}(\boldsymbol{a}, r)\big) \geq \varepsilon \quad \text{for any} \quad \boldsymbol{u}, \boldsymbol{a} \in \mathbf{A}. \tag{5.15}$$

The proof of (5.14) is based on the two observations below.

- With positive probability, the random variable $\boldsymbol{\Theta}^k(\boldsymbol{u})$ belongs to an arbitrarily small ball centred at $\mathbf{0}$ if k is sufficiently large. More exactly, for any $\delta > 0$ there is $\varepsilon_1 > 0$ and an integer $L_1 > 0$ such that

$$\mathbf{P}\big(L_1, \boldsymbol{u}, B_{\mathbf{A}}(\delta)\big) \geq \varepsilon_1 \quad \text{for any} \quad \boldsymbol{u} \in \mathbf{A}. \tag{5.16}$$

- With positive probability, the random variable $\boldsymbol{\Theta}^k(\boldsymbol{u})$ belongs to an arbitrarily small ball centred at $\boldsymbol{a} \in \mathbf{A}$ if the initial point $\boldsymbol{u}$ is sufficiently close to zero. More exactly, for any $r > 0$ there are $\varepsilon_2 > 0$ and $\delta > 0$ and an integer $L_2 > 0$ such that

$$\mathbf{P}\big(L_2, \boldsymbol{u}, B_{\mathbf{A}}(\boldsymbol{a}, r)\big) \geq \varepsilon_2 \quad \text{for any} \quad \boldsymbol{u} \in B_{\mathbf{A}}(\delta) \quad \text{and} \quad \boldsymbol{a} \in \mathbf{A}. \tag{5.17}$$

The proof of the first assertion is based on the dissipativity of the operator S (see inequality (2.2)) and the fact that the random variables η_k take small values with positive probability, while the second assertion follows from the definition of the set of attainability and the "continuous dependence" of the Markov chain $\boldsymbol{\Theta}^k(\boldsymbol{u})$ on the initial point. If (5.16) and (5.17) are proved, then the required inequality (5.15) with $l = L_1 + L_2$ and $\varepsilon = \varepsilon_1\varepsilon_2$ is easily implied by the Chapman–Kolmogorov equation. Indeed,

$$\begin{aligned}
\mathbf{P}\big(L_1 + L_2, \boldsymbol{u}, B_{\mathbf{A}}(\boldsymbol{a}, r)\big) &= \int_{\mathbf{A}} \mathbf{P}\big(L_1, \boldsymbol{u}, d\boldsymbol{v}\big)\mathbf{P}\big(L_2, \boldsymbol{v}, B_{\mathbf{A}}(\boldsymbol{a}, r)\big) \\
&\geq \int_{B_{\mathbf{A}}(\delta)} \mathbf{P}\big(L_1, \boldsymbol{u}, d\boldsymbol{v}\big)\mathbf{P}\big(L_2, \boldsymbol{v}, B_{\mathbf{A}}(\boldsymbol{a}, r)\big) \\
&\geq \mathbf{P}\big(L_1, \boldsymbol{u}, B_{\mathbf{A}}(\delta)\big) \inf_{\boldsymbol{v} \in B_{\mathbf{A}}(\delta)} \mathbf{P}\big(L_2, \boldsymbol{v}, B_{\mathbf{A}}(\boldsymbol{a}, r)\big) \\
&\geq \varepsilon_1\varepsilon_2.
\end{aligned} \tag{5.18}$$

Let us now turn to the accurate proof.

Step 1. We first check (5.16). To this end, we note that, with probability 1, we have

$$\boldsymbol{\Theta}^k(\boldsymbol{u}) = \bigl(\boldsymbol{u}, \Theta^1(u), \dots, \Theta^k(u)\bigr), \tag{5.19}$$

where $\boldsymbol{u} \in \mathbf{H}$ and $u = u_0$ is the zeroth component of $\boldsymbol{u}$. In view of the definition of the Tikhonov topology, inequality (5.17) will be proved once we show that for any $\delta_1 > 0$ and any integer $l \geq 0$ there is $\varepsilon_1 > 0$ and an integer $L_1 \geq l$ such that

$$\mathbb{P}\bigl\{\bigl\|\Theta^j(u)\bigr\| \leq 2\delta_1 \text{ for } L_1 - l \leq j \leq L_1\bigr\} \geq \varepsilon_1, \tag{5.20}$$

where u is an arbitrary element of A. Note that $S^k(u) \in A \subset B_H(R)$ for any $k \geq 0$ if $R > 0$ is sufficiently large. By condition (B), for given $r = \delta_1$ and $R > 0$, there are $n_0 \geq 1$ and a, $0 < a < 1$, such that (2.2) holds. Denote by $K \geq 1$ the smallest integer for which $a^k R < r$. Iterating K times inequality (2.2), we obtain

$$\|S^k(u)\| \leq r = \delta_1 \quad \text{for} \quad k \geq K n_0. \tag{5.21}$$

We claim that (5.20) holds for $L_1 = K n_0 + l$. Indeed, define a controllable system $\Theta^k(u; \xi_1, \dots, \xi_k)$ by the formulas (cf. (1.8), (5.1), (5.2))

$$\Theta^0(u) = u, \quad \Theta^k(u; \xi_1, \dots, \xi_k) = T\bigl(\Theta^{k-1}(u; \xi_1, \dots, \xi_{k-1})\bigr) + \xi_k,$$

where $\xi_j \in \operatorname{supp} \nu$. It follows from (5.21) and the continuity of S that if

$$\|\xi_k\| \leq \delta_2 \quad \text{for} \quad 1 \leq k \leq L_1, \tag{5.22}$$

where $\delta_2 > 0$ is sufficiently small, then

$$\bigl\|\Theta^j(u; \xi_1, \dots, \xi_k)\bigr\| \leq 2r = 2\delta_1 \quad \text{for} \quad L_1 - l \leq j \leq L_1. \tag{5.23}$$

Let

$$\Omega_k = \bigl\{\omega \in \Omega : \|\eta_k\| \leq \delta_2\bigr\}, \quad \Omega^{L_1} = \bigcap_{k=1}^{L_1} \Omega_k. \tag{5.24}$$

Condition (D) implies that $\mathbb{P}(\Omega_k) \geq p_0$ for any k and, therefore, in view of the independence, $\mathbb{P}(\Omega^{L_1}) \geq p_0^{L_1} = \varepsilon_1$. (See Lemma 5.4 below for a stronger result.) Now note that $\Theta^k(u) = \Theta^k(u; \eta_1, \dots, \eta_k)$. By virtue of (5.22) – (5.24), the event in braces in (5.20) contains Ω^{L_1}, and hence its probability is no less that ε_1.

Step 2. We now prove inequality (5.17). To this end, we need two auxiliary assertions.

Lemma 5.4. *For any $\rho > 0$ and any integer $M \geq 1$ there is $p_0 = p_0(\rho, M) > 0$ such that*

$$\mathbb{P}\bigl\{\|\eta_j - x_j\| < \rho, \, 1 \leq j \leq M\bigr\} \geq p_0 \tag{5.25}$$

uniformly in $x_1, \dots, x_M \in \operatorname{supp} \nu$.

Proof. Denote by $h(x)$, $x = (x_1, \dots, x_M) \in H^M = \prod_{j=1}^M$, the right-hand side of (5.25). It follows from the Fatou lemma that $h(x)$ is a lower semi-continuous function of x, i. e.,

$$\liminf_{x \to x^0} h(x) \geq h(x^0) \quad \text{for any} \quad x^0 \in H^M.$$

Since $h(x)$ is positive on the compact set $K = \prod_{j=1}^M \operatorname{supp} \nu$, it attains its positive minimum on K. Hence, (5.25) holds with a positive p_0. □

Denote by $\mathbf{A}_k$ the set of attainability from zero by the time k for Equation (3.11) (see (1.6)).

Lemma 5.5. *For any $r > 0$ there is an integer $k \geq 0$ such that $\mathbf{A}$ is contained in the r-neighbourhood of $\mathbf{A}_k$, i. e., for any $\boldsymbol{a} \in \mathbf{A}$ there exists $\boldsymbol{a}_k \in \mathbf{A}_k$ such that $\boldsymbol{a}_k \in B_{\mathbf{A}}(r, \boldsymbol{a})$.*

Proof. Since $\mathbf{A}$ is the closure of $\bigcup_{j=0}^\infty \mathbf{A}_j$, for any $r > 0$ we have

$$\mathbf{A} \subset \bigcup_{j=0}^\infty \mathbf{O}_j,$$

where $\mathbf{O}_j$ is the open r-neighbourhood of $\mathbf{A}_j$ in $\mathbf{H}$. Thus, we have an open covering of the compact set $\mathbf{A}$. Therefore there exists a finite subcovering. It remains to note that the sets $\mathbf{O}_j$ form an increasing sequence, and hence $\mathbf{A} \subset \mathbf{O}_k$ for some $k \geq 1$. □

We now fix arbitrary $r > 0$ and $\boldsymbol{a} \in \mathbf{A}$. By Lemma 5.5, there is an integer $k \geq 0$ not depending on $\boldsymbol{a}$ and an element $\boldsymbol{a}_k \in \mathbf{A}_k$ such that $d_\varkappa(\boldsymbol{a}, \boldsymbol{a}_k) \leq r/2$. Since $B_{\mathbf{A}}(\boldsymbol{a}_k, r/2) \subset B_{\mathbf{A}}(\boldsymbol{a}, r)$, we can assume from the very beginning that $\boldsymbol{a} \in \mathbf{A}_k$. We claim that (5.17) holds with $L_2 = k$.

Indeed, we must check that

$$\mathbb{P}\big\{\boldsymbol{\Theta}^k(\boldsymbol{u}) \in B_{\mathbf{A}}(\boldsymbol{a}, r)\big\} \geq \varepsilon_2 \tag{5.26}$$

if $\boldsymbol{a} \in \mathbf{A}_k$ and $\boldsymbol{u} \in B_{\mathbf{A}}(\delta)$ for a sufficiently small $\delta > 0$. Define a controllable system $\boldsymbol{\Theta}^k(\boldsymbol{u}; \xi_1, \dots, \xi_k)$ by formulas (cf. (1.8), (2.14), (2.15))

$$\boldsymbol{\Theta}^0(\boldsymbol{u}) = \boldsymbol{u}, \quad \boldsymbol{\Theta}^k(\boldsymbol{u}; \xi_1, \dots, \xi_k) = \mathbf{S}\big(\boldsymbol{\Theta}^{k-1}(\boldsymbol{u}; \xi_1, \dots, \xi_{k-1})\big) + \boldsymbol{\xi}_0^k,$$

where $\boldsymbol{u} \in \mathbf{A}$, $\xi_j \in \operatorname{supp} \nu$, $\boldsymbol{\xi}_0^k = (\dots, 0, 0, \xi_k)$ and $\mathbf{S}$ is given by (2.16). Since $\boldsymbol{a} \in \mathbf{A}_k$, there are $\xi_j^0 \in \operatorname{supp} \nu$, $j = 1, \dots, k$, such that

$$\boldsymbol{\Theta}^k(\mathbf{0}; \xi_1^0, \dots, \xi_k^0) = \boldsymbol{a}.$$

It follows from continuity of $\mathbf{S}$ that

$$\boldsymbol{\Theta}^k(\boldsymbol{u}; \xi_1, \dots, \xi_k) \in B_{\mathbf{A}}(\boldsymbol{a}, r)$$

if

$$d_\varkappa(\boldsymbol{u}, \mathbf{0}) < \delta, \quad \|\xi_j - \xi_j^0\| < \delta, \quad j = 1, \dots, k,$$

where δ is sufficiently small. Therefore,

$$\mathbb{P}\{\boldsymbol{\Theta}^k(\boldsymbol{u}) \in B_{\mathbf{A}}(\boldsymbol{a}, r)\} \geq \mathbb{P}\{\|\eta_j - \xi_j^0\| < \delta,\ j = 1, \dots, k\}. \tag{5.27}$$

It remains to note that, in view of Lemma 5.4, the right-hand side of (5.27) is bounded from below by a constant not depending on $\xi_j^0 \in \operatorname{supp} \nu$, $j = 1, \dots, k$. The proof of Proposition 5.3 is complete. $\square$

6 Ergodic properties of the invariant measure

6.1 Support of the invariant measure

Theorem 6.1. *Let the conditions of Theorem 2.2 hold and let $\lambda \in \mathcal{P}(H, A)$ be the invariant measure for P_k^*. Then* $\operatorname{supp} \lambda = A$.

Proof. Let $\boldsymbol{\lambda} \in \mathcal{P}(\mathbf{H}, \mathbf{A})$ and $\mu \in \mathcal{P}(\mathfrak{A})$ be the invariant measures for the semigroups $\mathbf{P}_k^*$ and $\mathfrak{P}_k^*$, respectively (see Subsections 2.2.2 and 3.2). By Theorem 4.2, we have $\operatorname{supp} \mu = \mathfrak{A}$. According to Step 3 in Section 5.1, we have

$$\operatorname{supp} \boldsymbol{\lambda} = \varPhi(\operatorname{supp} \mu) = \varPhi(\mathfrak{A}) = \mathbf{A}.$$

Now note that the projection

$$\pi_0 : \mathbf{H} \to H, \quad \boldsymbol{u} = (u_l, l \in \mathbb{Z}_0) \mapsto u_0,$$

maps the measure $\boldsymbol{\lambda}$ to λ. Therefore,

$$\pi_0(\operatorname{supp} \boldsymbol{\lambda}) = \pi_0(\mathbf{A}) = \operatorname{supp} \lambda.$$

Thus, the theorem will be proved once we show that $\pi_0(\mathbf{A}) = A$, i. e., for any $u \in A$ there is $\boldsymbol{u} \in \mathbf{A}$ such that $\pi_0(\boldsymbol{u}) = u$. This assertion is obvious if $u \in A_k$ and can easily be proved with the help of approximation of a given element $u \in A$ by a sequence $u^k \in A_k$. $\square$

6.2 Convergence to the invariant measure and mixing

Theorem 6.2. *Let the conditions of Theorem 2.2 hold and let $f \in \mathbf{C}(H)$. Then*

$$P_k f(u) \to (\lambda, f) \quad \text{as} \quad k \to \infty \tag{6.1}$$

for any $u \in A$. Moreover, the convergence is uniform in $u \in A$.

Proof. First note that the measure $P(k, u, \cdot)$ is supported by the set of attainability A for any $k \geq 0$ and $u \in A$, and therefore we can redefine the function f outside A without changing $P_k f(u)$ for $u \in A$. Since f is uniformly bounded on the compact set A, we can assume that $f \in \mathbf{C}_b(H)$.

Given $f \in \mathbf{C}_b(H)$, we define a function $\boldsymbol{f} \in \mathbf{C}_b(\mathbf{H})$ by the formula

$$\boldsymbol{f}(\boldsymbol{u}) = f(u_0), \quad \boldsymbol{u} \in \mathbf{H},$$

where u_0 is the zeroth component of $\boldsymbol{u}$. Let $\lambda \in \mathcal{P}(H, A)$ be an invariant measure for P_k^* and let $\boldsymbol{\lambda} \in \mathcal{P}(\mathbf{H}, \mathbf{A})$ be the corresponding invariant measure for $\mathbf{P}_k^*$ (see Step 2 in Section 5.1). Since the projection $\boldsymbol{u} = (\dots, u_{-1}, u_0) \mapsto u_0$ sends the measure $\boldsymbol{\lambda}$ to λ, we have

$$(\lambda, f) = (\boldsymbol{\lambda}, \boldsymbol{f}).$$

As was shown in the proof of Theorem 6.1, for any $u \in A$ there is $\boldsymbol{u} \in \mathbf{A}$ such that $u_0 = u$. Under this choice of $\boldsymbol{u} \in \mathbf{A}$, we have

$$\mathbf{P}_k \boldsymbol{f}(\boldsymbol{u}) = P_k f(u).$$

Therefore, it suffices to prove convergence (6.1) with P_k, f, and u replaced by $\mathbf{P}_k$, $\boldsymbol{f}$, and $\boldsymbol{u}$, respectively.

The map Ψ defined in Theorem 3.2 transforms the measure $\boldsymbol{\lambda}$ to the measure $\mu = \Psi_* \boldsymbol{\lambda}$, which is invariant for the family of Markov chains (3.10), (3.11). Using (3.17), we see that it remains to check that

$$\mathfrak{P}_k g(\boldsymbol{U}) \to (\mu, g) \quad \text{as} \quad k \to \infty \quad \text{uniformly in} \quad \boldsymbol{u} \in \mathfrak{A},$$

where $g(\boldsymbol{U}) = \boldsymbol{f}\big(\Phi(\boldsymbol{U})\big)$. Since Theorem 4.2 applies to the Markov semigroup $\mathfrak{P}_k$ corresponding to (3.10), (3.11), the last convergence follows from (4.5). □

Theorem 6.2 has two important corollaries.

Corollary 6.3. *Let the conditions of Theorem 2.2 hold. Then the invariant measure $\lambda \in \mathcal{P}(H, A)$ is mixing, i. e.,*

$$\int_H P_k f(u) g(u)\, d\lambda(u) \to \int_H f(u)\, d\lambda(u) \int_H g(u)\, d\lambda(u) \quad \text{as} \quad k \to \infty$$

for any two functions $f, g \in \mathbf{C}(H)$. In particular, the measure λ is ergodic in A.

Corollary 6.4. *Let the conditions of Theorem 2.2 hold and let Θ^k be an arbitrary Markov chain in H that satisfies (2.9) for $k \geq 1$ and whose initial distribution λ_0 is supported by A. Then the distribution of Θ^k weakly converges to λ, i. e.,*

$$(P_k^* \lambda_0, f) \to (\lambda, f) \quad \text{as} \quad k \to \infty \quad \text{for any} \quad f \in \mathbf{C}_b(H).$$

7 Application to stochastic dissipative PDE's

7.1 Navier-Stokes equations in a bounded domain

Let $D \subset \mathbb{R}^2$ be a bounded domain with boundary $\partial D \in C^2$. Denote by $\mathcal{V}$ the space of vector functions $u = (u_1, u_2)$, $u_j \in C_0^\infty(D)$, such that $\operatorname{div} u = 0$, by H and V the closure of $\mathcal{V}$ in[11] $L^2(D)$ and $H^1(D)$, respectively, and by Π the orthogonal projection in $L^2(D)$ onto H.

On the domain D, let us consider the system of Navier–Stokes (NS) equations (0.1) with random right-hand side. We write it as a functional equation in H (for instance, see [CF, Chapter 8] or [BV, Section 1.6]):

$$\partial_t u + \delta L u + B(u, u) = \eta(t). \tag{7.1}$$

Here $\delta > 0$ is a parameter, L is the closure in H of the operator $L_0 = -\Pi\Delta$ with domain $\mathcal{V}$, $B(u, u) = \Pi(u, \nabla)u$, and $\eta(t)$ is a random process of the form

$$\eta(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT)\eta_k(x), \tag{7.2}$$

where $T > 0$, $\eta_k(x)$ are H-valued i.i.d. random variables, and $\delta(\cdot)$ is the Dirac measure concentrated at zero. In what follows, to simplify the notation, we shall assume that $T = 1$. Let us define what is meant by a solution of Equation (7.1).

Let $\|\cdot\|$ and $\|\cdot\|_1$ be the norms in the spaces H and V, respectively. For an open interval $I \subset \mathbb{R}$, denote by $L^2(I, V)$ the space of Borel functions $f(t) : I \to V$ such that

$$\|f\|_{L^2(I,V)} := \left(\int_I \|f(t)\|_1^2 dt \right)^{1/2} < \infty$$

and by $C(I, H)$ the space of functions on I with range in H that are extendible to a continuous function $f(t) : \bar{I} \to H$, where $\bar{I}$ is the closure of I.

Definition 7.1. Let m and n be some integers such that $m+1 < n$. A stochastic process $u(t) = u(t, x)$ defined on the interval $[m, n)$, is called a *solution of Equation* (7.1) if the following two properties hold with probability 1:

- For any $k = m+1, \dots, n$, the restriction of $u(t)$ to $I_k := (k-1, k)$ belongs to the space $L^2(I_k, V) \cap C(I_k, H)$ and satisfies the homogeneous equation

$$\partial_t u + \delta L u + B(u, u) = 0. \tag{7.3}$$

- For $k = m+1, \dots, n-1$,

$$u(k+0, x) - u(k-0, x) = \eta_k(x). \tag{7.4}$$

- The function $u(t)$ is continuous from the right at the points $t = m + 1, \dots, n-1$.

The following proposition is a trivial consequence of Definition 7.1.

Proposition 7.2. *Let a stochastic process* $u(t, x)$ *be a solution of* (7.1) *on an interval* $[m, n)$. *Then, with probability* 1, $u(t, x)$ *satisfies Equation* (7.1) *in the sense of distributions.*

[11] We use the same notation for spaces of scalar and vector functions.

Consider now the Cauchy problem for Equation (7.1):

$$u(0,x) = u_0(x), \tag{7.5}$$

where $u_0(x)$ is an H-valued random variable. A stochastic process $u(t,x)$ is called a *solution of the problem* (7.1), (7.5) if it is a solution of Equation (7.1) and relation (7.5) holds with probability 1.

It follows from Definition 7.1 and the classical result on the correctness of the initial-boundary value problem for the 2D Navier–Stokes system (for instance, see [L, CF, BV]) that the problem (7.1), (7.5) has a unique solution $u(t,x)$ defined for all $t \ge 0$. This solution can be constructed in the following way.

Let S_t be the solving semi-group for the Cauchy problem (7.3), (7.5). Thus, $S_t(u_0) = v(t)$, where $v(t) = v(t,x)$ is the solution of (7.3), (7.5). Define a random process $u(t)$ as

$$u(k) = S\big(u(k-1)\big) + \eta_k(x), \qquad k = 1, 2, \dots, \tag{7.6}$$

$$u(k+t) = S_t\big(u(k)\big), \quad 0 \le t < 1, \qquad k = 0, 1, 2, \dots, \tag{7.7}$$

where $S = S_1$. It is easy to see that $u(t)$ is the required solution.

Consider now the sequence $u_k = u(k) \in H$, $k \ge 0$. Since the random variables $u_0, \eta_1, \eta_2, \dots$ are independent, we conclude that $\{u_k\}$ is a Markov chain. Hence, we can define a family of Markov chains by the formulas (cf. (2.8), (2.9))

$$\Theta^0(v) = v, \quad \Theta^k(v) = S(\Theta^{k-1}(v)) + \eta_k, \quad k \ge 1, \tag{7.8}$$

where $v \in H$. Denote by $P_k(v,\Gamma)$, P_k and P_k^* the corresponding transition function and the Markov operators (see (1.3) – (1.5)) and by A the set of attainability from zero for (7.8) (see (1.7)).

Assume that $\eta_k(x)$, $k \in \mathbb{Z}$, have the form (2.6), i. e.,

$$\eta_k = \sum_{j=1}^{\infty} b_j \xi_{jk} e_j(x), \tag{7.9}$$

where $b_j \ge 0$ are some constants satisfying (2.7), $\{\xi_{jk}\}$ is a family of independent random variables for which condition (D) holds, and $\{e_j = e_j(x)\}$ is the complete set of L^2-normalized eigenvectors of the operator L with the corresponding eigenvalues $\{\alpha_j\}$.

Theorem 7.3. *Under the above conditions, the Markov semi-group P_k^* has a unique invariant measure $\lambda \in \mathcal{P}(H,A)$ if (2.10) holds with a sufficiently large $N \ge 1$. This measure is concentrated on the domain of definition $D(L)$ of the operator L if* [12]

$$\sum_{j=1}^{\infty} \alpha_j^2 b_j^2 < \infty. \tag{7.10}$$

[12] Clearly, condition (7.10) implies that $\eta_k \in D(L)$ with probability 1.

Furthermore, the measure λ is mixing, and for any initial distribution $\lambda_0 \in \mathcal{P}(H,A)$ the sequence $P_k^*\lambda_0$ weakly converges to λ. In particular, if $u(t,x)$ is the solution of (7.1) *starting from any point in A (e. g., from zero), then the measures $\mathcal{D}\big(u(k,\cdot)\big)$ tends to λ as $k \to \infty$, $k \in \mathbb{Z}$.*

Proof. In view of Theorems 2.2, 6.1, 6.2 and Corollaries 6.3, 6.4, to prove the existence, uniqueness, and ergodic properties of an invariant measure, it suffices to check that conditions (A) – (C) are satisfied for the operator $S = S_1$.

The uniform Lipschitz property of S on any ball $B_H(R)$ is well known (for instance, see [CF, Chapter 10], [BV, Theorem 1.6.1], or [G, Section 3.2]). It is also a classical result that

$$\big\|S(u)\big\| \le a\|u\| \quad \text{for any} \quad u \in H, \tag{7.11}$$

where $a = e^{-\delta\alpha_1}$. Inequality (7.11) immediately implies (2.2) (with $n_0 = 1$) and condition (B). Finally, to check (C), we note that (for instance, see [BV, Theorem 1.6.2])

$$\big\|S(u_1) - S(u_2)\big\|_1 \le C(R)\|u_1 - u_2\| \quad \text{for any} \quad u_1, u_2 \in B_H(R). \tag{7.12}$$

We also have

$$\|\mathsf{Q}_N v\|^2 = \sum_{j=N+1}^{\infty} |v_j|^2 \le \alpha_{N+1}^{-1} \sum_{j=N+1}^{\infty} \alpha_j |v_j|^2 \le \alpha_{N+1}^{-1}\|v\|_1^2, \tag{7.13}$$

where $v_j = (v, e_j)$ are the Fourier components of v. The required inequality (2.5) with $\gamma_N(R) = C(R)\,\alpha_{N+1}^{-1}$ follows from (7.12) and (7.13).

We now show that the invariant measure $\lambda \in \mathcal{P}(H,A)$ is concentrated on $D(L)$ if (7.10) holds. It is well known that $S(H) \subset D(L)$ and that $D(L)$ is a Borel subset in H. Hence,

$$P\big(1, u, D(L)\big) = \mathbb{P}\big\{S(u) + \eta_1 \in D(L)\big\} = 1 \quad \text{for any} \quad u \in H.$$

It follows that

$$\lambda\big(D(L)\big) = \int_H P\big(1, u, D(L)\big)\, d\lambda(u) = 1,$$

which completes the proof of Theorem 7.3. □

7.2 Navier-Stokes equations on a torus

We now assume that $x \in \mathbb{T}^2$, where $\mathbb{T}^2$ is a two-dimensional torus, and that

$$\int_{\mathbb{T}^2} u(t,x)\, dx = 0, \qquad \int_{\mathbb{T}^2} \eta(t,x)\, dx = 0. \tag{7.14}$$

Let $\mathcal{H}^s$ be the space of divergence-free vector fields on $\mathbb{T}^2$ that belong to the Sobolev space $H^s(\mathbb{T}, \mathbb{R}^2)$ and whose mean value is zero. We fix an arbitrary integer $s \ge 0$ and denote by $\{e_j\}$ the complete set of L_2-normalised eigenvectors of the operator L. As before, S_t stands for the solving semi-group corresponding to the non-forced NS equations. It is well known that S_t is a continuous operator in $\mathcal{H}^s$ for any integer $s \geq 0$. Applying standard arguments (see [BV, CF, FT, L]), it is not difficult to show that the operator $S = S_1$ satisfies conditions (A) – (C). Besides,

$$\big\|S(u)\big\|_{s+k} \leq C_k\big(\|u\|_s\big) \quad \text{for any} \quad k \geq 0. \tag{7.15}$$

We assume that the forcing $\eta(t)$ has the form (7.9) and is smooth, i. e., the coefficients $b_j \geq 0$ satisfy the inequality

$$|b_j| \leq C_m j^{-m} \quad \text{for any} \quad j, m \geq 1, \tag{7.16}$$

where $C_m > 0$ does not depend on j. If condition (D) is satisfied, then Theorems 2.2, 6.1, and 6.2 apply to the space-periodic 2D NS equations in the space $H = \mathcal{H}^s$ provided that

$$b_j > 0 \quad \text{for} \quad j = 1, \dots, N = N(\delta, s). \tag{7.17}$$

By Theorem 2.2, there is a unique invariant measure λ supported by the set of attainability from zero in the space $\mathcal{H}^s$. It follows from (7.15) and (7.16) that the measure λ is concentrated on infinitely smooth functions.

Let $u(t,x)$ be a solution of (7.1), (7.2) (with $x \in \mathbb{T}^2$) such that

$$u(0,x) = 0.$$

The sequence of distributions of the random variables $u(k) \in \mathcal{H}^s$ weakly converges to λ. Hence,

$$\mathbb{E} f\big(u(k)\big) \to \int_{\mathcal{H}^s} f(u)\, d\lambda(u) \quad \text{as} \quad k \to \infty$$

for any nonlinear continuous functional f on $\mathcal{H}^s$.

7.3 A nonlinear Schrödinger equation on a torus

Consider the Schrödinger equation

$$\dot u = (\Delta - 1)u + i|u|^2 u + \eta(t), \quad x \in \mathbb{T}^n, \tag{7.18}$$

where $u = u(t,x)$ is an unknown complex-valued function, $\mathbb{T}^n$ is an n-dimensional torus, and $\eta(t)$ is a random process of the form (7.2) with $T = 1$. We regard (7.18) as a system of two equations for the real and imaginary parts of $u(t,x)$.

Assume that the random variables η_k in (7.2) have the form (7.9), where $b_j \geq 0$ are some constants, ξ_{jk} are independent random variables satisfying condition (D), and $\{e_j\}$ is the complete set of eigenvectors (which are pairs of real-valued functions) of the operator $1 - \Delta$ on the torus $\mathbb{T}^n$ with corresponding eigenvalues $\{\alpha_j\}$. It can be proved that if the inequality

$$\sum_{j=1}^{\infty} \alpha_j^s b_j^2 < \infty$$

holds with some integer $s > n/2$, then the Cauchy problem for Equation (7.18) is well-posed in the Sobolev space $H^s = H^s(\mathbb{T}^n, \mathbb{R}^2)$. More precisely, for any random variable $u_0(x)$ with values in H^s the problem (7.18), (7.5) has a unique solution $u(t,x) \in H^s$, $t \geq 0$, given by formulas (7.6), (7.7), where $S = S_1$ and S_t is the solving semi-group for the homogeneous equation.

We now define a family of Markov chains $\Theta^k(v)$, $v \in H^s$, by formulas (7.8). Let $P(k, v, \Gamma)$, P_k, and P_k^* be the transition function and the Markov semi-groups associated with the family $\Theta^k(v)$ and let $A = A^s \subset H^s$ be the corresponding set of attainability from zero. The proof of the following assertion is similar to that of Theorem 7.3.

Theorem 7.4. *Under the above conditions, the Markov semi-group P_k^* has a unique invariant measure $\lambda \in \mathcal{P}(H^s, A)$ if* (2.10) *holds with a sufficiently large $N \geq 1$. The measure λ is mixing, and for any initial distribution $\lambda_0 \in \mathcal{P}(H^s, A)$ the sequence $P_k^* \lambda_0$ weakly converges to λ. In particular, if $u(t,x)$ is the solution of* (7.18) *starting from any point in A (e. g., from zero), then the measures $\mathcal{D}\big(u(k,\cdot)\big)$ tends to λ as $k \to \infty$, $k \in \mathbb{Z}$.*

8 Appendix: proof of Theorem 3.1

The solvability of Equation (3.3) will be proved by the contraction mapping principle. For given $\boldsymbol{v} \in \mathbf{B}_N(R)$ and $\boldsymbol{\psi} \in \mathbf{B}_N^{\perp}(b)$, consider an operator K that is defined on $\mathbf{B}_N^{\perp}(r)$, where r is the constant in (3.5), and maps $\tilde{\boldsymbol{v}} = (\tilde{v}_l, l \leq 0)$ to $\tilde{\boldsymbol{v}}' = (\tilde{v}_l', l \leq 0)$, where

$$\tilde{v}_l' = \mathsf{Q}_N S(v_{l-1} + \tilde{v}_{l-1}) + \psi_l, \quad l \leq 0.$$

It is clear that an element $\tilde{\boldsymbol{v}} \in \mathbf{B}_N^{\perp}(r)$ is a solution of (3.3) if and only if it is a fixed point of K. We claim that for sufficiently large N the operator K maps the set $\mathbf{B}_N^{\perp}(r)$ into itself and is a contraction if $\mathbf{B}_N^{\perp}(r)$ is endowed with the norm $\|\cdot\|_\infty$.

Indeed, in view of inequality (2.5) with $u_2 = 0$, for any $\tilde{\boldsymbol{v}} \in \mathbf{B}_N^{\perp}(r)$, we have

$$\begin{aligned} \big\|K(\tilde{\boldsymbol{v}})\big\|_\infty &\leq \sup_{l \leq 0} \big\|\mathsf{Q}_N S(v_{l-1} + \tilde{v}_{l-1})\big\| + \sup_{l \leq 0} \|\psi_l\| \\ &\leq \gamma_N(\rho) \sup_{l \leq 0} \|v_{l-1} + \tilde{v}_{l-1}\| + b \leq \gamma_N(\rho)(R + r) + b. \end{aligned} \tag{8.1}$$

Choose an integer N such that $\gamma_N(\rho) \leq \gamma$. By (3.5) and (8.1),

$$\big\|K(\tilde{\boldsymbol{v}})\big\|_\infty \leq \gamma(R + r) + b \leq r,$$

which means that K maps the space $\mathbf{B}_N^{\perp}(r)$ into itself.

To prove that K is a contraction, we take arbitrary $\tilde{\boldsymbol{v}}^i = (\tilde{v}_l^i, l \leq 0) \in \mathbf{B}_N^{\perp}(r)$, $i = 1, 2$, and note that, in view of (2.5),

$$\begin{aligned} \big\|K(\tilde{\boldsymbol{v}}^1) - K(\tilde{\boldsymbol{v}}^2)\big\|_\infty &\leq \sup_{l \leq 0} \Big\|\mathsf{Q}_N\big(S(v_{l-1} + \tilde{v}_{l-1}^1) - S(v_{l-1} + \tilde{v}_{l-1}^2)\big)\Big\| \\ &\leq \gamma_N(\rho) \|\tilde{\boldsymbol{v}}^1 - \tilde{\boldsymbol{v}}^2\|_\infty. \end{aligned}$$

It follows that if $\gamma_N(\rho) \leq \gamma < 1$, then K is a contraction and hence has a unique fixed point in $\mathbf{B}_N^{\perp}(r)$.

We now prove (3.7). To this end, choose arbitrary $\boldsymbol{v}^i \in \mathbf{B}_N(R)$ and $\boldsymbol{\psi}^i \in \mathbf{B}_N^{\perp}(b)$, $i = 1, 2$, and set $\tilde{\boldsymbol{v}}^i = \mathcal{W}(\boldsymbol{v}^i, \boldsymbol{\psi}^i)$. In view of (3.3) and (2.5), we have

$$
\begin{aligned}
&\Big\| \mathcal{M}(\varkappa)\big(\mathcal{W}(\boldsymbol{v}^1, \boldsymbol{\psi}^1) - \mathcal{W}(\boldsymbol{v}^2, \boldsymbol{\psi}^2)\big) \Big\|_\infty \\
&\leq \sup_{l \leq 0} e^{\varkappa l} \Big(\big\| \mathsf{Q}_N \big(S(v_{l-1}^1 + \tilde{v}_{l-1}^1) - S(v_{l-1}^2 + \tilde{v}_{l-1}^2) \big) \big\| + \|\psi_l^1 - \psi_l^2\| \Big) \\
&\leq e^{\varkappa} \gamma \sup_{l \leq -1} e^{\varkappa l} \big(\|v_l^1 - v_l^2\| + \|\tilde{v}_l^1 - \tilde{v}_l^2\| \big) + \sup_{l \leq 0} e^{\varkappa l} \|\psi_l^1 - \psi_l^2\| \\
&\leq e^{\varkappa} \gamma \Big(\big\| \mathcal{M}(\varkappa)(\boldsymbol{v}^1 - \boldsymbol{v}^2) \big\|_\infty + \big\| \mathcal{M}(\varkappa)(\tilde{\boldsymbol{v}}^1 - \tilde{\boldsymbol{v}}^2) \big\|_\infty \Big) + \big\| \mathcal{M}(\varkappa)(\boldsymbol{\psi}^1 - \boldsymbol{\psi}^2) \big\|_\infty,
\end{aligned}
$$

whence we derive (3.7). The proof of the theorem is complete.

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