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STOCHASTIC DOMINANCE WITHOUT TRANSITIVE PREFERENCES.(U)
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Traditional definitions of stochastic dominance for decision analysis assume that the decision agent's preference-or-indifference relation on outcomes of risky decisions is transitive. This report proposes a stochastic dominance relation for the comparison of risky decisions that is applicable to any complete and reflexive preference-or-indifference relation, or to any asymmetric preference relation. The new dominance relation possesses a number of intuitively desirable properties and is equivalent to the usual stochastic dominance relation when preferences are transitive.

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TECHNICAL REPORT NO. 24

STOCHASTIC DOMINANCE WITHOUT TRANSITIVE PREFERENCES

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1. Introduction

Stochastic dominance has traditionally been associated with the comparison of probability measures p,q,... defined on the real line when it is presumed that the decision agent likes outcome x as much as outcome y whenever $x \geq y$. In the non-strict first-degree version, p stochastically dominates q just when the graph of the cumulative distribution for p never rises above the graph of the cumulative distribution for q, and this occurs if and only if the expected utility of p is as great as the expected utility of q for every nondecreasing utility function on the line [2, 6, 7, 8, 11, 14]. Less restrictive stochastic dominance concepts have been developed for other classes of utility functions. These include the class of risk averse utility functions [2, 7, 8, 11, 14], the class of risk averse utility functions that exhibit decreasing absolute risk aversion [16, 17], and several other classes [5, 20]. Most of these developments along with various applications of stochastic dominance to investment decisions and other risky choice problems are discussed in [20].

As shown in [2, 4, 6], stochastic dominance concepts also apply to risky decision situations with arbitrary outcomes--qualitative, multiattribute, or whatever--when the decision agent's preference-or-indifference relation R on the outcome set X is a weak order (transitive, reflexive and complete). In this case R replaces the natural order on the real line in defining first-degree stochastic dominance. In other words, if p and q are simple probability measures on X, so that $p(A) = q(B) = 1$ for finite $A,B \subseteq X$, and if we let SD(R) denote the non-strict first-degree stochastic dominance relation based on R, then

$$p \text{ SD}(R)\ q \text{ if and only if } p(\{x\colon yRx\}) \leq q(\{x\colon yRx\}) \text{ for all } y \in X.$$

It follows that p SD(R) q if, and only if, $\Sigma p(x)u(x) \geq \Sigma q(x)u(x)$ for every real valued function u on $A \cup B$ that preserves R in the sense that, for all $x,y \in A \cup B$, $u(x) \geq u(y)$ if and only if xRy.

My aim here is to propose and analyze a generalization of SD(R) that applies to every reflexive and complete preference-or-indifference relation R on X, whether or not it is transitive. This aim is motivated by discussions of intransitive indifference [3, 9, 12] and intransitive preference [1, 10, 13, 15, 18] which suggest that these phenomena can arise very naturally from factors such as sensory thresholds, discriminatory vagueness, and multiattribute outcomes. The problem that these phenomena pose for the stochastic dominance approach to the analysis of risky decisions can be put in the form of a question: Is there any reasonable or defensible notion of stochastic dominance for situations in which the decision agent's binary preferences on outcomes are not transitive? The answer given in this paper is a qualified "yes".

The basiç proposal for stochastic dominance with "unordered" preferences is presented in the next section along with an interpretation of its content. The third section then shows that the new stochastic dominance relation satisfies a number of conditions that seem like reasonable requirements for a dominance relation in the unordered context. The paper concludes with a brief summary.

2. Definitions

Throughout the paper, X is the outcome set, Π is the set of all simple probability measures on X, and $\mathcal{R}$ is the set of reflexive and complete binary relations on X. The indifference relation I and strict preference relation P that are induced by $R \in \mathcal{R}$ are defined by xIy iff (if and only if) xRy and yRx, and xPy iff xRy and not (yRx). We shall let $Px = \{y \in X: yPx\}$ and $xP = \{y \in X: xPy\}$. Hence Px is the set of outcomes preferred to x, and xP is the set of outcomes that x is preferred to. In like manner, $Rx = \{y \in X: yRx\}$, $xR = \{y \in X: xRy\}$, and $xI = \{y \in X: xIy\}$. Thus p(Rx) is the probability that a risky decision whose outcome probabilities are given by p will yield an outcome that is preferred or indifferent to x.

Our primary stochastic dominance relation D(R) on Π is based on two other relations on Π for each $R \in \mathcal{R}$. These are defined as follows for all $p,q \in \Pi$:

$$p\ D_1(R)\ q \text{ iff } p\{x: q(xP) \leq \lambda\} \leq q\{x: p(xP) \leq \lambda\} \text{ for all } \lambda \in [0,1]$$

$$p\ D_2(R)\ q \text{ iff } p\{x: q(Px) \geq \lambda\} \leq q\{x: p(Px) \geq \lambda\} \text{ for all } \lambda \in [0,1],$$

where $p\{x: \ldots\}$ is short for $p(\{x: \ldots\})$. These definitions can also be used for nonsimple probability measures when the probabilities are well defined. Since $q(xP) + q(Rx) = 1$, $p\{x: q(xP) \leq \lambda\} + p\{x: q(xP) > \lambda\} = 1$, and so forth, it follows easily that for all $p,q \in \Pi$,

$$p\ D_1(R)\ q \text{ iff } p\{x: q(Rx) \geq \lambda\} \leq q\{x: p(Rx) \geq \lambda\} \text{ for all } \lambda \in [0,1]$$

$$\text{iff } p\{x: q(xP) \geq \lambda\} \geq q\{x: p(xP) \geq \lambda\} \text{ for all } \lambda \in [0,1]$$

and

$$p\ D_2(R)\ q \text{ iff } p\{x: q(xR) \leq \lambda\} \leq q\{x: p(xR) \leq \lambda\} \text{ for all } \lambda \in [0,1]$$

$$\text{iff } p\{x: q(Px) \leq \lambda\} \geq q\{x: p(Px) \leq \lambda\} \text{ for all } \lambda \in [0,1].$$

It will be shown in the next section that each of D_1 and D_2 has a number of properties that we would desire for a stochastic dominance relation, including the fact that $D_1(R) = D_2(R) = SD(R)$ when R is a weak order. However, since--as will be shown shortly--$D_1(R)$ and $D_2(R)$ are not generally equal, and since both seem consistent with the general theme of stochastic dominance, I shall propose the intersection of the two as the primary dominance relation for "unordered" preferences:

$$p\ D(R)\ q \text{ iff } p\ D_1(R)\ q \text{ and } p\ D_2(R)\ q.$$

One could also consider the union of $D_1(R)$ and $D_2(R)$ as a viable candidate for stochastic dominance in the present context although I find this less intuitively appealing than the intersection proposal. Since $D_1(R) \cap D_2(R) \subseteq D_i(R) \subseteq D_1(R) \cup D_2(R)$ for $i = 1,2$, $D(R)$ is the most demanding relation in this little hierarchy.

To appreciate the spirit of $D(R)$ it is necessary to interpret $D_1(R)$ and $D_2(R)$. We examine $D_1(R)$ first. From its definition and equivalent characterizations, it is clear that $D_1(R)$ depends on interrelationships between p and q in a way that is not evident from the definition of $SD(R)$ for ordered outcomes. Consider the third characterization of $D_1(R)$, where we have

$$p\{x: q(xP) \geq \lambda\} \text{ and } q\{x: p(xP) \geq \lambda\},$$

and let A and B be respectively the minimal supports of p and q so that $p(A) = q(B) = 1$ with $p(a) > 0$ for all $a \in A$ and $q(b) > 0$ for all $b \in B$. Then

$$p\{x: q(xP) \geq \lambda\} = p\{a \in A: q(aP) \geq \lambda\}$$

$$q\{x: p(xP) \geq \lambda\} = q\{b \in B: p(bP) \geq \lambda\}.$$

Since these equal 1 when $\lambda = 0$, consider a fixed positive λ. Then $q(aP) \geq \lambda$ means that q has a probability of at least λ of yielding an outcome that is worse than $a \in A$; hence the higher the value of $p\{a \in A: q(aP) \geq \lambda\}$ the more attractive is p relative to q at level λ. But this is only half of the picture. In the other half, $p(bP) \geq \lambda$ means that p has a probability of at least λ of yielding an outcome that is worse than $b \in B$, so that the higher the value of $q\{b \in B: p(bP) \geq \lambda\}$ the more attractive is q relative to p at level λ. If the difference $p\{a \in A: q(aP) \geq \lambda\} - q\{b \in B: p(bP) \geq \lambda\}$ is nonnegative, then we might say that p is at least as attractive as q at level λ, and this is precisely what is required for every $\lambda \in [0,1]$ to have $p\ D_1(R)\ q$.

In the third characterization of $D_2(R)$ with A and B as in the preceding paragraph we have

$$p\{x: q(Px) \leq \lambda\} = p\{a \in A: q(Pa) \leq \lambda\}$$

$$q\{x: p(Px) \leq \lambda\} = q\{b \in B: p(Pb) \leq \lambda\}.$$

Here $q(Pa) \leq \lambda$ means that q has a probability of no more than λ of yielding an outcome that is preferred to $a \in A$, and $p\{a \in A: q(Pa) \leq \lambda\}$ is the probability that p will produce such an outcome. Similarly, $p(Pb) \leq \lambda$ means that p has a probability of no more than λ of yielding an outcome that is preferred to $b \in B$, and $q\{b \in B: p(Pb) \leq \lambda\}$ is the probability that q will produce such an outcome. As before, $p\{a \in A: q(Pa) \leq \lambda\} - q\{b \in B: p(Pb) \leq \lambda\}$ is a measure of the attractiveness of p relative to q at level λ, and this must be nonnegative for all $\lambda \in [0,1]$ to have $p\ D_2(R)\ q$.

Given $p(A) = q(B) = 1$, the preceding discussion shows that it is only necessary to examine the λ values where $q(aP) = \lambda$ or $p(bP) = \lambda$ in order to check $D_1(R)$ for p versus q. In like manner, $D_2(R)$ for p versus q is completely determined by what happens at the λ values for which $q(Pa) = \lambda$ or $p(Pb) = \lambda$. This will be illustrated in the proof of the following lemma.

Lemma 1. <u>If</u> P <u>is not transitive then</u> $D_1(R) \neq D_2(R)$.

Proof. Suppose P is not transitive on X. Then there must be distinct $x_1, x_2, x_3 \in X$ such that either $x_1 P x_2 P x_3 P x_1$ (a cycle) or $x_1 P x_2 P x_3 I x_1$. With $p_i = p(x_i)$ and $q_i = q(x_i)$, let $p = (p_1, p_2, p_3) = (.3, .5, .2)$ and $q = (q_1, q_2, q_3) = (.1, .6, .3)$, and suppose that $x_1 P x_2 P x_3 I x_1$. Then, to examine $D_1(R)$, we need to look at $p\{x: q(xP) \leq \lambda\}$ and $q\{x: p(xP) \leq \lambda\}$ at the $\lambda \in \{p(x_i P), q(x_i P)\}$ for $i = 1,2,3$. These λ values are $0 = p(x_3 P) = q(x_3 P)$, $.2 = p(x_2 P)$, $.3 = q(x_2 P)$, $.5 = p(x_1 P)$, and $.6 = q(x_1 P)$. The values of $p\{x: q(xP) \leq \lambda\}$ and $q\{x: p(xP) \leq \lambda\}$ for each critical λ value are as follows:

critical λ value:	0	.2	.3	.5	.6
$p\{x: q(xP) \leq \lambda\}$:	.2	.2	.7	.7	1.0
$q\{x: p(xP) \leq \lambda\}$:	.3	.9	.9	1.0	1.0.

Since $p\{x: q(xP) \leq \lambda\} \leq q\{x: p(xP) \leq \lambda\}$ for each critical λ, we get $p\ D_1(R)\ q$. To examine $D_2(R)$ we look at the values of λ for which $\lambda \in \{p(Px_i), q(Px_i)\}$ for $i = 1,2,3$. These are $0 = p(Px_1) = q(Px_1)$, $.1 = q(Px_2)$, $.3 = p(Px_2)$, $.5 = p(Px_3)$ and $.6 = q(Px_3)$:

critical λ value:	0	.1	.3	.5	.6
$p\{x: q(Px) \leq \lambda\}$:	.3	.8	.8	.8	1.0
$q\{x: p(Px) \leq \lambda\}$:	.1	.1	.7	1.0	1.0.

According to the third characterization of $D_2(R)$, which requires $p\{x: q(Px) \leq \lambda\} \geq q\{x: p(Px) \leq \lambda\}$ for all $\lambda \in [0,1]$, not $(p\ D_2(R)\ q)$ since $p\{x: q(Px) \leq .5\} < q\{x: p(Px) \leq .5\}$. Hence $D_1(R) \neq D_2(R)$ when $x_1 P x_2 P x_3 I x_1$.

The cyclic case of $x_1 P x_2 P x_3 P x_1$ is handled with $p = (.1,.5,.4)$ and $q = (.2,.2,.6)$. The computations for $D_1(R)$ are:

critical λ value:	.1	.2	.4	.5	.6
$p\{x: q(xP) \leq \lambda\}$:	0	.5	.5	.5	1.0
$q\{x: p(xP) \leq \lambda\}$:	.6	.6	.8	1.0	1.0

so that $p\ D_1(R)\ q$. The computations for $D_2(R)$ are

critical λ value:	.1	.2	.4	.5	.6
$p\{x: q(Px) \leq \lambda\}$:	0	.9	.9	.9	1.0
$q\{x: p(Px) \leq \lambda\}$:	.2	.2	.4	1.0	1.0

which imply not $(p\ D_2(R)\ q)$. Hence $D_1(R) \neq D_2(R)$ in this case also and the proof of the lemma is complete.

The next lemma will be needed in ensuing developments. For all $\lambda \in [0,1]$ and all $p,q \in \Pi$ and $R \in \mathcal{R}$ let

$$\Delta(\lambda,p;\ q,R) = p\{x: q(xP) \leq \lambda\} - p\{x: q(xR) \leq \lambda\}$$

$$= p\{x: q(xP) \leq \lambda < q(xP) + q(xI)\}.$$

Lemma 2. <u>For all</u> $p,q \in \Pi$ <u>and all</u> $R \in \mathcal{R}$,

$$\int_{\lambda=0}^{1} \Delta(\lambda,p;\ q,R)d\lambda = \int_{\lambda=0}^{1} \Delta(\lambda,q;\ p,R)d\lambda.$$

Proof. Let A and B be respectively the minimal supports of p and q. Since the total contribution of $a \in A$ to $\int \Delta(\lambda,p;\ q,R)d\lambda$ is $p(a)q(aI)$, i.e. a contributes $p(a)$ to Δ over an interval of length $q(aI)$,

$$\int_{\lambda=0}^{1} \Delta(\lambda, p;\ q, R)\, d\lambda = \Sigma_A\ p(a)q(aI).$$

For a similar reason

$$\int_{\lambda=0}^{1} \Delta(\lambda, q;\ p, R)\, d\lambda = \Sigma_B\ q(b)p(bI).$$

The term $p(a)q(b)$ is in Σ_A iff aIb, and $q(b)p(a)$ is in Σ_B iff bIa. Since I is symmetric, $\Sigma_A = \Sigma_B$ and the proof is complete.

Although Lemma 1 shows that $D_1(R)$ and $D_2(R)$ can be different, it does not answer the question of whether it is possible to have both $p\ D_1(R)\ q$ and $q\ D_2(R)\ p$ without also having the converses of these two. The following theorem shows that this cannot happen.

Theorem 1. <u>For all</u> $p, q \in \Pi$ <u>and all</u> $R \in \mathcal{R}$, <u>if</u> $p\ D_1(R)\ q$ <u>and</u> $q\ D_2(R)\ p$ <u>then</u> $q\ D_1(R)\ p$ <u>and</u> $p\ D_2(R)\ q$.

Proof. Suppose $p\ D_1(R)\ q$ and $q\ D_2(R)\ p$. Then, by the initial definition of $D_1(R)$ and the second characterization of $D_2(R)$,

$$q\{x\colon p(xR) \leq \lambda\} \leq p\{x\colon q(xR) \leq \lambda\} \leq p\{x\colon q(xP) \leq \lambda\} \leq q\{x\colon p(xP) \leq \lambda\} \tag{1}$$

for all $\lambda \in [0,1]$, where the middle inequality results from the fact that if $q(xR) \leq \lambda$ then $q(xP) \leq \lambda$. Now (1) implies that $\Delta(\lambda, p;\ q, R) \leq \Delta(\lambda, q;\ p, R)$ for all $\lambda \in [0,1]$. Hence, by Lemma 2, $\Delta(\lambda, p;\ q, R) = \Delta(\lambda, q;\ p, R)$ for all $\lambda \in [0,1]$. Therefore

$$q\{x\colon p(xR) \leq \lambda\} = p\{x\colon q(xR) \leq \lambda\} \text{ for all } \lambda \in [0,1],$$

$$p\{x\colon q(xP) \leq \lambda\} = q\{x\colon p(xP) \leq \lambda\} \text{ for all } \lambda \in [0,1],$$

and these imply respectively that $p\ D_2(R)\ q$ and that $q\ D_1(R)\ p$.

For any "as good as" dominance relation $D^*(R)$ on Π let $E^*(R)$ be defined by $p\ E^*(R)\ q$ iff $p\ D^*(R)\ q$ and $q\ D^*(R)\ p$. Thus $E^*(R)$ is a stochastic indifference relation. When R is a weak order, we have $p\ SE(R)\ q$ iff $p\ SD(R)\ q$ and $q\ SD(R)\ p$, which is true iff $p(xR) = q(xR)$ for all $x \in X$. The stochastic indifference functions for D_1, D_2 and D are respectively E_1, E_2 and E with

$$p\ E_1(R)\ q \text{ iff } p\{x: q(xP) \leq \lambda\} = q\{x: p(xP) \leq \lambda\} \text{ for all } \lambda \in [0,1],$$

$$p\ E_2(R)\ q \text{ iff } p\{x: q(Px) \leq \lambda\} = q\{x: p(Px) \leq \lambda\} \text{ for all } \lambda \in [0,1],$$

and $E(R) = E_1(R) \cap E_2(R)$. Moreover, it follows easily from Theorem 1 that the stochastic indifference relation for $D_1(R) \cup D_2(R)$ is $E_1(R) \cup E_2(R)$.

When A and B are respectively the minimal supports of p and q, it is readily verified that

$$p\ E_1(R)\ q \text{ iff } p\{a \in A: q(aP) = \lambda\} = q\{b \in B: p(bP) = \lambda\} \text{ for all } \lambda \in [0,1],$$

$$p\ E_2(R)\ q \text{ iff } p\{a \in A: q(Pa) = \lambda\} = q\{b \in B: p(Pb) = \lambda\} \text{ for all } \lambda \in [0,1].$$

These can be checked using only the λ values in $\{q(aP), p(bP), q(Pa), p(Pb)\}$, and both must be checked for $E(R)$. That is, it is not generally true that $E_1(R) = E_2(R)$. For example, when $A = \{a_1, a_2, a_3, a_4\}$, $B = \{b_1, b_2, b_3, b_4\}$, $A \cap B = \emptyset$, $p = (.3,.2,.1,.4)$, $q = (.1,.1,.3,.5)$, and $P = \{(a_1,b_1), (a_2,b_2), (b_1,a_3), (b_2,a_3), (b_3,a_3)\}$, we get $p\ E_1(R)\ q$ and not $(p\ E_2(R)\ q)$.

A simple and nontrivial example of $p\ E(R)\ q$ is given by $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$, $p = (.5,.5)$, $q = (.5,.5)$ and $P = \{(a_1,b_1), (b_1,a_2)\}$.

3. Properties of the Dominance Functions

To support the claim that $D(R)$ is a reasonable stochastic dominance relation in the $\mathcal{R}$ context we shall prove that it satisfies nine conditions that seem like desirable requirements for "as good as" dominance relations in this context. Each of $D_1(R)$ and $D_2(R)$ satisfies the first eight conditions, but they violate the final condition.

In the statements of the conditions, D^* denotes a generic function on $\mathcal{R}$ that assigns a binary relation $D^*(R)$ on Π to each $R \in \mathcal{R}$, $p \wedge A$ means that $A \subseteq X$ is the minimal support of $p \in \Pi$, ARB signifies that aRb for all $(a,b) \in A\times B$, bPA means that bPa for all $a \in A$, and BPa means that bPa for all $b \in B$. In C6 through C8, xP_iy iff xR_iy and not (yR_ix). In C9, R' is the converse or dual of R with $xR'y$ iff yRx. The conditions apply to all $R \in \mathcal{R}$ and all $p,q \in \Pi$.

C1. If $p \wedge A$, $q \wedge B$ and if the restriction of R to $A\cup B$ is a weak order on $A\cup B$, then $p\ D^*(R)\ q$ iff $p\ SD(R)\ q$.

C2. If $p \wedge A$, $q \wedge B$ and ARB, then $p\ D^*(R)\ q$.

C3. If $p \wedge A$, $q \wedge B$, ARB and not (BRA), then not $(q\ D^*(R)\ p)$.

C4. If $p \wedge A$, $q \wedge B$ and bPA for some $b \in B$, then not $(p\ D^*(R)\ q)$.

C5. If $p \wedge A$, $q \wedge B$ and BPa for some $a \in A$, then not $(p\ D^*(R)\ q)$.

C6. If $p \wedge A$, $q \wedge B$ and $P_1 \cap (A\times B \cup B\times A) = P_2 \cap (A\times B\cup B\times A)$, then $p\ D^*(R_1)\ q$ iff $p\ D^*(R_2) q$.

C7. If $p \wedge A$, $q \wedge B$, P_1 and P_2 are the same on $A\cup B$ except that $P_1 \cap (A\times B) \subseteq P_2 \cap (A\times B)$ and $P_2 \cap (B\times A) \subseteq P_1 \cap (B\times A)$, and if $p\ D^*(R_1)\ q$, then $p\ D^*(R_2)\ q$.

C8. If $p \wedge A$, $q \wedge B$, if the conditions of C7 on P_1 and P_2 hold along with $P_1 \neq P_2$ on $A \cup B$, and if $p\ D^*(R_1)\ q$, then not $(q\ D^*(R_2)\ p)$.

C9. $p\ D^*(R)\ q$ iff $q\ D^*(R')\ p$.

Condition C1 requires $D^*(R)$ to be identical to the basic stochastic dominance relation SD(R) for all pairs of simple measures in Π for which R on the union of the minimal supports of these measures is a weak order. Conditions C2 and C3 are fundamental dominance conditions. For example, C2 says that if every possible outcome under p is as preferable as every possible outcome under q, then $p\ D^*(R)\ q$; if, in addition, some p outcome is strictly preferred to some q outcome, then C3 requires that q not stochastically dominate p.

Conditions C4 and C5 might respectively be referred to as the "heaven" and "hell" conditions. If bPA for some $b \in B$, then b could be "infinitely" better than everything in A and it would not be reasonable to assert that p dominates q. Similarly, if BPa for some $a \in A$, then a could be "infinitely" worse than everything in B and again it would not be reasonable to assert that p dominates q.

Condition C6 is an independence condition. It says that $D^*(R)$ between p and q shall depend only on the ordered pairs in P whose first and second components are respectively in A and in B or else in B and in A. Equivalently, C6 says that the behavior of P in $A \backslash B = \{a \in A: a \notin B\}$ and in $B \backslash A = \{b \in B: b \notin A\}$ is irrelevant to the determination of $D^*(R)$ between p and q. Since this may seem odd at first, it should be remarked that the dominance comparison between p and q is concerned with the interrelationships <u>between</u> p and q and, in my judgment, should not depend on aspects of the risky decisions that are not directly related to one another. This says, for example, that if a and a' are in A but neither is in B, then the one of aPa', $a'Pa$ and aIa' that obtains should have no bearing on whether p dominates q.

The seventh condition is a monotonicity condition. It says that if P_1 and P_2 are identical within $A\backslash B$ and within $B\backslash A$, and if $aP_1b \Rightarrow aP_2b$ and $bP_2a \Rightarrow bP_1a$ for all $(a,b) \in A\times B$, then $p\ D^*(R_2)\ q$ whenever $p\ D^*(R_1)\ q$. In other words, C7 asserts that if P is changed by adding a-over-b preferences and/or deleting b-over-a preferences, then p dominates q after the changes if p dominates q before the changes. Together, C6 and C7 are equivalent to

C7*. If $p \wedge A$, $q \wedge B$ and if $aP_1b \Rightarrow aP_2b$ and $bP_2a \Rightarrow bP_1a$ for all $a \in A$ and all $b \in B$, then $p\ D^*(R_1)\ q \Rightarrow p\ D^*(R_2)\ q$.

Condition C8 is a strong monotonicity condition. It says that if p dominates q under P_1, and if p is improved relative to q by adding one or more a-over-b preferences and/or deleting one or more b-over-a preferences to give P_2 (with $a \in A\backslash B$ or $b \in B\backslash A$ required by the P_i conditions), then q shall not dominate p under P_2. In other words, if p dominates q before the change, then p will <u>strictly</u> dominate q after the change.

The final condition is a symmetry or duality condition. It says that if p dominates q under the preference relation P and if P' is formed from P by reversing all preferences, then q dominates p under P'. Thus reversing all preferences reverses all dominance comparisons.

Although other conditions might be proposed for D*, these nine form a core that, in my opinion, should hold for any satisfactory conception of stochastic dominance in the $\mathcal{R}$ context. We shall now prove that D_1 and D_2 satisfy the first eight conditions but do not satisfy the ninth. We note later that D satisfies all nine.

Theorem 2. <u>Conditions</u> C1 <u>through</u> C8 <u>hold for</u> $D_1(R)$ <u>and</u> $D_2(R)$ <u>for every</u> $R \in \mathcal{R}$. <u>If</u> X <u>has more than two outcomes then neither</u> D_1 <u>nor</u> D_2 <u>satisfies</u> C9.

Proof. The theorem will be verified only for D_1 since the proofs for D_2 are similar to those for D_1. Thus, we wish to show first that C1 through C8 hold when D* therein is replaced by D_1. The failure of D_1 for C9 will then be noted.

C1. Assume that R on $A \cup B$ is a weak order. Suppose first that p SD(R) q, and fix $\lambda \in [0,1]$. It is easily verified that p SD(R) q iff $[p(yR) \leq q(yR)$ for all $y \in A \cup B]$ iff $[p(xP) \leq q(xP)$ for all $x \in A \cup B]$. The last of these implies that $\{x: q(xP) \leq \lambda\} \subseteq \{x: p(xP) \leq \lambda\}$ within $A \cup B$ under the weak order hypothesis. Therefore, with y an element in $A \cup B$ that maximizes p(xP) subject to $p(xP) \leq \lambda$, we obtain $\{x: p(xP) \leq \lambda\} = yR$ and $p\{x: q(xP) \leq \lambda\} \leq p(yR) \leq q(yR) = q\{x: p(xP) \leq \lambda\}$, so that $p\{x: q(xP) \leq \lambda\} \leq q\{x: p(xP) \leq \lambda\}$. Since this is true for every $\lambda \in [0,1]$, p $D_1(R)$ q. To prove the converse suppose that p SD(R) q is false and let y be such that $p(yP) > q(yP)$. Then take $\lambda = q(yP)$. Since $\{x: p(xP) \leq \lambda\}$ doesn't contain y, and since $yR = \{x: q(xP) \leq \lambda\}$, we get $q\{x: p(xP) \leq \lambda\} \leq q(yP) < p(yP) \leq p(yR) = p\{x: q(xP) \leq \lambda\}$, or $q\{x: p(xP) \leq \lambda\} < p\{x: q(xP) \leq \lambda\}$, and this shows that p $D_1(R)$ q is false.

C2. If ARB then $p(bP) = 0$ for all $b \in B$ so that $q\{x: p(xP) \leq \lambda\} = 1$ for all $\lambda \in [0,1]$. Hence p $D_1(R)$ q.

C3. If ARB and aPb for some $a \in A$ and $b \in B$ then $q\{x: p(xP) \leq \lambda\} = 1$ for all $\lambda \in [0,1]$ but $p\{x: q(xP) \leq 0\} < 1$ since $q(aP) > 0$ and $p(a) > 0$. Hence not (q $D_1(R)$ p).

C4. Suppose bPA for some $b \in B$. Then $p(bP) = 1$ so that $q\{x: p(xP) \leq \lambda\} < 1$ for all $\lambda < 1$. But $q(aP) < 1$ for every $a \in A$ and therefore $p\{x: q(xP) \leq \lambda\} = 1$ for some $\lambda < 1$. Hence not $(p\ D_1(R)\ q)$.

C5. Suppose BPa for some $a \in A$. Then $q(aP) = 0$ so that $p\{x: q(xP) = 0\} > 0$. But $q\{x: p(xP) = 0\} = 0$ since $p(bP) \geq p(a)$ for every $b \in B$. Therefore not $(p\ D_1(R)\ q)$.

C7*. When $P_1 \cap (A \times B) \subseteq P_2 \cap (A \times B)$ and $P_2 \cap (B \times A) \subseteq P_1 \cap (B \times A)$, we get $q(aP_1) \leq q(aP_2)$ for all $a \in A$, and $p(bP_2) \leq p(bP_1)$ for all $b \in B$. Therefore $\{a \in A: q(aP_2) \leq \lambda\} \subseteq \{a \in A: q(aP_1) \leq \lambda\}$ and $\{b \in B: p(bP_1) \leq \lambda\} \subseteq \{b \in B: p(bP_2) \leq \lambda\}$ for all $\lambda \in [0,1]$. Hence, if $p\{a \in A: q(aP_1) \leq \lambda\} \leq q\{b \in B: p(bP_1) \leq \lambda\}$ for all $\lambda \in [0,1]$, or if $p\ D_1(R_1)\ q$, then $p\{a \in A: q(aP_2) \leq \lambda\} \leq q\{b \in B: p(bP_2) \leq \lambda\}$ for all $\lambda \in [0,1]$, or $p\ D_1(R_2)\ q$.

C8. In addition to the conditions on P_1 and P_2 of the preceding proof, the hypotheses of C8 imply that $q(aP_1) < q(aP_2)$ for some $a \in A$ or $p(bP_2) < p(bP_1)$ for some $b \in B$. Suppose for definiteness that $q(a_0P_1) < q(a_0P_2)$, and take $\lambda = q(a_0P_1)$. Then $\{a \in A: q(aP_2) \leq \lambda\} \subset \{a \in A: q(aP_1) \leq \lambda\}$ and therefore $p\{a \in A: q(aP_2) \leq \lambda\} < p\{a \in A: q(aP_1) \leq \lambda\}$. It then follows from the inequalities of the preceding proof that $p\{a \in A: q(aP_2) \leq \lambda\} < q\{b \in B: p(bP_2) \leq \lambda\}$, and therefore not $(q\ D_1(R_2)\ p)$. A similar conclusion obtains when $p(bP_2) < p(bP_1)$ for some $b \in B$.

C9. The proof of Lemma 1 gives examples of R on three outcomes for which $p\ D_1(R)\ q$ and not $(p\ D_2(R)\ q)$. Since $p\ D_2(R)\ q$ iff $q\ D_1(R')\ p$ when R' is the dual of R, the Lemma 1 examples imply not $(q\ D_1(R')\ p)$. Therefore C9 does not generally hold for D_1 when X has three or more elements.

We now observe that D satisfies all of C1 through C9 and that the union function introduced in section 2 also satisfies these conditions.

Theorem 3. Conditions C1 through C9 hold for $D(R) = D_1(R) \cap D_2(R)$ and for $D_1(R) \cup D_2(R)$ for all $R \in \mathcal{R}$.

Proof. It follows readily from Theorem 2 and the statements of the conditions that C1 through C7 hold for both $D_1 \cap D_2$ and $D_1 \cup D_2$. Condition C9 also holds for both since $p\ D_1(R)\ q$ iff $q\ D_2(R')\ p$, and $p\ D_2(R)\ q$ iff $q\ D_1(R')\ p$. Since C8 holds for each of D_1 and D_2, it must hold for their intersection. To violate C8 for $D_1 \cup D_2$ we need $p,q \in \Pi$ and $R_1, R_2 \in \mathcal{R}$ that satisfy the hypotheses of C8 such that either $p\ D_1(R_1)\ q$ or $p\ D_2(R_1)\ q$ along with either $q\ D_1(R_2)\ p$ or $q\ D_2(R_2)\ p$. Suppose for definiteness that $p\ D_1(R_1)\ q$. Then not $(q\ D_1(R_2)\ p)$ by Theorem 2, so that $q\ D_2(R_2)\ p$ is needed to violate C8 for $D_1 \cup D_2$. But if $q\ D_2(R_2)\ p$ then, in view of the fact that $p\ D_1(R_2)\ q$ by C7 for D_1, Theorem 1 implies that $q\ D_1(R_2)\ p$, a contradiction. Therefore $D_1(R) \cup D_2(R)$ satisfies C8 for all $R \in \mathcal{R}$.

4. Summary

The purpose of this paper has been to introduce a definition of stochastic dominance for "unordered" preferences and to argue that it is a reasonable definition. Although the presentation was based on reflexive and complete preference-or-indifference relations $R \in \mathcal{R}$, we could just as well have begun with asymmetric strict preference relations P with I and R defined respectively as the symmetric complement of P and the union of P and I.

The basic relations $D_1(R)$ and $D_2(R)$, whose intersection was suggested as the primary stochastic dominance relation for the $\mathcal{R}$ context, are based on interrelationships between probability measures that are implicit but by no means obvious in the usual stochastic dominance relation for ordered preferences. It was shown that $D_1(R)$ and $D_2(R)$ are equal to each other and to the usual stochastic dominance relation when R is a weak order. Moreover, D_1 and D_2 satisfy several reasonable conditions for stochastic dominance in the unordered context. However, neither one satisfies an appealing symmetry condition which says that dominance reverses when all preferences reverse. The latter condition holds for both the intersection relation $D(R) = D_1(R) \cap D_2(R)$ and the union relation $D_1(R) \cup D_2(R)$, and both of these satisfy the other conditions.

The development of D_1 and D_2 was motivated by intransitivity phenomena whose occurrence may be due to natural factors that have little if anything to do with "irrationality." I would argue that the proposed relation D(R) shows that one can make reasonable dominance comparisons between risky alternatives even when the individual's preferences on outcomes violate the "rationality" axiom of transitivity in the most flagrant ways.

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