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Stochastic Dynamics of an Inflationary Model and Initial Distribution of Universes

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We investigate the stationary solution of the modified Fokker-Planck equation which governs the global dynamics of the inflation. Contrary to the original FP equation which is for a Hubble horizon size region, we found that the normalizable stationary solution can exist for modified Fokker-Planck equation which is for many Hubble horizon size regions. For a chaotic inflationary model with the potential $\lambda\varphi^{2n}$, we get initial distribution of classical universes using this solution, and discussed the physical meaning of it. Especially for $n=2$, this distribution obeys power-law and classical universes which, created from the Planck energy region, make the fractal structure. In other cases $n\neq 2$, creation of large classical universes is strongly suppressed.

§ 1. Introduction

An inflationary model was proposed to solve many problems in the standard cosmological model. In this model, background geometry experiences exponentially expanding de Sitter stage. By this exponential expansion, problems contained in the standard cosmological model can be solved. Furthermore, a natural explanation for the origin of the density fluctuations is obtained by considering quantum fluctuations of the scalar field in de Sitter expanding space-time. When one considers dynamics of an inflationary model, quantum aspect of a scalar field becomes very important. In a classical picture, dynamics of the model is as follows: At some time, potential energy of the scalar field dominates the energy momentum tensor, and this becomes an effective cosmological constant. Then background space-time begins to expand exponentially. At this time, owing to a large friction force arising from exponential expansion, the acceleration of the scalar field becomes negligible compared to the velocity and the slow rolling condition is attained. The value of the scalar field rolls down a hill of potential very slowly. Due to this slow rolling, we can get sufficient duration of de Sitter expansion to solve problems in standard cosmology. But this picture must be changed when inflation is treated with quantum theory and we must be careful about difference, what variable can be treated as classical. But it is very difficult to solve full quantum field theory and get dynamics from it.

Recently stochastic treatment of an inflationary model is studied by many people.[1)~13)] This method describes dynamics of a long wave mode of the scalar field which drives de Sitter expansion. Starting from the operator Heisenberg equation of the scalar field on de Sitter space-time, one gets a "classical" Langevin equation by coarse graining the short wave length modes of the scalar field. In this equation, quantum effect enters as a Gaussian white noise and dynamics of the long wave mode of the scalar field becomes a stochastic process driven by this noise. In spite of treating a quantum system, the only thing one has to do is to solve a c-number stochastic equation. Therefore it becomes fairly easy to understand the dynamics of quantum inflation.

From this approach, an important aspect of an inflationary model has been understood. As domains of Hubble horizon size (called h-region) are causally independent, each h-region evolves independently, driven by quantum fluctuations and the potential force. Therefore, viewed on a large scale, an inflationary model can be recognized as a stochastic random process with each h-region being an equivalent sample. But it differs from an ordinary stochastic process such as a Brownian particle, since number of h-regions increases in time due to the expansion of the universe. As time goes on, the probability that one h-region is under inflation becomes very small, but the number of h-regions becomes very large. Therefore viewed on a large scale, there necessarily exists some h-regions which are under inflation. So an inflationary process continues forever on a global scale.

To investigate the stochastic feature of inflation, we must solve Langevin equation or equivalent Fokker-Planck equation (FP equation) and obtain a probability distribution $P_c(\varphi, t)$ for the scalar field. It describes probability of finding a value of φ of the scalar field in only one of h-regions in the universe. However the size of an h-region varies with the expansion rate, hence with the value of φ. Therefore we cannot say enough thing about the global structure of an inflationary universe by P_c. We have to investigate probability P_p for a fixed proper volume which contains many h-regions which are of various sizes. Owing to different expansion rate of each h-region, P_p satisfies a modified FP equation.[11]

In this paper, we investigate a chaotic inflationary model by the stochastic method. The chaotic inflationary model is considered to be the most natural model which needs no artificial initial condition for the scalar field. At Planck energy scale, the scalar field is excited quantumly, and gets the vacuum energy which becomes an effective cosmological constant. One of the most interesting approaches of stochastic method is to find initial condition of the inflationary universe. In the chaotic inflation, due to quantum tunneling or quantum diffusion, some universes escape from chaotic Planck energy scale to under Planck energy scale, and begin to evolve classically. The stationary solution of the FP equation is expected to give the probability of this stationary process. The stationary solution for P_c is discussed by several authors and its relation to a quantum cosmological wave function is suggested.[1),6);14)] But this solution is not normalizable, so the physical meaning is unclear. Furthermore P_c does not reflect the global structure of the inflationary universe. Hence we pay attention to the modified FP equation for P_p and search for the stationary solution of it. This modified FP equation has a different structure from the original one. Difference of expansion rate owing to that of the value φ enters as a source term of probability and this term acts the scalar field to pull up its value against the potential force. Hence we may expect the existence of a normalizable stationary solution. Furthermore since P_p describes the global structure of the inflationary universe, the stationary solution, if it exists, is expected to be closely related to the initial condition of the classical universe. These expectations were found to be true recently in Ref. 15). In this paper, we extend the approach given in Ref. 15) and perform a more general and complete analysis of the modified FP equation and its stationary solution.

The paper is organized as follows. In § 2, we review our formulation. In § 3, we treat a chaotic inflationary model by stochastic approach and discuss dynamics of the scalar field. In § 4, we search for a stationary solution for the modified FP equation. Finally, § 5 is devoted to conclusion.

§ 2. Review of formulation

In this section, we briefly summarize the stochastic approach to inflation. Our basic equations are the following Langevin equation on phase space (φ, v) with a Gaussian white noise,[11)]

$$\dot{\varphi}=v+\sigma\,,$$
$$\dot{v}=-3Hv+\frac{1}{a^2}\Delta\varphi-V'(\varphi)-\frac{M^2}{3H}\sigma\,, \tag{2·1}$$

where

$$M^2=\langle V''(\varphi)\rangle\,,$$
$$\langle\sigma(x_1)\sigma(x_2)\rangle=\frac{H^3}{4\pi^2}j_0(\epsilon aH|\boldsymbol{x}_1-\boldsymbol{x}_2|)\delta(t_1-t_2)\,. \tag{2·2}$$

These equations are obtained by coarse-graining the short wave length modes of the Heisenberg equation for the scalar field on de Sitter space-time. Equation (2·1) contains degrees of freedom for the velocity of the field φ. But due to rapid de Sitter expansion, a large friction force acts to the field and degrees of freedom for the velocity are erased adiabatically within a Hubble time scale.[10)] After the time, we recognize that the slow rolling condition is realized and we can use the following equation,

$$\dot{\varphi}=-\frac{V'(\varphi)}{3H}+\eta\,,$$
$$\langle\eta(t_1)\eta(t_2)\rangle=\frac{H^3}{4\pi^2}\delta(t_1-t_2)\,. \tag{2·3}$$

In addition, we ignored the spatial degrees of freedom and dropped out the gradient term of φ. This corresponds to paying our attention only to dynamics of the scalar field in a single h-region, in which field is regarded as homogeneous. Instead of treating this Langevin equation, we can use the equivalent FP equation,

$$\frac{\partial P_c}{\partial t}=\frac{\partial}{\partial\varphi}\left(\frac{V'}{3H}P_c+\frac{1}{8\pi^2}H^{3/2}\frac{\partial}{\partial\varphi}H^{3/2}P_c\right), \tag{2·4}$$

where $P_c(\varphi, t)$ represents a probability of finding the value of the scalar field φ in an h-region. The Hubble expansion rate H is connected to the value of the scalar field through the Einstein equation,

$$H^2(\varphi)\approx\frac{8\pi}{3m_{\rm pl}^2}V(\varphi)\,. \tag{2·5}$$

Therefore, the evolution of the background metric is influenced by the evolution of the scalar field and the FP equation (2·4) includes this back reaction effect implicitly.

As already mentioned in the Introduction, we cannot get information on the global structure of an inflationary universe by investigating a single h-region. When one considers only one h-region, the probability of finding values of the scalar field for which $V(\varphi)$ is large becomes small as time goes on, because the classical rolling force always acts on the scalar field to pull it down. But if the value of the potential is large, the physical volume becomes very large due to large Hubble expansion rate. It turns out that the average number of h-regions which are under inflation never becomes zero, and the inflationary process never ends, viewed on the global scale. We can see this feature explicitly by the following consideration. If the Hubble pamameter is sufficiently large (corresponds to the large value of the potential), time evolution of the dispersion of the scalar field is given by

$$\langle\varphi^2\rangle=\frac{H^3t}{4\pi^2}. \tag{2·6}$$

This is the same expression as the usual Brownian particle. Characteristic time scale of the system is Hubble time scale H^{-1}, and the step number of the Brownian particle is $n=Ht$. Therefore the probability of finding the system in the large potential value and the system is under inflation at time t is roughly given by

$$p\sim\left(\frac{1}{2}\right)^n=e^{-Ht\ln 2}. \tag{2·7}$$

This value becomes zero after sufficient time. On the other hand, the physical volume or the number of h-regions of the system increases as

$$N\sim e^{3Ht}. \tag{2·8}$$

Therefore, average number of h-region which is under inflation is given by

$$pN\sim e^{Ht(3-\ln 2)}, \tag{2·9}$$

and this value increases with time and the inflationary process never ends. To investigate this feature, we must observe the distribution of the values of the scalar field over a physical volume which contains many h-regions. Different h-regions are statistically independent and each h-region has different Hubble expansion rate. The number of h-region is proportional to the physical volume, and considering the infinitesimal duration of time Δt, each h-region splits into $e^{3H\Delta t}$ h-regions.

On the basis of the above consideration, we analyze a weighted distribution function defined by

$$\tilde{P}(\varphi,t)=\left\langle\delta(\varphi-\varphi_\eta(t))\exp\left(3\int^t dt' H(\varphi_\eta(t'))\right)\right\rangle_\eta, \tag{2·10}$$

where φ_η is a solution of the Langevin equation (2·3) with a fixed noise η, and the average is taken over the noise distribution with a weight factor representing the physical volume. From the above distribution, we can derive a FP equation for $\tilde{P}$. We start from the following distribution function for a fixed noise,

$$f(\varphi, t)=\delta(\varphi-\varphi_\eta(t))\exp\left(3\int^t dt' H(\varphi_\eta(t'))\right). \tag{2·11}$$

This distribution function traces a classical orbit on phase space for a given function of noise and satisfies the following Liouville equation:

$$\frac{\partial f}{\partial t}=\hat{\Omega} f\,,$$
$$\hat{\Omega}=-\frac{\partial}{\partial\varphi}\left(-\frac{V'}{3H}+\frac{H^{3/2}}{2\pi}\eta\right)+3H\,. \tag{2·12}$$

We can solve this equation by iteration,

$$f(\varphi, t+\varDelta t)=f(\varphi, t)+\int_t^{t+\varDelta t} dt_1 \hat{\Omega}(t_1) f(\varphi, t)$$
$$+\int_t^{t+\varDelta t} dt_1 \int_t^{t_1} dt_2 \hat{\Omega}(t_1)\hat{\Omega}(t_2) f(\varphi, t)+\cdots\,. \tag{2·13}$$

Then by averaging over the noise η and taking the limit $\varDelta t\to 0$, we get the evolution equation for $\tilde{P}=\langle\delta(\varphi-\varphi_\eta)\exp(3\int dtH)\rangle_\eta$,

$$\frac{\partial\tilde{P}}{\partial t}=\frac{\partial}{\partial\varphi}\left(\frac{V'}{3H}\tilde{P}+\frac{1}{8\pi^2}H^{3/2}\frac{\partial}{\partial\varphi}H^{3/2}\tilde{P}\right)+3H\tilde{P}\,. \tag{2·14}$$

To normalize the total probability to unity, we define

$$P_p(\varphi, t)=\frac{\tilde{P}(\varphi, t)}{\langle\exp(3\int dtH)\rangle}\,. \tag{2·15}$$

For this probability distribution function P_p, we get

$$\frac{\partial P_p}{\partial t}=\frac{\partial}{\partial\varphi}\left(\frac{V'}{3H}P_p+\frac{1}{8\pi^2}H^{3/2}\frac{\partial}{\partial\varphi}H^{3/2}P_p\right)+3(H(\varphi)-\langle H\rangle)P_p\,. \tag{2·16}$$

The 3rd and the 4th term on the right-hand side of this equation represent the difference of expansion rates among each h-region. If every h-region has the same expansion rate, then this equation reduces to the original FP equation (2·4).

§ 3. Chaotic inflation and stochastic approach

In this section, we treat a chaotic inflationary model by stochastic approach. First we briefly review the scenario of a chaotic model using $\lambda\varphi^4$ model.[17] Potential of a scalar field is $V(\varphi)=\lambda/4\varphi^4$ and the equation of motion of the scalar field and the Einstein equation are

$$\ddot{\varphi}+3H\dot{\varphi}=-\lambda\varphi^3\,,$$
$$H^2+\frac{k}{a^2}=\frac{8\pi}{3m_{\rm pl}^2}\left(\frac{1}{2}\dot{\varphi}^2+V(\varphi)\right), \tag{3·1}$$

where $k=+1, 0, -1$. By solving these equations, we can get purely classical scenario of a chaotic inflationary model but this scenario is changed if we consider the quantum effect of the scalar field and treat this system by the stochastic approach.

If the initial value of a scalar field is sufficiently large and is in the region $\varphi \gg m_{\rm pl}$, the solution for the above equations approaches the following asymptotic form within one Hubble time $t \sim H^{-1}$.

$$\varphi(t) \sim \varphi_0 \exp\left(-\sqrt{\frac{\lambda}{6\pi}} m_{\rm pl} t\right),$$

$$a(t) \sim a_0 \exp\left(\frac{\pi}{m_{\rm pl}^2}(\varphi_0{}^2 - \varphi^2(t))\right). \tag{3·2}$$

And at this time, the field satisfies the conditions

$$|\ddot{\varphi}| \ll 3H|\dot{\varphi}|,$$

$$\dot{H} \ll H^2. \tag{3·3}$$

Therefore the slow rolling condition is achieved and the background space-time undergoes quasi-de Sitter expansion. When the value of the scalar field comes down to near $m_{\rm pl}$, the field begins to oscillate rapidly and entropy is produced by particle creation and inflationary era ends.

In the following, we pay attention to the region $\varphi \gg m_{\rm pl}$ and investigate time evolution of the scalar field using the modified FP equation. First we pay attention to the dynamics of the scalar field in a single h-region and see how the classical behavior described above changes. Our basic equations are the FP equation (2·4) and the Einstein equation (2·5) and we consider the $\lambda\varphi^4$ model. Substituting Eq. (2·5) into Eq.(2·4) and introducing dimensionless variables $x = \varphi/m_{\rm pl}$ and $u = m_{\rm pl}/2\pi^2(2\pi\lambda/3)^{3/2}t$, we get the following FP equation,

$$\frac{\partial P}{\partial u} = \frac{\partial}{\partial x}\left(\frac{3}{\lambda}xP + \frac{1}{2}x^3\frac{\partial}{\partial x}x^3 P\right). \tag{3·4}$$

Range of x is $1 \le x \le \lambda^{-1/4}$. The lower bound corresponds to the end of inflationary stage and the upper bound to the Planck energy scale (i.e., $V(\varphi) = m_{\rm pl}^4$), beyond which quantum gravitational effects will be dominant. In spite of the complicated structure, we can solve Eq. (3·4) by changing the variables,

$$y = \frac{1}{x^2}, \quad P(x, u) = 2y^{3/2}\tilde{P}(y, u). \quad \left(\frac{1}{2\lambda^{1/2}} \le y < +\infty\right) \tag{3·5}$$

Then our FP equation becomes

$$\frac{\partial \tilde{P}}{\partial u} = \frac{\partial}{\partial y}\left(-\frac{3}{\lambda}y\tilde{P} + \frac{\partial \tilde{P}}{\partial y}\right). \tag{3·6}$$

This equation is with a linear force and the diffusion coefficient is constant. Therefore its solution is given by a Gaussian function. As we are interested in the time evolution of the scalar field in a single h-region now, we consider a delta function $\tilde{P}(y; u=0) = \delta(y - y_0)$ for the initial distribution. As the value of y is constrained to be positive, the probability current $J(y)$ of $\tilde{P}$ must be zero at $y=0$. The solution that satisfies this boundary condition is

$$\tilde{P}(y;\, u)=N\left[e^{-((y-y_c(u))^2)/(2\sigma(u)^2)}+e^{-((y+y_c(u))^2)/(2\sigma(u)^2)}\right], \tag{3·7}$$

where

$$N=\left[\frac{6}{\pi\lambda(e^{12u/\lambda}-1)}\right]^{1/2},$$

$$y_c(u)=e^{6u/\lambda}y_0\,, \quad \sigma(u)^2=\frac{\lambda}{3}(e^{12u/\lambda}-1)\,. \tag{3·8}$$

If necessary, we can form a wave packet from the above solution to obtain the time evolution of a general initial distribution.

Characteristic time scale that the location of the center of the wave packet (3·7) changes is given by $\Delta u \sim \lambda/12$. And within this time scale, the shape of the solution changes as follows:

$$\Delta_{\text{peak}}(\text{change of center of the wave packet}) \sim y_0\,,$$

$$\Delta_{\text{disp}}(\text{change of dispersion of the wave packet}) \sim \sqrt{\lambda}\,. \tag{3·9}$$

Returning to the original variable x, we get

$$\left|\frac{\Delta_{\text{peak}}}{\Delta_{\text{disp}}}\right| \sim \begin{cases} \ll 1 & \text{for } x_0 \gg \lambda^{-1/4}\,, \\ \gg 1 & \text{for } x_0 \ll \lambda^{-1/4}\,. \end{cases} \tag{3·10}$$

If this ratio is greater than unity, dispersion of the wave packet does not spread so much and the center of the wave packet rolls down the hill of potential obeying the classical equation of motion. We can call this situation "classical". On the other hand, if this ratio is smaller than unity, the wave packet spreads rapidly. In this case the quantum fluctuation (stochastic noise) dominates the classical rolling force, and the scalar field behaves as Brownian particles. In this situation, we can say the system is "quantum". The boundary of quantum and classical regions is determined by the balance of quantum and classical forces. If the system starts from the quantum region ($x_0 \gg \lambda^{-1/4}$), quantum noise dominates the system. But potential force becomes dominant gradually, and in final the system completely behaves classically and rolls down into the bottom of the potential.

But the situation completely changes if the observer looks the inflationary universe globally. Due to the exponential expansion, number of h-regions in a physical volume increases in time and each h-region has a different value of Hubble expansion rate. Therefore to get information of the global structure of the inflationary universe, we must investigate the physical volume distribution function. From now on, we investigate the FP equation (2·16) which is for a physical volume.

For the moment, we assume that the contribution of the volume effect is small compared to other terms in the modified FP equation. Setting the initial distribution as the delta function and within the time in which dispersion of the wave packet does not spread so much, we can replace $\langle H(\varphi)\rangle$ by $\langle H(\varphi_c)\rangle$ in Eq. (2·16). Then an approximate solution of FP equation (2·16) is given by

$$P_p(\varphi, t) \approx P_c(\varphi, t) e^{3t(H(\varphi)-H(\varphi_c))}, \tag{3·11}$$

where P_c is a solution of the original FP equation.

Now consider a potential of the following general form:

$$V(\varphi)=\frac{\lambda}{2n} m_{\rm pl}^4 \left(\frac{\varphi}{m_{\rm pl}}\right)^{2n}. \qquad (n=1, 2, 3, \cdots) \tag{3·12}$$

Substituting Eq.(3·12) and Einstein equation (2·5) into FP equation (2·4) and introducing dimensionless variables:

$$x=\frac{\varphi}{m_{\rm pl}}, \qquad u=\frac{m_{\rm pl}}{8\pi^2}\left(\frac{4\pi\lambda}{3n}\right)^{3/2} t, \tag{3·13}$$

we obtain

$$\frac{\partial P_c}{\partial u}=\frac{\partial}{\partial x}\left(\frac{3n^2}{2\lambda} x^{n-1} P_c + x^{3n/2} \frac{\partial}{\partial x} x^{3n/2} P_c\right). \tag{3·14}$$

Since the potential (classical) force and diffusion (quantum) force as well as the terms representing the volume effect are all x-dependent, qualitative behavior of $P_p(x, u)$ changes with variation of the initial value x_0 of $\delta(x-x_0)$. This FP equation can be tranformed to FP equation with constant diffusion coefficient by using the following variables:

$$y=x^{-(3n-2)/2}, \qquad v=\left(\frac{3n-2}{2}\right)^2 u. \tag{3·15}$$

Under the above tranformation, the conservation of probability implies that the distribution function transforms as

$$P_c(x) \to \tilde{P}(y)=\frac{2}{3n-2} x^{3n/2} P_c(x). \tag{3·16}$$

Then our FP equation becomes

$$\frac{\partial \tilde{P}}{\partial v}=\frac{\partial}{\partial y}\left(-\frac{3}{\lambda}\left(\frac{n^2}{3n-2}\right) y^{(n+2)/(3n-2)} \tilde{P} + \frac{\partial \tilde{P}}{\partial y}\right). \tag{3·17}$$

In the case $n=2$, the force term becomes linear and we already know the exact solution (3·7). So we start with this case. The approximate solution (3·11) becomes

$$P_p(y, u) \sim \exp\left[-\frac{(y-y_c(u))^2}{2\sigma(u)^2}\right] \exp\left[\frac{3\Delta u}{\lambda}\left(\frac{1}{y}-\frac{1}{y_c(u)}\right)\right]. \tag{3·18}$$

Within the characteristic time scale $\Delta u \lesssim \lambda/12$, we can approximately replace y_c by y_0. Then for $\Delta u \sim \lambda/12$,

$$P_p \sim \exp\left[-24\frac{y_0^2}{\lambda}\left(\frac{y}{y_0}-1\right)^2 + \frac{1}{4y_0}\left(\left(\frac{y}{y_0}\right)^{-1/2}-1\right)\right]. \tag{3·19}$$

From the above expression, it is easily observed that the qualitative feature of the region under consideration varies, depending on whether initial value y_0 is greater or smaller than the following critical value:

$$y_* \sim \lambda^{1/3} . \tag{3·20}$$

This value corresponds to $x_* \sim \lambda^{-1/6}$ in the variable x and larger than the scale given by Eq. (3·10), i.e., $x \sim \lambda^{1/4}$. From this, if one observes that a scalar field evolves classically (i.e., $x \lesssim \lambda^{-1/4}$) in a single h-region, there necessarily exist some h-regions which are in the "quantum state" (i.e., $x \gtrsim \lambda^{-1/6}$) and noise is dominant. If $x_0 < x_*$, the distribution dose not spread so much and the center of wave packet moves obeying the classical equation of motion. Hence those h-regions which happened to have the values of x smaller than x_* would become large classical universes like our universe. On the other hand, if $x_0 > x_*$, the shape of the wave packet deviates from the Gaussian form and the center of the wave packet does not roll down the potential hill but actually goes up and the dispersion increases rapidly. Thus, as long as there is at least one h-region in which $x_0 > x_*$, the universe viewed on a global scale will be eventually dominated by exponentially expanding h-regions and inflation never ends.

We can easily generalize the above discussion for any value of n. We first solve the FP equation (3·17) approximately. Following Goncharov et al.,[3),6)] let us introduce new variables as

$$s = v ,$$

$$\tilde{y} = y - y_c(u) , \tag{3·21}$$

where $y_c(u)$ is a classical solution which satisfies the classical equation of motion of the scalar field,

$$\dot{y}_c = \frac{3n^2}{\lambda(3n-2)} y_c^{(n+2)/(3n-2)} . \tag{3·22}$$

Then the FP equation (3·17) becomes

$$\frac{\partial \tilde{P}}{\partial s} = \frac{\partial}{\partial \tilde{y}} \left(\left(-\frac{3n^2}{\lambda(3n-2)} (\tilde{y} + y_c)^{(n+2)/(3n-2)} \frac{3n^2}{\lambda(3n-2)} y_c^{(n+2)/(3n-2)} \right) \tilde{P} + \frac{\partial \tilde{P}}{\partial \tilde{y}} \right) . \tag{3·23}$$

As we are searching for an approximate solution for a delta function initial condition and investigate how the shape of the wave packet deviates from the Gaussian form, it is enough to take leading terms of $\tilde{y}$ in this FP equation. Therefore

$$\frac{\partial \tilde{P}}{\partial s} = -\frac{3n^2(4-2n)}{\lambda(3n-2)^2} y_c^{(4-2n)/(3n-2)} \frac{\partial}{\partial \tilde{y}} (\tilde{y} \tilde{P}) + \frac{\partial^2}{\partial \tilde{y}^2} \tilde{P} .$$

In this equation, the potential force is linear and the diffusion coefficient is constant. So its solution for a delta function initial condition is given by the Gaussian form,

$$\tilde{P}(\tilde{y}, u) \sim \exp\left(-\frac{\tilde{y}^2}{2\sigma(u)} \right) ; \quad \sigma(u) = \int d\tilde{y}\, \tilde{y}^2 P_c(\tilde{y}, s) . \tag{3·24}$$

Dispersion $\sigma(u)$ is given by

$$\sigma(u) = \frac{(3n-2)\lambda}{24n^2} y_c^{(2n+4)/(3n-2)} \left(y_0^{(-8)/(3n-2)} - y_c^{(-8)/(3n-2)} \right) . \tag{3·25}$$

Now we estimate the probability P_p. From Eq. (3·11), P_p has the following approximate form:

$$P_p(y, u) \sim \tilde{P}(y, u) e^{3\Delta u(h(y)-h(y_c))}, \tag{3·26}$$

where $h(y)=y^{-2n/(3n-2)}/\lambda$. In the present case, the solution of the classical equation of motion (3·22) is

$$y_c \approx y_0 \left(1+\frac{3n^2}{\lambda(3n-2)} y_0^{-(2n-4)/(3n-2)} u\right), \tag{3·27}$$

and the characteristic time scale that the center of the wave packet changes is given by

$$\Delta u \sim \frac{3n-2}{3n^2} \lambda y_0^{(2n-4)/(3n-2)}. \tag{3·28}$$

Within this characteristic time, we can replace y_c as y_0. After all, the shape of the wave packet after Δu is approximately given by

$$P_p \sim \exp\left[-\frac{24n^2}{\lambda(3n-2)} y_0^{4n/(3n-2)} \left(\frac{y}{y_0}-1\right)^2 + \frac{1}{n^2} y_0^{-4/(3n-2)} \left(\left(\frac{y}{y_0}\right)^{-2/(3n-2)} - 1\right)\right]. \tag{3·29}$$

From this expression, it is easily observed that the qualitative feature of the region under consideration varies, depending on whether the initial value y_0 is greater or smaller than the following critical value:

$$y_* \sim \lambda^{(3n-2)/(4n+4)}. \tag{3·30}$$

Returning to the original variable x, this critical value corresponds to

$$x_* \sim \lambda^{-1/(2n+2)}. \tag{3·31}$$

We will see in the next section that this scale is closely related to the form of potential in the modified FP equation.

§ 4. Stationary solution of the modified FP equation

One question about chaotic inflation is its initial condition. At Planck energy scale, quantum gravitational effects dominate and space-time foams are created and annihilated as Hawking stated.[16] Classical universe appears from these quantum gravitational regions by quantum tunneling or quantum diffusion. Therefore the distribution of classical universes carries some information about a process of Planck energy scale. In stochastic approach, the FP equation includes the effect of back-reaction to the geometry through the φ-dependence of H. If we assume that a process of classical universe creation is stationary, the corresponding solution which represents this process is stationary. This assumption is not so unreasonable because it is a possible view that the inflationary universe has no end and no beginning on a global scale.[19] In this section, we investigate the stationary solution of the modified FP equation which includes the volume effect. Naive thinking suggests that the volume effect term in the FP equation due to large fluctuation of the scalar field balances the classical rolling force, and the normalizable stationary solution will exist.

We first consider a stationary solution of the original FP equation of P_c for $n=2$ case, which is the probability for a single h-region. FP equation is

$$0=\frac{\partial}{\partial x}\left(\frac{6}{\lambda}xP_c+x^3\frac{\partial}{\partial x}x^3P_c\right). \tag{4·1}$$

Solution of this equation is easily obtained,

$$P_c(x)=x^{-3}e^{3/(2\lambda)x^{-4}}\left[C_1+C_2\int^x dx x^{-3}e^{-3/(2\lambda)x^{-4}}\right], \tag{4·2}$$

where C_1 and C_2 are integral constants. This solution becomes singular as $x\to 0$ and is not normalizable. This is because after a sufficient time, a scalar field in a single h-region rolls down into $x=0$ and accumulates there. We study this feature using a time dependent exact solution of the previous section (Eq. (3·7)). The range of variable y is constrained in the region $y\lesssim 1$. Therefore we must set cutoff at $y=1$ and this equivalent to put the reflecting boundary condition at $y\lesssim 1$ for $\tilde{P}(y, u)$. We do this by adding ϵy^3 term to the force term in FP equation, and taking $\epsilon\to 0$ in the end. By the original variable x, this corresponds to putting a reflecting barrier near $x\sim 0$. Then FP describes the system of the potential $-(3/2\lambda)y^2+(\epsilon/4)y^4$ with a constant diffusion, and it is well known that the solution of this FP equation for $P_c(y)$ is given by the scaling solution.[18] Starting from an initial delta function distribution, this solution evolves keeping its Gaussian form until the effect of non-linear force (ϵy^3) becomes dominant. After that the distribution function forms double peaks and these peaks roll down into minima of the double well potential. Although the scaling solution does not approach the stationary solution at $u\to\infty$, after a sufficient lapse of time, dispersion of this solution is known to give the correct answer for the stationary solution which is given by

$$\tilde{P}_{st}(y)=C\exp\left[\frac{3}{2\lambda}y^2-\frac{\epsilon}{y^4}\right], \tag{4·3}$$

where C is a normalization constant. In variable x, this probability distribution is $P_{st}(x)=x^{-3}\exp(3/(2\lambda)x^{-4}-\epsilon/4x^{-8})$ and its value becomes zero at $x=0$, $+\infty$ and has peak at $x=\lambda\epsilon/3$. In $\epsilon\to 0$ limit, this peak approaches $x=0$ and its height becomes infinite; $P_{st}=x^{-3}\exp(3/(2\lambda)x^{-4})$. This is the same form as the stationary solution Eq. (4·2) with the boundary condition $J|_{x=+\infty}=0$, which corresponds to $C_2=0$.

We can interpret the behavior of the above stationary solution as follows. For any given initial distribution, $P(x)$ approaches this stationary solution after sufficient long time provided the value of ϵ keeps finite. Therefore the observer who can only view the value of the scalar field in one h-region observes that the probability of finding the scalar field near $x\sim 0$, which is the bottom of the potential, is very high. For general n, the stationary solution of FP equation (3·14) can be obtained,

$$P_c(x)=x^{-3n/2}e^{3n/4\lambda x^{-2n}}\left[C_1+C_2\left(\frac{3n}{4\lambda}\right)^{1/2n-3/4}F\left(\left(\frac{3n}{4\lambda}\right)^{-1/2n}x\right)\right],$$

$$F(x)=\int^{x} dx x^{-3n/2} e^{-x^{-2n}}, \tag{4·4}$$

and the physical situation is the same as $n=2$ case.

What is the modified FP equation for P_p which is for a physical volume? We investigate the modified FP equation (2·16) for a physical volume in which there exist many h-regions. From the structure of this equation, we can observe the following behavior of P_p: If the value of φ is sufficiently large, terms which represent the volume effect enhance the probability and will balance the classical rolling force. So we can expect the existence of a stationary solution.

Using potential of the form (3·12) and the Einstein equation (2·5) and the dimensionless variables (3·13), the modified FP equation becomes

$$\frac{\partial P_p}{\partial u}=\frac{\partial}{\partial x}\left(\frac{3n^2}{2\lambda}x^{n-1}P_p+x^{3n/2}\frac{\partial}{\partial x}x^{3n/2}P_p\right)+\frac{18\pi n}{\lambda}(x^n-\langle x^n\rangle)P_p\,. \tag{4·5}$$

This is an integro-differential equation and cannot be solved in general. However, by further transforming it into a Schrödinger-type equation, we can obtain sufficient qualitative information about the stationary solution for P_p. We use the dimensionless variables y, v introduced in Eq. (3·15). Note that for $n\geq 1$, which one generally assumes, $x\to\infty$ corresponds to $y\to 0$. Then by defining $\psi(y, v)$ as

$$\psi(y, v)=\exp\left[-\frac{3n}{8\lambda}y^{4n/(3n-2)}\right]\tilde{P}(y, v)\,, \tag{4·6}$$

we get a Schrödinger equation of the form,*)

$$-\frac{\partial\psi}{\partial v}=-\frac{\partial^2\psi}{\partial y^2}+(U_0(y)-E)\psi\,, \tag{4·7}$$

where

$$U_0(y)=\frac{3n^2(n+2)}{2\lambda(3n-2)^2}y^{(-2n+4)/(3n-2)}+\frac{9n^4}{4\lambda^2(3n-2)^2}y^{(2n+4)/(3n-2)}$$
$$-\frac{72\pi n}{\lambda(3n-2)^2}y^{-2n/(3n-2)}\,,$$

$$E=-\frac{72\pi n}{\lambda(3n-2)^2}\langle y^{-2n/(3n-2)}\rangle\,. \tag{4·8}$$

A normalizable stationary solution should correspond to a "negative" energy bound state solution of Eq. (4·7), i.e., $E=-\langle U_0(y)\rangle=-\int dy\psi U_0\psi<0$, and the existence of the latter is expected from the behavior of potential $U_0(y)$ (see Fig. 1). If there were no volume effect, the stationary solution corresponds to "zero" energy bound state, i.e., $E=-\langle U_0\rangle=0$. But U_0 would be positive everywhere and no zero energy bound state would exist. However, due to the volume effect, $U_0(y)$ goes to $-\infty$ as $y\to 0$ and a negative energy solution may exist. However, the existence of a negative energy bound state solution does not directly imply that of a normalizable stationary solution, since E is not just an eigenvalue but is an expectation value which must be

*) We use the opposite sign convention for the definition of the energy E used in the previous paper.[16] In this paper, the value of E is always negative.

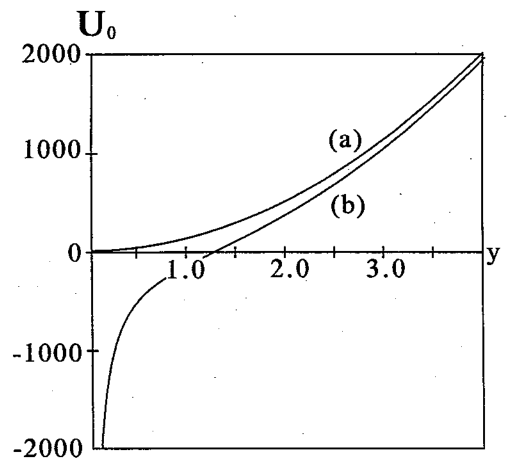

Fig. 1. Shape of the FP potentials for $n=2$ case and $\lambda=0.1$. (a) The potential of the original FP equation. At $y=0$, its value is positive and zero energy state which corresponds to the stationary solution cannot exist. (b) The potential of the modified FP equation. Owing to the volume effect term, the potential becomes $-\infty$ at $y=0$ and a negative energy state corresponds to the stationary solution exists.

consistent with certain eigenvalue.

Observing the form of $U_0(y)$, we can get one charcacteristic scale of y, which comes to appear due to the volume effect. One readily sees that the second term of $U_0(y)$ determines the behavior of ψ at $y\to\infty$ and the third term at $y\to 0$. By equating these two terms, we find the critical value of y as

$$y_* \sim \lambda^{(3n-2)/(4n+4)} . \qquad (4\cdot 9)$$

The second term of the potential represents the classical rolling force, and the third one represents the volume effect. Therefore this scale characterizes the boundary of the classical region and the quantum region, and coincides with the critical value obtained in Eq. (3·31) in terms of the variable x.

Now, we first analyze the behavior of ψ in the region $y\sim 0$. For $y \ll y_*$, $U_0(y)$ is given approximately by

$$U_0(y) \simeq -\frac{18\pi n}{\lambda}\left(\frac{2}{3n-2}\right)^2 y^{-2n/(3n-2)} . \qquad (4\cdot 10)$$

For $n\geq 2$, $U_0(y)$ goes to $-\infty$ slower than $-y^{-2}$, so it is known from quantum mechanics that a regular eigenfunction can exist. But for $n=1$, the potential behaves as $-y^{-2}$ and eigenfunctions become singular at $y=0$. Therefore without some cutoff of the value of y near $y\sim 0$, a normalizable stationary solution does not exist. By tracing back the origin of this singularity in Eq. (4·5), one finds that it is due to the weakness of the potential force for $n=1$ at energy greater than the Planck scale. The effect of quantum fluctuations dominates and almost all physical volumes of the universe are blown away beyond Planck energy; $V(\varphi) > m_{\rm pl}^4$ $(y < \lambda^{1/2})$. Since we cannot say much about the physics beyond the Planck energy, we do not go into discussion further for the $n=1$ case.

We start with the case $n=2$. For the moment we ignore the consistency condition for E and regard it as a free parameter. Near $y\sim 0$, potential $U_0(y)$ becomes

$$U_0(y) \sim -\frac{9\pi}{\lambda y} + \frac{3}{2\lambda} . \qquad (4\cdot 11)$$

This is just a Coulomb potential and its solution is already known. As probability distribution is non-negative, a desirable solution is the ground state eigenfunction because it has no nodes. It is given by

$$\phi = N_0 y \exp\left[-\frac{9\pi}{2\lambda}y\right], \qquad E = -\left(\frac{9\pi}{2\lambda}\right)^2 + \frac{3}{2\lambda}, \tag{4·12}$$

where N_0 is a constant. Hence for $\lambda \ll 1$, the negativity of E which is necessary by its definition is guaranteed and it is possible that the above solution is the consistent stationary solution.

Given the value of E, the behavior of ϕ at $y \to \infty$ can be determined by the standard WKB method. An inspection of Eq. (4·7) shows that the WKB method is applicable for $y \gg \lambda$ (turning point). It is easy to see that the WKB solution in the range $\lambda \ll y \lesssim \lambda^{1/3}$ coincides with Eq. (4·12). For $y \gtrsim \lambda^{1/3}$, the solution takes the form,

$$\phi \approx (y^2+9\pi^2)^{-1/4}(y+(y^2+9\pi^2)^{1/2})^{-27\pi^2/4\lambda} \exp\left[-\frac{3y(y^2+9\pi^2)^{1/2}}{4\lambda}\right]. \tag{4·13}$$

Therefore,

$$\tilde{P}_E(y) \sim \begin{cases} y\exp\left[-\dfrac{9\pi y}{2\lambda}\right] & \text{for } y \lesssim \lambda^{1/3}, \\ \lambda^{1/3}\exp\left[-\dfrac{9\pi y}{2\lambda} + \dfrac{3y^2}{4\lambda}\right] & \text{for } \lambda^{1/3} \lesssim y \lesssim 1, \\ \lambda^{1/3} y^{-(1/2)-(27\pi^2/4\lambda)} & \text{for } 1 \lesssim y, \end{cases} \tag{4·14}$$

where the subscript E denotes that the solution is obtained by regarding E as a pamameter, i. e., ignoring the consistency condition.

We next investigate $n \geq 3$. Let y_t be the turning point which satisfies $U_0(y_t) = E$. Then in the region $y < y_t$, a WKB solution which becomes zero at $y=0$ is given by

$$\phi(y) = \frac{A}{\sqrt{p(y)}} \sin\left[\int_0^y dy p(y)\right],$$

$$; p(y)^2 = E - U_0(y), \tag{4·15}$$

where A is some constant. On the other hand in the region $y > y_t$, a WKB solution which becomes zero at $y \to \infty$ is

$$\phi(y) = \frac{B}{\sqrt{\pi(y)}} \exp\left[-\int_{y_t}^y dy \pi(y)\right],$$

$$; \pi(y)^2 = U_0(y) - E, \tag{4·16}$$

where B is some constant. The condition that the above two solutions are connected smoothly at turning point $y = y_t$ is

$$\int_0^{y_t} dy p(y) = m\pi + \frac{3\pi}{4}, \tag{4·17}$$

where m is integer, and to obtain the ground state solution, we must set $m=0$ whose solution has no nodes. Using this condition, we can obtain the value of energy E and asymptotic behavior of ϕ at $y > y_t$. By evaluating the above integral, we get the approximate value of ground state energy E and y_t,

$$E \approx -\lambda^{(-2n+2)/(n-2)},$$

$$y_t \approx \lambda^{(3n-2)/2(n-2)}\,. \tag{4·18}$$

The turning point y_t is smaller than the Planck energy scale $y_{\rm pl}=\lambda^{(3n-2)/4n}$ provided that λ is much smaller than 1. Near $y\sim 0$, Eq. (4·15) gives

$$\psi(y)\sim y^{(3n-4)/2(3n-2)}\,. \tag{4·19}$$

On the other hand, in the region $y\gg y_t$, the potential is approximated by

$$U_0(y)\approx\frac{9n^4}{4\lambda^2(3n-2)^2}y^{(2n+4)/(3n-2)}\,,$$

and Eq. (4·16) gives

$$\psi\sim y^{-(n+2)/2(3n-2)}\left(1-\frac{1}{4}y^{-(2n+4)/(3n-2)}\right)$$
$$\times\exp\left(-\frac{3n}{2(3n-2)}y^{4n/(3n-2)}+\frac{\lambda(3n-2)^2E}{6n^2(n-2)}y^{(2n-4)/(3n-2)}\right). \tag{4·20}$$

The expression is correct in the region $y\gtrsim 1$ and in this region, inflation has already ended (at $y\sim 1$, inflation ends). After all for $n\geq 3$, $\tilde{P}(y)$ is given by

$$\tilde{P}_E(y)\sim\begin{cases} y^{(3n-4)/2(3n-2)} & \text{for } y\ll\lambda^{-4(n+2)/(n-2)(3n-2)}\,,\\ y^{-(n+2)/2(3n-2)}\exp\left(-\dfrac{(3n-2)^2}{6n^2(n-2)}\lambda^{-2n/(n-2)}y^{(2n-4)/(3n-2)}\right) & \\ \qquad\qquad \text{for } y\gtrsim 1\,. \end{cases} \tag{4·21}$$

Now as we have got the WKB solution for $n\geq 2$, we must check if it is a consistent solution of Eq. (4·5), i.e., whether it satisfies

$$\frac{\partial}{\partial y}\left(-\frac{3n^2}{\lambda(3n-2)}y^{(n+2)/(3n-2)}\tilde{P}+\frac{\partial\tilde{P}}{\partial y}\right)+\left(\frac{72\pi n}{\lambda(3n-2)^2}y^{-2/(3n-2)}+E\right)\tilde{P}=0\,,$$

$$E=-\frac{72\pi n}{\lambda(3n-2)^2}\langle y^{-2n/(3n-2)}\rangle\,. \tag{4·22}$$

By estimating the probability current $\tilde{J}=(-3n^2/\lambda(3n-2))y^{(n+2)/(3n-2)}\tilde{P}+(\partial/\partial y)\tilde{P}$ of the solution, we see that $\tilde{J}|_{y\to 0}\neq 0$ and $\tilde{J}|_{y\to\infty}=0$ for $n=2$ and $\tilde{J}|_{y\to 0}=0$ and $\tilde{J}|_{y\to\infty}=0$ for $n\geq 3$. Therefore for $n=2$, if we regard the range of definition for y as $[0,\infty)$, Eq. (4·14) cannot be a consistent solution. However, Eq. (4·22) would not be quite meaningful far beyond the Planck energy scale. Because in this scale, quantum gravitational effect becomes very dominant and space-time fluctuates too much and our formulation cannot apply. Hence it is not unnatural to introduce a cutoff at some non-zero value, $y=\delta>0$. Fortunately, it turns out that the probability current has a zero at $y\sim\lambda$, which is well beyond the Planck energy scale. This suggests that we should choose δ such that $\tilde{J}|_{y=\delta}=0$. With this choice, it can be shown that a consistent solution does exist and given by Eq. (4·14). The proof is as follows.

Using the solution $\tilde{P}_E$, we construct a new probability distribution function defined in the range $\delta\leq y<\infty$ as

$$\tilde{P}(y)=\frac{1}{N}\tilde{P}_E(y)\,, \qquad (\delta\le y<\infty) \tag{4·23}$$

where N is the normalization constant given by

$$N=\int_\delta^\infty dy\tilde{P}_E\,. \tag{4·24}$$

Now, since $\tilde{P}_E$ is the solution which yields $\tilde{J}=0$ at both ends of y, it follows from the integration of Eq. (4·22) and from Eqs. (4·23) and (4·24) that

$$\frac{\lambda}{9\pi}E=-\int_\delta^\infty\frac{dy}{y}\tilde{P}(y)\,. \tag{4·25}$$

Thus we have shown that $\tilde{P}$ does satisfy the consistency condition.

For $n\ge 3$, $\tilde{J}$ becomes zero at $y=0$, and a consistent solution can exist without cutoff. But because $\psi(y)$ is a ground state wave function, it has no node and has a maximum value near the turning point y_t which is far beyond Planck energy scale. At this point the current $\tilde{J}$ becomes zero similar to $n=2$ case. Therefore we may set a cutoff even for $n\ge 3$ if it is more natural. As we are interested in the region below the Planck energy scale, the behavior of the distribution does not so much depend on the cutoff at beyond the Planck energy scale.

Let us consider the physical meaning of the stationary solution. Transforming back to the original variable x and taking account of the normalization factor approximately, the probability distribution $P_p(x)$ takes the form for $n=2$,

$$P_p(x)\sim\frac{1}{\lambda^2}\times\begin{cases}\dfrac{1}{x^5}\exp\left[-\dfrac{9\pi}{2\lambda x^2}\right] & \text{for } \lambda^{-1/6}\lesssim x\,,\\ \dfrac{\lambda^{1/3}}{x^3}\exp\left[-\dfrac{9\pi}{2\lambda x^2}+\dfrac{3}{4\lambda x^4}\right] & \text{for } 1\lesssim x\lesssim\lambda^{-1/6}\,,\\ \lambda^{1/3}x^{-2+(1/3)(9\pi/2)^2(1/\lambda)} & \text{for } x\lesssim 1\,,\end{cases} \tag{4·26}$$

with the upper cutoff at $x\sim\lambda^{-1/2}$ (we assume $\lambda\ll 1$ throughout the paper). And for $n\ge 3$,

$$P_p(x)\sim\begin{cases}e^{-\lambda^{-2c/(n-2)}}x^{(-7n/4)-(1/2)} & \text{for } \lambda^{-1/(n-2)}<x\,,\\ x^{(-5n/4)+(1/2)}\exp\left[-\dfrac{(3n-2)^2}{6n^2(n-2)}\lambda^{-n/(n-2)}x^{-n+2}\right] & \text{for } x\lesssim 1\,,\end{cases} \tag{4·27}$$

where c is a numerical factor of order unity. It is then readily observed that P_p is a monotonically increasing function of x up to the cutoff or to the turning point and vanishes rapidly as $x\to 0$ and this behavior is completely different from the stationary solution of $P_c(x)$. This implies that the most physical volume of the entire universe is in a highly chaotic quantum gravitational state. The regions having the values of x in the range $1\lesssim x\lesssim x_*$ are undergoing classical slow rolling inflation and those in the range $x\lesssim 1$ are large classical universes which have gone through inflation already. Although our analysis cannot apply to $x\lesssim 1$ where the exponential expansion is no longer on, we expect the form of P_p given in Eq. (4·26) and Eq. (4·27) is still qualitively correct. This follows from the fact that the expansion rate is extremely small in regions with $x \sim 1$ as compared with the dominant part of the universe and as a result no serious error arises from the incorrectly assumed de Sitter expansion, i.e., it does not matter whether those regions with $x \lesssim 1$ are exponentially expanding or not.

It is worthwhile to mention the power-law behavior of P_p for $n=2$ at $x \lesssim 1$. This means that the distribution has no characteristic scale. However, this is a special feature for the $n=2$ case. Let us see what is happening in the FP equation. In the variable y, consider our FP equation (4·22). We search for the condition that $\tilde{P}$ obeys power-law as $y \to \infty$, i.e.,

$$\tilde{P}(y) \propto y^p . \qquad (y \to \infty, \ p < 0) \tag{4·28}$$

Substituting Eq. (4·28) into Eq. (4·22), we compare the degree of power of each term. For the case $n \neq 2$, the 1st term on the right-hand side dominates over the last term as $y \to \infty$. The 1st term is the classical rolling force and the last term is the volume effect due to the fluctuations of the field. In the region $y \to \infty$ the classical force dominates and the probability function could not have the power-law behavior. In contrast, for $n=2$, the 1st term and the last term give the same contribution when $y \to \infty$ and balances with each other, therefore the probability obeys power-law. This power-law behavior is related to the scale invariance of $\lambda\varphi^4$-theory. As we have seen above, the existence of a power-law solution is guaranteed by the balance between the volume effect term and the potential force term. But $\lambda\varphi^4$-theory is invariant in the level of the classical equation of motion under the following scale transformation:

$$\begin{cases} \varphi \to c\varphi , \\ t \to c^{-1}t \end{cases} \tag{4·29}$$

with c being a constant. Therefore the balance is preserved under this transformation. Thus as long as one is interested in the stationary distribution function, there appears no characteristic scale of φ. For $n \neq 2$, $P_p(x)$ has an exponential part and a characteristic scale appears.

As each h-region of the universe evolves effectively independently, we may regard each h-region as an independent universe. Then transforming the variable x (or φ) to $V = H^{-3}$, we can study the distribution of volumes of large classical universes which have gone through inflation. For V greater than $\lambda^{-3/2} l_{\rm pl}^3$, corresponding to $x \sim 1$, one finds

$$\begin{aligned} n=2\,; \qquad & P(V) \sim V^{(-1/18\lambda)(9\pi/2)^2} , \\ n \geq 3\,; \qquad & P(V) \sim V^{-7/12} \exp\left[-c\lambda^{-1/2} \left(\frac{V}{l_{\rm pl}^3} \right)^{1/3} \right], \end{aligned} \tag{4·30}$$

where c is a numerical factor. For $n=2$, various sizes of universes are distributed in the way that they form a fractal structure, i.e., the probability to find an arbitrarily large universe is not exponentially suppressed. In Fig. 2, we visualized this fractal structure given by the probability distribution function (4·30). The size of universes is expressed in terms of the sizes of squares. We must be careful that this figure is

Fig. 2. Fractal distribution of universes. The size of each square reresents the size of each h-region. As the distribution function has no characteristic scale, the size of universes also has no characteristic scale.

idealized one and does not necessarily represent the shape of real universes. This is because we treat each h-region which is spatially distributed as statistically independent, and do not know by what way they are connected spatially. For $n \geq 3$, the probability to find the large univeres is exponentially suppressed. The characteristic scale is $V \sim \lambda^{3/2} l_{\rm pl}^3$ and much smaller than Planck scale size. Therefore we expect that creation of large classical universes like ours by quantum fluctuation is a very rare event for $n \geq 3$.

Now an interpretation of the distribution (4·30) in the picture of creation-from-nothing[22] would be as follows.

On scales smaller than the Planck volume $V_{\rm pl} = l_{\rm pl}^3$, quantum gravitational fluctuations play a dominant role and one can hardly say that universes (h-regions) do exist in the classical sense; spontaneous creation and annihilation of them must be occurring violently. In addition, since the probability current vanishes at $x \sim x_t$ (turning point corresponds to y_t), one can regard them as created from "nothing".

It is important to note that this process is stationary. One can estimate the rate of creation of classically behaving universes like ours provided that the obtained solutions can be used in the region $x \lesssim 1$. For $n=2$,

$$J = \frac{6}{\lambda} x P_p(x) + x^3 \frac{\partial}{\partial x} x^3 P_p |_{x \sim 1}$$

$$\sim \exp\left(-\frac{c}{\lambda}\right), \qquad (4\cdot31)$$

per unit time $\varDelta u = 1$, where c is a constant of order unity. On the other hand, for $n \geq 3$,

$$J = \frac{3n^2}{2\lambda} x^{n-1} P_c + x^{3n/2} \frac{\partial}{\partial x} x^{3n/2} P_c |_{x \sim 1}$$

$$\sim \exp\left(-\frac{c}{\lambda^{n/(n-2)}}\right). \qquad (4\cdot32)$$

As one can see from the expression of J at $x \sim 1$, creation rate of the classical universe for $n \geq 3$ is much suppressed compared with $n=2$ case.

§ 5. Conclusions

In this paper, we investigated the global structure of an inflationary model. Using the modified FP equation which represents the probability distribution of the scalar field for a physical volume, we found that a normalizable stationary solution can exist. Existence of the stationary solution is guaranteed by the volume effect term due to large fluctuation of a scalar field which balances classical rolling force.

As the inflationary process becomes stationary on a global scale, the stationary solution represents the initial distribution of classical universes. We interpret that the stationary solution gives the probability of creation of classical universes which was produced from the chaotic quantum gravitational regime to the classical regime by quantum tunneling or quantum diffusion. From the behavior of the solution, it is found that below the scale at which inflation ends, the probability distribution obeys power-law for a $\lambda\varphi^4$ model. This means universes which begin classical evolution have a fractal distribution in space. This result is originated from the scale invariance of the $\lambda\varphi^4$ model. It is very interesting that the microscopic symmetry results in the macroscopic order; fractal distribution of classical universes.

In spite of our lack of knowledge about Planck energy scale physics, our result may reflect quantum gravitational effect implicitly. Several people suggested the relation between the stationary solution for P_c and Hartle-Hawking cosmological wave function Ψ_{HH}.[21),22)] But it is considered that this relation depends on how one includes the dependence of a scalar field φ in the FP equation, i.e., it depends on the way of ordering "Hubble constant" in FP equation, in other words, the way to include the effect of back-reaction to the geometry. From the start, stochastic approach to inflation is some approximate method of quantum field theory on curved space-time and it is unclear what relation it has to canonical quantum gravity on mini-super space. Stochastic approach is based on the Heisenberg picture and treat mainly flnctuations of a scalar field. On the other hand, canonical qnantum gravity (Wheeler-Dewitt equation) on mini-super space is based on Schrödinger picture and treat fluctuations of a scale factor. We do not know by what way quantum gravitational effect comes into the stochastic approach. Therefore to clarify the relation between stochastic approach and canonical quantum gravity, we must investigate how the information of the state of a qunatum field is included in stochastic approach. This is one important problem left to us.

There is another important problem. Stochastic approach treats each h-region independently and this makes treatment of the system very simple. But real universes are different. Due to the spatial degrees of freedom of the field, h-regions interact with each other. In addition, background geometry receives these effects and will have very inhomogeneous space-time structure and this affects the evolution of a scalar field on it. Especially in the chaotic inflation, the fluctuation of the metric becomes very large at the Planck energy scale, and mini-universes are created continuously and space-time forms "self-reproducing" structure.[19)] To observe these complicated and non-trivial space-time structure more precisely and dynamically, we must include many degrees of freedom of space-time metric into the stochastic approach and solve the geometrical evolution in a general relativistic way. Or we must develop some approximate method which describes stochastically (dynamically) evolving h-regions (space-times) interacting with each other. This is our future problem.

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References

1) A. A. Starobinsky, in *Fundamental Interactions*, ed. V. N. Ponomarev (MGPI press, Moscow, 1984), p. 54; in *Current Topics in Field Theory, Quantum Gravity, and Strings, Lecture Notes in Physics*, **246**, ed. H. J. de Vega and N. Sanchez (Springer, Berlin, 1986), p. 107.
2) G. Mazenko, Phys. Rev. Lett. **54** (1985), 2163; Phys. Rev. **D34** (1986), 2223.
3) A. S. Goncharov and A. D. Linde, Sov. J. Part. Nucl. **17** (1986), 369.
4) F. Graziani and K. Olynyk, Fermilab Preprint Pub-85/175-T (1985).
5) S-J. Rey, Nucl. Phys. **B284** (1987), 706.
6) A. S. Goncharov, A. D. Linde and V. F. Mukhanov, Int. J. Mod. Phys. **A2** (1987), 561.
7) J. Bardeen and G. J. Bublik, Class. Quant. Grav. **4** (1987), 573.
8) R. Brandenberger, H. A. Feldman and J. H. MacGibbon, Stony Brook Preprint ITP-SB-87-73 (1987).
9) A. Ortolan, F. Lucchin and S. Matarrese, Phys. Rev. **D38** (1988), 465.
10) Y. Nambu and M. Sasaki, Phys. Lett. **B205** (1988), 441.
11) M. Sasaki, Y. Nambu and K. Nakao, Phys. Lett. **B209** (1988), 197; Nucl. Phys. **B308** (1988), 868.
12) A. Hosoya, M. Morikawa and K. Nakayama, Preprint OU-HET **114** (1988).
13) F. R. Graziani, Phys. Rev. **D38** (1988), 1122, 1131, 1808.
14) K. Nakao, Y. Nambu and M. Sasaki, Prog. Theor. Phys. **80** (1989), 1041.
15) M. D. Pollock, Nucl. Phys. **B298** (1988), 673.
16) Y. Nambu and M. Sasaki, Phys. Lett. **B219** (1989), 240.
17) S. W. Hawking, in *General Relativity: An Einstein Centenary Survey*, ed. S. W. Hawking and W. Israel (Cambridge, 1979).
18) A. D. Linde, Phys. Lett. **129B** (1983), 177; Mod. Phys. Lett. **1A** (1986), 81; Phys. Lett. **175B** (1986), 395.
19) M. Suzuki, in *Systems Far from Equilibrium, Lecture Notes in Physics* **132**, ed. L. Garrido (Springer, Berlin, 1980), p. 48.
20) A. D. Linde, Physc. Scr. **T15** (1987), 169.
21) J. B. Hartle and S. W. Hawking, Phys. Rev. **D28** (1983), 2960.
22) S. W. Hawking, Nucl. Phys. **B239** (1984), 257.
23) A. Vilenkin, Phys. Rev. **D27** (1983), 2848; **D30** (1984), 509; **D33** (1986), 3560.