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Stochastic Dynamics of New Inflation

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We investigate thoroughly the dynamics of an inflation-driving scalar field in terms of an extended version of the stochastic approach proposed by Starobinsky and discuss the spacetime structure of the inflationary universe. To avoid any complications which might arise due to quantum gravity, we concentrate our discussion on the new inflationary universe scenario in which all the energy scales involved are well below the Planck mass. The investigation is done both analytically and numerically. In particular, we present a full numerical analysis of the stochastic scalar field dynamics on the phase space. Then implications of the results are discussed.

§ 1. Introduction

The inflationary universe scenario has been successful in explaining the fundamental problems of the standard model (e.g., the horizon and flatness problems). In this scenario a scalar field plays an essential role. It is generally assumed that the scalar field has a sufficiently flat potential and there exists an era in which the field slowly rolls down the potential hill toward a minimum of the potential. During that era, the potential energy acts as an effective cosmological constant and the universe expands exponentially.

However the above picture is essentially classical and our understanding of quantitative, quantum mechanical features of inflation is far from complete. One of the early papers which demonstrated the inaccuracy of the classical picture was Mazenko et al.,[1] in which it was pointed out that the use of the effective potential and the expectation value of the scalar field in discussing the dynamics of inflation can lead to a very erroneous result. The reason is that the effective potential is a global notion which is valid for an equilibrium state in the strict sense, while inflation is intrinsically a dynamical, non-equilibrium process.

The stochastic approach recently developed by many authors[2)~9)] seems to be particularly suited for investigations of the scalar field dynamics during the inflationary phase of the universe. The idea was first advocated by Vilenkin.[2] Then under the assumption of slow rolling motion, the basic stochastic equation was derived by Starobinsky.[3] The derivation is based on the fact that the inflation is a macroscopic phenomenon and the spacetime structure of the universe is determined by the behavior of the scalar field on large spatial scales. Thus one divides the modes of the scalar field into two parts in the momentum space; the long wavelength part and the short wavelength part, and focuses on the former. Then the short wavelength modes can be regarded as a quantum noise to the equation of motion for the long wavelength modes, yielding a *quantum* Langevin equation for the scalar field on large spatial scales. Further if the wavelength which divides the modes of the scalar field is larger than the Hubble horizon scale, the equation reduces essentially to a *classical* Langevin equation. The stochastic approach makes it relatively easy to calculate various physical quantities and investigate the quantum effect on the evolution of the large-scale scalar field compared with the conventional field theoretic method. In our recent papers, we have elaborated the stochastic approach[7)] and investigated various issues of inflation in the context of the new inflationary universe scenario.[7)~9)] Our formulation does not assume the slow rolling motion. Consequently, the basic equations turn out to be coupled Langevin equations for the phase space variables (φ, v), where v is the velocity variable corresponding to $\dot{\varphi}$, contrary to Starobinsky's original equation which involves only φ. Thus it has an advantage that it can deal with more general situations such as the very initial stage of inflation or the final stage at which the scalar field undergoes damped oscillations. The main issues studied in Refs. 7)~9) are:

(a) To derive Starobinsky's stochastic equation from the first principle and to examine its range of applicability.
(b) Relation to the conventional field theoretic method.
(c) Evolution of the scalar field at the early stage when the slow rolling assumption fails.
(d) Condition for the realization of classical slow roll-over phase at which the quantum noise due to the short wavelength modes ceases to dominate the evolution of the scalar field.

Issues (a) and (b) were investigated in Ref. 7), (c) in Ref. 8), and (d) in Ref. 9). In this paper, we shall briefly review our formulation of the stochastic approach to inflation and extend our analyses done in Refs. 8) and 9) in more details. In particular, we shall present a detailed analysis on the behavior of the large-scale scalar field by integrating numerically the coupled Langevin equations.

The paper is organized as follows. In § 2, a review on our formulation of the stochastic approach to inflation is given. In § 3, we take up a simple double-well potential model as a typical example and analyze the behavior of the scalar field at the early stage of inflation. In § 4, we investigate the behavior of the scalar field at the intermediate stage of inflation and discuss its implications to the spacetime structure of the inflationary universe. In § 5, the results of numerical simulation of the stochastic scalar field are presented. The simulation is done for the double-well potential model by means of the Monte Carlo method. Finally, § 6 is devoted to conclusions.

§ 2. Formulation

We consider the dynamics of a scalar field in de-Sitter background. The Lagrangian of the scalar fields is

$$\mathcal{L}=-\sqrt{-g}\left[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi+V(\phi)\right]. \tag{2·1}$$

The background metric is assumed to have the form,

$$ds^2 = -dt^2 + a(t)^2 d\boldsymbol{x}^2\,, \tag{2·2}$$

where

$$a(t) = e^{Ht} \quad \text{and} \quad H \approx \text{const.}$$

Then taking the Heisenberg picture, the equation for the scalar field operator $\phi(\boldsymbol{x}, t)$ on this background is

$$\left(\frac{\partial^2}{\partial t^2} + 3H\frac{\partial}{\partial t} - \frac{1}{a(t)^2}\Delta\right)\phi(\boldsymbol{x}, t) + \frac{\partial V}{\partial \phi}(\boldsymbol{x}, t) = 0\,. \tag{2·3}$$

As we have mentioned in the Introduction, the long wavelength part of the scalar field is what we are interested in. Therefore we split the field operator $\phi(\boldsymbol{x}, t)$ and its time derivative $\dot{\phi}(\boldsymbol{x}, t)$ into the long wavelength modes and the short wavelength modes as

$$\begin{aligned}\phi(\boldsymbol{x}, t) &= \varphi(\boldsymbol{x}, t) + \sqrt{\hbar}\,\varphi_s(\boldsymbol{x}, t)\,,\\ \dot{\phi}(\boldsymbol{x}, t) &= v(\boldsymbol{x}, t) + \sqrt{\hbar}\,v_s(\boldsymbol{x}, t)\,,\end{aligned} \tag{2·4}$$

where φ_s and v_s are defined by

$$\begin{aligned}\varphi_s(\boldsymbol{x}, t) &= \int \frac{d^3k}{(2\pi)^{3/2}}\theta(k - \epsilon a(t)H)\phi_{\boldsymbol{k}}(t)e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\,,\\ v_s(\boldsymbol{x}, t) &= \int \frac{d^3k}{(2\pi)^{3/2}}\theta(k - \epsilon a(t)H)\dot{\phi}_{\boldsymbol{k}}(t)e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\,;\\ \phi_{\boldsymbol{k}}(t) &= a_{\boldsymbol{k}}\varphi_{\boldsymbol{k}}(t) + a_{\boldsymbol{k}}^{\dagger}\varphi^*_{-\boldsymbol{k}}(t)\,.\end{aligned} \tag{2·5}$$

In the above, $\varphi_{\boldsymbol{k}}(t)$ is a positive frequency mode function which satisfies

$$\ddot{\varphi}_{\boldsymbol{k}}(t) + 3H\dot{\varphi}_{\boldsymbol{k}}(t) + \left(\frac{k^2}{a(t)^2} + M^2\right)\varphi_{\boldsymbol{k}}(t) = 0\,, \tag{2·6}$$

where $M^2 = \langle V''(\varphi)\rangle$ and $a_{\boldsymbol{k}}$ and $a_{\boldsymbol{k}}^{\dagger}$ are the annihilation and creation operators, respectively, with respect to a suitably chosen vacuum state. In what follows, we assume the so-called Bunch-Davies vacuum,[10),11)] which is the de Sitter invariant Euclidean vacuum if $M^2 > 0$.[12)] In Eq. (2·4), we inserted $\sqrt{\hbar}$ in front of φ_s and v_s in order to exhibit the quantum effect explicitly. The parameter ϵ in Eq. (2·5) is a constant smaller than unity and determines a scale at which we split the scalar field into the two parts. The expectation value $\langle V''(\varphi)\rangle$, which plays the role of mass square for the short wavelength components, should be determined self-consistently when we solve the dynamics of the large-scale stochastic scalar field φ. However, for a reason we shall argue at the end of § 3, it seems more reasonable to replace M^2 by $V''(\varphi)$ itself without taking the expectation value of it, though unfortunately we do not know any logical justification of the procedure at the moment. In any case, it is enough to assume that M^2 varies sufficiently slowly in time compared with the expansion time H^{-1} and $|M^2| \ll H^2$ for the present purpose.

Now we expand Eq. (2·3) around φ and v and keep the terms up to the lowest order with respect to $\sqrt{\hbar}$. Then we obtain

$$\dot{\varphi}=v+\sqrt{\hbar}\,\sigma\,,$$

$$\dot{v}=-3Hv+\frac{1}{a^2}\Delta\varphi-V'(\varphi)+\sqrt{\hbar}\,\tau\,, \tag{2·7}$$

where σ and τ are given by

$$\sigma(\boldsymbol{x},t)=\epsilon aH^2\int\frac{d^3k}{(2\pi)^{3/2}}\delta(k-\epsilon aH)\phi_{\boldsymbol{k}}(t)e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\,,$$

$$\tau(\boldsymbol{x},t)=\epsilon aH^2\int\frac{d^3k}{(2\pi)^{3/2}}\delta(k-\epsilon aH)\dot{\phi}_{\boldsymbol{k}}(t)e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\,. \tag{2·8}$$

Obviously these equations show that φ_s and v_s act as sources for generating nonvanishing values of φ and v. This follows from the fact that the short wavelength modes are continuously redshifted and their wavelengths become eventually greater than the size $(\epsilon H)^{-1}$, hence generate the large-scale components of the scalar field. We note that because we have kept the terms up to the lowest order in $\sqrt{\hbar}$, the above prescription essentially corresponds to one-loop approximation.[7] We also note that φ and v are not classical variables, but they are quantum quantities. The quantum nature of φ and v arises from the fact that σ and τ are quantum operators. In this sense, regarding σ and τ as quantum noises, Eq. (2·7) can be called quantum Langevin equations.

In order to solve Eq. (2·7), we need to know the correlations of the noises σ and τ. For the Bunch-Davies vacuum $|0\rangle$, we find

$$\langle 0|\sigma(x_1)\sigma(x_2)|0\rangle\approx\epsilon^{(2M^2/3H^2)}\frac{H^3}{4\pi^2}j_0(\epsilon aH|\boldsymbol{x}_1-\boldsymbol{x}_2|)\delta(t_1-t_2)\,,$$

$$\langle 0|\tau(x_1)\tau(x_2)|0\rangle\approx\epsilon^{(2M^2/3H^2)}\left(\frac{M^2}{3H^2}+\epsilon^2\right)^2\frac{H^5}{4\pi^2}j_0(\epsilon aH|\boldsymbol{x}_1-\boldsymbol{x}_2|)\delta(t_1-t_2)\,,$$

$$\langle 0|\sigma(x_1)\tau(x_2)+\tau(x_2)\sigma(x_1)|0\rangle$$
$$\approx-2\epsilon^{(2M^2/3H^2)}\left(\frac{M^2}{3H^2}+\epsilon^2\right)\frac{H^4}{4\pi^2}j_0(\epsilon aH|\boldsymbol{x}_1-\boldsymbol{x}_2|)\delta(t_1-t_2)\,, \tag{2·9}$$

where j_0 is the 0-th order spherical Bessel function and we have assumed $|M^2|\ll H^2$ and $\epsilon^2\ll 1$.

The quantum nature of the noises σ and τ shows up in the commutation relations,

$$[\sigma(x_1),\sigma(x_2)]=[\tau(x_1),\tau(x_2)]=0\,,$$

$$[\sigma(x_1),\tau(x_2)]=i\epsilon^3\frac{H^4}{4\pi^2}j_0(\epsilon aH|\boldsymbol{x}_1-\boldsymbol{x}_2|)\delta(t_1-t_2)\,. \tag{2·10}$$

From Eqs. (2·9) and (2·10), one finds that for the parameter ϵ in the range,

$$\exp\left(-\frac{3H^2}{|M^2|}\right)\ll\epsilon^2\ll\frac{|M^2|}{3H^2}\,, \tag{2·11}$$

not only the quantum nature of σ and τ becomes negligible but also the explicit ϵ-dependence disappears from Eq. (2·9). Moreover we have

$$\tau \approx -\frac{M^2}{3H}\sigma . \tag{2·12}$$

Hence Eq. (2·7) reduces to a set of coupled *classical* Langevin equations,

$$\dot{\varphi} = v + \sqrt{\hbar}\,\sigma ,$$

$$\dot{v} = -3Hv + \frac{1}{a^2}\Delta\varphi - V'(\varphi) - \frac{M^2}{3H}\sqrt{\hbar}\,\dot{\sigma} \tag{2·13}$$

with

$$\langle \sigma(x_1)\sigma(x_2)\rangle = \frac{H^3}{4\pi^2} j_0(\epsilon a H |\boldsymbol{x}_1 - \boldsymbol{x}_2|)\delta(t_1 - t_2) , \tag{2·14}$$

where one should keep in mind that "*classical*" merely means that these equations can be treated as ordinary Langevin equations, in spite of the fact that they intrinsically describe quantum processes.

This completes a review on our formulation of the stochastic approach.[7] Quantum effects on the large-scale scalar field are determined solely by the property of the "classical" noise σ. This noise is Gaussian because the short wavelength part of the field is treated essentially as a free field. The Markov property is a result of splitting the modes of the scalar field sharply by the step function at a fixed physical wavelength.

§ 3. The early stage of inflation

For definiteness, let us consider a Higgs type double-well potential for $V(\phi)$, which is a typical example in the new inflationary universe scenario,

$$V(\phi) = \frac{\lambda}{4}\left(\phi^2 - \frac{m^2}{\lambda}\right)^2 . \tag{3·1}$$

We require the parameters λ and m of this model to satisfy

$$\frac{m^4}{m_{\rm pl}^4} \ll \lambda \ll \frac{m^2}{m_{\rm pl}^2} \ll 1 , \tag{3·2}$$

where $m_{\rm pl}$ is the Planck mass. The above condition is necessary for our approach to be successful.[7]

Following our previous papers,[7)~9)] we introduce the non-dimensional quantities,

$$\varphi = \frac{m}{\sqrt{\lambda}}x , \qquad v = \frac{m^3}{3\sqrt{\lambda}H}y ,$$

$$t = \frac{3H}{m^2}u , \qquad \boldsymbol{r} = \frac{3H}{m^2}\boldsymbol{z} . \tag{3·3}$$

Then Eqs. (2·13) and (2·14) are rewritten as

$$\frac{d}{du}x=y+f\,,$$

$$\frac{d}{du}y=-3\gamma(y-x(1-x^2))+e^{-2\gamma u}\Delta_z x+(1-3\langle x^2\rangle)f \tag{3·4}$$

and

$$\langle f(z_1,\,u_1)f(z_2,\,u_2)\rangle=2\alpha j_0(\epsilon\gamma e^{\gamma u}|z_1-z_2|)\delta(u_1-u_2)\,, \tag{3·5}$$

where the parameters α and γ are given by

$$\alpha=\frac{3\lambda}{8\pi^2}\left(\frac{H^2}{m^2}\right)^2\hbar\,,\qquad \gamma=\frac{3H^2}{m^2}\,. \tag{3·6}$$

Using the fact that $H^2\approx 8\pi V(0)/3m_{\rm pl}^2$, the requirement (3·2) is equivalent to the following conditions on α and γ,

$$\alpha\ll 1\qquad\text{and}\qquad \gamma\gg 1\,. \tag{3·7}$$

Now we are in a position to study the behavior of the large-scale scalar field φ. Starobinsky[3] derived the Langevin equation from the equation of motion for the scalar field, neglecting the acceleration term $\ddot{\phi}$, i.e., taking the slow rolling approximation. However to study the effect of the initial value of the velocity $\dot{\varphi}(0)$ on the later evolution of the scalar field, it is necessary to retain the term $\ddot{\phi}$. Thus our formulation makes it possible to investigate the early stage behavior of φ and its connection to the dynamics at the intermediate stage of inflation.

Assuming the universe was in a thermal equilibrium before inflation, as commonly regarded so in the new inflationary universe scenario, one expects that $\langle\varphi^2\rangle\lesssim\langle\phi^2\rangle\approx T_c{}^2\approx m^2/\sqrt{\lambda}$ and $\langle\dot{\varphi}^2\rangle\lesssim\langle\dot{\phi}^2\rangle\approx T_c{}^4$. Or in terms of the dimensionless variables x and y,

$$\langle x^2\rangle\lesssim\lambda^{1/2}\qquad\text{and}\qquad\langle y(0)^2\rangle\lesssim\gamma\,, \tag{3·8}$$

at the initial stage of inflation. Thus, in particular, we may neglect the terms non-linear in x in Eq. (3·4) during the early stage of inflation, since $\lambda\ll 1$. Further, for simplicity, let us only consider the behavior of x and y averaged over a spatial volume of radius $L\lesssim(\epsilon H)^{-1}$. Then Eqs. (3·4) and (3·5) reduce to

$$\frac{d}{du}x=y+f(u)\,,$$

$$\frac{d}{du}y=-3\gamma(y-x)+f(u) \tag{3·9}$$

and

$$\langle f(u_1)f(u_2)\rangle=2\alpha\delta(u_1-u_2)\,. \tag{3·10}$$

The process described by the above equations is known as an Orstein-Uhlenbeck process[13] and can be solved exactly. The solution is

$$x(u)=\frac{1}{\lambda_+-\lambda_-}[(\lambda_+x(0)+y(0))e^{-\lambda_-u}-(\lambda_-x(0)+y(0))e^{-\lambda_+u}]$$
$$+\frac{1}{\lambda_+-\lambda_-}\int_0^u du'[(\lambda_++1)e^{-\lambda_-(u-u')}-(\lambda_-+1)e^{-\lambda_+(u-u')}]f(u'),$$
$$y(u)=\frac{1}{\lambda_+-\lambda_-}[\lambda_+(y(0)+\lambda_-x(0))e^{-\lambda_+u}-\lambda_-(y(0)+\lambda_+x(0))e^{-\lambda_-u}]$$
$$+\frac{1}{\lambda_+-\lambda_-}\int_0^u du'[\lambda_-(\lambda_++1)e^{-\lambda_-(u-u')}-\lambda_+(\lambda_-+1)e^{-\lambda_+(u-u')}]f(u'), \qquad (3\cdot11)$$

where

$$\lambda_\pm=\frac{3\gamma}{2}\left\{-1\pm\sqrt{1+\frac{4}{3\gamma}}\right\}$$
$$\approx\begin{cases}3\gamma+1\\-1+(3\gamma)^{-1}\end{cases} ;\quad \text{for } \gamma\gg1. \qquad (3\cdot12)$$

After a lapse of time $\Delta u\gtrsim1/\gamma$, the above solution reduces approximately, in the leading order of γ, to

$$y(u)\approx x(u)\approx\left(x(0)+\frac{y(0)}{3\gamma}\right)e^u+\int_0^u du'e^{u-u'}f(u'). \qquad (3\cdot13)$$

Thus independent of initial conditions, the solution approaches that of the slow roll-over version of the stochastic equation,

$$\frac{d}{du}x=x+f(u); \qquad y=x, \qquad (3\cdot14)$$

within the timescale $\Delta u=O(1/\gamma)$, i.e., the Hubble expansion timescale, which is apparently due to the large friction force (the term proportional to 3γ in Eq. (3·9)). Further from Eq. (3·13), one finds that the range of initial conditions which are consistent with the realization of inflation is given by

$$|x(0)|\ll1 \quad \text{and} \quad |y(0)|\ll3\gamma. \qquad (3\cdot15)$$

Comparing this with Eq. (3·8), we see that an inflationary stage of sufficient duration is realized in our model. Therefore if the quantum effect, i.e., the noise $f(u)$ could be neglected, the slow roll-over phase would be realized within a few expansion times for the parameters of our model in the assumed range.[14),15)] However, as it is apparent from Eq. (3·13), the realization of the stage described by the slow rolling approximation does not imply the realization of the actual slow roll-over phase in which the motion of x can be described by the classical slow roll-over equation of motion; rather, the noise $f(u)$ plays an important role and the motion of x is quite stochastic at $u\gtrsim1/\gamma$. Thus one may call this era *the stochastic inflationary stage.* The stage at which x undergoes the actual classical slow rolling and the condition for the realization of such a stage will be discussed in the next section.

To understand the quantitative behavior of the scalar field, it is sometimes more convenient to consider the probability distribution function than to deal directly with the solution of stochastic Langevin equations. Formally, the probability distribution function $W(x, y; u)$ can be expressed as[16]

$$W(x, y; u)=\langle\delta(x-x_f(u))\delta(y-y_f(u))\rangle\,, \tag{3·16}$$

where $x_f(u)$ and $y_f(u)$ are the solutions of Langevin equation (3·9) and $\langle\cdots\rangle$ is the average with respect to the distribution of the noise f determined by Eq. (3·10). Then using the method of Kubo's stochastic Liouville equations,[16] one can derive the Fokker-Planck equation corresponding to the stochastic process (3·9). In the present case, because Eq. (3·9) is linear in x and y, it is straightforward to solve for the probability distribution function $W(x, y; u)$.[13] For $u \gtrsim 1/\gamma$, and retaining the terms up to the next leading order of γ this time, we find[8]

$$W(x, y; u)\approx\frac{1}{2\pi}\left(\frac{(3\gamma)^3}{a^2(e^{2u}-1)}\right)^{1/2}\exp\left[-\frac{(3\gamma)^3}{2a}(x-y)^2-\frac{1}{2a(e^{2u}-1)}x^2\right], \tag{3·17}$$

where we have assumed $x(0)=y(0)=0$ for simplicity. This manifestly shows that the dispersion perpendicular to the $y=x$ line stays at a very small value $\sim a/(3\gamma)^3$ independent of time, while that in the direction of x-axis grows as $\sim a(e^{2u}-1)$. Note that in the limit of perfect slow rolling ($\gamma\to\infty$), Eq. (3·17) reduces to

$$W(x, y; u)\to\delta(y-x)P(x; u)\,, \tag{3·18}$$

where

$$P(x; u)=\frac{1}{\sqrt{2\pi a(e^{2u}-1)}}\exp\left[-\frac{x^2}{2a(e^{2u}-1)}\right]. \tag{3·19}$$

From the distribution function $P(x; u)$ (or equivalently the dispersion $\langle x^2\rangle=a(e^{2u}-1)$, which determines the form of $P(x; u)$ completely), one can gain an insight into an important feature of the stochastic inflationary stage. At the stage $u\ll 1$, the behavior of $\langle x^2\rangle$ is identical to the case of a free Brownian particle.[2),17)] In terms of the original variable φ, one has

$$\langle\varphi^2(t)\rangle=\frac{H^3}{4\pi^2}t\,, \tag{3·20}$$

which is the well-known result for a massless minimally coupled scalar field in de Sitter space.[11),12)] Thus φ, which is the mean value of the scalar field in a region of size $L\lesssim(\epsilon H)^{-1}$, changes by $\pm H/2\pi$ in a time step H^{-1}.[2),17)] However there is one important difference between the present case and a simple Brownian motion. Consider a region of size $L\lesssim(\epsilon H)^{-1}$ in the universe at $t=0$ in which the mean value of the scalar field is φ_0. In every time interval H^{-1}, the universe expands e times and the number of regions of size L increases e^3 times. In each of these regions, the scalar field takes either of the values $\varphi_0\pm H/2\pi$ randomly.[5),6)] Thus the process is much more similar to the multiplication of cells having certain genic information: When the fission of a cell occurs, a part of the genic information would be varied randomly due to external disturbances and transmitted to daughter cells. In this sense, the probability distribution function P (or W in general), initially having a delta function peak at a certain value of φ (or at a certain point in the phase space (φ, v)), can be interpreted as representing the spatial structure of a comoving region of initial size L, and the expectation value $\langle Q\rangle$ can be regarded as the spatial average of a quantity Q over this comoving region; the ensemble consists of regions of size L filling out this comoving volume.

In principle, we should analyze the original equations which involve the spatial dependence of φ (i.e., Eqs. (3·4) and (3·5)) in order to investigate the spatial structure of the inflationary universe. However, an approximate but quite accurate description of the spatial structure can be obtained from the spatially averaged, slow roll-over version of them (i.e., Eqs. (3·14) and (3·10)), if one takes into account the picture of cellular fission mentioned above. Let us show why this seemingly rough approximation turns out to be so good. Since the following arguments apply to general models of inflation, equations will be expressed in terms of the original variables with dimensions.

The picture that the universe consists of mutually independent regions of size $L \lesssim (\epsilon H)^{-1}$ corresponds to approximating the noise correlation function (2·14) by

$$\langle\sigma(x_1)\sigma(x_2)\rangle = \frac{H^3}{4\pi^2}\delta(t_1-t_2)\theta(L-a(t)|\boldsymbol{x}_1-\boldsymbol{x}_2|)\,, \tag{3·21}$$

that is, replacing the 0-th order spherical Bessel function j_0 by the step function θ with ϵH being replaced by L^{-1} at the same time. On the other hand, the spatially averaged, slow roll-over version of the basic stochastic equation (2·13) is

$$\dot{\varphi} = -\frac{V'(\varphi)}{3H} + \sigma\,. \tag{3·22}$$

Now assuming $V'(\varphi) \approx M^2\varphi$, where M^2 is nearly constant in time, it is straightforward to derive the spatial correlation function of φ from Eqs. (3·21) and (3·22). The result is

$$\langle\varphi(\boldsymbol{x}+\boldsymbol{r}, t)\varphi(\boldsymbol{x}, t)\rangle = \frac{3H^4}{8\pi^2 M^2}\theta(a(t)L-r_p)$$
$$\times\left[\theta(L-r_p)(1-e^{-(2M^2/3H)t}) + \theta(r_p-L)\left\{\left(\frac{L}{r_p}\right)^{2M^2/3H^2} - e^{-(2M^2/3H)t}\right\}\right]. \tag{3·23}$$

Here $r_p = a(t)r$ is the physical separation length and the length L plays the role of a regularization scale; the short distance singularity is removed since we have smeared the field φ over the scale L. In addition, there arises the over-all factor $\theta(a(t)L-r_p)$ which plays the role of an infra-red cutoff at a constant comoving scale. However, since what we are interested in is the dynamics within a fixed comoving scale, this factor plays no essential role. Hence it will be ignored hereafter. Interestingly enough, in the limit $t\to\infty$, Eq. (3·23) agrees with the behavior of the exact two-point function for Bunch-Davies vacuum at large r_p,[12] provided that one chooses $L=H^{-1}$. One can also see that Eq. (3·23) with $L=H^{-1}$ yields the correct form in the limit $M^2\to 0$,

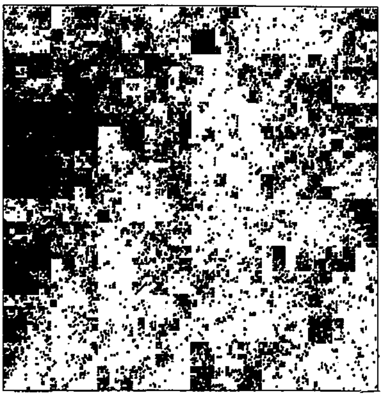


Fig. 1. A characteristic spatial pattern of the scalar field at the early stage of the inflationary universe. The black regions are where the absolute value of the scalar field is greater than $\sqrt{\langle\varphi^2\rangle}\simeq 0.6H$. See § 5 for details.

$$\langle\varphi(\boldsymbol{x}+\boldsymbol{r},\ t)\varphi(\boldsymbol{x},\ t)\rangle=\frac{H^2}{4\pi^2}\times[Ht-\theta(r_p-H^{-1})\ln(Hr_p)]\ . \qquad (3\cdot 24)$$

This is in agreement with the result obtained in Refs. 12) and 18).

Thus, at least at a stage when one can neglect the time variations of H and M due to that of φ (i.e., at the early stage of inflation at which $|M^2|\varphi^2\ll V(0)$ and $|M^2|\ll H^2$ are valid), the spacetime structure of the inflationary universe is accurately described by the approximation expressed in Eqs. (3·21) and (3·22). Then we may naturally expect the same approximation to be valid even under situations such that the non-linearity of $V(\varphi)$ and/or the φ-dependence of H could not be neglected. This implies that one can visualize the actual spacetime structure by dividing a fixed comoving region into regions of horizon size and carrying out a simple numerical simulation of stochastic solutions to Eq. (3·22) in each of these regions. Such a simulation has been done by Aryal and Vilenkin[6)] recently. Following them, we call these regions of horizon size "h-regions" in the rest of the paper. We have also carried out a similar but more elaborate numerical simulation. A typical spatial structure at the early stage of inflation is depicted in Fig. 1. Details about our numerical simulation will be presented in § 5.

Finally, we mention a bit delicate problem which one encounters if one attempts to extend the above picture to situations in which the slow rolling approximation fails. In such cases, we must return to the original equation (2·13) defined on the phase space (φ, v). Then we find that the noise term for v involves $M^2=\langle V''(\varphi)\rangle$. However, if each h-region of a given comoving region should be regarded as independent of the other h-regions, it is quite unnatural that the expectation value, which is interpreted as the spatial average over the comoving region, should play any role. Hence it seems more natural to think $M^2=V''(\varphi)$, i.e., to regard M^2 as determined locally in each h-region. Nevertheless, it seems that results would not depend too much on the choice of M^2. This is because the situations in which the slow rolling approximation is invalid are either the stage right after the beginning of inflation when $M^2\approx V''(0)=$const is a good approximation or the final stage of inflation at which the scalar field should be coherent over a large number of h-regions so that $\langle V''(\varphi)\rangle\approx V''(\varphi)$ holds. Hence, the present issue is probably inessential in practice.

§ 4. Universe in the midst of inflation

In this section, we consider the universe in the midst of inflation at which both the potential force $V'(\varphi)$ and the φ-dependence of the Hubble parameter H are nonnegligible but actually play important roles. In order to avoid inessential complications, we adapt the slow rolling approximation. Thus our starting point would be Eqs. (3·21) and (3·22). However, as discussed in the previous section, the probability distribution of φ at a fixed spatial point may be interpreted as the probability distribution of the number of h-regions characterized by the value φ in a given comoving region of the universe. Therefore, we can gain a good knowledge of spatial features of the inflationary universe by putting $\boldsymbol{x}_1=\boldsymbol{x}_2$ in Eq. (3·21) and ignoring the spatial dependence of φ in Eq. (3·22). On the other hand, we take into account the φ-dependence of H in Eqs. (3·21) and (3·22). Hence the basic equations for the following discussions are

$$\dot{\varphi}=-\frac{V'(\varphi)}{3H(\varphi)}+\eta\,, \tag{4·1}$$

$$\langle\eta(t_1)\eta(t_2)\rangle=\frac{H(\varphi)^3}{4\pi^2}\delta(t_1-t_2)\,, \tag{4·2}$$

where and in the rest of this section, we use the symbol η for the noise originally denoted by σ, since the symbol σ will be used for a measure of dispersion of the scalar field below.

We focus on a comoving region in which the scalar field had a certain value φ_0 and whose size was H^{-1} initially. In particular, we are interested in the condition for the region to enter the actual slow roll-over phase (i.e., the stage at which the time evolution of the scalar field is well approximated by the classical equation of motion under the slow rolling approximation) through which the region would eventually develop into a domain of the universe which is sufficiently large and homogeneous, like the universe observed today. Note that "initially" here does not mean the beginning of inflation: At the beginning of inflation, the dispersion of the scalar field would have grown as Eq. (3·20) and after a while there would be a number of h-regions having various values of φ. We are focusing on one of such h-regions and reset the time to zero, which is what we mean by "initial".

Before analyzing the stochastic process given by Eqs. (4·1) and (4·2), we must realize an important effect of the φ-dependence of H to the spacetime structure of the inflationary universe. In the case of the double-well potential model, Eq. (3·1), H is expressed in the form,

$$H^2\approx H_*{}^2\left(1-\frac{\lambda}{m^2}\varphi^2\right)^2, \tag{4·3}$$

where H_* corresponds to the Hubble parameter appeared in the previous discussions (i.e., $H_*{}^2=8\pi V(0)/3m_{\rm pl}^2$) and the slow rolling approximation has been employed to neglect the v-dependence of H. It is apparent that the expansion is faster for a region with a smaller value of φ, which is generally true for all models of new inflation. Therefore the physical volume of a comoving region would be eventually dominated by those h-regions with smaller values of φ and taking this effect into account is crucial for a correct understanding of the spacetime structure of the inflationary universe.[5),9)]

The above comment implies that the probability distribution function P considered in the previous section would not be as useful as one might have originally expected. Consider, for example, the mean value of φ^2 in a comoving region of the universe in which there are $N(\gg 1)$ h-regions at time t. Following our interpretation of P, we would calculate $\langle\varphi^2(t)\rangle$ as

$$\langle\varphi^2(t)\rangle=\int\varphi^2 P(\varphi;\, t)d\varphi=\frac{1}{N}\sum_{i=1}^{N}\varphi_i{}^2\,, \tag{4·4}$$

where the index i is the number assigned to each h-region and φ_i is the value of the scalar field in the i-th h-region. Obviously, however, if H varies appreciably from h-regions to h-regions due to differences in the values of φ, Eq. (4·4) does not describe the true spatial average over the region under consideration. Instead, the correct spatial average should be given in the form,

$$\langle\varphi^2(t)\rangle_p=\frac{\sum_{i=1}^{N}\varphi_i{}^2(t)a^3[\varphi_i(t)]}{\sum_{i=1}^{N}a^3[\varphi_i(t)]}\,, \tag{4·5}$$

where the suffix p indicates that the average takes into account the weight of proper (physical) volume and the weight factor $a^3[\varphi(t)]$ is given by

$$a^3[\varphi(t)]=\exp\left[3\int_0^t dt' H(\varphi(t'))\right]. \tag{4·6}$$

As it is apparent from this expression, the weight factor a^3 depends not only on the value of the scalar field at time t but also on the evolutionary history of it.

Comparing the expression (4·4) with (4·5) and recalling the formal expression of a conventional probability distribution function as given in Eq. (3·16), one can easily guess the corresponding form of a new probability distribution function P_p which yields the average such as Eq. (4·5). It is

$$P_p(\varphi;\, t)=\frac{\langle\delta(\varphi-\varphi_\eta(t))\exp[3\int_0^t dt' H(\varphi_\eta(t'))]\rangle}{\langle\exp[3\int_0^t dt' H(\varphi_\eta(t'))]\rangle}\,, \tag{4·7}$$

where φ_η is the solution of Eq. (4·1). Then following the standard method,[16] a modified Fokker-Planck equation for $P_p(\varphi;\, t)$ is obtained to be[9]

$$\frac{\partial}{\partial t}P_p(\varphi;\, t)=\frac{\partial}{\partial\varphi}\left(\frac{V'(\varphi)}{3H(\varphi)}P_p+\frac{1}{8\pi^2}H(\varphi)^{3/2}\frac{\partial}{\partial\varphi}H(\varphi)^{3/2}P_p\right)$$
$$+3(H(\varphi)-\langle H(t)\rangle_p)P_p\,, \tag{4·8}$$

where

$$\langle H(t)\rangle_p=\int H(\varphi)P_p(\varphi;\, t)d\varphi\,.$$

The above equation is both non-linear in P_p and non-local in φ. Hence it is almost impossible to solve it analytically in general. However, in the case of the double-well potential model, Eq. (3·1), it can be solved approximately by an iterative method and the solution turns out to describe characteristic features of the inflationary universe quite accurately.

For convenience, let us rewrite Eq. (4·8) in terms of the dimensionless variables introduced in the previous section. Noting the expression (4·3) for the Hubble parameter and regarding H appeared in the definitions (3·3) and (3·6) of the dimensionless variables and parameters as H_*, Eq. (4·8) becomes

$$\frac{\partial}{\partial u}P_p(x;\,u)=\frac{\partial}{\partial x}\left(-xP_p+\alpha(1-x^2)^{3/2}\frac{\partial}{\partial x}(1-x^2)^{3/2}P_p\right)$$
$$-3\gamma(x^2-\langle x^2(u)\rangle_p)P_p\,, \tag{4·9}$$

where

$$\langle x^2(u)\rangle=\int x^2P_p(x;\,u)dx\,.$$

This equation for P_p can be further simplified by noting that $|x|$ must be sufficiently smaller than unity during the phase in which the slow rolling approximation is valid. This is because the points $x=\pm 1$ are the absolute minima of the potential and x approaching one of the minima would start rolling down the potential slope rapidly (and eventually undergo damped oscillations around the minimum) to invalidate the slow rolling approximation. Hence in accordance with our basic assumption, we can replace the factor $(1-x^2)^{3/2}$ in Eq. (4·9) by 1. In other words, we consider only those distribution functions that have their supports at $x^2\ll 1$, hence are meaningful under the slow rolling approximation. Further, we restrict our discussion to distribution functions which have a delta function peak at $u=0$; $P_p(x;\,0)=\delta(x-x_0)$, since we are interested in the time evolution of a comoving region which has a particular value of the scalar field initially.

With the simplification mentioned above, Eq. (4·9) now involves only terms up to quadratic order in x. Hence we may expect the solution to have a Gaussian form. Note that if the terms proportional to γ in Eq. (4·9), which represents the proper volume effect, were absent, the solution would have the form,[13]

$$P(x;\,u)=(2\pi\sigma_0(u))^{-1/2}\exp\left[-\frac{(x-x_c)^2}{2\sigma_0(u)}\right], \tag{4·10}$$

where $x_c=x_0e^u$ is the classical solution of the slow roll-over equation of motion and σ_0 is given by

$$\sigma_0(u)=\alpha(e^{2u}-1)\,. \tag{4·11}$$

Thus the mean value and dispersion of x would be

$$\langle x\rangle=x_c(u)\qquad\text{and}\qquad\langle x^2\rangle=\sigma_0(u)\,. \tag{4·12}$$

In reality, the presence of the terms proportional to γ makes the values of $\langle x\rangle$ and $\langle x^2\rangle$ deviate from the above. In order to take this effect into account, we postulate the following form for P_p,

$$P_p(x;\,u)=N(u)\exp\left[-\frac{(x-x_c(u))^2}{2\sigma(u)}-\frac{1}{2}g(u)x^2\right], \tag{4·13}$$

where $N(u)$, $\sigma(u)$ and $g(u)$ are some unknown functions. This form has been guessed since the probability should be smaller for larger values of x^2. Note that, from the normalization condition, Eq. (4·13) can be rewritten in the form,

$$P_p(x;u)=\left(\frac{1+g\sigma}{2\pi\sigma}\right)^{1/2}\exp\left[-\frac{(1+g\sigma)}{2\sigma}\left(x-\frac{x_c}{1+g\sigma}\right)^2\right]. \tag{4·14}$$

This implies that $\langle x\rangle_p$ and $\langle x^2\rangle_p$ are expressed as

$$\langle x\rangle_p=\frac{x_c}{1+g\sigma},$$

$$\langle x^2\rangle_p=\frac{\sigma}{1+g\sigma}+\langle x\rangle_p{}^2. \tag{4·15}$$

Hence the ansatz (4·13) corresponds to the parametrization of $\langle x\rangle_p$ and $\langle x^2\rangle_p$ in terms of two unknown functions σ and g.

Substituting Eq. (4·13) into Eq. (4·9) and equating the coefficients of equal powers of x on both sides of the equation, we find

$$x^1:\quad \dot\sigma x_c-\sigma\dot x_c=(2\alpha+\sigma+2\alpha g\sigma)x_c, \tag{4·16}$$

$$x^2;\quad \dot\sigma-\dot g\sigma^2=2\alpha+\sigma+4\alpha\sigma g+\sigma^2(2g-6\gamma+\alpha g^2). \tag{4·17}$$

In addition, there is an equation for the coefficients of the 0-th power of x. However it is redundant if the Gaussian postulate is valid. Actually we found it is consistent with the above equations. Hence we leave it aside from the argument. Recalling that α is a very small parameter, we can solve the above equations by the perturbation expansion with respect to α. First note that σ is at most of order α. Then Eq. (4·16) implies that σ takes the form,

$$\sigma(u)=\sigma_0(u)+\alpha^2 s(u)+O(\alpha^3), \tag{4·18}$$

where $\sigma_0(u)$ has been given in Eq. (4·11) and $s(u)$ is a function independent of α. Inserting Eq. (4·18) into Eqs. (4·16) and (4·17), we obtain the equations for g and s:

$$\dot g=-\frac{2e^{2u}}{e^{2u}-1}g+6\gamma,$$

$$\dot s=2s+2(e^{2u}-1)g. \tag{4·19}$$

These equations can be readily solved. Then σ and g are found to be

$$\sigma(u)=\alpha[(1-6\alpha\gamma)(e^{2u}-1)+6\alpha\gamma u(e^{2u}+1)+O(\alpha^2)],$$

$$g(u)=3\gamma\left[\left(1-\frac{2u}{e^{2u}-1}\right)+O(\alpha)\right]. \tag{4·20}$$

Thus the solution for P_p is given by Eq. (4·14) with σ and g given by Eq. (4·20).

As discussed in the previous section, most of h-regions in the universe would have been in the stochastic inflationary phase and the scalar field would have been random walking during the early stage of inflation. Then soon or later there would appear such h-regions as those having a sufficiently large value of the scalar field. In such regions, the scalar field would begin to roll down the potential slope slowly, following the classical equation of motion. Let us call this stage *classical slow roll-over phase.* We note that, since the quantum noise force would be negligible, the value of φ would evolve coherently over a comoving region of the universe which entered this phase. This implies that the region would have a potentiality to become sufficiently homogeneous and isotropic. Thus it is at least necessary that this stage should last long enough in order to solve the horizon problem.

Hence, we would like to know a critical value of φ beyond which the classical slow roll-over phase is realized. That is, we consider the probability distribution function $P_p(x;0)=\delta(x-x_0)$ and derive the condition for the initial value x_0 such that the peak of $P_p(x;u)$ remains sufficiently close to the classical trajectory $x_c=x_0e^u$ and the dispersion of x around it remains sufficiently small for all times until x reaches one of the absolute minima where inflation ends. These requirements are expressed as

$$\langle(x-\langle x\rangle_p)^2\rangle_p \ll x_c^2 \quad \text{and} \quad |\langle x\rangle_p - x_c| \ll x_c\,. \tag{4·21}$$

From Eq. (4·15) or (4·14), they simply imply $g\sigma\ll 1$. From Eq. (4·20), one can readily observe that it is satisfied if

$$x_0^2 \gg 3\alpha\gamma x_c^2 \qquad \text{for } u\gg 1\,. \tag{4·22}$$

Then, since the inflation ends at $|x_c|\sim 1$, the condition for a comoving region of the universe to enter the classical slow roll-over phase as a whole is finally expressed as[7),9)]

$$x_0^2 \gg 3\alpha\gamma\,, \tag{4·23}$$

or in terms of the original parameters,

$$\varphi_0^2 \gg \frac{m^2}{8\pi^2}\left(\frac{3H_*^2}{m^2}\right)^3 = \frac{\pi m^8}{\lambda^3 m_{\rm pl}^6}\,. \tag{4·24}$$

Incidentally, since x_0^2 must be much smaller than unity, the condition (4·23) also implies that the parameters of a successful inflationary universe model must satisfy the additional condition,

$$3\alpha\gamma = \frac{\lambda}{8\pi^2}\left(\frac{3H_*^2}{m^2}\right)^3 \ll 1\,. \tag{4·25}$$

We mentioned before that the realization of the classical slow roll-over phase in a comoving region of the universe should be necessary for that region to become a Friedmann-like homogeneous and isotropic domain. It is then natural to ask what the exact relation between the classical slow rolling condition and the degree of homogeneity and isotropy is. To answer this question, let us recall that the gauge-invariant amplitude of inhomogeneity of a universe is characterized by the amplitude of density perturbations on the Hubble horizon scale.[19)] In an inflationary universe model, the conventional argument yields[20)]

$$\left(\frac{\delta\rho}{\rho}\right)_k \approx \left(\frac{H^2}{2\pi|\dot{\varphi}|}\right)_H, \tag{4·26}$$

where $(\delta\rho/\rho)_k$ is the amplitude of a density perturbation with comoving wavenumber k at the horizon crossing time in the Friedmann era and the subscript H on the right-hand side denotes the value evaluated at the time at which the wavelength leaves the horizon during the inflationary stage. Using the slow roll-over equation of motion for φ and rewriting Eq. (4·26) in terms of α, γ and x, we find

$$\left(\frac{\delta\rho}{\rho}\right)_k \approx \sqrt{2\alpha\gamma}|x_H|^{-1}. \tag{4·27}$$

Comparing Eqs. (4·23) and (4·27) and noting that $x_H \geq x_0$, one immediately sees that $(\delta\rho/\rho)_k \ll 1$ is always satisfied in domains of the universe which went through the classical slow roll-over phase. Taking another point of view, this fact implies that the requirement on the parameters of the model, Eq. (4·25), is a necessary condition for $(\delta\rho/\rho)_k \ll 1$ (if one requires $(\delta\rho/\rho)_k \lesssim 10^{-4}$ as to be consistent with observations of the cosmic microwave anisotropies,[21] Eq. (4·25) should be correspondingly modified). Although we have discussed a particular model of inflation here, it can be shown that the present conclusion applies also to more general models.[9]

We now consider the dynamics of a comoving region with x_0 smaller than the critical value $x_* = O(\sqrt{\alpha\gamma})$. In such a case, $g\sigma$ would become eventually greater than unity and $\langle x\rangle_p$ would tend to zero, since $x_c \propto e^u$ while $g\sigma \propto e^{2u}$; see Eq. (4·15). Thus one expects the probability distribution to become independent of the initial condition and approaches the one with $x_0=0$ asymptotically at large u. Hence let us put $x_0=0$ for simplicity. Note that this initial condition would describe the global feature of the inflationary universe, since at the very beginning of inflation the probability distribution of x over the whole universe would have been narrowly peaked at $x=0$ (see Eq. (3·8)). In this case, in the limit $u \to \infty$, the dispersion of x approaches a constant value,

$$\langle x^2\rangle = \frac{\sigma}{1+g\sigma} \simeq g^{-1} \to (3\gamma)^{-1}. \tag{4·28}$$

This implies that the dominant physical volume of the universe would continue to expand exponentially forever and the inflation would never end as a whole. That is, there is continuous generation of mini-universe which would eventually turn into Friedmann-like universes out of the highly stochastic, quantum fluctuation-dominated, never-ending inflationary universe. This feature of the inflationary universes has been found by Linde[22] in the context of chaotic inflation. However, we note that the feature has been originally found in the old scenario.[23] Thus it seems to be a common feature of all inflationary universe scenarios.

Returning to the new inflationary universe scenario, the fact that inflating regions in the whole universe form a self-similar fractal structure has been pointed out by Aryal and Vilenkin.[6] Using the asymptotic solution for P_p considered above, we can gain more information about the nature of inflating regions. The physical volume fraction of the universe occupied by regions in which the scalar field is in the stochastic inflationary stage is estimated as

$$f(|x|<x_*)=2\int_0^{x_*}P(x;\,+\infty)dx$$
$$=\frac{2}{\sqrt{\pi}}\int_0^{C\alpha^{1/2}\gamma}\exp[-t^2]dt\,, \tag{4·29}$$

where the critical value x_* is set equal to $C\sqrt{2\alpha\gamma/3}$ with C being a constant of order unity. Thus the dominant volume fraction of the universe is in the stochastic inflationary phase if $\alpha\gamma^2\gg1$, while that is in the classical slow roll-over phase if $\alpha\gamma^2\ll1$.

Finally let us estimate the duration of the classical slow roll-over phase in a comoving region which entered that phase. Since the scalar field would start rolling down classically once it passed over the critical value x_*, it can be regarded as the initial value of x which undergoes the classical slow roll-over phase. Hence we may put $x(u)=x_*e^{u-u_i}$ where u_i is the (dimensionless) time a region of size H^{-1} entered the slow roll-over phase. On the other hand, since the slow roll-over phase would end when $H^2\simeq m^2$, the final time u_f can be estimated from the equation,

$$\frac{3H(t_f)^2}{m^2}=\gamma[1-x(u_f)^2]^2=O(1)\,. \tag{4·30}$$

Thus, apart from an inessential logarithmic ambiguity, the duration of the slow roll-over phase is estimated as

$$N\equiv\int_{t_i}^{t_f}H(t)dt\simeq\frac{3H_*^2}{2m^2}\ln\left[\frac{8\pi^2}{\lambda}\left(\frac{m^2}{3H_*^2}\right)^3\right], \tag{4·31}$$

where N is the number of e-folds of the classical slow roll-over phase a domain of the universe would undergo. Moreover, if one requires that the density perturbation amplitude on the comoving scale corresponding to the initial h-region at $u=u_i$ should be equal to a specified value $|\delta_k|$, we should rather set $x(u)=|\delta_k|^{-1}x_*e^{u-u_i}$ and the corresponding number of e-folds is expressed as

$$N(\delta_k)\simeq\frac{3H_*^2}{2m^2}\ln\left[\frac{8\pi^2}{\lambda}\left(\frac{m^2}{3H_*^2}\right)^3|\delta_k|^2\right]. \tag{4·32}$$

In order to solve the homogeneity problem of the presently observed universe, the number $N(\delta_k)$ is required to be greater than ~65 for $|\delta_k|^2\lesssim10^{-8}$ (see Ref. 20) for further details on the required number of e-folds).

§ 5. Numerical results

In this section, we present the results of numerical simulation of the stochastic scalar field for the double-well potential model. For convenience, we use the dimensionless variables and parameters exclusively in what follows. Following the arguments in § 3, we adapt the picture of cellular fission. Thus we focus on the behavior of the scalar field averaged over an h-region and investigate the spacetime structure of a comoving region of initial size H^{-1}.

The stochastic equations we have solved are

$$\frac{dx}{du}=y+f(u)\,,$$

$$\frac{dy}{du}=-3\gamma\left[y\sqrt{(1-x^2)^2+\frac{2}{3\gamma}y^2}-x(1-x^2)\right]+f(u) \tag{5·1}$$

with

$$\langle f(u_1)f(u_2)\rangle=2\alpha\delta(u_1-u_2)\,. \tag{5·2}$$

In addition to the neglect of the spatial dependence, these equations differ from the ones given in § 3 (Eqs. (3·4) and (3·5)) in the two respects: First, the (φ, v)-dependence of the Hubble parameter H,

$$h^2(x, y)\equiv\frac{H^2}{H_*{}^2}=\left((1-x^2)+\frac{2}{3\gamma}y^2\right), \tag{5·3}$$

has been taken into account in the friction term of the equation of motion. Thus Eq. (5·1) would yield the exact solution of the classical equation of motion for the scalar field if the noise f were absent. Second, the term $\langle x^2\rangle$ in the coefficient of f on the second line of Eq. (3·4) has been put equal to zero. This approximation should be valid since $\langle x^2\rangle\ll1$ at the stochastic inflationary stage (i.e., when the quantum noise dominates over the potential force) while the potential force dominates when $\langle x^2\rangle$ becomes large. The only possible failure of the approximation would occur at the final stage of inflation when the scalar field is close to one of the potential minima, *if* reheating processes were so efficient that a huge friction force would suppress the potential force and the noise term would become important again. However, since we take no account of any reheating processes, the $\langle x^2\rangle$ term can be consistently neglected.

Strictly speaking, corresponding to the first point mentioned above we should take into account the (φ, v)-dependence of f through that of H in Eq. (5·1) (see e.g., Eq. (4·2) or (4·9)). However, by a reasoning similar to the second point above, it has been neglected.

The algorithm by which we have solved Eq. (5·1) is based on the method proposed by Helfand.[24] It is a Monte Carlo method combined with a Runge-Kutta method for integrations of ordinary differential equations. As for the latter, we have incorporated the second order Runge-Kutta method. We have chosen a time step for numerical integration to be $\Delta u=0.001$. In most cases, the number of samples for an individual simulation was chosen to be 3×10^3. However, in some cases it was increased up to 10^4 in order to obtain a result with a better accuracy. Although these numbers are not satisfactorily large enough, we are forced to adopt them due to the limitations of the computation time and the size of the memory. For each run, the volume factor a^3 has been calculated at each time step and stored in the memory for the evaluation of distribution functions and expectation values $\langle\cdots\rangle_p$ of various quantities.

In order to understand the effect of the (x, y)-dependence of physical volume of an h-region, we have evaluated conventional distribution functions in several cases as well, in addition to "proper" distribution functions which take into account the volume factor. The conventional distribution function is evaluated by the formula,

$$W_{(N)}(x, y, u)=\frac{1}{N}\sum_{i=1}^{N}\bar{\delta}(x-x_i(u))\,\bar{\delta}(y-y_i(u))\,, \tag{5·4}$$

while the proper distribution function, by

$$W_{P(N)}(x, y, u)=\frac{\sum_{i=1}^{N}\bar{\delta}(x-x_i(u))\,\bar{\delta}(y-y_i(u))a_i^3(u)}{\sum_{j=1}^{N}a_j^3(u)}\,, \tag{5·5}$$

where

$$a_i(u)=\exp\left[\gamma\int_0^u ds\; h(x_i(s), y_i(s))\right]$$

with h being the dimensionless expansion rate defined in Eq. (5·3), x_i and y_i are the values associated with the i-th sample, N is the number of samples, and $\bar{\delta}$ is a smeared delta function over a small phase space volume. The distribution functions $W_{(N)}$ and $W_{P(N)}$ would approach the exact distribution functions W and W_P, respectively, as N goes to infinity.

In order to confirm, as well as to improve the analytical estimate of the critical value x_* for realization of the classical slow roll-over phase, we have calculated $\langle x^2\rangle_P$ in various models under various initial values of x (in accordance with the discussions of §§ 3 and 4, in each case the initial condition is chosen as $(x, y)=(x_0, 0)$ for all the samples). A typical example of the two different temporal behaviors of $\langle x^2\rangle_P$ is

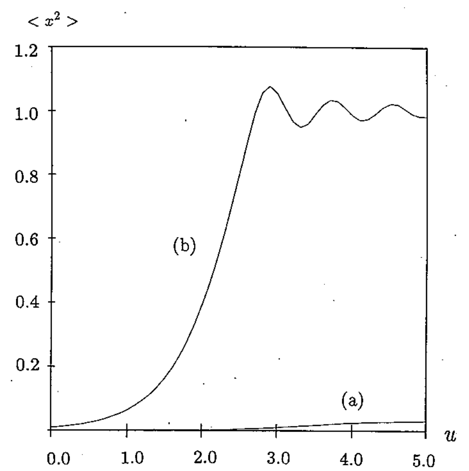


Fig. 2. The temporal behavior of $\langle x^2\rangle_P$ in the two typically different cases of the initial data; (a) $(x_0, y_0)=(0, 0)$ and (b) $(x_0, y_0)=(0.1, 0)$. The model parameters are $\alpha=5\times10^{-5}$ and $\gamma=10$, which imply the critical initial value of x to be somewhere around 0.03. The number of samples for each simulation is 10^4.

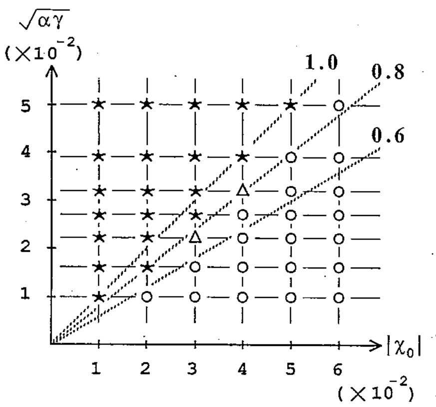


Fig. 3. Classification of various models and initial data. The cases in which the classical slow roll-over phase was realized are denoted by the circle ○ and those in which it was not, by the star ★. The cases denoted by the triangle △ are ambiguous ones due to the limited number of samples and the limited computation time. The dotted lines denoted by 1.0, 0.8 and 0.6 are the lines $\sqrt{\alpha\gamma}=A|x_0|$ with the respective values of A.

shown in Fig. 2 for a model with the parameters $\alpha=5\times10^{-5}$ and $\gamma=10$. For this model, the analytical estimate of the critical value is $x_*\sim\sqrt{3\alpha\gamma}\simeq0.039$. Figure 2(a) shows the case the initial value x_0 is smaller than x_*. The variance of the scalar field approaches asymptotically the value $(3\gamma)^{-1}\simeq0.033$, which agrees with our analytical estimate. On the other hand, Fig. 2(b) shows the case the initial value x_0 is larger than x_*. The variance of the scalar field increases monotonically up to $\langle x^2\rangle_p\simeq1$ and then oscillates around the potential minimum. Thus the classical slow roll-over phase is realized in this example.

In Fig. 3, the classification of various models with various initial values of x is shown according to whether the classical slow roll-over phase is realized or not.

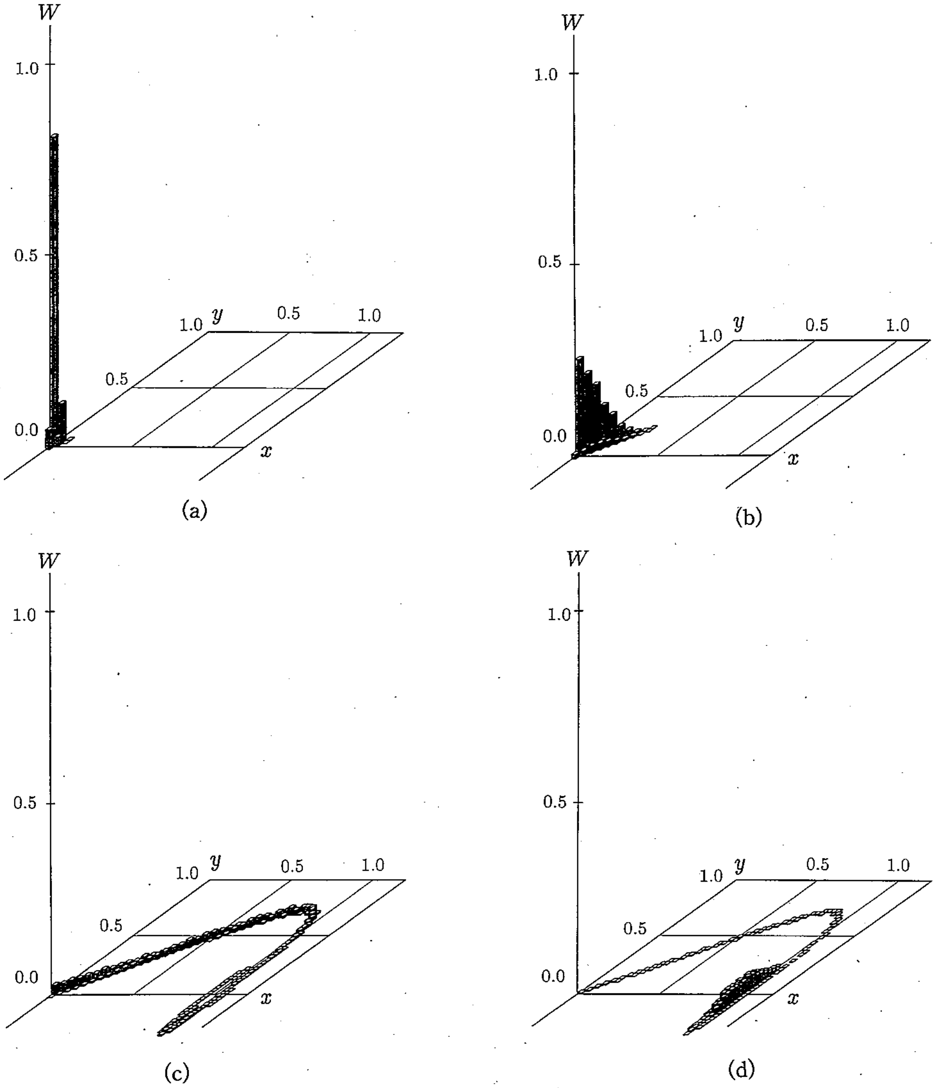


Fig. 4. The probability distribution function $W(x, y; u)$ with the initial distribution $W(x, y; 0) =\delta(x)\delta(y)$; W at (a)$=u=1.0$, (b) $u=2.5$, (c) $u=5.0$ and (d) $u=7.5$ are shown. Because of the symmetry $W(-x, -y; u)=W(x, y; u)$, only a half of the phase space ($x\geq0$) is shown. The model parameters are $\alpha=5\times10^{-5}$ and $\gamma=10$ and the number of samples is 5×10^3.

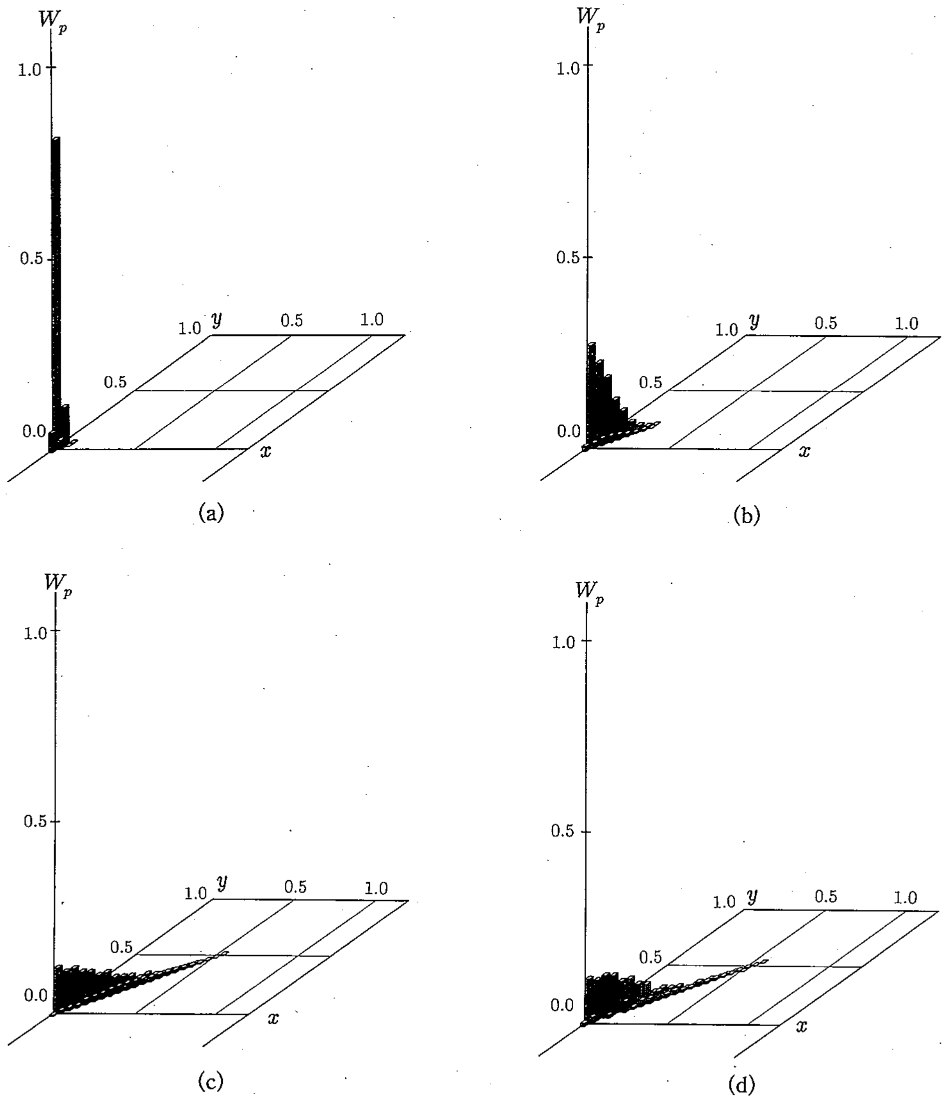
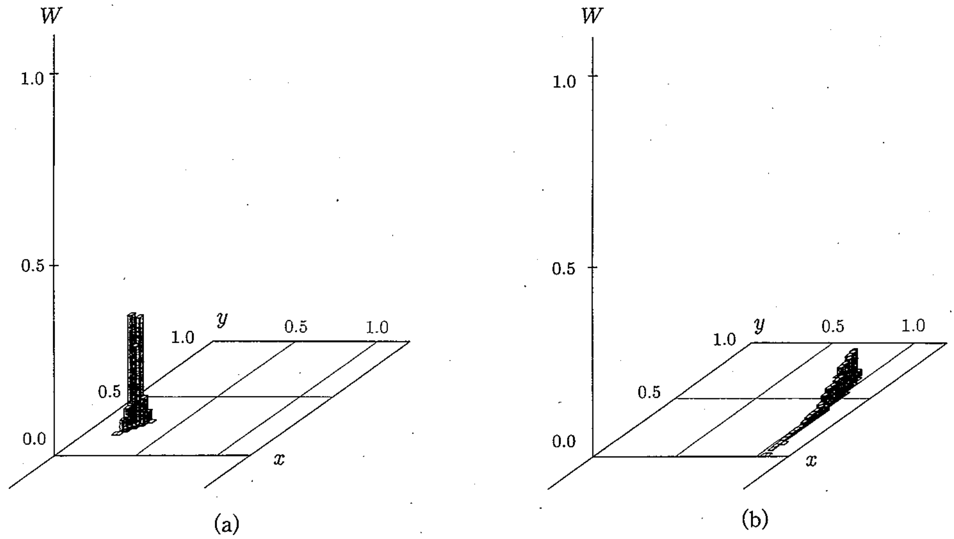


Fig. 5. The same as Fig. 4, but for the probability distribution function $W_p(x, y; u)$, which takes into account the volume effect.

Specifically, the circle ○ denotes the case when the value of the ratio $\langle x^2\rangle_p/\langle x^2\rangle$ at time $u \doteq 5$ is larger than 0.7 (i.e., the classical slow roll-over phase has been or is about to be realized; see Figs. 6 and 7 and discussions associated with it for a justification of this criterion), the star ★ when smaller than 0.1 (i.e., the never-ending inflation is undergoing) and the triangle △, otherwise. Due to the presence of the ambiguous cases marked by △, we were unable to determine the precise numerical factor for the critical value x_*. In principle, one could run the simulation as far as one wishes and eventually all the cases would fall into either of the class ○ or ★. However, due to the limited number of samples (3×10^3 in the present case) and the exponential growth of the volume factor a, the time $u=5$ is approximately the furthest one could go with a sufficient numerical reliability. Nevertheless, one may conclude from Fig. 3 that



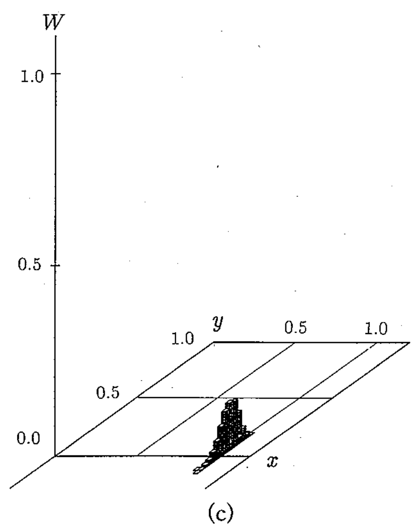
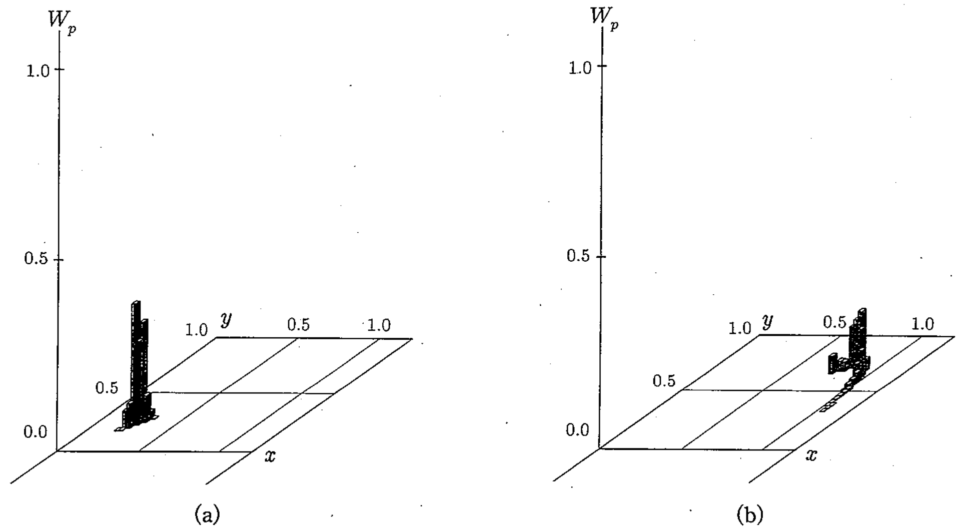


Fig. 6. The probability distribution function $W(x, y; u)$ with the initial condition $W(x, y; 0) = \delta(x-0.1)\delta(y)$, which corresponds to the case the classical slow roll-over phase is realized; W at (a) $u=1.0$, (b) $u=2.5$ and (c) $u=5.0$ are shown. The model parameters and the number of samples are the same as in Fig. 4.

$$x_* \simeq (1.3 \pm 0.3) \times \sqrt{\alpha\gamma}$$
$$\left(\Longleftrightarrow \varphi_* \simeq (0.5 \pm 0.1) \times \frac{H^3}{m^2}\right). \quad (5\cdot6)$$

To demonstrate manifestly the importance of the effect due to the volume factor a, the conventionally defined probability distribution (5·4) and the properly defined one (5·5) are shown in Figs. 4 and 5, respectivery, for the same parameters as in the case of Fig. 2 ($\alpha=5\times10^{-5}$ and $\gamma=10$), under the initial condition $x=y=0$ for all the samples. In both figures, the respective distribution functions at four different epochs are shown chronologically from (a) to (d). The characteristic difference between the distribution functions with and without the volume effect can be seen by comparing those at a late time, Figs. 4(d) and 5(d) ($u=7.5$). In Fig. 4(d), the peak of the distribution is well away from the origin; the calculated variance $\langle x^2 \rangle$ turns out to be about 0.95. On the other hand, the peak of the distribution shown in Fig. 5(d) is still rather close to the origin; actually for this distribution we found $\langle x^2 \rangle_p \simeq 0.033$, i.e., the universe is already in the stationary never-ending inflationary state.

On the other hand, in the case $x_0 > x_*$, the difference between $W_{(N)}$ and $W_{p(N)}$ is negligible. Such a circumstance is depicted in Figs. 6 and 7. One can also see that the dispersion is kept small and all the samples fall into the bottom of the potential



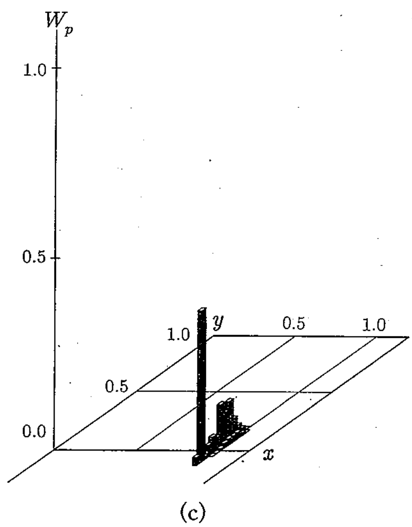


Fig. 7. The same as Fig. 6, but for the probability distribution function $W_p(x, y; u)$.

valley together.

Up until now, we have not dealt with cases with non-zero initial values of y in our numerical simulation, since y is expected to become vanishingly small within a couple of expansion times as discussed in § 3. To examine this is indeed the case, we have investigated a case with non-zero initial values of y numerically. Specifically, we have considered a case with the same model parameters as those in Fig. 2, 4, 5, 6 or 7, but with y_0 distributed over an interval $[-0.5, +0.5]$ uniformly while x_0 being set equal to zero. The resulting probability distributions $W_{P(N)}$ at several different epochs are depicted in Fig. 8. One can clearly see that the memory of non-zero y_0 is erased within one expansion time ($u=0.1$ in the present case) before the dispersion in the direction of x grows.

We have also simulated an actual spatial feature of the scalar field by following essentially the prescription given by Aryal and Vilenkin.[6] The simulation has been done in the following manner. For simplicity, the spatial dimension is restricted to two. Initially, the whole region of the simulated space corresponds to one h-region and we evolve one sample of the stochastic scalar field with certain initial data ($x_{(1,0)}$, $y_{(1,0)}$) for an interval of time $\Delta_* u=\ln 2/\gamma(H\Delta_* t=\ln 2)$ according to Eq. (5·1). At this time $u_1=\Delta_* u$, the universe is twice as big as it was initially. Hence we divide the space into 2×2 cells, each of which is regarded as a new h-region, assign the number $i(=1, 2, 3, 4)$ and the values $(x_{(i,1)}, y_{(i,1)})=(x_{(1,1)}, y_{(1,1)})$ to the i-th cell, where $(x_{(1,1)}, y_{(1,1)})$ are the values obtained for the original sample at u_1. Then the scalar field in each

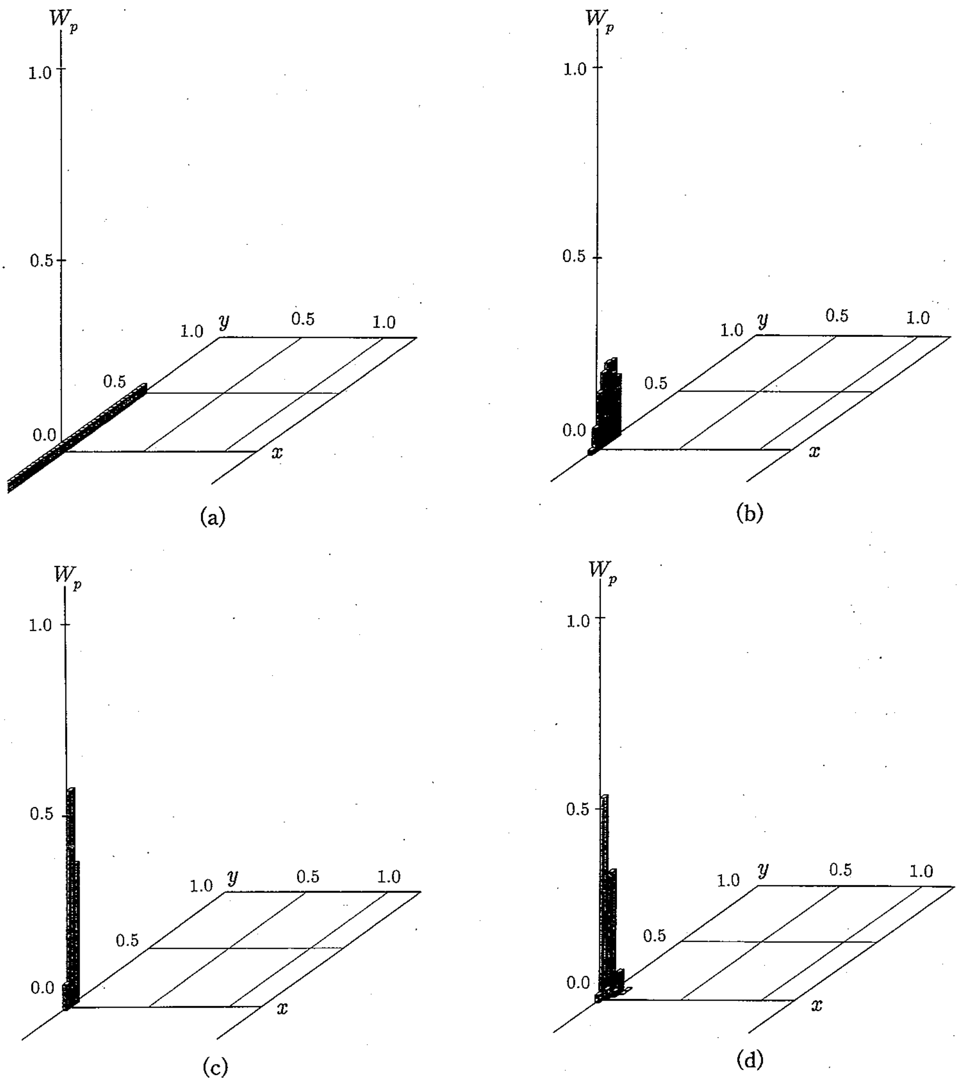


Fig. 8. The probability distribution function $W_p(x, y; u)$ in the case of non-zero initial dispersion in the velocity, y; W_p at (a) $u=0$, (b) $u=0.05$, (c)$=0.1$ and (d) $u=1.0$ are shown. The model parameters and the number of samples are the same as in Fig. 4.

cell is evolved independently for another interval $\Delta_* u$, each cell is divided further into 2×2 cells at $u_2=2\Delta_* u$ and the new number $j=i+i'(i'=0, 1, 2, 3)$ and the values $(x_{(i,2)}, y_{(i,2)})$ are assigned to every new cell in the old i-th cell where $(x_{(i,2)}, y_{(i,2)})$ are the values in the old cell at u_2. Then the same procedure is repeated at every time step $u_k=k\Delta_* u(k=3, 4, \cdots)$.

As for this simulation, we have chosen the model parameters as $\alpha=5\times10^{-5}$ and $\gamma=10$, the initial data as $(x_{(1,0)}, y_{(1,0)})=(0, 0)$ and a time step for the stochastic numerical integration of Eq. (5·1) as $\Delta u=(\ln 2)/10^3\simeq6.93\times10^{-4}$. The simulation has been stopped at time just before $u=u_9=(\ln 512)/10$, i.e., the final number of independent cells is $2^8\times2^8=65{,}536$. The resulting spatial feature of the universe has been shown in Fig. 1. In this simulation, all the h-regions are forced to have the same size.

Hence the volume effect discussed before is neglected. However, the neglect of the volume effect is justified since the total lapse of time for the simulation is still rather small; $u_9 \sim 0.62$ (i.e., $Ht \sim 6.2$). In other words, it would be very difficult to visualize the spatial feature of the inflationary universe on its global scale by this kind of simulation, since the number of h-regions increases exponentially and the volume effect becomes increasingly important as time goes on. Nevertheless, one can gain some feeling about how inflation proceeds. Here we would like to examine the power spectrum of the spatial fluctuations of φ, which would clarify the limit of applicability of this kind of numerical simulation.

Since the initial data for the above simulation are $x=y=0$ and it is run for only a few expansion times, the slow rolling approximation must be valid. Hence we can analytically evaluate the fluctuation spectrum from Eqs. (3·21) and (3·22) with $L = H^{-1}$. In 2-dimensional space, the Fourier spectrum is evaluated to be

$$\langle \delta\tilde{\varphi}(\boldsymbol{k}, t)\delta\tilde{\varphi}^*(\boldsymbol{k}', t)\rangle = |\delta\varphi(k, t)|^2(2\pi)^2\delta^{(2)}(\boldsymbol{k}-\boldsymbol{k}')$$

$$= \frac{2\pi H^2}{k^2}\left(\frac{k}{a(t)H}\right)^{-2m^2/3H^2}\int_{k/a(t)H}^{k/H} dz\, J_1(z)z^{2m^2/3H^2}\delta^{(2)}(\boldsymbol{k}-\boldsymbol{k}'), \tag{5·7}$$

where

$$\delta\tilde{\varphi}(\boldsymbol{k}, t) \equiv \int d^2x[\varphi(\boldsymbol{x}, t) - \langle\varphi(\boldsymbol{x}, t)\rangle]e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}, \tag{5·8}$$

and $J_1(z)$ is the first order Bessel function. Hence the power of fluctuations with respect to the logarithmic interval of k is given by

$$|\delta\varphi(k, t)|^2\frac{2\pi k^2}{(2\pi)^2} = \frac{H^2}{(2\pi)^2}\left(\frac{k}{a(t)H}\right)^{-2m^2/3H^2}\int_{k/a(t)H}^{k/H} dz\, J_1(z)z^{2m^2/3H^2}. \tag{5·9}$$

We note that, in the massless limit $m^2 \to 0$, the last integral can be done explicitly to yield

$$|\delta\varphi(k, t)|^2\frac{2\pi k^2}{(2\pi)^2} \to \frac{H^2}{(2\pi)^2}\left[J_0\left(\frac{k}{a(t)H}\right) - J_0\left(\frac{k}{H}\right)\right]; \qquad m^2 \to 0, \tag{5·10}$$

where $J_0(z)$ is the 0-th order Bessel function. On the other hand, in the realistic 3-dimensional case, one obtains from Eqs. (2·13) and (2·14),

$$\langle \delta\tilde{\varphi}(\boldsymbol{k}, t)\delta\tilde{\varphi}^*(\boldsymbol{k}', t)\rangle = |\delta\varphi(k, t)|^2(2\pi)^3\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}')$$

$$= \frac{4\pi^3 H^2}{k^3}\left(\frac{k}{\epsilon a(t)H}\right)^{-2m^3/3H^2}\theta(\epsilon a(t)H - k)\theta(k - \epsilon H)\delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}'), \tag{5·11}$$

and the power in the logarithmic interval of k is given by

$$|\delta\varphi(k, t)|^2\frac{4\pi k^3}{(2\pi)^3} = \frac{H^2}{(2\pi)^2}\left(\frac{k}{\epsilon a(t)H}\right)^{-2m^2/3H^2}\theta(\epsilon a(t)H - k)\theta(k - \epsilon H). \tag{5·12}$$

As mentioned previously, the factor $\theta(k-\epsilon H)$ in the above serves as an infra-red cutoff at a fixed comoving scale and is not of much importance for us. In the massless 2-dimensional case, this factor is replaced by the term $J_0(k/H)$, which gives

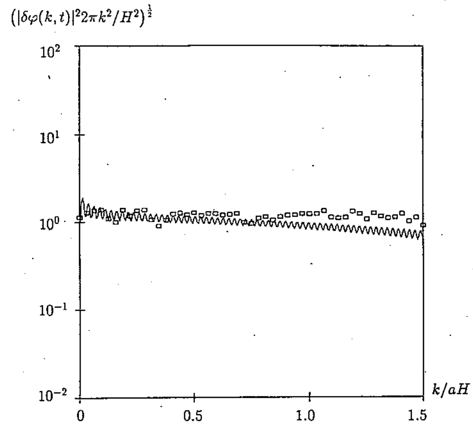


Fig. 9. The power spectrum of spatial fluctuations of the scalar field shown in Fig. 1. The real line denotes the analytic power spectrum in the corresponding 2-dimensional model.

rise to an unnatural oscillatory behavior on large scales $k/aH \ll 1$. Although implicit in Eq. (5·9), a similar oscillatory behavior appears also for the massive case. However, in any case, the amplitude becomes negligibly small for $t \to \infty$. Thus on scales of our interest $H/a \ll k/a \ll H$, the spectrum of the 2-dimensional model, Eq. (5·9), agrees well with the realistic 3-dimensional case, Eq. (5·12). This implies that a 2-dimensional simulation as depicted in Fig. 1 reproduces the actual pattern of the scalar field fairly accurately, provided that a good numerical accuracy is guaranteed.

It should be mentioned, however, that the spectrum given by Eq. (5·9) does not correspond exactly to the true spectrum which would result from our numerical simulation, mainly because there exist two special directions in the numerical case which would lead to a direction-dependent Fourier amplitude in $\boldsymbol{k}$-space, while the analytic answer does not depend on the direction of $\boldsymbol{k}$. Nevertheless, this disagreement would be inessential if the final number of h-regions is sufficiently large.

For an examination of the numerical accuracy, we have calculated the power spectrum from our numerical data and compared it with Eq. (5·9). The result is shown in Fig. 9. One can see that the spectrum calculated from the numerical data agrees quite well with the analytical one on scales $k/a \lesssim H$. Although we do not know how to evaluate quantitatively the numerical accuracy of our simulation from this result, we can at least say that Fig. 1 represents a 0-th order approximation to the actual spatial feature of the inflationary universe.

§ 6. Conclusions

In this paper, we have investigated the dynamics and the spatial feature of the new inflationary universe both analytically and numerically by using the stochastic approach to inflation. The stochastic approach used in the present paper is an extended version of the original one formulated by Starobinsky.[3] In our extended version, the basic equations are defined on the phase space of the scalar field so that there is no need to appeal to the slow rolling approximation and a unified treatment of inflation throughout its whole stage became possible. For definiteness, we have analyzed a model of a single component real scalar field with a double-well potential. For this model, we have found the following.

The initial velocity (i.e., the time derivative) of the scalar field v does not affect the later evolution of the scalar field as long as the kinetic energy is comparable to or less than the potential energy; v would almost vanish within about an expansion time H^{-1} if $v^2/V(0) \lesssim 1 \ll 24\pi\lambda^{-1}(m^2/m_{\rm Pl}^2)$ (see Eqs. (3·2) and (3·15)) and the validity of the slow rolling approximation is recovered. However it is important to keep in mind that the validity of the slow rolling approximation does not mean the realization of the actual slow roll-over phase in which the scalar field evolves according to the slow roll-over version of the classical equation of motion.

The condition for the realization of the actual slow roll-over phases is determined by competition between the quantum noise force and the potential force. However, a naive comparison of these two forces does not give the correct criterion. To obtain the correct criterion for the slow roll-over phase, one must take into account the difference in physical volume of an h-region (i.e., a region of horizon size) due to the (φ, v)-dependence of the expansion rate H. In particular, a region having a smaller value of the scalar field has a larger expansion rate and hence eventually dominates the physical volume of the whole universe.

Thus, as a result of the volume effect, the universe on its global scale would stay in the inflationary stage forever. However if the value of the scalar field in some h-region becomes larger than some critical value φ_*, the scalar field begins to evolve toward one of the potential minima and the region enters the classical slow roll-over phase. Then that region would inflate to a size large enough to become a homogeneous and isotropic domain of the universe which resembles the universe we observe today.

We have found this critical value by solving for probability distribution functions of the scalar field both analytically and numerically with the volume effect taken into account. In particular, the numerical simulation has been done for various cases in the parameter space of the model to fix the numerical factor to the critical value which could not be determined analytically. It is given by $\varphi_* \simeq (0.5 \pm 0.1) \times H^3/m^2$ (see Eq. (5·6)). This value is to be regarded as the initial value for the classical slow roll-over phase. An important point to be mentioned here is that the density perturbation amplitude at the horizon crossing time in the later Friedmann era is automatically suppressed to be smaller than unity if a comoving region under consideration went through the classical slow roll-over phase.

We have also visualized the spatial feature of the scalar field in a comoving region which undergoes the never-ending inflation by solving the stochastic scalar field equations in the 2-dimensional space. By analyzing the spectrum of spatial fluctuations, we have found that the 2-dimensional simulation is a fairly good representative of an actual spatial pattern of the scalar field.

However, of course, there exist many unsatisfactory points in the present numerical simulation which should be improved in future work. One direction of future work we are planning to pursue is to include the spatial derivative term (i.e., $a^{-2}\Delta\varphi$) of the basic stochastic equation in numerical simulation. This term has a tendency to make the spatial distribution of the scalar field uniform on the horizon scale and would affect the evolution of the scalar field in the early stage. It would be also necessary to include the volume effect in a simulation of the spatial feature of the inflationary universe. However, this implies that we have to do a general relativistic simulation in which the metric degrees of freedom are properly taken into account. Therefore an accurate simulation of the spacetime structure of the inflationary universe may not be easy.

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