STOCHASTIC ESTIMATION OF THE MAXIMUM OF A REGRESSION FUNCTION[1]

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1. Summary. Let $M(x)$ be a regression function which has a maximum at the unknown point θ. $M(x)$ is itself unknown to the statistician who, however, can take observations at any level x. This paper gives a scheme whereby, starting from an arbitrary point x_1, one obtains successively $x_2, x_3, \cdots$ such that x_n converges to θ in probability as $n \to \infty$.

2. Introduction. Let $H(y \mid x)$ be a family of distribution functions which depend on a parameter x, and let

$$M(x) = \int_{-\infty}^{\infty} y \, dH(y \mid x). \tag{2.1}$$

We suppose that

$$\int_{-\infty}^{\infty} (y - M(x))^2 \, dH(y \mid x) \leqq S < \infty, \tag{2.2}$$

and that $M(x)$ is strictly increasing for $x < \theta$, and $M(x)$ is strictly decreasing for $x > \theta$. Let $\{a_n\}$ and $\{c_n\}$ be infinite sequences of positive numbers such that

$$c_n \to 0, \tag{2.3}$$

$$\sum a_n = \infty, \tag{2.4}$$

$$\sum a_n c_n < \infty, \tag{2.5}$$

$$\sum a_n^2 c_n^{-2} < \infty. \tag{2.6}$$

(For example, $a_n = n^{-1}$, $c_n = n^{-1/3}$.)

We can now describe a recursive scheme as follows. Let z_1 be an arbitrary number. For all positive integral n we have

$$z_{n+1} = z_n + a_n \frac{(y_{2n} - y_{2n-1})}{c_n}, \tag{2.7}$$

where y_{2n-1} and y_{2n} are independent chance variables with respective distributions $H(y \mid z_n - c_n)$ and $H(y \mid z_n + c_n)$. Under regularity conditions on $M(x)$ which we shall state below we will prove that z_n converges stochastically to θ (as $n \to \infty$).

The statistical importance of this problem is obvious and need not be discussed. The stimulus for this paper came from the interesting paper by Robbins and Monro [1] (see also Wolfowitz [2]).

[1] Research under contract with the Office of Naval Research. Presented to the American Mathematical Society at New York on April 25, 1952.

While we have no need to postulate the existence of the derivative of $M(x)$ (indeed, $M(x)$ can be discontinuous), the spirit of our regularity assumptions postulated below is as follows. (a) If $M(x)$ did have a derivative it would be zero at $x = \theta$. Hence we would have expected the derivative not to be too large in a neighborhood of $x = \theta$. (b) If, at a distance from θ, $M(x)$ were very flat, then movement towards θ would be too slow. Hence outside of a neighborhood of $x = \theta$ we would have liked the absolute value of the derivative to be bounded below by a positive number. (c) If $M(x)$ rose too steeply in places we might through mischance get a movement of z_n which would throw us far out from θ. If there were many such steep places z_n could be made to approach $+\infty$ or $-\infty$ with positive probability. We would therefore have postulated a Lipschitz condition.

From the mathematical point of view it would be aesthetic to weaken the conditions. From the practical point of view it might be objected that these conditions prevent $M(x)$ from being a function which flattens out toward the x-axis, for example, $M(x) = e^{-x^2}$, or from being a function which drops off steadily faster to $-\infty$, for example, $M(x) = -x^2$. Now in any practical situation one can always give a priori an interval $[C_1, C_2]$ such that $C_1 \leqq \theta \leqq C_2$. It will be sufficient if our conditions are fulfilled in this interval.

Suppose, however, that some $z_n \pm c_n$ falls outside the interval $[C_1, C_2]$ and one cannot take an observation at that level. If one then moves z_n so that the offending $z_n \pm c_n$ is at C_1 or C_2, as the case may be, and proceeds as directed by (2.7), then our conclusion remains valid.

We postulate the following regularity conditions on $M(x)$.

CONDITION 1. There exist positive β and B such that

$$|x' - \theta| + |x'' - \theta| < \beta \text{ implies } |M(x') - M(x'')| < B\,|x' - x''|. \tag{2.8}$$

CONDITION 2. There exist positive ρ and R such that

$$|x' - x''| < \rho \text{ implies } |M(x') - M(x'')| < R. \tag{2.9}$$

CONDITION 3. For every $\delta > 0$ there exists a positive $\pi(\delta)$ such that

$$|z - \theta| > \delta \text{ implies } \inf_{\frac{1}{2}\delta > \epsilon > 0} \frac{|M(z + \epsilon) - M(z - \epsilon)|}{\epsilon} > \pi(\delta). \tag{2.10}$$

3. Proof that z_n converges stochastically to 0. Let

$$b_n = E(z_n - \theta)^2, \tag{3.1}$$

$$U_n(z) = (z - \theta)\, E\{y_{2n} - y_{2n-1} \mid z_n = z\}, \tag{3.2}$$

$$U_n^+(z) = \tfrac{1}{2}(U_n(z) + |U_n(z)|),\quad U_n^-(z) = \tfrac{1}{2}(U_n(z) - |U_n(z)|), \tag{3.3}$$

$$P_n = E(U_n^+(z_n)),\quad N_n = E(U_n^-(z_n)), \tag{3.4}$$

$$e_n = E(y_{2n} - y_{2n-1})^2. \tag{3.5}$$

From (2.7) we have

$$b_{n+1} = b_n + 2\frac{a_n}{c_n}(P_n + N_n) + \frac{a_n^2}{c_n^2}e_n. \tag{3.6}$$

Adding the expressions obtained from (3.6) for $b_{j+1} - b_j$ for $1 \leqq j \leqq n$, we obtain

$$b_{n+1} = b_1 + 2\sum_{j=1}^{n}\frac{a_j}{c_j}P_j + 2\sum_{j=1}^{n}\frac{a_j}{c_j}N_j + \sum_{j=1}^{n}\frac{a_j^2}{c_j^2}e_j. \tag{3.7}$$

Noting that $U_n^+(z) \geqq 0$ and that $U_n^+(z) > 0$ implies that $|z - \theta| < c_n$ because $M(x)$ is monotonic for $x < \theta$ and for $x > \theta$, it follows from (2.8) that, for all n for which $c_n < \frac{1}{2}\beta$, we have

$$0 \leqq U_n^+(z) < 2Bc_n^2 \tag{3.8}$$

It follows from (2.5) and (3.8) that the positive-term series

$$\sum_{n=1}^{\infty}\frac{a_n}{c_n}P_n \tag{3.9}$$

converges, say to α. From (2.9) we have

$$[M(z_n + c_n) - M(z_n - c_n)]^2 < R^2 \tag{3.10}$$

for n sufficiently large. Also for large enough n,

$$\begin{aligned} &E\{(y_{2n} - y_{2n-1})^2 \mid z_n\} \\ &= E\{(y_{2n} - M(z_n + c_n))^2 + (y_{2n-1} - M(z_n - c_n))^2 \mid z_n\} \\ &\qquad + [M(z_n + c_n) - M(z_n - c_n)]^2 \leqq 2S + R^2 \end{aligned} \tag{3.11}$$

by (2.2) and (3.10). Hence for large enough n

$$E[y_{2n} - y_{2n-1}]^2 \leqq 2S + R^2. \tag{3.12}$$

Consequently from (2.6) we obtain that the positive-term series

$$\sum_{n=1}^{\infty}\frac{a_n^2}{c_n^2}e_n \tag{3.13}$$

converges, say to γ. Hence, since $b_{n+1} \geqq 0$, it follows from (3.7) that

$$2\sum_{j=1}^{n}\frac{a_j}{c_j}N_j \geqq -b_1 - 2\alpha - \gamma > -\infty, \tag{3.14}$$

so that the negative-term series

$$\sum_{n=1}^{\infty}\frac{a_n}{c_n}N_n \tag{3.15}$$

converges.

Let

$$K_n = \left| \frac{M(z_n + c_n) - M(z_n - c_n)}{c_n} \right|. \tag{3.16}$$

Then

$$E\{K_n \mid z_n - \theta \mid \} = \frac{P_n - N_n}{c_n}. \tag{3.17}$$

From the convergence of (3.9) and (3.15) and the divergence of $\sum a_n$, it follows that

$$\liminf_{n\to\infty} E\{K_n \mid z_n - \theta \mid \} = 0. \tag{3.18}$$

Let $n_1 < n_2 < \cdots$ be an infinite sequence of positive integers such that

$$\lim_{j\to\infty} E\{K_{n_j} \mid z_{n_j} - \theta \mid \} = 0. \tag{3.19}$$

We assert that $(z_{n_j} - \theta)$ converges stochastically to zero as $j \to \infty$. For if not, there would exist two positive numbers δ and ϵ and a subsequence $\{t_j\}$ of $\{n_j\}$ such that, for all j,

$$P\{ \mid z_{t_j} - \theta \mid > \delta\} > \epsilon, \tag{3.20}$$

which implies that

$$E\{K_{t_j} \mid z_{t_j} - \theta \mid \} \geqq \delta\epsilon\pi\left(\frac{\delta}{2}\right) > 0 \tag{3.21}$$

for all j for which $c_{t_j} < \frac{1}{2}\delta$. But (3.21) contradicts (3.19) and the stochastic convergence to zero of $(z_{n_j} - \theta)$ is proved.

Let η and ϵ be arbitrary positive numbers. The proof of the theorem will be complete if we can show the existence of an integer $N(\eta, \epsilon)$ such that

$$P\{ \mid z_n - \theta \mid > \eta\} \leqq \epsilon \text{ for } n > N(\eta, \epsilon). \tag{3.22}$$

Let s be a positive number such that

$$\frac{s^2 + s}{\eta^2} < \frac{\epsilon}{2}. \tag{3.23}$$

Because z_{n_j} converges stochastically to θ there exists an integer N_0 such that

$$P\{\mid z_{N_0} - \theta \mid \geqq s\} < \frac{\epsilon}{2}. \tag{3.24}$$

We may also choose N_0 so large that

$$c_n < \min\left(\frac{\rho}{2}, \frac{\beta}{2}\right) \text{ for all } n \geqq N_0, \tag{3.25}$$

and

$$\sum_{n=N_0}^{\infty} \frac{a_n^2}{c_n^2} < \frac{s}{2R^2 + 4S}, \tag{3.26}$$

and

$$\sum_{n=N_0}^{\infty} a_n c_n < \frac{s}{8B}. \tag{3.27}$$

Proceeding in a manner similar to that used to obtain (3.7), we have, for each $n > N_0$,

$$\begin{aligned} E\{(z_n - \theta)^2 \mid z_{N_0} = z\} &= (z-\theta)^2 + 2 \sum_{j=N_0}^{n-1} \frac{a_j}{c_j} E\{U_j \mid z_{N_0} = z\} \\ &+ \sum_{j=N_0}^{n-1} \frac{a_j^2}{c_j^2} E\{(y_{2j} - y_{2j-1})^2 \mid z_{N_0} = z\} \end{aligned} \tag{3.28}$$

$$\leqq (z-\theta)^2 + 2 \sum_{j=N_0}^{\infty} \frac{a_j}{c_j} E\{U_j^+ \mid z_{N_0} = z\} + (R^2 + 2S) \sum_{j=N_0}^{\infty} \frac{a_j^2}{c_j^2} < (z-\theta)^2 + s.$$

Using (3.23), (3.28), and Tchebycheff's inequality, we have

$$P\{| z_n - \theta | > \eta \,\big|\, | z_{N_0} - \theta | < s\} < \frac{\epsilon}{2}. \tag{3.29}$$

The inequalities (3.24) and (3.29) show that (3.22) holds for $N(\eta, \epsilon) = N_0$, and the proof is complete.

4. Further problems. The following remarks about further problems apply also to [1].

A. An obvious problem is to determine sequences $\{c_n\}$ and $\{a_n\}$ which would be optimal in some reasonable sense.

B. An important problem is to determine a stopping rule, that is, a rule by which the statistician decides when he is sufficiently close to θ.

C. This problem is a combination of *B* and a generalization of *A*, that is, to determine an optimal procedure with its stopping rule.

REFERENCES

[1] H. Robbins and S. Monro, "A stochastic approximation method," *Annals of Math. Stat.*, Vol. 22 (1951), pp. 400–407.
[2] J. Wolfowitz, "On the stochastic approximation method of Robbins and Monro," *Annals of Math. Stat.*, Vol. 23 (1952), pp. 457–461.