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STOCHASTIC EVOLUTION EQUATIONS DRIVEN BY LIOUVILLE FRACTIONAL BROWNIAN MOTION

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Abstract. Let H be a Hilbert space and E a Banach space. We set up a theory of stochastic integration of $\mathscr{L}(H,E)$-valued functions with respect to H-cylindrical Liouville fractional Brownian motion with arbitrary Hurst parameter $0<\beta<1$. For $0<\beta<\frac{1}{2}$ we show that a function $\Phi\colon (0,T)\to\mathscr{L}(H,E)$ is stochastically integrable with respect to an H-cylindrical Liouville fractional Brownian motion if and only if it is stochastically integrable with respect to an H-cylindrical fractional Brownian motion.

We apply our results to stochastic evolution equations

$$\mathrm{d}U(t)=AU(t)\,\mathrm{d}t+B\,\mathrm{d}W_H^\beta(t)$$

driven by an H-cylindrical Liouville fractional Brownian motion, and prove existence, uniqueness and space-time regularity of mild solutions under various assumptions on the Banach space E, the operators $A\colon\mathscr{D}(A)\to E$ and $B\colon H\to E$, and the Hurst parameter β.

As an application it is shown that second-order parabolic SPDEs on bounded domains in $\mathbb{R}^d$, driven by space-time noise which is white in space and Liouville fractional in time, admit a mild solution if $\frac{1}{4}d<\beta<1$.

Keywords: (Liouville) fractional Brownian motion, fractional integration, stochastic evolution equations

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1. Introduction

Since the pioneering paper of Mandelbrot and Van Ness [27], fractional Brownian motion (fBm) has been proposed as a model to a variety of phenomena in population

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dynamics (e.g. [22], [23]), random long-time influences in climate systems (e.g. [38], [39]), mathematical finance (e.g. [3], [8], [16], [17], [37], and the references therein), random dynamical systems (e.g. [29]), and telecommunications (e.g. [26], [46]). This has motivated many studies of stochastic partial differential equations driven by fBm, among them [1], [7], [12], [16], [19], [20], [21], [40], [44]. Following the approach of Da Prato and Zabczyk [9], such equations may often be formulated as abstract stochastic evolution equations on an infinite-dimensional state space. This naturally leads to the problem of defining a stochastic integral with respect to cylindrical fBm in such spaces. This problem has been considered by various authors, among them Duncan, Maslowski, Pasik-Duncan [12] (in Hilbert spaces for cylindrical fBm with Hurst parameter $0 < \beta < \frac{1}{2}$) and Duncan, Maslowski, Pasik-Duncan [40] (in Hilbert spaces for fBm with Hurst parameter $\frac{1}{2} < \beta < 1$). The stochastic integral constructed in these papers was used to prove existence of mild solutions for stochastic abstract Cauchy problems of the form

$$\mathrm{d}U(t) = AU(t) + B\,\mathrm{d}W_H^\beta(t), \quad U(0) = u_0,$$

where A is the generator of a C_0-semigroup on a Hilbert space E, B is a bounded operator from another Hilbert space H to E, and W_H^β is an H-cylindrical fBm with Hurst parameter β (in contrast with the literature on fBm, but in line with the literature of SPDEs, we use the letter H for the Hilbert space associated with the cylindrical noise). Among other things, for $\frac{1}{2} < \beta < 1$ it was shown that a mild solution always exists if B is a Hilbert-Schmidt operator, and for $0 < \beta < \frac{1}{2}$ the same conclusion holds if one assumes that the semigroup generated by A is analytic.

The purpose of this paper is to prove analogues of the above-mentioned results for cylindrical Liouville fBm and to extend the setting to Banach spaces E. Stochastic integration with respect to Liouville fBm turns out to be both simpler and more symmetric with respect to the choice of the Hurst parameter below or above the critical value $\beta = \frac{1}{2}$. In many respects this allows a unified treatment of both cases. For $0 < \beta < \frac{1}{2}$ it turns out that an operator-valued function $\Phi\colon (0,T) \to \mathscr{L}(H,E)$ is stochastically integrable with respect to an H-cylindrical fBm if and only if it is stochastically integrable with respect to an H-cylindrical Liouville fBm.

Our theory is applied to stochastic evolution equations driven by an H-cylindrical fBm. We show that second-order parabolic SPDEs on bounded domains in $\mathbb{R}^d$, driven by space-time noise which is white in space and Liouville fractional in time, admit a mild solution if $\frac{1}{4}d < \beta < 1$.

We conclude this introduction with a brief comparison of our results with the existing literature. In [44] the authors study stochastic evolution equations driven by additive cylindrical fBm in a Hilbert space framework for self-adjoint operators A.

When applied to the stochastic heat equation, their regularity results appear to be weaker than ours. This seems to be an intrinsic feature of the fact that the method is limited to the Hilbert space framework; in our Banach space framework we are able to use L^p-techniques.

The method of Young integrals employed in [18] leads to the same regularity results as ours in the case of one-dimensional stochastic heat equation. The approach taken in that paper is purely pathwise while ours is stochastic.

Let us finally mention that semilinear stochastic evolution equations in Hilbert spaces driven by multiplicative cylindrical fBm have been studied in [28] for Hurst parameter $\frac{1}{2} < \beta < 1$. We believe that the results obtained here can be extended to this class of equations in a Banach space framework by following the approach of, e.g., [34].

2. Fractional integration spaces

For $\alpha > 0$ the *left Liouville fractional integral* and the *right Liouville fractional integral* of order α of a function $f \in L^2(a, b)$ are defined by

$$
\begin{aligned}
(I^\alpha_{a+} f)(t) &:= \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\,\mathrm{d}s, \quad t \in [a,b],\\
(I^\alpha_{b-} f)(t) &:= \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s)\,\mathrm{d}s, \quad t \in [a,b].
\end{aligned}
$$

By Young's inequality, the functions $I^\alpha_{a+} f$ and $I^\alpha_{b-} f$ belong to $L^2(a, b)$. The operators I^α_{a+} and I^α_{b-} are bounded and injective on $L^2(a, b)$, with dense ranges denoted by

$$H^\alpha_{a+}(a, b) := I^\alpha_{a+}(L^2(a, b)), \quad H^\alpha_{b-}(a, b) := I^\alpha_{b-}(L^2(a, b)).$$

These spaces are Hilbert spaces with respect to the norms

$$\|I^\alpha_{a+} f\|_{H^\alpha_{a+}} := \|f\|_{L^2(a,b)}, \quad \|I^\alpha_{b-} f\|_{H^\alpha_{b-}} := \|f\|_{L^2(a,b)}.$$

We have continuous inclusions

$$H^\alpha_{a+}(a, b) \hookrightarrow L^2(a, b), \quad H^\alpha_{b-}(a, b) \hookrightarrow L^2(a, b).$$

The following simple observation will be used frequently.

Lemma 2.1. *Let $\alpha > 0$ and $a < x < b$.*

(1) (*Restriction with left boundary condition*). *If $f \in H^\alpha_{a+}(a,b)$, then $f|_{(a,x)} \in H^\alpha_{a+}(a,x)$ and*
$$\|f|_{(a,x)}\|_{H^\alpha_{a+}(a,x)} \leqslant \|f\|_{H^\alpha_{a+}(a,b)}.$$

(2) (*Extension with right boundary condition*). *If $f \in H^\alpha_{x-}(a,x)$ and we define $f_x(s) = f(s)$ for $s \in (a,x)$ and $f_x(s) = 0$ otherwise, then $f_x \in H^\alpha_{b-}(a,b)$ and*
$$\|f_x\|_{H^\alpha_{b-}(a,b)} = \|f\|_{H^\alpha_{x-}(a,x)}.$$

(3) (*Reflection*). *We have $f \in H^\alpha_{a+}(a,b)$ if and only if $\check{f} \in H^\alpha_{b-}(a,b)$, where $\check{f}(t) := f(b-(t-a))$, and in this situation we have*
$$\|f\|_{H^\alpha_{a+}(a,b)} = \|\check{f}\|_{H^\alpha_{b-}(a,b)}.$$

Proof. We only prove (1); the proofs of (2) and (3) are similar. By assumption, $f = I^\alpha_{a+}g$ for some $g \in L^2(a,b)$. Clearly, for $s \in (a,x)$ we have $(I^\alpha_{a+}(g|_{(a,x)}))(s) = f(s)$, where by slight abuse of notation we also use the notation I^α_{a+} for the fractional integration operator acting on $L^2(a,x)$. It follows that $f|_{(a,x)} \in H^\alpha_{a+}(a,x)$ and $I^\alpha_{a+}(g|_{(a,x)}) = f|_{(a,x)}$. Moreover, $\|f|_{(a,x)}\|_{H^\alpha_{a+}(a,x)} = \|g|_{(a,x)}\|_{L^2(a,x)} \leqslant \|g\|_{L^2(a,x)} = \|f\|_{H^\alpha_{a+}(a,b)}$. □

The following result is less elementary; for a proof we refer to [42, Chapter 3, Section 13.3].

Lemma 2.2. *Let $0 < \alpha < \frac{1}{2}$. Then we have $H^\alpha_{a+}(a,b) = H^\alpha_{b-}(a,b)$ with equivalent norms. As a consequence, for all $a < x < b$ there is a constant $C_{\alpha,x}$ such that:*

(1) (*Restriction with right boundary condition*). *If $f \in H^\alpha_{b-}(a,b)$, then $f|_{(a,x)} \in H^\alpha_{x-}(a,x)$ and*
$$\|f|_{(a,x)}\|_{H^\alpha_{x-}(a,x)} \leqslant C_{\alpha,x}\|f\|_{H^\alpha_{b-}(a,b)}.$$

(2) (*Extension with left boundary condition*). *If $f \in H^\alpha_{a+}(a,x)$ and we define $f^x(s) = f(s)$ for $s \in (a,x)$ and $f^x(s) = 0$ otherwise, then $f^x \in H^\alpha_{a+}(a,b)$ and*
$$\|f^x\|_{H^\alpha_{a+}(a,b)} \leqslant C_{\alpha,x}\|f\|_{H^\alpha_{a+}(a,x)}.$$

This lemma allows us to write, for $0 < \alpha < \frac{1}{2}$,
$$H^\alpha(a,b) := H^\alpha_{a+}(a,b) = H^\alpha_{b-}(a,b)$$

with equivalent norms. We will use this simplified notation whenever the precise choice of the norm is irrelevant; when the choice of the norm does matter we stick to the original notation with subscripts.

The next result is formulated for the right fractional integral; a similar result holds for the left fractional integral.

Lemma 2.3. *Let $0 < \alpha < \frac{1}{2}$. For all $a \leqslant x < y \leqslant b$, the indicator function $1_{[x,y)}$ defines an element in $H^\alpha(a,b)$. Moreover, the linear span of the indicator functions is dense in $H^\alpha(a,b)$.*

Proof. To prove the first assertion, by linearity we may assume that $x = a$. Define

$$g_y(t) := \frac{1}{\Gamma(1-\alpha)}(y-t)^{-\alpha}\, 1_{(a,y)}(t), \quad t \in (a,b).$$

Then $g_y \in L^2(a,b)$ and

$$(I^\alpha_{b-} g_y)(t) = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_t^b (s-t)^{\alpha-1}(y-s)^{-\alpha}\, 1_{(a,y)}(s)\, \mathrm{d}s, \quad t \in (a,b).$$

For $t \in [y,b)$ it is clear that $(I^\alpha_{b-} g_y)(t) = 0$, whereas for $t \in (a,y)$ we have, by a change of variable $\sigma = (s-t)/(y-t)$,

$$(I^\alpha_{b-} g_y)(t) = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^1 \sigma^{\alpha-1}(1-\sigma)^{-\alpha}\, \mathrm{d}\sigma = 1.$$

This shows that

$$I^\alpha_{b-} g_y = 1_{(a,y)} \tag{2.1}$$

and therefore $1_{(a,y)} \in H^\alpha(a,b)$.

Next we prove that the linear span of all indicator functions of the form $1_{[x,y)}$ is dense in $H^\alpha(a,b)$. Since I^α_{b-} is an isomorphism from $L^2(a,b)$ onto $H^\alpha(a,b)$, it is enough to prove that the linear span of the set of functions g_y introduced above is dense in $L^2(a,b)$. To this end let us assume that $f \in L^2(a,b)$ is a function such that for all $a < y \leqslant b$ we have $[f, g_y]_{L^2(a,b)} = 0$. In view of

$$0 = [f, g_y]_{L^2(a,b)} = \frac{1}{\Gamma(1-\alpha)} \int_a^y (y-t)^{-\alpha} f(t)\, \mathrm{d}t = (I^{1-\alpha}_{a+} f)(y), \quad a < y \leqslant b,$$

this implies that $I^{1-\alpha}_{a+} f = 0$ in $H^\alpha(a,b)$. Therefore $f = 0$ in $L^2(a,b)$ and the proof is complete. □

For $\alpha > 0$ we define the negative fractional integral spaces $H^{-\alpha}_{0+}(a,b)$ and $H^{-\alpha}_{T-}(a,b)$ as the completions of $L^2(a,b)$ with respect to the norms

$$\|f\|_{H^{-\alpha}_{a+}(a,b)} := \|I^{\alpha}_{a+}f\|_{L^2(a,b)}, \quad \|f\|_{H^{-\alpha}_{b-}(a,b)} := \|I^{\alpha}_{b-}f\|_{L^2(a,b)}.$$

It is an easy consequence of Lemma 2.2 that for $0 < \alpha < \frac{1}{2}$ we have $H^{-\alpha}_{0+}(a,b) = H^{-\alpha}_{T-}(a,b)$ with equivalent norms. Accordingly we will write

$$H^{-\alpha}(a,b) := H^{-\alpha}_{0+}(a,b) = H^{-\alpha}_{T-}(a,b)$$

as long as the precise choice of the norm is unimportant.

We further define, for $\alpha > 0$,

$$I^{-\alpha}_{a+} := (I^{\alpha}_{a+})^{-1} = D^{\alpha}_{a+}, \quad I^{-\alpha}_{b-} := (I^{\alpha}_{b-})^{-1} = D^{\alpha}_{b-},$$

where D^{α}_{a+} and D^{α}_{b-} are the left- and right fractional derivatives of order α. With these definitions, we have isometric isomorphisms

$$I^{-\alpha}_{a+}\colon\ L^2(a,b) \simeq H^{-\alpha}_{a+}(a,b), \quad I^{-\alpha}_{b-}\colon\ L^2(a,b) \simeq H^{-\alpha}_{b-}(a,b).$$

Finally, we let $H^0_{a+}(a,b) = H^0_{b-}(a,b) := L^2(a,b)$ and agree that $I^0_{a+} = I^0_{b-} := I$, the identity mapping on $L^2(a,b)$.

3. Stochastic integration

Throughout the rest of this paper we fix a number $T > 0$. We shall write $L^2 := L^2(0,T)$ and

$$H^{\alpha}_{0+} := H^{\alpha}_{0+}(0,T), \quad H^{\alpha}_{T-} := H^{\alpha}_{T-}(0,T),$$
$$C_{0+} := C_{0+}[0,T], \qquad C_{T-} := C_{T-}[0,T],$$

where $C_{0+}[0,T] = \{f \in C[0,T]\colon\ f(0) = 0\}$ and $C_{T-}[0,T] = \{f \in C[0,T]\colon\ f(T) = 0\}$. For $\alpha > \frac{1}{2}$ we denote the inclusion mappings $H^{\alpha}_{0} \hookrightarrow C_{0+}$ and $H^{\alpha}_{T-} \hookrightarrow C_{T-}$ by i^{α}_{0+} and i^{α}_{T-}, respectively.

For $0 < \beta < 1$ and $0 \leqslant s,t \leqslant T$ we set

$$\Gamma^{\beta}_{s,t} := \left[(i^{\beta+\frac{1}{2}}_{0+})^*\delta_s, (i^{\beta+\frac{1}{2}}_{0+})^*\delta_t\right]_{H^{\beta+\frac{1}{2}}_{0+}},$$

where δ_s and δ_t denote the Dirac measures concentrated at s and t (which we identify with functionals in the dual of C_{0+} in the natural way). An easy computation, cf. [14, Swection 6.2], gives

$$\Gamma^{\beta}_{s,t} = \frac{1}{(\Gamma(\beta+\frac{1}{2}))^2}\int_0^{s\wedge t}(s-u)^{\beta-\frac{1}{2}}(t-u)^{\beta-\frac{1}{2}}\,du, \quad s,t \in [0,T].$$

Definition 3.1. A *Liouville fractional Brownian motion (Liouville fBm) of order* $0 < \beta < 1$, indexed by $[0,T]$, is a Gaussian process $W^\beta = (W^\beta(t))_{t\in[0,T]}$ such that

$$\mathbb{E}(W^\beta(s)W^\beta(t)) = \Gamma^\beta_{s,t}, \quad s,t \in [0,T].$$

By the general theory of Gaussian processes, Liouville fBm exists. Note that $\Gamma^{\frac{1}{2}}_{s,t} = s \wedge t$, so a Liouville fBm of order $\frac{1}{2}$ is just a standard Brownian motion.

The *stochastic integral* of a real-valued step function $f = \sum_{j=1}^{N} c_j\, 1_{(a_j,b_j]}$ with respect to a Liouville fBm W^β is defined by

$$\int_0^T f \,\mathrm{d}W^\beta := \sum_{j=1}^{N} c_j (W^\beta(b_j) - W^\beta(a_j)).$$

One easily checks that this definition does not depend on the representation of f. We proceed with analogues, for $0 < \beta < \frac{1}{2}$ and $\frac{1}{2} < \beta < 1$, of the classical Itô isometry (which corresponds to $\beta = \frac{1}{2}$). These cases require different treatments and are therefore considered separately.

3.1. The case $0 < \beta < \frac{1}{2}$

Proposition 3.2 (Itô isometry I). *Let* $0 < \beta < \frac{1}{2}$. *If* $f\colon (0,T) \to \mathbb{R}$ *is a step function, then* $\int_0^T f \,\mathrm{d}W^\beta$ *is Gaussian and*

$$\mathbb{E}\left|\int_0^T f \,\mathrm{d}W^\beta\right|^2 = \|f\|^2_{H^{\frac{1}{2}-\beta}_{T-}}. \tag{3.1}$$

As a result, the mapping $f \mapsto \int_0^T f \,\mathrm{d}W^\beta$ *has a unique extension to an isometry from* $H^{\frac{1}{2}-\beta}_{T-}$ *into* $L^2(\Omega)$.

P r o o f. Suppose that $f = \sum_{j=1}^{N} c_j\, 1_{(a_j,b_j]}$ is a step function with real coefficients c_j. We may assume that the intervals $(a_j, b_j]$ are disjoint. Then,

$$\begin{aligned}\mathbb{E}\left|\int_0^T f \,\mathrm{d}W^\beta\right|^2 &= \mathbb{E}\left|\sum_{j=1}^{N} c_j (W^\beta(b_j) - W^\beta(a_j))\right|^2 \\ &= \sum_{i,j=1}^{N} c_i c_j (\Gamma^\beta_{b_i,b_j} - \Gamma^\beta_{a_i,b_j} - \Gamma^\beta_{a_j,b_i} + \Gamma^\beta_{a_i,a_j}) = \int_0^T |g(s)|^2 \,\mathrm{d}s,\end{aligned}$$

where

$$g(s) := \frac{1}{\Gamma(\beta+\frac{1}{2})}\sum_{j=1}^{N} c_j((b_j - s)^{\beta-\frac{1}{2}}\,1_{(0,b_j]}(s) - (a_j - s)^{\beta-\frac{1}{2}}\,1_{(0,a_j]}(s)).$$

Since $0 < \beta < \frac{1}{2}$, (2.1) shows that

$$I_{T-}^{\frac{1}{2}-\beta} g = \sum_{j=1}^{N} c_j\,1_{(a_j,b_j]} = f.$$

In view of the identity $\|g\|_{L^2} = \|f\|_{H_{T-}^{\frac{1}{2}-\beta}}$, the isometry (3.1) is proved. The final assertion concerning the unique extendability of the integral follows from the density of the step functions in $H_{T-}^{\frac{1}{2}-\beta}$ as proved in Lemma 2.3. □

3.2. The case $\frac{1}{2} < \beta < 1$

Proposition 3.3 (Itô isometry II)**.** *Let $\frac{1}{2} < \beta < 1$. If $f\colon (0,T) \to \mathbb{R}$ is a step function, then $\int_0^T f\,\mathrm{d}W^\beta$ is Gaussian and*

$$\mathbb{E}\left|\int_0^T f\,\mathrm{d}W^\beta\right|^2 = \|f\|^2_{H_{T-}^{\frac{1}{2}-\beta}}.$$

As a result, the mapping $f \mapsto \int_0^T f\,\mathrm{d}W^\beta$ has a unique extension to an isometry from $H_{T-}^{\frac{1}{2}-\beta}$ into $L^2(\Omega)$.

P r o o f. First let $f = 1_{(0,s)}$ and $g = 1_{(0,t)}$ be left indicator functions with $s < t$. Then

$$\mathbb{E}\left[\int_0^T 1_{(0,s)}\,\mathrm{d}W^\beta \cdot \int_0^T 1_{(0,t)}\,\mathrm{d}W^\beta\right]$$
$$= \mathbb{E}[W^\beta(s)W^\beta(t)] = \frac{1}{(\Gamma(\beta+\frac{1}{2}))^2}\int_0^s (s-u)^{\beta-\frac{1}{2}}(t-u)^{\beta-\frac{1}{2}}\,\mathrm{d}u.$$

On the other hand, for $\tau \in \{s,t\}$,

$$\begin{aligned} I_{T-}^{\beta-\frac{1}{2}}\,1_{(0,\tau)}(u) &= \frac{1}{\Gamma(\beta-\frac{1}{2})}\int_u^T (r-u)^{\beta-\frac{3}{2}}\,1_{(0,\tau)}(r)\,\mathrm{d}r \\ &= \frac{1}{\Gamma(\beta-\frac{1}{2})}\int_{u\wedge\tau}^{\tau} (r-u)^{\beta-\frac{3}{2}}\,\mathrm{d}r \\ &= \frac{1}{\Gamma(\beta+\frac{1}{2})}[(\tau-u)^{\beta-\frac{1}{2}} - ((u\wedge\tau)-u)^{\beta-\frac{1}{2}}]. \end{aligned}$$

Hence,

$$
\begin{aligned}
&[I_{T-}^{\beta-\frac{1}{2}} 1_{(0,s)}, I_{T-}^{\beta-\frac{1}{2}} 1_{(0,t)}]_{L^2} \\
&\quad = \frac{1}{(\Gamma(\beta+\frac{1}{2}))^2} \int_0^T [(s-u)^{\beta-\frac{1}{2}} - ((u\wedge s)-u)^{\beta-\frac{1}{2}}] \\
&\qquad \times [(t-u)^{\beta-\frac{1}{2}} - ((u\wedge t)-u)^{\beta-\frac{1}{2}}]\,\mathrm{d}u \\
&\quad = \frac{1}{(\Gamma(\beta+\frac{1}{2}))^2} \int_0^{s\wedge t} (s-u)^{\beta-\frac{1}{2}}(t-u)^{\beta-\frac{1}{2}}\,\mathrm{d}u.
\end{aligned}
$$

Putting things together we obtain

$$
\mathbb{E}\Big[\int_0^T 1_{(0,s)}\,\mathrm{d}W^\beta \cdot \int_0^T 1_{(0,t)}\,\mathrm{d}W^\beta\Big] = [I_{T-}^{\beta-\frac{1}{2}} 1_{(0,s)}, I_{T-}^{\beta-\frac{1}{2}} 1_{(0,t)}]_{L^2}^2.
$$

By linearity, this identity extends to linear combinations of left indicator functions. Therefore we obtain, for all step functions φ,

$$
\mathbb{E}\Big|\int_0^T \varphi\,\mathrm{d}W^\beta\Big|^2 = \|I_{T-}^{\beta-\frac{1}{2}}\varphi\|_{L^2}^2.
$$

Since step functions are dense in L^2, this proves the result. □

3.3. Lemma

We close this section with a lemma that will be needed in Section 5.

Lemma 3.4. *Let W^β be a Liouville fBm of order $0<\beta<1$. For all $0\leqslant\alpha<\min\{\beta+\frac{1}{2},1\}$ and $0\leqslant s<t<\infty$,*

$$
\Big(\mathbb{E}\Big|\int_s^t (t-r)^{-\alpha}\,\mathrm{d}W^\beta(r)\Big|^2\Big)^{\frac{1}{2}} = c_{\alpha,\beta}(t-s)^{\beta-\alpha},
$$

where $c_{\alpha,\beta}$ is a constant depending only on α and β.

Proof. For $\beta=\frac{1}{2}$ the result is immediate from the classical Itô isometry. Next, let $0<\beta<\frac{1}{2}$. For $g_s(r) := (r-s)^{\beta-\alpha-\frac{1}{2}}$ we have

$$
I_{s+}^{\frac{1}{2}-\beta} g_s(u) = \frac{1}{\Gamma(\frac{1}{2}-\beta)} \int_s^u (u-r)^{-\frac{1}{2}-\beta}(r-s)^{\beta-\alpha-\frac{1}{2}}\,\mathrm{d}r = C_{\alpha,\beta}(u-s)^{-\alpha},
$$

where $C_{\alpha,\beta}$ is a constant depending only on α and β. Hence by Lemma 2.1 (3) and Proposition 3.2,

$$\begin{aligned}\Big(\mathbb{E}\Big|\int_s^t (t-r)^{-\alpha}\,\mathrm{d}W^\beta(r)\Big|^2\Big)^{\frac12} \\ &= \|r \mapsto (t-r)^{-\alpha}\|_{H_{t-}^{\frac12-\beta}} = \|r \mapsto (r-s)^{-\alpha}\|_{H_{s+}^{\frac12-\beta}} \\ &= \frac{1}{C_{\alpha,\beta}}\|u \mapsto (u-s)^{\beta-\alpha-\frac12}\|_{L^2(s,t)} = c_{\alpha,\beta}(t-s)^{\beta-\alpha}.\end{aligned}$$

Next, let $\frac12 < \beta < 1$. By Lemma 2.1 (3) and Proposition 3.3,

$$\begin{aligned}\Big(\mathbb{E}\Big|\int_s^t (t-r)^{-\alpha}\,\mathrm{d}W^\beta(r)\Big|^2\Big)^{\frac12} \\ &= \|r \mapsto (t-r)^{-\alpha}\|_{H_{t-}^{\frac12-\beta}} = \|r \mapsto (r-s)^{-\alpha}\|_{H_{s+}^{\frac12-\beta}} \\ &= \Big\|u \mapsto \frac{1}{\Gamma(\beta-\frac12)}\int_s^u (u-r)^{\beta-\frac32}(r-s)^{-\alpha}\,\mathrm{d}r\Big\|_{L^2(s,t)} \\ &= C'_{\alpha,\beta}\|u \mapsto (u-s)^{\beta-\alpha-\frac12}\|_{L^2(s,t)} = c'_{a,b}(t-s)^{\beta-\alpha},\end{aligned}$$

where $C'_{\alpha,\beta}$ and $c'_{\alpha,\beta}$ are constants depending only on α and β. □

3.4. γ-radonifying operators

In order to prepare for the results on vector-valued stochastic integration we need a couple of preliminaries on spaces of γ-radonifying operators. For the rest of this paper we fix a real Hilbert space H and a real Banach space E. Unless otherwise stated, $[\cdot,\cdot]_H$ and $\|\cdot\|_H$ refer to the inner product and norm of H, and $\|\cdot\|$ refers to the norm of E.

Any finite rank operator $S\colon H \to E$ can be represented in the form

$$S = \sum_{n=1}^N h_n \otimes x_n$$

with $h_1, \dots, h_N$ orthonormal in H and $x_1, \dots, x_N$ taken from E. The *γ-radonifying norm* of S is then defined by

$$\Big\|\sum_{n=1}^N h_n \otimes x_n\Big\|_{\gamma(H,E)}^2 := \mathbb{E}\Big\|\sum_{n=1}^N \gamma_n x_n\Big\|^2,$$

where $(\gamma_n)_{n\geqslant 1}$ is a sequence of independent standard Gaussian random variables on some probability space $(\Omega, \mathscr{A}, \mathbb{P})$. It is easy to check that this definition does not depend on the particular representation of S. The completion of the space of finite rank operators with respect to this norm is denoted by $\gamma(H, E)$. This space is continuously embedded in $\mathscr{L}(H, E)$, and a bounded operator $R \in \mathscr{L}(H, E)$ is called *γ-radonifying* if it belongs to $\gamma(H, E)$.

The space $\gamma(H, E)$ is an *operator ideal* in $\mathscr{L}(H, E)$ in the sense that whenever H' is another real Hilbert space, E' is another real Banach space, and $R\colon H' \to H$ and $T\colon E \to E'$ are bounded operators, then $S \in \gamma(H, E)$ implies $TSR \in \gamma(H', E')$ and

$$\|TSR\|_{\gamma(H',E')} \leqslant \|T\|\|S\|_{\gamma(H,E)}\|R\|.$$

If E is a Hilbert space, then $\gamma(H, E)$ is isometrically isomorphic to the Hilbert space of Hilbert-Schmidt operators from H to E.

Example 3.5 ([5], [32])**.** For $E = L^q(X, \mu)$ with $q \in [1, \infty)$ and (X, μ) a σ-finite measure space we have a natural isomorphism of Banach spaces

$$\gamma(H, L^q(X, \mu)) \simeq L^q(X, \mu; H)$$

obtained by assigning to a function $f \in L^p(X, \mu; H)$ the operator $S_f\colon H \to L^p(X, \mu)$, $S_f h := [f(\cdot), h]$.

Lemma 3.6. *Let $S = \sum_{n=1}^{N} h_n \otimes x_n$ be a finite rank operator from H to E, with $h_1, \dots, h_N$ orthonormal in H and $x_1, \dots, x_N$ taken from E, and suppose H' is another Hilbert space. For all $h' \in H'$ we have*

$$\Big\|\sum_{n=1}^{N}(h' \otimes h_n) \otimes x_n\Big\|_{\gamma(H'\widehat{\otimes}H,E)} = \|h'\|\Big\|\sum_{n=1}^{N} h_n \otimes x_n\Big\|_{\gamma(H,E)}.$$

Here $H'\widehat{\otimes}H$ denotes the Hilbert space tensor product of H' and H.

For the proof, just note that if $\|h'\|$ is normalised to 1, then the vectors $h' \otimes h_n$ are orthonormal in $H'\widehat{\otimes}H$.

A bounded operator $T \in \mathscr{L}(H, E)$ is said to be *γ-summing* if

$$\|T\|^2_{\gamma_\infty(H,E)} := \sup_h \mathbb{E}\Big\|\sum_{n=1}^{N} \gamma_n T h_n\Big\|^2 < \infty,$$

the supremum being taken over all finite orthonormal systems $h = (h_n)_{n=1}^N$ in H. Endowed with the above norm, the space $\gamma_\infty(H, E)$ of all γ-summing operators from H to E is a Banach space. Every γ-radonifying operator is γ-summing, and the inclusion $\gamma(H,E) \subseteq \gamma_\infty(H,E)$ is isometric. It follows from a theorem of Hoffmann-Jørgensen and Kwapień that equality $\gamma(H,E) = \gamma_\infty(H,E)$ holds when E does not contain a closed subspace isomorphic to c_0. For proofs and more information we refer to the survey paper [30] and the references given therein.

Let $\Phi\colon \mathbb{R}_+ \to \mathscr{L}(H,E)$ be an *H-strongly measurable function*, i.e. Φh is strongly measurable for all $h \in H$, and suppose that $\Phi^* x^* \in L^2(\mathbb{R}_+;H)$ for all $x^* \in E^*$. We say that an operator $R \in \mathscr{L}(L^2(\mathbb{R}_+;H),E)$ is *represented* by Φ if we have

$$R^* x^* = \Phi^* x^*$$

in $L^2(\mathbb{R}_+;H)$ for all $x^* \in E^*$.

A family $\mathscr{T}$ of bounded linear operators from a Banach space E to another Banach space F is called *γ-bounded* if there exists a finite constant C such that for all finite sequences $(x_n)_{n=1}^N$ in E and $(T_n)_{n=1}^N$ in $\mathscr{T}$ we have

$$\mathbb{E}\Big\|\sum_{n=1}^N \gamma_n T_n x_n\Big\|^2 \leqslant C^2\,\mathbb{E}\Big\|\sum_{n=1}^N \gamma_n x_n\Big\|^2.$$

The least admissible constant C is called the *γ-bound* of $\mathscr{T}$, notation $\gamma(\mathscr{T})$. An important way of generating γ-bounded families is due to Weis [45] who showed that if $f\colon (0,T) \to \mathscr{L}(E,F)$ is continuously differentiable with integrable derivative, then $\mathscr{T}_f = \{f(t)\colon f \in (0,T)\}$ is γ-bounded and

$$\gamma(\mathscr{T}_f) \leqslant \|f(0+)\| + \int_0^T \|f'(t)\|\,\mathrm{d}t. \tag{3.2}$$

An application of this result is contained in Lemma 5.4 below.

We continue with a multiplier result of Kalton and Weis [25] (see [30] for a proof) which connects the notions of radonification and γ-boundedness.

Lemma 3.7. *Let $M\colon (0,T) \to \mathscr{L}(E,F)$ be a function with the following properties:*
(1) *for all $x \in E$ the function Mx is strongly measurable;*
(2) *the range $\mathscr{M} = \{M(t)\colon t \in (0,T)\}$ is γ-bounded.*
Then for all functions $\Phi\colon (0,T) \to \mathscr{L}(H,E)$ representing an operator $S_\Phi \in \gamma(L^2(0,T;H),E)$, the function $M\Phi\colon (0,T) \to \mathscr{L}(H,F)$ represents an operator $S_{M\Phi} \in \gamma_\infty(L^2(0,T;H),F)$ and

$$\|S_{M\Phi}\|_{\gamma_\infty(L^2(0,T;H),F)} \leqslant \gamma(\mathscr{M})\|S_\Phi\|_{\gamma(L^2(0,T;H),E)}.$$

In many situations (such as in the application of this lemma in Section 5) one actually has $S_{M\Phi} \in \gamma(L^2(0,T;H),F)$, for instance by an application of Theorem 3.11. In view of the isometric inclusion $\gamma(L^2(0,T;H),F) \subseteq \gamma_\infty(L^2(0,T;H),F)$, the estimate of Lemma 3.7 then takes the form

$$\|S_{M\Phi}\|_{\gamma(L^2(0,T;H),F)} \leqslant \gamma(\mathscr{M})\|S_\Phi\|_{\gamma(L^2(0,T;H),E)}.$$

3.5. Stochastic integration in Banach spaces

Let $\mathscr{H}$ be a Hilbert space and $(\Omega,\mathscr{F},\mathbb{P})$ a probability space. A mapping $W\colon \mathscr{H} \to L^2(\Omega)$ is called an *$\mathscr{H}$-isonormal process* if $W(h)$ is centred Gaussian for all $h \in \mathscr{H}$ and

$$\mathbb{E}W(h_1)W(h_2) = [h_1,h_2]_{\mathscr{H}}, \quad h_1,h_2 \in \mathscr{H}.$$

By Proposition 3.2 (for $0<\beta<\frac12$), Proposition 3.3 (for $\frac12<\beta<1$) and the classical Itô isometry (for $\beta=\frac12$), for all $0<\beta<1$ the mapping

$$W^\beta\colon\, f \mapsto \int_0^T f\,\mathrm{d}W^\beta,$$

initially defined for step functions f, has a unique extension to an $H_{T-}^{\frac12-\beta}$-isonormal process. This observation suggests the following definition.

Definition 3.8. Let H be a Hilbert space and let $0<\beta<1$. An *H-cylindrical Liouville fBm of order β*, indexed by $[0,T]$, is an $H_{T-}^{\frac12-\beta}(H)$-isonormal process.

Here the Hilbert space

$$H_{T-}^{\frac12-\beta}(H) := H_{T-}^{\frac12-\beta}(0,T;H)$$

is defined in the obvious way using the right fractional integral operators acting in $L^2(H) := L^2(0,T;H)$. It is easy to see that $H_{T-}^{\frac12-\beta}(H)$ can be identified isometrically with the Hilbert space completion of the tensor product $H_{T-}^{\frac12-\beta}\otimes H$.

Our next task is to define an integral for E-valued functions with respect to a Liouville fBm, and more generally for $\mathscr{L}(H,E)$-valued functions with respect to an H-cylindrical Liouville fBm W_H^β, where H is a real Hilbert space. We shall proceed directly with the latter, as the former corresponds to the special case $H=\mathbb{R}$. We follow [30], which puts the approach of [35] into an abstract format.

For an *elementary rank one* function $\Phi\colon\,(0,T)\to\mathscr{L}(H,E)$, i.e. a function of the form

$$\Phi = f\otimes(h\otimes x),$$

where $f \in H^{\frac{1}{2}-\beta}_{T-}$ and $h \otimes x \in \mathscr{L}(H,E)$ is the rank one operator $h' \mapsto [h',h]x$, we define

$$\int_0^T \Phi \, \mathrm{d}W_H^\beta := W_H^\beta(f \otimes h) \otimes x.$$

This definition is extended by linearity to all *finite rank elementary functions* $\Phi\colon (0,T) \to \mathscr{L}(H,E)$, i.e. linear combinations of elementary rank one functions. Any such function

$$\Phi = \sum_{n=1}^N f_n \otimes (h_n \otimes x_n)$$

defines a finite rank operator $R_\Phi\colon\ H^{\frac{1}{2}-\beta}_{T-}(H) \to E$ by

$$R_\Phi := \sum_{n=1}^N (f_n \otimes h_n) \otimes x_n.$$

It is immediate to verify that for all $x^* \in E^*$ we have

$$R_\Phi^* x^* = \Phi^* x^* = \sum_{n=1}^N \langle x_n, x^* \rangle (f_n \otimes h_n)$$

as elements of $H^{\frac{1}{2}-\beta}_{T-}(H)$. Applying the results of [30], [35] to the Hilbert space $H^{\frac{1}{2}-\beta}_{T-}(H)$ we obtain

Theorem 3.9 (Itô isometry)**.** *Let W_H^β be a cylindrical Liouville fBm of order $0 < \beta < 1$. For all elementary finite rank functions $\Phi\colon\ (0,T) \to \mathscr{L}(H,E)$ we have*

$$\mathbb{E}\left\| \int_0^T \Phi \, \mathrm{d}W_H^\beta \right\|^2 = \|R_\Phi\|^2_{\gamma(H^{\frac{1}{2}-\beta}_{T-}(H),E)}.$$

As a result, the E-valued stochastic integral with respect to W_H^β has a unique extension to an isometry from $\gamma(H^{\frac{1}{2}-\beta}_{T-}(H),E)$ into $L^2(\Omega;E)$.

Motivated by (3.3) we shall call a function $\Phi\colon\ (0,T) \to \mathscr{L}(H,E)$ *stochastically integrable* with respect to W_H^β if $\Phi^* x^* \in H^{\frac{1}{2}-\beta}_{T-}(H)$ for all $x^* \in E^*$, and there exists an operator $R \in \gamma(H^{\frac{1}{2}-\beta}_{T-}(H),E)$ such that

$$R^* x^* = \Phi^* x^*$$

in $H^{\frac{1}{2}-\beta}_{T-}(H)$ for all $x^* \in E^*$. The operator R, if it exists, is uniquely determined. In this situation we say that Φ *represents* R.

Using the right ideal property for spaces of γ-radonifying operators, applied to the embeddings $H^{\frac{1}{2}-\beta}_{T-}(H) \hookrightarrow L^2(0,T;H)$ (for $0<\beta<\frac{1}{2}$) and $L^2(0,T;H) \hookrightarrow H^{\frac{1}{2}-\beta}_{T-}(H)$ (for $\frac{1}{2}<\beta<1$) we obtain the first part of the following simple consequence of Theorem 3.9; the second part is proved similarly.

Corollary 3.10. *Let W^β_H be an H-cylindrical Liouville fBm, W_H an H-cylindrical Brownian motion, and consider a function $\Phi\colon (0,T) \to \mathscr{L}(H,E)$.*
(1) *If $0<\beta<\frac{1}{2}$ and Φ is stochastically integrable with respect to W^β_H, then Φ is stochastically integrable with respect to W_H as well.*
(2) *If $\frac{1}{2}<\beta<1$ and Φ is stochastically integrable with respect to W_H, then Φ is stochastically integrable with respect to W^β_H as well.*
In fact, for any two numbers $0<\beta_1<\beta_2<1$, stochastic integrability with respect to $W^{\beta_1}_H$ implies stochastic integrability with respect to $W^{\beta_2}_H$.

We proceed with two further sufficient conditions for stochastic integrability. In both we assume that W^β_H is an H-cylindrical Liouville fBm of order β and W_H is an H-cylindrical Brownian motion.

The first theorem is a simple adaptation of a result due to Kalton and Weis for $\beta=\frac{1}{2}$ and $H=\mathbb{R}$ [25].

Theorem 3.11. *Let $0<\beta<\frac{1}{2}$. If $\Phi\colon (0,T) \to \gamma(H,E)$ is a continuously differentiable function that satisfies*

$$\int_0^T t^\beta \|\Phi'(t)\|_{\gamma(H,E)} \,\mathrm{d}t < \infty,$$

then Φ is stochastically integrable with respect to W^β_H and we have

$$\left(\mathbb{E}\left\|\int_0^T \Phi \,\mathrm{d}W^\beta_H\right\|^2\right)^{\frac{1}{2}} \leqslant C_\beta T^\beta \|\Phi(T-)\|_{\gamma(H,E)} + C_\beta \int_0^T t^\beta \|\Phi'(t)\|_{\gamma(H,E)} \,\mathrm{d}t,$$

where $C_\beta = 1/\sqrt{2\beta}\,\Gamma(\frac{1}{2}+\beta)$.

P r o o f. Put $g(s,t) := 1_{(t,T)}(s) f'(s)$ for $s,t \in (0,T)$. Then,

$$f(t) = f(T-) - \int_0^T g(s,t) \,\mathrm{d}s$$

for all $t \in (0,T)$. Due to Lemma 2.3, for almost all $s \in (0,T)$ the function $t \mapsto g(s,t) = 1_{(t,T)}(s) f'(s) = 1_{(0,s)}(t) f'(s)$ belongs to $\gamma(H^{\frac{1}{2}-\beta}_{T-}(H),E)$ with the norm

$$\begin{aligned}\|1_{(\cdot,T)}(s) f'(s)\|_{\gamma(H^{\frac{1}{2}-\beta}_{T-}(H),E)} &= \|1_{(0,s)}\|_{H^{\frac{1}{2}-\beta}_{T-}} \|f'(s)\|_{\gamma(H,E)} \\ &= C_\beta s^\beta \|f'(s)\|_{\gamma(H,E)}.\end{aligned}$$

It follows that the $\gamma(H^{\frac{1}{2}-\beta}_{T-}(H),E)$-valued function $s\mapsto g(s,\cdot)$ is Bochner integrable. Identifying the operator $f(T-)\in\gamma(H,E)$ with the constant function $1_{(0,T)}f(T-)\in\gamma(H^{\frac{1}{2}-\beta}_{T-}(H),E)$, we find that $f\in\gamma(H^{\frac{1}{2}-\beta}_{T-}(H),E)$ and

$$\begin{aligned}\|f\|_{\gamma(H^{\frac{1}{2}-\beta}_{T-}(H),E)} &\leqslant C_\beta T^\beta\|f(T-)\|_{\gamma(H,E)}+\int_0^T\|g(s,\cdot)\|_{\gamma(H^{\frac{1}{2}-\beta}_{T-}(H),E)}\,\mathrm{d}s\\ &= C_\beta T^\beta\|f(T-)\|_{\gamma(H,E)}+C_\beta\int_0^T s^\beta\|f'(s)\|_{\gamma(H,E)}\,\mathrm{d}s.\end{aligned}$$

□

The second theorem gives an improvement to Corollary 3.10 (2).

Theorem 3.12. *Let $\frac{1}{2}<\beta<1$ and $0\leqslant\alpha<\beta-\frac{1}{2}$. If $\Phi\colon (0,T)\to\mathscr{L}(H,E)$ is stochastically integrable with respect to W_H, then*

$$t\mapsto t^\alpha\Phi(t)$$

is stochastically integrable with respect to W^β_H.

Proof. This is proved in the same way as Corollary 3.10 (2), except that now we apply the right ideal property of γ-radonifying operators, now applied to the bounded operator $K_{\alpha,\beta}$ on $L^2(0,T;H)$,

$$K_{\alpha,\beta}f(t):=t^{-\alpha}I^{\beta-\frac{1}{2}}_{0+}f(t).$$

□

4. Comparison with classical fBm

In this section we compare the Liouville fBm W^β with the classical fBm, that is, a Gaussian process $(\tilde{W}^\beta(t))_{t\in[0,T]}$ with covariance

$$\mathbb{E}\tilde{W}^\beta(s)\tilde{W}^\beta(t)=s^{2\beta}+t^{2\beta}-|t-s|^{2\beta},$$

where $\beta\in(0,1)$ is the so-called *Hurst parameter*. Brownian motion again corresponds to the case $\beta=\frac{1}{2}$. For a review of the theory of stochastic integration with respect to the classical fBm we refer to [3], [36].

Let $\mathscr{H}^\beta$ and $\tilde{\mathscr{H}}^\beta$ be the Hilbert spaces obtained as the completions of the step functions with respect to the scalar products

$$[1_{(0,s)},1_{(0,t)}]_{\mathscr{H}^\beta}:=\mathbb{E}W^\beta(s)W^\beta(t)$$

and

$$[1_{(0,s)}, 1_{(0,t)}]_{\tilde{\mathscr{H}}^\beta} := \mathbb{E}\tilde{W}^\beta(s)\tilde{W}^\beta(t)$$

for the Liouville fBm and the classical fBm, respectively.

Proposition 4.1. *For all $0 < \beta < \frac{1}{2}$ we have $\mathscr{H}^\beta = \tilde{\mathscr{H}}^\beta = H_{T-}^{\frac{1}{2}-\beta}$ with equivalent norms.*

Proof. We have already seen that $\mathscr{H}^\beta = H_{T-}^{\frac{1}{2}-\beta}$ isometrically. The fact that $\tilde{\mathscr{H}}^\beta = H_{T-}^{\frac{1}{2}-\beta}$ up to an equivalent norm is well known; see [2, Proposition 8], [3, Formulas (2.27)], and [10]. □

By the very definition of an isonormal process we have the Itô isometry

$$\mathbb{E}\Big|\int_0^T f \, \mathrm{d}W_H^\beta\Big|^2 = \|f\|^2_{\mathscr{H}^\beta}.$$

Similarly, it is well known [3], [36] that

$$\mathbb{E}\Big|\int_0^T f \, \mathrm{d}\tilde{W}_H^\beta\Big|^2 = \|f\|^2_{\tilde{\mathscr{H}}^\beta},$$

where $\tilde{W}_H^\beta$ is the H-cylindrical classical fBm. Having observed the latter, we can repeat the constructions of the previous section and obtain analogues of our results for the classical fBm with Hurst parameter $0 < \beta < \frac{1}{2}$. The following result relates the two stochastic integrals.

Theorem 4.2. *Let $0 < \beta < \frac{1}{2}$. For a function $\Phi\colon (0,T) \to \mathscr{L}(H,E)$ the following are equivalent:*

(1) *Φ is stochastically integrable with respect to $\tilde{W}_H^\beta$;*

(2) *Φ is stochastically integrable with respect to W_H^β.*

In this situation we have

$$\mathbb{E}\Big\|\int_0^T \Phi \, \mathrm{d}\tilde{W}_H^\beta\Big\|^2 \simeq \mathbb{E}\Big\|\int_0^T \Phi \, \mathrm{d}W_H^\beta\Big\|^2$$

with two-sided constants independent of Φ.

Proof. In view of Proposition 4.1, the first assertion is immediate from Theorem 3.9 and its counterpart for the classical fBm (which again holds by the abstract results of [30], now applied to the Hilbert space $\tilde{H}^\beta$). To prove equivalence of the norms, we first observe that for all $x^* \in E^*$,

$$\mathbb{E}\Big|\int_0^T \Phi^* x^* \, \mathrm{d}W_H^\beta\Big|^2 = \|\Phi^* x^*\|^2_{\tilde{H}^\beta}$$

and similarly

$$\mathbb{E}\left|\int_0^T \Phi^* x^* \,\mathrm{d}\tilde{W}_H^\beta\right|^2 = \|\Phi^* x^*\|^2_{\tilde{H}^\beta}.$$

Hence by Proposition 4.1,

$$\mathbb{E}\left|\int_0^T \Phi^* x^* \,\mathrm{d}W_H^\beta\right|^2 \simeq \mathbb{E}\left|\int_0^T \Phi^* x^* \,\mathrm{d}\tilde{W}_H^\beta\right|^2$$

with two-sided constants independent of Φ and x^*. The result now follows from a standard comparison result for Banach space-valued Gaussian random variables.

□

Remark 4.3. For $\frac{1}{2} < \beta < 1$, the spaces $\mathscr{H}^\beta$ and $\tilde{\mathscr{H}}^\beta$ are different: the former consists of all distributions ψ such that $I_{T-}^{\beta-\frac{1}{2}}\psi \in L^2$, whereas the latter consists of those distributions for which

$$s \mapsto s^{\frac{1}{2}-\beta}(I_{T-}^{\beta-\frac{1}{2}}(u \mapsto u^{\beta-\frac{1}{2}}\psi(u)))(s) \in L^2.$$

See [3, Formula 2.18].

5. Evolution equations driven by Liouville fBm

In this section we shall apply the results of the previous sections to study the stochastic abstract Cauchy problem

$$\text{(sACP)} \qquad \begin{cases} \mathrm{d}U(t) = AU(t)\,\mathrm{d}t + B\,\mathrm{d}W_H^\beta(t), & t \in [0,T], \\ U(0) = x. \end{cases}$$

Here A is the generator of a C_0-semigroup $S = \{S(t)\}_{t\geqslant 0}$ on E, $B \in \mathscr{L}(H,E)$ is a given bounded linear operator, and W_H^β is a Liouville cylindrical fBm of order $0 < \beta < 1$ on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

If, for all $t > 0$, the $\mathscr{L}(H,E)$-valued function $S(t-\cdot)B$ is stochastically integrable on $(0,t)$ with respect to W_H^β, the process

$$U^x(t) = S(t)x + \int_0^t S(t-s)B\,\mathrm{d}W_H^\beta(s) \qquad (5.1)$$

is called the *mild solution* of (sACP). It is an easy consequence of the definition of the stochastic integral that the process U^x is strongly measurable as a mapping from $[0,\infty) \times \Omega$ into E.

The next theorem asserts the existence of a mild solution in the case where the Banach space E is of type 2 and the operator B is γ-radonifying. To some extent this could be seen as a generalization of some results from [4], [35]; see also [6] where the case of equations driven by a non-Gaussian Lévy process is considered.

Theorem 5.1. *Let S be a C_0-semigroup on a Banach space E with type $p \in (1, 2]$. Then for all $\beta \in (1/p, 1)$ and $B \in \gamma(H, E)$ the function $S(t - \cdot)B$ is stochastically integrable on $(0, t)$ with respect to W_H^β for all $t > 0$. As a consequence, the problem (sACP) has a unique mild solution U which is given by* (5.1).

P r o o f. First assume that E has type 2. In that case, we have a continuous embedding

$$L^2(0, T; \gamma(H, E)) \hookrightarrow \gamma(L^2(0, T; H), E).$$

Evidently, $S(\cdot)B$ belongs to $L^2(0, T; \gamma(H, E))$, and therefore this function is stochastically integrable with respect to H-cylindrical Brownian motions W_H. The result then follows from Corollary 3.10.

Next, assume that $1 < p < 2$. By the results of [24], [33], for a Banach space E with type p we have a continuous embedding

$$B_{p,p}^{\frac{1}{p}-\frac{1}{2}}(0, T; \gamma(H, E)) \hookrightarrow \gamma(L^2(0, T; H), E).$$

By [43, Theorem 4.6.1] one has continuous embeddings

$$H^{\beta-\frac{1}{2}}(0, T; \gamma(H, E)) \hookrightarrow B_{2,2}^{\beta-\frac{1}{2}}(0, T; \gamma(H, E)) \hookrightarrow B_{p,p}^{\frac{1}{p}-\frac{1}{2}}(0, T; \gamma(H, E)).$$

Combining them, we obtain a continuous embedding

$$H^{\beta-\frac{1}{2}}(0, T; \gamma(H, E)) \hookrightarrow \gamma(L^2(0, T; H), E).$$

Recalling that this embedding is given, for finite element rank functions, by

$$f \otimes (h \otimes x) \mapsto (f \otimes h) \otimes x,$$

the isometry $I_{T-}^{\beta-\frac{1}{2}} : L^2(0, T) \mapsto H_{T-}^{\beta-\frac{1}{2}}$ induces a continuous embedding

$$L^2(0, T; \gamma(H, E)) \hookrightarrow \gamma(H^{\frac{1}{2}-\beta}(0, T; H), E). \tag{5.2}$$

Now we may apply Theorem 3.9. □

This result is sharp in the following sense.

Example 5.2. Suppose $\frac{1}{2} < \beta < 1$ is given. Then for any $1 \leqslant p < 1/\beta$ there exists a Banach space E with type p, a C_0-semigroup S with a generator A on E, and a vector $x \in E$ for which the problem

$$\begin{cases} \mathrm{d}U(t) = AU(t)\,\mathrm{d}t + x\,\mathrm{d}w^\beta(t), & t \in [0,T], \\ U(0) = 0, \end{cases} \tag{5.3}$$

fails to have a mild solution. Here, w^β is a real-valued Liouville fBm with Hurst parameter β. Note that (5.3) corresponds to the special case of (sACP) for $H = \mathbb{R}$ (identifying $x \in E$ with $1 \otimes x \in \gamma(\mathbb{R}; E)$).

Indeed, let $1 \leqslant p < 1/\beta$. On $E := L^p(0,T)$, let S denote the left translation semigroup on E,

$$S(t)x(s) = \begin{cases} x(s+t), & s+t < T, \\ 0, & \text{otherwise.} \end{cases}$$

By combining Theorem 3.9 and [5, Theorem 2.3] (see also [34, Lemma 2.1]), for a given $x \in L^p(0,T)$ the problem (5.3) has a weak solution if only if $S(\cdot)x$ defines an element of $\gamma(H^{\frac{1}{2}-\beta}, L^p(0,T)) \simeq L^p(0,T; H^{\frac{1}{2}-\beta})$ (we need not worry about boundary conditions since $0 < \beta - \frac{1}{2} < \frac{1}{2}$; see the remarks at the end of Section 2).

Let us now suppose that this is true for all $x \in L^p(0,T)$. Fix a number $0 < \delta < T$ and consider an arbitrary function $x \in L^p(0,T)$ with support in (δ, T). For almost all $t \in (0,T)$ it follows that $s \mapsto S(t)x(s) = 1_{\{s+t<T\}}x(s+t)$ belongs to $H^{\frac{1}{2}-\beta}$. In particular, it follows that $s \mapsto x(s+t)$ belongs to $H^{\frac{1}{2}-\beta}$ for almost all $t \in (0,\delta)$. This function being identically zero on the intervals $(0, \delta - t)$ and $(T-t, T)$, it is immediate to see that its restriction to $(\delta - t, T - t)$ belongs to $H^{\frac{1}{2}-\beta}(\delta - t, T - t)$. This implies that $s \mapsto x(s)$ is in $H^{\frac{1}{2}-\beta}(\delta, T)$.

By the closed graph theorem, this proves that we have a continuous inclusion $L^p(\delta,T) \hookrightarrow H^{\frac{1}{2}-\beta}(\delta,T)$. This is the same as saying that the fractional integral operator $I_{T-}^{\beta-\frac{1}{2}}$ acts boundedly from $L^p(\delta,T)$ to $L^2(\delta,T)$. The latter is known to be false if $\beta < 1/p$.

Hence there must exist $x \in L^p(0,T)$ for which the problem (5.3) has no mild solution.

Remark 5.3. By a result of Veraar (in preparation), for p-concave Banach lattices E (such spaces have type p) one has a continuous embedding

$$L^p(0,T;\gamma(H,E)) \hookrightarrow \gamma(H^{\frac{1}{2}-\frac{1}{p}}(0,T;H),E)$$

and therefore (5.2) holds with $\beta = 1/p$. As a consequence, for such spaces E, Theorem 5.1 also holds for the critical exponent $\beta = 1/p$. We do not know whether this extends to arbitrary Banach spaces with type p.

We return to the setting where E is an arbitrary real Banach space. In the proof of the next theorem we will need the following result, which is a direct consequence of (3.2) combined with standard estimates for analytic semigroups (cf. [41]):

Lemma 5.4. *Let A generate an analytic C_0-semigroup E. Then for all $0 \leqslant \theta < \eta$ and large enough $w \in \mathbb{R}$ the set*

$$\{t^{\eta}(w-A)^{\theta}S(t)\}: \ t \in (0,T)\}$$

is γ-bounded.

The main result of this section is an extension of a result of [40], where it was assumed that $0 < \beta < \frac{1}{2}$ and that E is a Hilbert space.

Theorem 5.5. *Let $0 < \beta < 1$. If S is an analytic C_0-semigroup on an arbitrary Banach space E, then for all $B \in \gamma(H,E)$ the function $S(t-\cdot)B$ is stochastically integrable on $(0,t)$ with respect to W_H^{β}. As a consequence, the problem* (sACP) *has a mild solution U given by* (5.1). *Moreover, for all $1 \leqslant p < \infty$ and all $\alpha, \theta \geqslant 0$ satisfying $\alpha + \theta < \beta$ we have*

$$U \in L^p(\Omega; C^{\alpha}([0,T]; E_{\theta})),$$

where E_{θ} denotes the fractional domain space of exponent θ associated with A.

Remark 5.6. If $0 < \beta < \frac{1}{2}$, Theorem 4.2 enables us to replace the cylindrical Liouville fBm by a cylindrical fBm.

P r o o f. For $\frac{1}{2} \leqslant \beta < 1$, the existence of a mild solution follows from the fact that $S(t \mapsto \cdot)B$ is stochastically integrable on $(0,t)$ with respect to every H-cylindrical Brownian motion W_H and from Corollary 3.10. □

For $0 < \beta < \frac{1}{2}$ we verify the condition stated in Theorem 3.11. With $\Phi(t) = S(t)B$ we have

$$t^{\beta}\|\Phi'(t)\|_{\gamma(H,E)} = t^{\beta}\|AS(t)B\|_{\gamma(H,E)} \leqslant Ct^{-1+\beta}\|B\|_{\gamma(H,E)},$$

where C is a constant depending only on T and the semigroup S. Since the function on the right-hand side is integrable the result follows from Theorem 3.11.

Next we prove the space-time regularity assertion. We follow the proof of [31, Theorem 10.19]; for the reader's convenience we include the details. By the Kahane-Khintchine inequality we may assume that $p = 2$.

Fix $\theta \geqslant 0$ and $\alpha \geqslant 0$ such that $\alpha + \theta < \beta$. We claim that for all $t \in [0,T]$ the random $U(t)$ takes its values in E_{θ} almost surely. We prove this by showing that the functions $S(t-\cdot)B$ are stochastically integrable on $(0,t)$ with respect to W_H^β as $\mathscr{L}(H,E_\theta)$-valued functions. Indeed, this follows from Lemmas 5.4 and 3.7, once we realise three things:

(i) For all $0<\eta<\theta$ the set $\{r^\eta S(r)\colon r\in(0,T)\}$ is γ-bounded in $\mathscr{L}(E,E_\theta)$ by Lemma 5.4;

(ii) the function $s\mapsto(t-s)^{-\eta}B$ represents an operator in $\gamma(H^{\frac{1}{2}-\beta}(0,t;H),E)$ of the norm $\|s\mapsto(t-s)^{-\eta}\|_{H^{\frac{1}{2}-\beta}(0,t)}\|B\|_{\gamma(H,E)}$ by Lemmas 3.4 and 3.6;

(iii) $S(t-s)B=[(t-s)^\eta S(t-s)][(t-s)^{-\eta}B]$.

Variations of this argument will be used repeatedly below.

Fix $0\leqslant s\leqslant t\leqslant T$. By the triangle inequality in $L^2(\Omega;E)$,

$$(\mathbb{E}\|U(t)-U(s)\|_{E_\theta}^2)^{\frac{1}{2}}\leqslant\Big(\mathbb{E}\Big\|\int_0^s[S(t-r)-S(s-r)]B\,\mathrm{d}W^\beta(r)\Big\|_{E_\theta}^2\Big)^{\frac{1}{2}}+\Big(\mathbb{E}\Big\|\int_s^t S(t-r)B\,\mathrm{d}W^\beta(r)\Big\|_{E_\theta}^2\Big)^{\frac{1}{2}}.$$

Choose $\lambda\in\mathbb{R}$ sufficiently large in order that the fractional powers of $\lambda-A$ exist. For the first term we have, for any choice of $\varepsilon>0$ such that $\alpha+\theta+\varepsilon<\beta$,

$$\begin{aligned}
&\mathbb{E}\Big\|\int_0^s[S(t-r)-S(s-r)]B\,\mathrm{d}W_H^\beta(r)\Big\|_{E_\theta}^2\\
&\quad\simeq\mathbb{E}\Big\|\int_0^s(s-r)^{\alpha+\theta+\varepsilon}(\lambda-A)^{\alpha+\theta}S(s-r)\\
&\qquad\times(s-r)^{-\alpha-\theta-\varepsilon}[S(t-s)-I](\lambda-A)^{-\alpha}B\,\mathrm{d}W_H^\beta(r)\Big\|^2\\
&\quad\overset{\text{(i)}}{\leqslant}C^2\mathbb{E}\Big\|\int_0^s(s-r)^{-\alpha-\theta-\varepsilon}[S(t-s)-I](\lambda-A)^{-\alpha}B\,\mathrm{d}W_H^\beta(r)\Big\|^2\\
&\quad\overset{\text{(ii)}}{=}C^2\|[S(t-s)-I](\lambda-A)^{-\alpha}B\|_{\gamma(H,E)}^2\mathbb{E}\Big|\int_0^s(s-r)^{-\alpha-\theta-\varepsilon}\,\mathrm{d}W^\beta(r)\Big|^2\\
&\quad\overset{\text{(iii)}}{=}C^2s^{2\beta-2\alpha-2\theta-2\varepsilon}\|[S(t-s)-I](\lambda-A)^{-\alpha}\|^2\|B\|_{\gamma(H,E)}^2\\
&\quad\overset{\text{(iv)}}{\leqslant}C^2T^2(t-s)^{2\alpha}\|B\|_{\gamma(H,E)}^2,
\end{aligned}$$

where the numerical value of C changes from line to line. In (i) we have used Lemmas 3.7 and 5.4 in combination with Theorem 3.9. In (ii) we have used Lemma 3.6 and Theorem 3.9 in combination with an approximation argument to see that if W^β is any real-valued Liouville fBm, then for all $f\in H_{T-}^{\frac{1}{2}-\beta}$ and $\tilde{B}\in\gamma(H,E)$,

$$\mathbb{E}\Big\|\int_0^T f(t)\tilde{B}\,\mathrm{d}W_H^\beta(t)\Big\|^2=\|\tilde{B}\|_{\gamma(H,E)}^2\mathbb{E}\Big|\int_0^T f(t)\,\mathrm{d}W^\beta(r)\Big|^2.$$

In (iii) we have used Lemma 3.4, and (iv) follows from standard estimates for analytic semigroups (see [41]).

Similarly,

$$
\begin{aligned}
&\mathbb{E}\bigg\|\int_s^t S(t-r)B\,\mathrm{d}W_H^\beta(r)\bigg\|_{E_\theta}^2 \\
&\qquad \simeq \mathbb{E}\bigg\|\int_s^t (t-r)^{\beta-\alpha}(\lambda-A)^\theta S(t-r)(t-r)^{-\beta+\alpha}B\,\mathrm{d}W_H^\beta(r)\bigg\|^2 \\
&\qquad \leqslant C^2\mathbb{E}\bigg\|\int_s^t (t-r)^{-\beta+\alpha}B\,\mathrm{d}W_H^\beta(r)\bigg\|^2 \\
&\qquad = C^2\|B\|_{\gamma(H,E)}^2\mathbb{E}\bigg|\int_s^t (t-r)^{-\beta+\alpha}\,\mathrm{d}W^\beta(r)\bigg|^2 \leqslant C_T^2\|B\|_{\gamma(H,E)}^2(t-s)^{2\alpha}.
\end{aligned}
$$

The first part of the theorem follows by combining these estimates.

For the second part, pick $\alpha < \alpha' < \beta-\theta$. Given $q \geqslant 1$, by the above we find a constant C such that for $0 \leqslant s, t \leqslant T$,

$$
\mathbb{E}\|U(t)-U(s)\|_{E_\theta}^q \leqslant C^q|t-s|^{\alpha' q}.
$$

For q large enough the existence of a version with α-Hölder continuous trajectories now follows from the Kolmogorov-Chentsov continuity theorem. Finally, $U \in L^p(\Omega; C^\alpha([0,T]; E_\theta))$ by Fernique's theorem [13].

6. An example

We will apply our results to prove existence and space-time Hölder regularity of mild solutions for stochastic partial differential equations of the form

$$
\begin{cases}
\dfrac{\partial u}{\partial t}(t,x) = \mathscr{A}u(t,x) + \dfrac{\partial W^\beta(t,x)}{\partial t\,\partial x}, & x \in \mathscr{O},\ t \in [0,T], \\
u(0,x) = 0, & x \in \mathscr{O},
\end{cases}
\tag{6.1}
$$

where $\mathscr{O}$ is a bounded C^2-domain in $\mathbb{R}^d$ and $\partial W^\beta(t,x)/\partial t\partial x$ denotes a space-time noise which is 'white' in space and 'Liouville fractional' of order $0 < \beta < 1$ in time.

We shall assume that $\mathscr{A}$ is a second-order uniformly elliptic operator on $\mathscr{O}$ of the form

$$
\mathscr{A}f(x) = \sum_{i,j=1}^d a_{ij}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_{j=1}^d b_j(x)\frac{\partial f}{\partial x_j}(x) + c(x)f(x).
$$

The problem (6.1) can be rewritten in the abstract form

$$\text{(6.2)} \qquad \begin{cases} \mathrm{d}U(t) = AU(t)\,\mathrm{d}t + \mathrm{d}W^{\beta}_{L^2(\mathscr{O})}(t), & t \in [0,T], \\ U(0) = 0, \end{cases}$$

where $W^{\beta}_{L^2(\mathscr{O})}$ is an $L^2(\mathscr{O})$-cylindrical fBm of order β on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Under mild boundedness and regularity assumptions on the coefficients (to be precise, $a_{ij} \in C^{\varepsilon}(\overline{\mathscr{O}})$ for some $\varepsilon > 0$ and $b_j, c \in L^{\infty}(\mathscr{O})$), which we shall henceforth assume to be satisfied, A generates an analytic C_0-semigroup S on $L^p(\mathscr{O})$. Moreover, A has bounded imaginary powers and hence, by [43], the fractional domain spaces of A are equal to the the complex interpolation spaces $(L^p(\mathscr{O}))_{\theta} = \mathscr{D}((-A)^{\theta})$ and given, up to equivalent norms, by

$$(L^p(\mathscr{O}))_{\theta} = [L^p(\mathscr{O}), \mathscr{D}(A)]_{\theta} = \begin{cases} H^{2\theta,p}(\mathscr{O}), & 0 < \theta < \frac{1}{2}, \\ H^{2\theta,p}_0(\mathscr{O}), & \frac{1}{2} < \theta < 1. \end{cases}$$

In the case $\beta = \frac{1}{2}$ the driving process $W^{\beta}_{L^2(\mathscr{O})}$ is an $L^2(\mathscr{O})$-cylindrical Brownian motion. In that case, in dimension $d = 1$ the mild solution U of (6.2) satisfies

$$U \in L^q(\Omega; C^{\alpha}([0,T]; C^{\gamma}(\overline{\mathscr{O}})))$$

for all $1 \leqslant q < \infty$ and $\alpha, \gamma \geqslant 0$ satisfying $2\alpha + \gamma < \frac{1}{2}$; see [4], [11]. Following the methods used in these papers (where also more details can be found), for general $0 < \beta < 1$ we may apply Theorem 5.5 in negative extrapolation spaces of $L^p(\mathscr{O})$ of exponent greater than $\frac{1}{4}d$. The regularising properties of the semigroup S then yield that problem (6.2) admits a mild solution U in $L^q(\Omega; C^{\alpha}([0,T]; (L^p(\mathscr{O}))_{\theta-\frac{1}{4}d})) \subseteq L^q(\Omega; C^{\alpha}([0,T]; H^{2\theta-\frac{1}{2}d,p}(\mathscr{O})))$ for all $1 \leqslant q < \infty$, provided $\alpha, \theta \geqslant 0$ satisfy $\frac{1}{4}d < \theta < 1$, $\theta \neq \frac{1}{2} + \frac{1}{4}d$, and $\alpha + \theta < \beta$. Combining this with the Sobolev embedding $H^{2\eta,p}(\mathscr{O}) \hookrightarrow C^{\gamma}(\overline{\mathscr{O}})$ for $\gamma + d/p < 2\eta$, by taking p large enough we obtain Hölder continuity of the solution jointly in space and time:

Theorem 6.1. *Under the above stated assumptions, the problem* (6.2) *has a mild solution U which belongs to $L^q(\Omega; C^{\alpha}([0,T]; C^{\gamma}(\overline{\mathscr{O}})))$ for all $1 \leqslant q < \infty$ and $\alpha, \gamma \geqslant 0$ satisfying $2\alpha + \gamma < 2\beta - \frac{1}{2}d$.*

In particular, we see that a mild solution with space-time Hölder regularity exists if $\frac{1}{4}d < \beta < 1$. This contrasts the cylindrical Brownian motion case $\beta = \frac{1}{2}$ where such solutions only exist in dimension $d = 1$. Note that in dimension $d = 2$ and $d = 3$ we obtain the existence of a space-time Hölder continuous solution for $\frac{1}{2} < \beta < 1$ and $\frac{3}{4} < \beta < 1$, respectively.

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