STOCHASTIC GAMES OF CONTROL AND STOPPING FOR A LINEAR DIFFUSION *

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Abstract

We study three stochastic differential games. In each game, two players control a process $X = \{X_t\,,\ 0 \le t < \infty\}$ which takes values in the interval $I = (0,1)$, is absorbed at the endpoints of I, and satisfies a stochastic differential equation

$$dX_t \,=\, \mu\big(X_t, \alpha(X_t), \beta(X_t)\big)\, dt \,+\, \sigma\big(X_t, \alpha(X_t), \beta(X_t)\big)\, dW_t\,, \quad X_0 = x \in I\,.$$

The *control functions* $\alpha(\cdot)$ and $\beta(\cdot)$ are chosen by players $\mathfrak{A}$ and $\mathfrak{B}$, respectively.

In the first of our games, which is zero-sum, player $\mathfrak{A}$ has a continuous reward function $u : [0,1] \to \mathbb{R}$. In addition to $\alpha(\cdot)$, player $\mathfrak{A}$ chooses a stopping rule τ and seeks to maximize the expectation of $u(X_\tau)$; whereas player $\mathfrak{B}$ chooses $\beta(\cdot)$ and aims to minimize this expectation.

In the second game, players $\mathfrak{A}$ and $\mathfrak{B}$ each have continuous reward functions $u(\cdot)$ and $v(\cdot)$, choose stopping rules τ and ρ, and seek to maximize the expectations of $u(X_\tau)$ and $v(X_\rho)$, respectively.

In the third game the two players again have continuous reward functions $u(\cdot)$ and $v(\cdot)$, now assumed to be unimodal, and choose stopping rules τ and ρ. This game terminates at the minimum $\tau \wedge \rho$ of the stopping rules τ and ρ, and players $\mathfrak{A}$, $\mathfrak{B}$ want to maximize the expectations of $u(X_{\tau\wedge\rho})$ and $v(X_{\tau\wedge\rho})$, respectively.

Under mild technical assumptions we show that the first game has a value, and find a saddle point of optimal strategies for the players. The other two games are not zero-sum, in general, and for them we construct Nash equilibria.

Key Words: Stochastic games, optimal control, optimal stopping, Nash equilibrium, linear diffusions, local time, generalized Itô rule.

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1 Preliminaries

The state space for the two-player games studied in this paper, is the interval $I = (0,1)$. For each $x \in [0,1]$, players $\mathfrak{A}$ and $\mathfrak{B}$ are given nonempty action sets $A(x)$ and $B(x)$, respectively. For simplicity, we assume that $A(x)$ and $B(x)$ are Borel sets of the real line $\mathbb{R}$.

Denote by $\mathfrak{Z} = C([0,\infty))$ the space of continuous functions $\xi : [0,\infty) \to \mathbb{R}$ with the topology of uniform convergence on compact sets. A `stopping rule` is a mapping $\tau : \mathfrak{Z} \to [0,\infty]$ such that

$$\{\xi \in \mathfrak{Z} \,:\, \tau(\xi) \le t\} \,\in\, \mathcal{B}_t \,:=\, \varphi_t^{-1}(\mathcal{B})$$

holds for every $t \in [0,\infty)$. Here $\mathcal{B}$ is the Borel σ-algebra generated by the open sets in $\mathfrak{Z}$, and $\varphi_t : \mathfrak{Z} \to \mathfrak{Z}$ is the mapping

$$(\varphi_t\, \xi)(s) \,:=\, \xi(t \wedge s)\,, \qquad 0 \le s < \infty\,.$$

We shall denote by $\mathcal{S}$ the set of all stopping rules.

The two players are allowed to choose measurable functions $\alpha : [0,1] \to \mathbb{R}$ and $\beta : [0,1] \to \mathbb{R}$ with $\alpha(x) \in A(x)$ and $\beta(x) \in B(x)$ for every $x \in [0,1]$. (We assume that the correspondences $A(\cdot)$ and $B(\cdot)$ are sufficiently regular, to guarantee the existence of such 'choice functions'.) For each such pair $\alpha(\cdot)$ and $\beta(\cdot)$, we have the dynamics

$$dX_t \,=\, \mu\big(X_t, \alpha(X_t), \beta(X_t)\big)\, dt \,+\, \sigma\big(X_t, \alpha(X_t), \beta(X_t)\big)\, dW_t\,, \qquad X_0 = x \in I \tag{1.1}$$

of a diffusion process $X \equiv X^{x,\alpha,\beta}$ that lives in the interval I and is absorbed the first time it reaches either of the endpoints. Here $W = \{W_t\,,\ 0 \le t < \infty\}$ is a standard, one-dimensional Brownian motion, and the functions $\mu(\,\cdot\,,\,\cdot\,,\,\cdot\,)$, $\sigma(\,\cdot\,,\,\cdot\,,\,\cdot\,)$ are measurable and map the product space $[0,1] \times \mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$ and $(0,\infty)$, respectively.

More precisely, we are assuming that for any pair of functions $\alpha(\cdot)$ and $\beta(\cdot)$ chosen as above, the local drift and variance

$$b(x) \,:=\, \mu\big(x, \alpha(x), \beta(x)\big)\,, \qquad s^2(x) \,:=\, \sigma^2\big(x, \alpha(x), \beta(x)\big)\,, \qquad x \in [0,1]$$

are measurable functions, which satisfy the conditions

$$s^2(x) \,>\, 0\,, \qquad \int_{x-\varepsilon}^{x+\varepsilon} \frac{1 + |b(y)|}{s^2(y)}\, dy \,<\, \infty \qquad \text{for some } \ \varepsilon = \varepsilon(x) > 0$$

for each $x \in I$, as well as

$$b(x) \,=\, s^2(x) \,=\, 0 \qquad \text{for} \qquad x \in \{0,1\}$$

(in other words, both endpoints are absorbing). The first of these conditions implies, in particular, that the resulting diffusion process

$$dX_t \,=\, b(X_t)\, dt \,+\, s(X_t)\, dW_t\,, \qquad X_0 = x$$

is *regular:* it exits in finite expected time from any open neighborhood $(x-\varepsilon, x+\varepsilon) \subset I$ (cf. Karatzas & Shreve (1991), p.344).

If τ is a stopping rule, we often write X_τ as abbreviation for a random variable that equals $X_{\tau(X)}$ when $\tau(X)$ is finite on the path $\{X_t\,,\, 0 \le t < \infty\}$. Also, for a bounded and measurable function $u : [0,1] \to \mathbb{R}$, we set

$$u(X_\tau) \equiv \limsup_{t\to\infty} u\left(X_{\tau(X)\wedge t}\right) = \left\{ \begin{array}{ccc} u\left(X_{\tau(X)}\right) & ; & \text{if } \tau(X) < \infty \\ \limsup_{t\to\infty} u\left(X_t\right) & ; & \text{if } \tau(X) = \infty \end{array} \right\}.$$

2 A Zero-Sum Game

Assume player $\mathfrak{A}$ selects the control function $\alpha(\cdot)$ and a stopping rule τ, whereas player $\mathfrak{B}$ has only the choice of the control function $\beta(\cdot)$. Let $u : [0,1] \to \mathbb{R}$ be a continuous function, which we regard as a *reward function* for player $\mathfrak{A}$. Then the expected payoff from player $\mathfrak{B}$ to player $\mathfrak{A}$ is

$$\mathbb{E}[\, u(X_\tau)\,]\,,$$

where $X \equiv X^{x,\alpha,\beta}$ and the game begins at $X_0 = x \in I$. Assume further that $u(\cdot)$ attains its maximum value at a unique position $m \in [0,1]$. Call this game $\mathfrak{G}_1(x)$.

Optimal strategies for the players have a simple intuitive description: In the interval $(0,m)$ to the left of the maximum, player $\mathfrak{A}$ chooses a control function $\alpha^*(\cdot)$ that maximizes the "signal-to-noise" (mean/variance) ratio μ/σ^2, whereas player $\mathfrak{B}$ chooses a control function $\beta^*(\cdot)$ that minimizes this ratio. In the interval $(m,1)$ to the right of the maximum, player $\mathfrak{A}$ chooses $\alpha^*(\cdot)$ to minimize μ/σ^2, whereas player $\mathfrak{B}$ chooses $\beta^*(\cdot)$ to maximize it. Finally, player $\mathfrak{A}$ takes the stopping rule τ^* to be optimal for maximizing $\mathbb{E}[\, u(Z_\tau)\,]$ over all stopping rules τ of the diffusion process $Z \equiv X^{x,\alpha^*,\beta^*}$, namely

$$dZ_t = b^*(Z_t)\,dt + s^*(Z_t)\,dW_t\,, \qquad Z_0 = x\,, \tag{2.1}$$

with drift and dispersion coëfficients given by

$$b^*(z) := \mu\big(z, \alpha^*(z), \beta^*(z)\big)\,, \qquad s^*(z) := \sigma\big(z, \alpha^*(z), \beta^*(z)\big)\,, \tag{2.2}$$

respectively. More precisely, we impose the following condition.

C.1: Local Saddle-Point Condition. *There exist measurable functions $\alpha^*(\cdot)$, $\beta^*(\cdot)$ such that $(\alpha^*(z),\, \beta^*(z)) \in A(z) \times B(z)$ holds for all $z \in (0,1)$;*

$$\left(\frac{\mu}{\sigma^2}\right)(z, a, \beta^*(z)) \le \left(\frac{\mu}{\sigma^2}\right)(z, \alpha^*(z), \beta^*(z)) \le \left(\frac{\mu}{\sigma^2}\right)(z, \alpha^*(z), b)\,, \quad \forall\ (a,b) \in A(z) \times B(z)$$

holds for every $z \in (0,m)$; and

$$\left(\frac{\mu}{\sigma^2}\right)(z, a, \beta^*(z)) \ge \left(\frac{\mu}{\sigma^2}\right)(z, \alpha^*(z), \beta^*(z)) \ge \left(\frac{\mu}{\sigma^2}\right)(z, \alpha^*(z), b)\,, \quad \forall\ (a,b) \in A(z) \times B(z)$$

holds for every $z \in (m,1)$.

Remark 1: For fixed z, the existence of $\alpha^*(z)$, $\beta^*(z)$ satisfying the condition $C.1$ corresponds to the existence of optimal strategies for a "local" (one-shot) game with action sets $A(z)$ and $B(z)$, and with payoff given by $(\mu/\sigma^2)(z,a,b)$ for $0<z<m$ and by $(-\mu/\sigma^2)(z,a,b)$ for $m<z<1$. Such strategies exist, for example, if $(\mu/\sigma^2)(z,\cdot,\cdot)$ is continuous, and the sets $A(z)$, $B(z)$ are compact for every $z \in I$.

Suppose that the local saddle-point condition $C.1$ holds, and recall the diffusion process $Z \equiv X^{x,\alpha^*,\beta^*}$ of (2.1), (2.2) for a fixed initial position $x \in I$. Fix an arbitrary point η of I, and write the *scale function* of Z as

$$p(z) := \int_\eta^z p'(u)\,du\,, \qquad z \in I\,,$$

where we have set

$$p'(z) := \exp\left\{-2\int_\eta^z \frac{b^*(u)}{(s^*(u))^2}\,du\right\} = 1+\int_\eta^z p''(u)\,du > 0$$

and

$$p''(z) := -2\,p'(z)\cdot\frac{b^*(z)}{(s^*(z))^2}\,.$$

C.2: Strong Non-degeneracy Condition. *We have*

$$\inf_{z\in I}\left(\inf_{\substack{a\in A(z)\\ b\in B(z)}} \sigma^2(z,a,b)\right) > 0\,.$$

Theorem 1: *Assume that $\alpha^*(\cdot)$, $\beta^*(\cdot)$ satisfy the local saddle-point condition $C.1$. For each $x \in I$, define the process $Z^x \equiv X^{x,\alpha^*,\beta^*}$ of (2.1), (2.2), and assume that the strong non-degeneracy condition C.2 holds. Let*

$$U(x) := \sup_{\tau\in\mathcal{S}} \mathbb{E}[\,u(Z^x_\tau)\,]\,, \quad x \in I \tag{2.3}$$

where τ ranges over the set $\mathcal{S}$ of all stopping rules, and consider the element τ^* of $\mathcal{S}$ given by

$$\tau^*(\xi) := \inf\{t \ge 0 : U(\xi_t) = u(\xi_t)\}\,, \quad \xi \in \mathfrak{Z}\,. \tag{2.4}$$

Then for each x, the game $\mathfrak{G}_1(x)$ has value $U(x)$ and saddle-point $\big((\alpha^*(\cdot),\tau^*),\beta^*(\cdot)\big)$.

Namely, the pair $(\alpha^*(\cdot),\tau^*)$ is optimal for player $\mathfrak{A}$, when player $\mathfrak{B}$ chooses $\beta^*(\cdot)$; and the function $\beta^*(\cdot)$ is optimal for player $\mathfrak{B}$, when player $\mathfrak{A}$ chooses $(\alpha^*(\cdot),\tau^*)$.

Before broaching the proof of this result, let us recall some well-known facts from the theory of optimal stopping (as developed, for instance, in Snell (1952), Chow et al. (1971), Neveu (1975) or Shiryaev (1978)). The function $U(\cdot)$ of (2.3) is the *value function* for the optimal stopping problem associated with the random process $u(Z^x) = \{\,u(Z^x_t),\, 0 \le t < \infty\,\}$; it is obviously bounded, because the reward function $u(\cdot)$ is continuous (therefore bounded) on the compact interval [0,1]. In addition, the function $U(\cdot)$ is known to be continuous, and the stopping rule τ^* to be optimal: i.e., $U(x) = \mathbb{E}[\,u(Z^x_{\tau^*})\,]$ (see, for example, Shiryaev (1978)). Furthermore,

$$U(x) = \widetilde{U}(p(x))\,, \qquad x \in I\,,$$

where $\widetilde{U}(\cdot)$ is the least concave majorant of the function $\widetilde{u}(\cdot)$ defined through

$$u(x) = \widetilde{u}(p(x))\,;$$

for instance, see Karatzas & Sudderth (1999) or Dayanik & Karatzas (2003), pp.178-181. The process $Y \equiv Y^y := p(Z^x)$ is easily seen to be a "diffusion in natural scale", that is,

$$dY_t = \widetilde{s}(Y_t)\,dW_t\,, \qquad Y_0 = y := p(x)\,,$$

where $\widetilde{s}(\cdot)$ is defined via $\widetilde{s}(p(x)) = s(x)\,p'(x)\,,\ x \in I$. The state-space of this diffusion is the open interval

$$\widetilde{I} = \big(p(0+),\, p(1-)\big)\,.$$

The function $\widetilde{U}(\cdot)$ is the value function of the optimal stopping problem for the process $\widetilde{u}(Y)$ with starting point $Y_0 = y \in \widetilde{I}$; to wit,

$$\widetilde{U}(y) = \sup_{\varrho \in \mathcal{S}} \mathbb{E}[\,\widetilde{u}(Y^y_\varrho)\,]\,, \quad Y^y_0 = y \in \widetilde{I}\,.$$

We shall refer to this as the "auxiliary optimal stopping problem".

As the least concave majorant of $\widetilde{u}(\cdot)$, the function $\widetilde{U}(\cdot)$ is clearly increasing on the interval $(p(0+), \widetilde{m})$ and decreasing on the interval $(\widetilde{m}, p(1-))$, where $\widetilde{m} := p(m)$ is the unique point at which the function $\widetilde{u}(\cdot)$ attains its maximum over the interval $\widetilde{I}$. Furthermore, the function $\widetilde{U}(\cdot)$ is affine on the connected components of the open set

$$\widetilde{\mathcal{O}} := \Big\{\, y \in \widetilde{I} \,:\, \widetilde{U}(y) > \widetilde{u}(y) \,\Big\}\,,$$

the optimal continuation region for the auxiliary stopping problem. Consequently, the positive measure $\widetilde{\nu}$ defined by

$$\widetilde{\nu}\big(\,[c,d\,)\big) := D^-\widetilde{U}(c) - D^-\widetilde{U}(d\,)\,, \qquad p(0+) < c < d < p(1-)\,,$$

assigns zero mass to the set $\widetilde{\mathcal{O}}$: $\widetilde{\nu}(\widetilde{\mathcal{O}}) = 0$. Note also that $\widetilde{\mathcal{O}} = p(\mathcal{O})$, where

$$\mathcal{O} := \{\, x \in I \,:\, U(x) > u(x) \,\}$$

is the optimal continuation region for the original stopping problem.

Proof of Theorem 1: Let $\alpha(\cdot)$ and $\beta(\cdot)$ be arbitrary control functions for the two players, and let $\tau \in \mathcal{S}$ be an arbitrary stopping rule for player $\mathfrak{A}$. Consider the processes

$$Z \equiv X^{x,\alpha^*,\beta^*}\,, \quad H \equiv X^{x,\alpha^*,\beta}\,, \quad \Theta \equiv X^{x,\alpha,\beta^*}\,.$$

It suffices to show

$$\mathbb{E}[\,u(\Theta_\tau)\,] \le \mathbb{E}[\,u(Z_{\tau^*})\,] \le \mathbb{E}[\,u(H_{\tau^*})\,]\,. \tag{2.5}$$

Indeed, these inequalities mean that the pair $(\alpha^*(\cdot), \tau^*)$ is optimal for player $\mathfrak{A}$, when player $\mathfrak{B}$ chooses $\beta^*(\cdot)$; and that $\beta^*(\cdot)$ is optimal for player $\mathfrak{B}$, when player $\mathfrak{A}$ chooses $(\alpha^*(\cdot), \tau^*)$.

In other words, $\big((\alpha^*(\cdot),\tau^*)\,,\,\beta^*(\cdot)\big)$ is a saddle-point for the game $\mathfrak{G}_1(x)$, and this game has value

$$\mathbb{E}\,[\,u(Z_{\tau^*})\,] \,=\, \mathbb{E}\,\Big[\,u\left(X^{x,\alpha^*,\beta^*}_{\tau^*}\right)\Big] \,=\, U(x)$$

equal to the middle term in (2.5). This is the same as the value $U(x)$ of the optimal stopping problem for the diffusion process Z, as in (2.3).

Furthermore, it suffices to prove (2.5) when $m=1$, to wit, when the maximum of $u(\cdot)$ occurs at the right-endpoint of I. This is because it is obviously optimal for player $\mathfrak{A}$ to stop at m, so the intervals $(0,m)$ and $(m,1)$ can be treated separately and similarly.

- To prove the second inequality of (2.5), consider the semimartingale $\widetilde{H} := p(H)$ which satisfies

$$d\widetilde{H}_t \,=\, p'(H_t)\,dH_t \,+\, \frac{1}{2}\,p''(H_t)\,\sigma^2_H(t)\,dt \,=\, p'(H_t)\,[\,\mu_H(t)\,dt + \sigma_H(t)\,dW_t\,] \,+\, \frac{1}{2}\,p''(H_t)\,\sigma^2_H(t)\,dt$$

or equivalently

$$d\widetilde{H}_t \,=\, \sigma^2_H(t)\,\left[\,\frac{1}{2}\,p''(H_t) + \frac{\mu_H(t)}{\sigma^2_H(t)}\cdot p'(H_t)\right]dt \,+\, p'(H_t)\,\sigma_H(t)\,dW_t\,, \tag{2.6}$$

where we have set

$$\mu_H(t) \,:=\, \mu\big(H_t,\alpha^*(H_t),\beta(H_t)\big) \qquad \text{and} \qquad \sigma_H(t) \,:=\, \sigma\big(H_t,\alpha^*(H_t),\beta(H_t)\big)\,.$$

By condition $C.1$ and the positivity of $p'(\cdot)$, we have

$$\begin{aligned}\frac{1}{2}\,p''(H_t) + \frac{\mu_H(t)}{\sigma^2_H(t)}\cdot p'(H_t) \,&\geq\, \frac{1}{2}\,p''(H_t) + \Big(\frac{\mu}{\sigma^2}\Big)\,\big(H_t,\alpha^*(H_t),\beta^*(H_t)\big)\cdot p'(H_t)\\ &=\, \frac{1}{2}\,p''(H_t) + \frac{b^*(H_t)}{\big(s^*(H_t)\big)^2}\cdot p'(H_t) \,=\, 0\end{aligned}$$

from (2.2). Therefore, the process $\widetilde{H} = p(H)$ is a local submartingale.

Now let us look at the semimartingale $U(H_t) = \widetilde{U}(\widetilde{H}_t)$, $0 \leq t < \infty$. By the generalized Itô rule for concave functions (cf. Karatzas & Shreve (1991), section 3.7), this process satisfies

$$\begin{aligned}U(H_T) \,&=\, \widetilde{U}(\widetilde{H}_T) \,=\, U(x) + \int_0^T D^-\widetilde{U}(\widetilde{H}_t)\cdot d\widetilde{H}_t \,-\, \int_{\widetilde{I}} L^{\widetilde{H}}_T(\zeta)\,\widetilde{\nu}(d\zeta)\\ &=\, U(x) + \int_0^T D^-\widetilde{U}(p(H_t))\cdot\sigma^2_H(t)\,\left[\,\frac{1}{2}\,p''(H_t) + \frac{\mu_H(t)}{\sigma^2_H(t)}\cdot p'(H_t)\right]dt\\ &\quad+ \int_0^T D^-\widetilde{U}(p(H_t))\cdot p'(H_t)\,\sigma_H(t)\,dW_t \,-\, \int_{\widetilde{I}} L^{\widetilde{H}}_T(\zeta)\,\widetilde{\nu}(d\zeta)\,.\end{aligned} \tag{2.7}$$

We are using the notation $L^{\Upsilon}_T(\zeta)$ for the local time of a continuous semimartingale Υ, accumulated by calendar time T at the site ζ. The last integral of (2.7) is equal to

$$\int_{\widetilde{I}\setminus\widetilde{\mathcal{O}}} L^{\widetilde{H}}_T(\zeta)\,\widetilde{\nu}(d\zeta)\,,$$

and therefore vanishes for $T = \tau^*(H)$; just recall that $\widetilde{\nu}(\widetilde{\mathcal{O}}) = 0$, $\widetilde{\mathcal{O}} = p(\mathcal{O})$, and observe that the stopping rule $\tau^*(H)$ of (2.4) can be written as

$$\tau^*(H) \,=\, \inf\{t \geq 0 \,:\, \widetilde{U}(\widetilde{H}_t) = \widetilde{u}(\widetilde{H}_t)\} \,=\, \inf\{t \geq 0 \,:\, \widetilde{H}_t \notin \widetilde{\mathcal{O}}\,\}\,. \tag{2.8}$$

Thus, the nonnegativity of $D^-\widetilde{U}(\cdot)$ and of the term in brackets on the second line of (2.7) guarantee that $U(H_{\cdot\wedge\tau^*})$ is a bounded local submartingale, hence a (bounded, and genuine) submartingale. Consequently, the optional sampling theorem gives

$$U(x) \,\leq\, \mathbb{E}[\,U(H_{\tau^*})\,] \,=\, \mathbb{E}[\,u(H_{\tau^*})\,]\,.$$

For this last equality, we have used (2.8) and the property

$$\tau^*(X) \,<\, \infty \quad \text{a.s., for every process } X \equiv X^{\,x,\alpha,\beta} \quad \text{as in (1.1);}$$

this is a consequence of the strong non-degeneracy condition C.2. Since $U(x) = \mathbb{E}[\,u(Z_{\tau^*})\,]$, the proof of the second inequality in (2.5) is now complete.

- To verify the first inequality of (2.5), namely

$$U(x) \,=\, \mathbb{E}[\,u(Z_{\tau^*})\,] \,\geq\, \mathbb{E}[\,u(\Theta_\tau)\,]\,,$$

we observe that an analysis of the processes $\widetilde{\Theta} := p(\Theta)$ and $\widetilde{U}(\widetilde{\Theta}) \equiv U(\Theta)$, similar to that carried out above, shows they are local supermartingales, and that $U(\Theta)$ is a genuine supermartingale because it is bounded. By the optional sampling theorem we obtain then

$$U(x) \,\geq\, \mathbb{E}[\,U(\Theta_\tau)\,] \,\geq\, \mathbb{E}[\,u(\Theta_\tau)\,]$$

for an arbitrary stopping rule $\tau \in \mathcal{S}$. □

Remark 2: Before proceeding to the next section it is useful to recall that, in the special case where player $\mathfrak{B}$ is a dummy, we have solved a one-person game control-and-stopping problem for player $\mathfrak{A}$. That is, against any fixed control function $\beta(\cdot)$ for player $\mathfrak{B}$, it is optimal for player $\mathfrak{A}$ to choose an $\alpha(\cdot)$ that maximizes (μ/σ^2) to the left of m and minimizes (μ/σ^2) to the right of m; and then to choose a rule τ which is optimal for stopping the process

$$u\big(X^{x,\alpha,\beta}\big)\,.$$

This essentially recovers, and slightly extends, the main result of Karatzas & Sudderth (1999).

General results about the existence of values for zero-sum stochastic differential games have been established in the literature, using both analytical and probabilistic tools; see, for instance, Friedman (1972), Elliott (1976), Nisio (1988) and Fleming & Souganidis (1989).

3 A Non-Zero-Sum Game of Control and Stopping

Suppose now that *both* players $\mathfrak{A}$ and $\mathfrak{B}$ have continuous reward functions, $u(\cdot)$ and $v(\cdot)$ respectively, which map the unit interval [0,1] into the real line. Will shall assume that $u(\cdot)$ and $v(\cdot)$ attain their maximum values over the interval [0,1] at unique points m and ℓ, respectively; and without loss of generality, we shall take $\ell \leq m$.

As in the previous section, the two players choose control functions $\alpha(\cdot)$ and $\beta(\cdot)$, but now we assume that *both* players can select stopping rules, say τ and ρ respectively. The dynamics of the process $X \equiv X^{x,\alpha,\beta}$ are given by (1.1), just as before. The expected rewards to the two players resulting from their choices of $(\alpha(\cdot),\tau)$ and $(\beta(\cdot),\rho)$ at the initial state $x \in I$ are defined to be

$$\mathbb{E}[\, u(X_\tau)\,] \qquad \text{and} \qquad \mathbb{E}[\, v(X_\rho)\,]\,,$$

respectively. Let $\mathfrak{G}_2(x)$ denote this game.

As this game is typically *not* zero-sum, we seek a *Nash equilibrium:* namely, pairs $(\alpha^*(\cdot),\tau^*)$ and $(\beta^*(\cdot),\rho^*)$ such that

$$\mathbb{E}\left[\, u\left(X_\tau^{x,\,\alpha,\,\beta^*}\right)\right] \leq \mathbb{E}\left[\, u\left(X_{\tau^*}^{x,\,\alpha^*,\,\beta^*}\right)\right]\,, \qquad \forall\;\; (\alpha(\cdot),\tau)$$

and

$$\mathbb{E}\left[\, v\left(X_\rho^{x,\,\alpha^*,\,\beta}\right)\right] \leq \mathbb{E}\left[\, v\left(X_{\rho^*}^{x,\,\alpha^*,\,\beta^*}\right)\right]\,, \qquad \forall\;\; (\beta(\cdot),\rho)\,.$$

This simply says that the choice $(\alpha^*(\cdot),\tau^*)$ is optimal for player $\mathfrak{A}$ when player $\mathfrak{B}$ uses $(\beta^*(\cdot),\rho^*)$, and vice-versa.

Speaking intuitively: in positions to the left of the point ℓ, both players want the process X to move to the right; in the interval (ℓ,m), player $\mathfrak{A}$ wants the process to move to the right, and player $\mathfrak{B}$ wants it to move to the left; and at positions to the right of m, both players prefer X to move to the left. Here, "moving to the right" (resp., to the left) is a euphemism for "trying to maximize (resp., to minimize) the signal-to-noise ratio (μ/σ^2)"; recall the discussion in Remark 4.1 of Karatzas & Sudderth (1999), in connection with the problem of that paper and with the *goal problem* of Pestien & Sudderth (1985).

These considerations suggest the following condition.

C.3: Local Nash Equilibrium Condition. *There exist measurable functions* $\alpha^*(\cdot)$, $\beta^*(\cdot)$ *such that* $(\alpha^*(z),\, \beta^*(z)) \in A(z) \times B(z)$ *holds for all* $z \in (0,1)$*;*

$$\left(\frac{\mu}{\sigma^2}\right)(z,a,b) \;\leq\; \left(\frac{\mu}{\sigma^2}\right)(z,\alpha^*(z),\beta^*(z))\,, \quad \forall\;(a,b) \in A(z) \times B(z)$$

holds for every $z \in (0,\ell)$*;*

$$\left(\frac{\mu}{\sigma^2}\right)(z,a,\beta^*(z)) \;\leq\; \left(\frac{\mu}{\sigma^2}\right)(z,\alpha^*(z),\beta^*(z)) \;\leq\; \left(\frac{\mu}{\sigma^2}\right)(z,\alpha^*(z),b)\,, \quad \forall\;(a,b) \in A(z) \times B(z)$$

holds for every $z \in (\ell,m)$*; and*

$$\left(\frac{\mu}{\sigma^2}\right)(z,\alpha^*(z),\beta^*(z)) \;\leq\; \left(\frac{\mu}{\sigma^2}\right)(z,a,b)\,, \quad \forall\;(a,b) \in A(z) \times B(z)$$

holds for every $z \in (m, 1)$.

Suppose there exist measurable functions $\alpha^*(\cdot)$ and $\beta^*(\cdot)$ that satisfy condition $C.3$, and construct, for each fixed $x \in I$, the diffusion process $Z^x \equiv X^{x,\alpha^*,\beta^*}$ of (2.1), (2.2). Next, consider the optimal stopping problems for these two players with value functions

$$U(x) := \sup_{\tau \in \mathcal{S}} \mathbb{E}[\, u(Z^x_\tau)\,] \qquad \text{and} \qquad V(x) := \sup_{\rho \in \mathcal{S}} \mathbb{E}[\, v(Z^x_\rho)\,] , \qquad x \in I ,$$

respectively. Define also the stopping rules

$$\tau^*(\xi) := \inf\{t \geq 0 \,:\, U(\xi_t) = u(\xi_t)\} \qquad \text{and} \qquad \rho^*(\xi) := \inf\{t \geq 0 \,:\, V(\xi_t) = v(\xi_t)\} , \quad \xi \in \mathfrak{Z} .$$

Theorem 2: *Assume that the strong non-degeneracy condition C.2 holds, and that there exist measurable functions* $\alpha^*(\cdot)$, $\beta^*(\cdot)$ *satisfying the local Nash equilibrium condition C.3.*

For each $x \in I$*, construct the process* $Z^x \equiv X^{x,\alpha^*,\beta^*}$ *of (2.1) and (2.2). Then the pairs* $(\alpha^*(\cdot), \tau^*)$ *and* $(\beta^*(\cdot), \rho^*)$ *form a Nash equilibrium for the game* $\mathfrak{G}_2(x)$.

Proof : Consider the control-and-stopping problem faced by player $\mathfrak{A}$, when player $\mathfrak{B}$ chooses the pair $(\beta^*(\cdot), \rho^*)$. By condition $C.3$, the control function $\alpha^*(\cdot)$ maximizes the ratio (μ/σ^2) to the left of m, and minimizes it to the right of m. Also, the rule τ^* is optimal for stopping the process $u(Z^x)$ (player $\mathfrak{A}$). Thus, by Remark 2 at the end of section 2, we see that the pair $(\alpha^*(\cdot), \tau^*)$ is optimal for player $\mathfrak{A}$ when player $\mathfrak{B}$ chooses the pair $(\beta^*(\cdot), \rho^*)$.

Similar considerations show that the pair $(\beta^*(\cdot), \rho^*)$ is optimal for player $\mathfrak{B}$ when player $\mathfrak{A}$ chooses the pair $(\alpha^*(\cdot), \tau^*)$. $\square$

It is not difficult to extend Theorem 2 to any finite number of players. General existence results for Nash equilibria in non-zero-sum stochastic differential games have been established in the literature; see, for instance, Uchida (1978, 1979), Bensoussan & Frehse (2000), Buckdhan et al. (2004). For examples of special games with explicit solutions, see Mazumdar & Radner (1991), Nilakantan (1993) and Brown (2000).

4 A General Game of Stopping

In this section we address a game of stopping, which will be crucial to our study of the control-and-stopping game in the next section. This stopping game is a cousin of the so-called *Dynkin (1967) Games,* which have been studied by a number of authors; see, for instance, Friedman (1973), Bensoussan & Friedman (1974, 1977), Neveu (1975), Bismut (1977), Stettner (1982), Alario-Nazaret et al. (1982), Morimoto (1984), Lepeltier & Maingueneau (1984), Ohtsubo (1987), Hamadène & Lepeltier (1995), Karatzas (1996), Cvitanić & Karatzas (1996), Karatzas & Wang (2001), Touzi & Vieille (2002), Fukushima & Taksar (2002) and Boetius (2005), among others.

Let $\mathfrak{X}$ be a locally compact metric space and suppose that, for each $x \in \mathfrak{X}$, the process $Z^x = \{Z^x_t, \, 0 \leq t < \infty\}$ is a continuous strong Markov process with $Z^x_0 = x$ and values in the space $\mathfrak{X}$. We shall assume that Z^x is *standard* in the sense of Shiryaev (1978), p.19. (For the application to the next section, Z^x will be a diffusion process in the interval (0,1) with absorption at the endpoints, just as it was in sections 2 and 3.)

Consider now, for each $x \in \mathfrak{X}$, a two-person game $\mathfrak{G}_3(x)$ in which players $\mathfrak{A}$ and $\mathfrak{B}$ have bounded, continuous reward functions $u(\cdot)$ and $v(\cdot)$, respectively, that map $\mathfrak{X}$ into the real line. Player $\mathfrak{A}$ chooses a stopping rule τ, player $\mathfrak{B}$ chooses a stopping rule ρ, and their respective expected rewards are

$$\mathbb{E}[\, u(Z^x_{\tau\wedge\rho})\,] \qquad \text{and} \qquad \mathbb{E}[\, v(Z^x_{\tau\wedge\rho})\,]\,.$$

The resulting non-zero-sum game of timing has a trivial Nash equilibrium $\check{\tau} = \check{\rho} = 0$, which leaves player $\mathfrak{A}$ with reward $u(x)$ and player $\mathfrak{B}$with reward $v(x)$. A somewhat less trivial Nash equilibrium is described in Theorem 3 and its proof below.

Theorem 3: *There exist stopping rules τ^* and ρ^* such that*

$$\mathbb{E}[\, u(Z^x_{\tau^*\wedge\rho^*})\,] \,\geq\, \mathbb{E}[\, u(Z^x_{\tau\wedge\rho^*})\,]\,, \qquad \forall \;\; \tau \in \mathcal{S}$$

and

$$\mathbb{E}[\, v(Z^x_{\tau^*\wedge\rho^*})\,] \,\geq\, \mathbb{E}[\, v(Z^x_{\tau^*\wedge\rho})\,]\,, \qquad \forall \;\; \rho \in \mathcal{S}$$

hold for each $x \in \mathfrak{X}$, to wit, the game $\mathfrak{G}_3(x)$ has (τ^*, ρ^*) as a Nash equilibrium; in this equilibrium the expected rewards of players $\mathfrak{A}$ and $\mathfrak{B}$ satisfy

$$\mathbb{E}[\, u(Z^x_{\tau^*\wedge\rho^*})\,] \,\geq\, u(x) \qquad \text{and} \qquad \mathbb{E}[\, v(Z^x_{\tau^*\wedge\rho^*})\,] \,\geq\, v(x)\,, \qquad \forall \;\; x \in \mathfrak{X}\,, \tag{4.1}$$

respectively; and the inequalities of (4.1) can typically be strict.

The stopping rules τ^*, ρ^* of the theorem will be constructed as limits of two decreasing sequences $\{\tau_n\}$ and $\{\rho_n\}$ of stopping rules. The specific structure of these sequences will be important for the application we undertake in the next section.

These stopping rules will be defined inductively, in the order $\tau_1,\, \rho_1,\, \tau_2,\, \rho_2,\, \cdots$, as solutions to a sequence of optimal stopping problems. First, let us define

$$U_1(x) \,:=\, \sup_{\tau\in\mathcal{S}} \mathbb{E}[\, u(Z^x_\tau)\,] \tag{4.2}$$

and

$$\tau_1(\xi) \,:=\, \inf\{t \geq 0 \,:\, \xi_t \in F_1\,\}\,, \quad \xi \in \mathfrak{Z}\,,$$

where

$$F_1 \,:=\, \{x \in \mathfrak{X} \,:\, U_1(x) = u(x)\}\,.$$

Next, let

$$V_1(x) \,:=\, \sup_{\rho\in\mathcal{S}} \mathbb{E}[\, v(Z^x_{\tau_1\wedge\rho})\,]\,,$$

$$\rho_1(\xi) \,:=\, \inf\{t \geq 0 \,:\, \xi_t \in G_1\,\}\,, \quad \xi \in \mathfrak{Z}\,,$$

where

$$G_1 \,:=\, \{x \in \mathfrak{X} \,:\, V_1(x) = v(x)\}\,.$$

It is well known (e.g. Shiryaev (1978)) that the stopping rules τ_1 and ρ_1 are optimal in their respective problems, in the sense that $U_1(x) = \mathbb{E}[\,u(Z^x_{\tau_1})\,]$ and $V_1(x) = \mathbb{E}[\,v(Z^x_{\tau_1 \wedge \rho_1})\,]$; and that the functions $U_1(\cdot)$, $V_1(\cdot)$ are continuous. Thus the optimal stopping regions for these two problems, namely, F_1 and G_1, are closed sets.

Suppose now that we have already constructed: the stopping rules $\tau_1,\ \rho_1, \cdots,\ \tau_n,\ \rho_n$; the continuous functions $U_1(\cdot)\,, V_1(\cdot)\,, \cdots,\ U_n(\cdot)\,, V_n(\cdot)$; and the closed sets $F_1,\, G_1,\, \cdots,\, F_n,\, G_n$. Let

$$U_{n+1}(x) \;:=\; \sup_{\tau \in \mathcal{S}} \mathbb{E}[\,u(Z^x_{\tau \wedge \rho_n})\,]\,,$$

$$F_{n+1} \;:=\; \{x \in \mathfrak{X}\,:\, U_{n+1}(x) = u(x)\}\,,$$

$$\tau_{n+1}(\xi) \;:=\; \inf\{t \ge 0\,:\, \xi_t \in F_{n+1}\,\}\,, \quad \xi \in \mathfrak{Z}\,;$$

as well as

$$V_{n+1}(x) \;:=\; \sup_{\rho \in \mathcal{S}} \mathbb{E}[\,v(Z^x_{\tau_{n+1} \wedge \rho})\,]\,,$$

$$G_{n+1} \;:=\; \{x \in \mathfrak{X}\,:\, V_{n+1}(x) = v(x)\}$$

and

$$\rho_{n+1}(\xi) \;:=\; \inf\{t \ge 0\,:\, \xi_t \in G_{n+1}\,\}\,, \quad \xi \in \mathfrak{Z}\,.$$

Observe that

$$u(x) \;\le\; U_2(x) \;=\; \sup_{\tau \in \mathcal{S}} \mathbb{E}[\,u(Z^x_{\tau \wedge \rho_1})\,] \;\le\; U_1(x)\,, \qquad x \in \mathfrak{X}\,.$$

Thus $F_1 \subseteq F_2$ and so $\tau_1 \ge \tau_2$. Similarly, $G_1 \subseteq G_2$ and $\rho_1 \ge \rho_2$. An argument by induction shows that $\tau_n \ge \tau_{n+1}$ and $\rho_n \ge \rho_{n+1}$, as well as

$$u(x) \;\le U_{n+1}(x) \;\le\; U_n(x)\,, \qquad v(x) \;\le V_{n+1}(x) \;\le\; V_n(x)\,,$$

hold for all $n \in \mathbb{N}$.

Also by construction, we have

$$\mathbb{E}[\,u(Z^x_{\tau_{n+1} \wedge \rho_n})\,] \;\ge\; \mathbb{E}[\,u(Z^x_{\tau \wedge \rho_n})\,] \tag{4.3}$$

and

$$\mathbb{E}[\,v(Z^x_{\tau_n \wedge \rho_n})\,] \;\ge\; \mathbb{E}[\,v(Z^x_{\tau_n \wedge \rho})\,]\,, \tag{4.4}$$

for all integers n and stopping rules τ, ρ. Define the decreasing limits

$$\tau^* \;:=\; \lim_{n \to \infty} \downarrow \tau_n\,, \qquad \rho^* \;:=\; \lim_{n \to \infty} \downarrow \rho_n\,,$$

and let $n \to \infty$ in (4.3), (4.4).

The above construction shows also that, for every integer $n \in \mathbb{N}$, the functions $U_n(\cdot)$, $V_n(\cdot)$ are continuous and the sets F_n, G_n closed.

On the other hand, it is clear that the inequalities in (4.1) can be strict. For instance, take $u(\cdot) \equiv v(\cdot)$ and suppose that the optimal stopping problem of (4.2) is not trivial, that is, has a non-empty optimal continuation region $\mathcal{O}_1 := \mathfrak{X} \setminus F_1 = \{ x \in \mathfrak{X} : U_1(x) > u(x) \}$. Then $\rho_1 = \tau_1$, $V_1(\cdot) \equiv U_1(\cdot)$ as well as $\rho_2 = \rho_1 = \tau_1$, $U_2(\cdot) \equiv U_1(\cdot)$, and by induction: $V_n(\cdot) \equiv U_n \equiv U_1(\cdot)$, $\tau_n = \rho_n = \tau_1$ for all n. For every $x \in \mathcal{O}_1$, the inequalities in (4.1) are then strict, since

$$\mathbb{E}[\, u(Z^x_{\tau^* \wedge \rho^*})\,] = \mathbb{E}[\, u(Z^x_{\tau_1})\,] = U_1(x) > u(x)\,. \qquad \Box$$

Remark 3: It is straightforward to generalize Theorem 3 to the case of $K \geq 2$ players with reward functions $u_1(\cdot), \cdots, u_K(\cdot)$, who choose stopping rules $\tau_1, \cdots, \tau_K$ and receive payoffs

$$\mathbb{E}[\, u_i(Z^x_{\tau_1 \wedge \cdots \wedge \tau_K})\,], \qquad i = 1, \cdots, K\,.$$

5 Another Non-Zero-Sum Game of Control and Stopping

For each point x in the interval $I = (0,1)$, let $\mathfrak{G}_4(x)$ be a two-player game which is the same as the game $\mathfrak{G}_2(x)$ of section 3, except that the payoffs to the two players, resulting from their choices $(\alpha(\cdot), \tau)$ and $(\beta(\cdot), \rho)$, are now, respectively,

$$\mathbb{E}\left[u\left(X^{x,\alpha,\beta}_{\tau\wedge\rho} \right) \right] \qquad \text{and} \qquad \mathbb{E}\left[v\left(X^{x,\alpha,\beta}_{\tau\wedge\rho} \right) \right].$$

Just like the game of the previous section, this one has the trivial Nash equilibrium $(\check{\alpha}(\cdot), \check{\tau})$ and $(\check{\beta}(\cdot), \check{\rho})$, with arbitrary $\check{\alpha}(\cdot)$, $\check{\beta}(\cdot)$ and with $\check{\tau} = \check{\rho} = 0$.

A somewhat less trivial equilibrium for this game is constructed in Theorem 4 below. It uses the additional assumption that the reward functions $u(\cdot)$ and $v(\cdot)$ are *unimodal* with unique maxima at m and ℓ, respectively, and with $0 \leq \ell \leq m \leq 1$. To say that $u(\cdot)$, for example, is unimodal, means that $u(\cdot)$ is increasing on the interval $(0, m)$ and decreasing on the interval $(m, 1)$.

Theorem 4: *Assume that the strong non-degeneracy condition C.2 holds; that there exist measurable functions $\alpha^*(\cdot)$, $\beta^*(\cdot)$ satisfying the local Nash equilibrium condition C.3; and that the reward functions $u(\cdot)$, $v(\cdot)$ are unimodal. For any given $x \in I$, define the process $Z^x \equiv X^{x,\alpha^*,\beta^*}$ as in (2.1) and (2.2).*

Let τ^*, ρ^* be the Nash equilibrium of Theorem 3 for the stopping game $\mathfrak{G}_3(x)$ of section 4. Then the pairs $(\alpha^*(\cdot), \tau^*)$ and $(\beta^*(\cdot), \rho^*)$ form a Nash equilibrium for the game $\mathfrak{G}_4(x)$.

Proof: We must establish the comparisons

$$\mathbb{E}\left[u\left(X^{x,\,\alpha,\,\beta^*}_{\tau\wedge\rho^*} \right) \right] \leq \mathbb{E}\left[u\left(X^{x,\,\alpha^*,\,\beta^*}_{\tau^*\wedge\rho^*} \right) \right], \qquad \forall\ (\alpha(\cdot), \tau) \tag{5.1}$$

and

$$\mathbb{E}\left[v\left(X^{x,\,\alpha^*,\,\beta}_{\tau^*\wedge\rho} \right) \right] \leq \mathbb{E}\left[v\left(X^{x,\,\alpha^*,\,\beta^*}_{\tau^*\wedge\rho^*} \right) \right], \qquad \forall\ (\beta(\cdot), \rho)\,. \tag{5.2}$$

It suffices to show that for all $n \in \mathbb{N}$, we have

$$\mathbb{E}\left[u\left(X^{x,\,\alpha,\,\beta^*}_{\tau\wedge\rho_n} \right) \right] \le \mathbb{E}\left[u\left(X^{x,\,\alpha^*,\,\beta^*}_{\tau_{n+1}\wedge\rho_n} \right) \right], \qquad \forall\ (\alpha(\cdot),\tau), \tag{5.3}$$

and

$$\mathbb{E}\left[v\left(X^{x,\,\alpha^*,\,\beta}_{\tau_n\wedge\rho} \right) \right] \le \mathbb{E}\left[v\left(X^{x,\,\alpha^*,\,\beta^*}_{\tau_n\wedge\rho_n} \right) \right], \qquad \forall\ (\beta(\cdot),\rho), \tag{5.4}$$

where the stopping rules $\tau_1,\, \rho_1,\, \tau_2,\, \rho_2,\, \cdots$ are as defined in section 4. The latter inequalities are sufficient, because we can obtain (5.1) and (5.2) from them by passing to the limit as $n \to \infty$.

Let us prove the first of these latter inequalities – the second, (5.4), will follow then by symmetry. This inequality (5.3) says that $(\alpha^*(\cdot), \tau_{n+1})$ is optimal for player $\mathfrak{A}$ in the one-person problem of control-and-stopping that occurs when player $\mathfrak{B}$ fixes the pair $(\beta^*(\cdot), \rho_n)$. (See Remark 2, at the end of section 2.)

Consider two cases: namely, when the initial state x belongs to G_n, the stopping region for the rule ρ_n, and when x is not an element of G_n.

- If $x \in G_n$, then ρ_n stops the process immediately, and every pair $(\alpha(\cdot), \tau)$ that player $\mathfrak{A}$ may choose is optimal .
- If x belongs to the open set $\mathcal{O}_n := [0,1] \setminus G_n$, then there exists an interval (q,r) such that $x \in (q,r) \subseteq \mathcal{O}_n$ and the process $X^{x,\,\alpha,\,\beta^*}_{\cdot\wedge\tau\wedge\rho_n}$ is absorbed at the endpoints of this interval, for all $(\alpha(\cdot), \tau)$.

It is clearly optimal for player $\mathfrak{A}$ to stop at the point m where $u(\cdot)$ attains its maximum, so we can assume that the interval (q,r) lies to one side of m. Suppose, to be precise, that $x \in (q,r) \subseteq \mathcal{O}_n \cap (0,m)$. Because it is unimodal, the function $u(\cdot)$ increases on (q,r) and attains its maximum over the interval $[q,r]$ at the point r. By Remark 2 and the condition $C.3$, it is optimal for player $\mathfrak{A}$ to use the control function $\alpha^*(\cdot)$, and then to choose a rule that is optimal for stopping the process $X^{x,\,\alpha^*,\,\beta^*}_{\cdot\wedge\rho_n} \equiv Z^{x}_{\cdot\wedge\rho_n}$, namely, the stopping rule τ_{n+1}.

This completes the proof. □

As in previous sections, we conjecture that Theorem 4 can be extended to a $K-$person game in a straightforward manner. However, it may be more difficult to generalize to reward functions that are not unimodal.

6 Generalizations

The results of the paper can easily be extended to much more general information structures on the part of the two players, than we have indicated so far.

Consider, for example, the zero-sum game $\mathfrak{G}_1(x)$ of section 2. Suppose that on some filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathfrak{F} = \{\mathcal{F}_t\}_{0\le t<\infty}$ we can construct, not just the diffusion process Z of (2.1), but also progressively measurable processes H and Θ satisfying the stochastic differential equations with random coëfficients

$$dH_t \,=\, \mu\left(H_t, \alpha^*(H_t), \mathfrak{b}_t\right)\, dt \,+\, \sigma\left(H_t, \alpha^*(H_t), \mathfrak{b}_t\right)\, dW_t\,, \qquad H_0 = x$$

and

$$d\Theta_t \,=\, \mu\left(\Theta_t, \mathfrak{a}_t, \beta^*(\Theta_t)\right)\, dt \,+\, \sigma\left(\Theta_t, \mathfrak{a}_t, \beta^*(\Theta_t)\right)\, dW_t\,, \qquad \Theta_0 = x\,,$$

respectively. Here $\mathfrak{a}$ and $\mathfrak{b}$ are progressively measurable processes, with

$$\mathfrak{a}_t \in A(\Theta_t) \qquad \text{and} \qquad \mathfrak{b}_t \in B(H_t)\,, \qquad \forall\; 0 \le t < \infty\,.$$

Then the same analysis as in the proof of Theorem 1, leads to the comparisons

$$\mathbb{E}[\, u(\Theta_{\mathfrak{t}})\,] \,\le\, \mathbb{E}[\, u(Z_{\tau^*})\,] \,\le\, \mathbb{E}[\, u(H_{\tau^*})\,]$$

as in (2.5), for an arbitrary stopping time $\mathfrak{t}$ of the filtration $\mathfrak{F}$.

In other words: the pair $(\alpha^*(\cdot), \tau^*)$ is optimal for player $\mathfrak{A}$ against all such pairs $(\mathfrak{a}, \mathfrak{t})$, and the function $\beta^*(\cdot)$ is optimal for player $\mathfrak{B}$ against all such $\mathfrak{b}$.

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