Stochastic Geometry and Wireless Networks: Volume I Theory

To Béatrice and Mira

Stochastic Geometry and Wireless Networks: Volume I Theory

François Baccelli

INRIA & Ecole Normale Supérieure
45 rue d'Ulm
Paris
francois.baccelli@ens.fr

Bartłomiej Błaszczyszyn

INRIA & Ecole Normale Superieure and
Math. Inst. University of Wrocław
45 rue d'Ulm
Paris
bartek.blaszczyszyn@ens.fr

the essence of knowledge

Boston – Delft

Foundations and Trends® in Networking

Published, sold and distributed by:
now Publishers Inc.
PO Box 1024
Hanover, MA 02339
USA
Tel. +1-781-985-4510
www.nowpublishers.com
sales@nowpublishers.com

Outside North America:
now Publishers Inc.
PO Box 179
2600 AD Delft
The Netherlands
Tel. +31-6-51115274

The preferred citation for this publication is F. Baccelli and B. Błaszczyszyn, Stochastic Geometry and Wireless Networks: Volume I Theory, Foundation and Trends® in Networking, vol 3, nos 3-4, pp 249–449, 2009

ISBN: 978-1-60198-264-3

Foundations and Trends® in Networking

Volume 3 Issues 3-4, 2009

Editorial Scope

Foundations and Trends® in Networking will publish survey and tutorial articles in the following topics:

- Ad Hoc Wireless Networks
- Sensor Networks
- Optical Networks
- Local Area Networks
- Satellite and Hybrid Networks
- Cellular Networks
- Internet and Web Services
- Protocols and Cross-Layer Design
- Network Coding
- Energy-Efficiency Incentives/Pricing/Utility-based
- Games (co-operative or not)
- Security
- Scalability
- Topology
- Control/Graph-theoretic models
- Dynamics and Asymptotic Behavior of Networks

Information for Librarians

Foundations and Trends® in Networking, 2009, Volume 3, 4 issues. ISSN paper version 1554-057X. ISSN online version 1554-0588. Also available as a combined paper and online subscription.

Foundations and Trends® in
Networking
Vol. 3, Nos. 3-4 (2009) 249–449

DOI: 10.1561/1300000006

now
the essence of knowledge

Stochastic Geometry and Wireless Networks: Volume I Theory

François Baccelli[1] and Bartłomiej Błaszczyszyn[2]

[1] *INRIA & Ecole Normale Supérieure, 45 rue d'Ulm, Paris, francois.baccelli@ens.fr*
[2] *INRIA & Ecole Normale Superieure and Math. Inst. University of Wrocław, 45 rue d'Ulm, Paris, bartek.blaszczyszyn@ens.fr*

Abstract

Volume I first provides a compact survey on classical stochastic geometry models, with a main focus on spatial shot-noise processes, coverage processes and random tessellations. It then focuses on *signal to interference noise ratio* (SINR) stochastic geometry, which is the basis for the modeling of wireless network protocols and architectures considered in Volume II. It also contains an appendix on mathematical tools used throughout Stochastic Geometry and Wireless Networks, Volumes I and II.

Contents

Preface

A wireless communication network can be viewed as a collection of nodes, located in some domain, which can in turn be transmitters or receivers (depending on the network considered, nodes may be mobile users, base stations in a cellular network, access points of a WiFi mesh, etc.). At a given time, several nodes transmit simultaneously, each toward its own receiver. Each transmitter–receiver pair requires its own wireless link. The signal received from the link transmitter may be jammed by the signals received from the other transmitters. Even in the simplest model where the signal power radiated from a point decays in an isotropic way with Euclidean distance, the geometry of the locations of the nodes plays a key role since it determines the *signal to interference and noise ratio* (SINR) at each receiver and hence the possibility of establishing simultaneously this collection of links at a given bit rate. The interference seen by a receiver is the sum of the signal powers received from all transmitters, except its own transmitter.

Stochastic geometry provides a natural way of defining and computing macroscopic properties of such networks, by averaging over all potential geometrical patterns for the nodes, in the same way as queuing theory provides response times or congestion, averaged over all potential arrival patterns within a given parametric class.

Modeling wireless communication networks in terms of stochastic geometry seems particularly relevant for large scale networks. In the simplest case, it consists in treating such a network as a snapshot of a stationary random model in the whole Euclidean plane or space and analyzing it in a probabilistic way. In particular the locations of the network elements are seen as the realizations of some point processes. When the underlying random model is ergodic, the probabilistic analysis also provides a way of estimating *spatial averages* which often capture the key dependencies of the network performance characteristics (connectivity, stability, capacity, etc.) as functions of a relatively small number of parameters. Typically, these are the densities of the underlying point processes and the parameters of the protocols involved. By spatial average, we mean an empirical average made over a large collection of 'locations' in the domain considered; depending on the cases, these locations will simply be certain points of the domain, or nodes located in the domain, or even nodes on a certain route defined on this domain. These various kinds of spatial averages are defined in precise terms in the monograph. This is a very natural approach, e.g. for ad hoc networks, or more generally to describe user positions, when these are best described by random processes. But it can also be applied to represent both irregular and regular network architectures as observed in cellular wireless networks. In all these cases, such a space average is performed on a large collection of nodes of the network executing some common protocol and considered at some common time when one takes a snapshot of the network. Simple examples of such averages are the fraction of nodes which transmit, the fraction of space which is covered or connected, the fraction of nodes which transmit their packet successfully, and the average geographic progress obtained by a node forwarding a packet towards some destination. This is rather new to classical performance evaluation, compared to time averages.

Stochastic geometry, which we use as a tool for the evaluation of such spatial averages, is a rich branch of applied probability particularly adapted to the study of random phenomena on the plane or in higher dimension. It is intrinsically related to the theory of point processes. Initially its development was stimulated by applications to biology, astronomy and material sciences. Nowadays, it is also used in image analysis and in the context of communication networks. In this latter case, its role is similar to that played by the theory of point processes on the real line in classical queuing theory.

The use of stochastic geometry for modeling communication networks is relatively new. The first papers appeared in the engineering literature shortly before 2000. One can consider Gilbert's paper of 1961 [19] both as the first paper on continuum and Boolean percolation and as the first paper on the analysis of the connectivity of large wireless networks by means of stochastic geometry. Similar observations can be made on [20] concerning Poisson–Voronoi tessellations. The number of papers using some form of stochastic geometry is increasing fast. One of the most important observed trends is to take better account in these models of specific mechanisms of wireless communications.

Time averages have been classical objects of performance evaluation since the work of Erlang (1917). Typical examples include the random delay to transmit a packet from a given node, the number of time steps required for a packet to be transported from source to destination on some multihop route, the frequency with which a transmission is not granted access due to some capacity limitations, etc. A classical reference on the matter is [28]. These time averages will be studied here either on their own or in conjunction with space averages. The combination of the two types of averages unveils interesting new phenomena and leads to challenging mathematical questions. As we shall see, the order in which the time and the space averages are performed matters and each order has a different physical meaning.

This monograph surveys recent results of this approach and is structured in two volumes.

Volume I focuses on the theory of spatial averages and contains three parts. Part I in Volume I provides a compact survey on *classical* stochastic geometry models. Part II in Volume I focuses on *SINR* stochastic geometry. Part III in Volume I is an appendix which contains mathematical tools used throughout the monograph. Volume II bears on more practical wireless network modeling and performance analysis. It is in this volume that the interplay between wireless communications and stochastic geometry is deepest and that the time–space framework alluded to above is the most important. The aim is to show how stochastic geometry can be used in a more or less systematic way to analyze the phenomena that arise in this context. Part IV in Volume II is focused on medium access control (MAC). We study MAC protocols used in ad hoc networks and in cellular networks. Part V in Volume II discusses the use of stochastic geometry for the quantitative analysis of routing algorithms in MANETs. Part VI in Volume II gives a concise summary of wireless communication principles and of the network architectures considered in the monograph. This part is self-contained and readers not familiar with wireless networking might either read it before reading the monograph itself, or refer to it when needed.

Here are some comments on what the reader will obtain from studying the material contained in this monograph and on possible ways of reading it.

For readers with a background in applied probability, this monograph provides direct access to an emerging and fast growing branch of spatial stochastic modeling (see, e.g., the proceedings of conferences such as IEEE Infocom, ACM Sigmetrics, ACM Mobicom, etc. or the special issue [22]). By mastering the basic principles of wireless links and the organization of communications in a wireless network, as summarized in Volume II and already alluded to in Volume I, these readers will be granted access to a rich field of new questions with high practical interest. SINR stochastic geometry opens new and interesting mathematical questions. The two categories of objects studied in Volume II, namely medium access and routing protocols, have a large number of variants and implications. Each of these could give birth to a new stochastic model to be understood and analyzed. Even for classical models of stochastic geometry, the new questions stemming from wireless networking often provide an original viewpoint. A typical example is that of route averages associated with a Poisson point process as discussed in Part V in Volume II. Reader already knowledgeable in basic stochastic geometry might skip Part I in Volume I and follow the path:

$$\begin{aligned}\text{Part II in Volume I} &\Rightarrow \text{Part IV in Volume II}\\ &\Rightarrow \text{Part V in Volume II},\end{aligned}$$

using Part VI in Volume II for understanding the physical meaning of the examples pertaining to wireless networks.

For readers whose main interest in wireless network design, the monograph aims to offer a new and comprehensive methodology for the performance evaluation of large scale wireless networks. This methodology consists in the computation of both time and space averages within a unified setting. This inherently addresses the scalability issue in that it poses the problems in an infinite domain/population case from the very beginning. We show that this methodology has the potential to provide both qualitative and quantitative results as below:

- Some of the most important qualitative results pertaining to these infinite population models are in terms of *phase transitions*. A typical example bears on the conditions under which the network is spatially connected. Another type of phase transition bears on the conditions under which the network delivers packets in a finite mean time for a given medium access and a given routing protocol. As we shall see, these phase transitions allow one to understand how to tune the protocol parameters to ensure that the network is in the desirable "phase" (i.e. well connected and with small mean delays). Other qualitative results are in terms of scaling laws: for instance, how do the overhead or the end-to-end delay on a route scale with the distance between the source and the destination, or with the density of nodes?
- Quantitative results are often in terms of closed form expressions for both time and space averages, and this for each variant of the involved protocols. The reader will hence be in a position to discuss and compare various protocols and more generally various wireless network organizations. Here are typical questions addressed and answered in Volume II: is it better to improve on Aloha by using a collision avoidance scheme of the CSMA type or by using a channel-aware extension of Aloha? Is Rayleigh fading beneficial or detrimental when using a given MAC scheme? How does geographic routing compare to shortest path routing in a mobile

ad hoc network? Is it better to separate the medium access and the routing decisions or to perform some cross layer joint optimization?

The reader with a wireless communication background could either read the monograph from beginning to end, or start with Volume II, i.e. follow the path

Part IV in Volume II $\Rightarrow$ Part V in Volume II $\Rightarrow$ Part II in Volume I

and use Volume I when needed to find the mathematical results which are needed to progress through Volume II.

We conclude with some comments on what the reader will *not* find in this monograph:

- We do not discuss statistical questions and give no measurement based validation of certain stochastic assumptions used in the monograph, e.g., when are Poisson-based models justified? When should one rather use point processes with some repulsion or attraction? When is the stationarity/ergodicity assumption valid? Our only aim is to show what can be done with stochastic geometry when assumptions of this kind can be made.
- We will not go beyond SINR models either. It is well known that considering interference as noise is not the only possible option in a wireless network. Other options (collaborative schemes, successive cancellation techniques) can offer better rates, though at the expense of more algorithmic overhead and the exchange of more information between nodes. We believe that the methodology discussed in this monograph has the potential of analyzing such techniques but we decided not to do this here.

Here are some final technical remarks. Some sections, marked with a * sign, can be skipped at the first reading as their results are not used in what follows; the index, which is common to the two volumes, is designed to be the main tool to navigate within and between the two volumes.

Acknowledgments

The authors would like to express their gratitude to Dietrich Stoyan, who first suggested them to write a monograph on this topic, as well as to Daryl Daley and Martin Haenggi for their very valuable proof-reading of the manuscript. They would also like to thank the anonymous reviewer of NOW for his/her suggestions, particularly so concerning the two volume format, as well as Paola Bermolen, Pierre Brémaud, Srikant Iyer, Mohamed Karray, Omid Mirsadeghi, Paul Muhlethaler, Barbara Staehle and Patrick Thiran for their useful comments on the manuscript.

Preface to Volume I

This volume focuses on the theory and contains three parts.

Part I provides a compact survey on *classical* stochastic geometry models. The basic models defined in this part will be used and extended throughout the whole monograph, and in particular to SINR based models. Note, however, that these classical stochastic models can be used in a variety of contexts which go far beyond the modeling of wireless networks. Chapter 1 reviews the definition and basic properties of Poisson point processes in Euclidean space. We review key operations on Poisson point processes (thinning, superposition, displacement) as well as key formulas like Campbell's formula. Chapter 2 is focused on properties of the spatial shot-noise process: its continuity properties, Laplace transform, moments, etc. Both additive and max shot-noise processes are studied. Chapter 3 bears on coverage processes, and in particular on the Boolean model. Its basic coverage characteristics are reviewed. We also give a brief account of its percolation properties. Chapter 4 studies random tessellations; the main focus is on Poisson–Voronoi tessellations and cells. We also discuss various random objects associated with bivariate point processes such as the set of points of the first point process that fall in a Voronoi cell w.r.t. the second point process.

Part II focuses on the stochastic geometry of SINR. The key new stochastic geometry model can be described as follows: consider a marked point process of the Euclidean space, where the mark of a point is a positive random variable that represents its "transmission power". The SINR cell of a point is then defined as the region of the space where the reception power from this point is larger than an affine function of the interference power. Chapter 5 analyzes a few basic stochastic geometry questions pertaining to such SINR cells in the case with independent marks, such as the volume and the shape of the typical cell. Chapter 6 focuses on the complex interactions that exist between cells. Chapter 7 studies the coverage process created by the collection of SINR cells. Chapter 8 studies the impact of interferences on the connectivity of large-scale mobile ad hoc networks using percolation theory on the SINR graph.

Part III is an appendix which contains mathematical tools used throughout the monograph.

It was our choice not to cover Gibbs point processes and the random closed sets that one can associate to them. And this in spite of the fact that these point processes already seem to be quite relevant within this wireless network context (see the bibliography of Chapter 18 in Volume II for instance). There are two main reasons for this decision: first, these models are rarely amenable to closed form analysis, at least in the case of systems with randomly located nodes as those considered here; second and more importantly, the amount of additional material needed to cover this part of the theory is not compatible with the format retained here.

References

[1] F. Baccelli and B. Błaszczyszyn, "On a coverage process ranging from the Boolean model to the Poisson Voronoi tessellation, with applications to wireless communications," *Advances in Applied Probability*, vol. 33, pp. 293–323, 2001.

[2] F. Baccelli, B. Błaszczyszyn, and F. Tournois, "Downlink capacity and admission/congestion control in CDMA networks," in *Proceedings of IEEE INFOCOM*, San Francisco, 2003.

[3] F. Baccelli, M. Klein, M. Lebourges, and S. Zuyev, "Stochastic geometry and architecture of communication networks," *Journal of Telecommunication Systems*, vol. 7, pp. 209–227, 1997.

[4] B. Błaszczyszyn, E. Merzbach, and V. Schmidt, "A note on expansion for functionals of spatial marked point processes," *Statistics and Probability Letters*, vol. 36, pp. 299–306, 1997.

[5] B. Błaszczyszyn and D. Yogeshwaran, "Directionally convex ordering of random measures, shot-noise fields and some applications to wireless networks," *Advances in Applied Probability*, 111, in press.

[6] H. Blichfeldt, "The minimum value of quadratic forms and the closest packing of spheres," *Math. Ann.*, vol. 101, pp. 605–608, 1929.

[7] P. Brémaud, *Mathematical Principles of Signal Processing*. New York: Springer, 2002.

[8] J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices and Groups*. New York: Springer-Verlag, 1998.

[9] D. J. Daley and D. Vere-Jones, *An Introduction to the Theory of Point Processes*. New York: Springer, 1988.

[10] D. J. Daley and D. Vere-Jones, *An Introduction to the Theory of Point Processes*. Probability and its Applications, New York: Springer. Volume I: Elementary Theory and Methods, 3, 2005.

[11] D. J. Daley and D. Vere-Jones, *An Introduction to the Theory of Point Processes*. Probability and its Applications, New York: Springer. Volume II: General Theory and Structure, 3, 2008.

[12] O. Dousse, F. Baccelli, and P. Thiran, "Impact of interferences on connectivity in ad-hoc networks," *IEEE/ACM Transactions on Networking*, vol. 13, pp. 425–543, 2005.

[13] O. Dousse, M. Franceschetti, N. Macris, R. Meester, and P. Thiran, "Percolation in the signal to interference ratio graph," *Journal of Applied Probability*, vol. 43, no. 2, pp. 552–562, 2006.

[14] W. Feller, *An Introduction to Probability Theory and its Applications,* vol. II. Wiley, 35, 60, 1971.

[15] M. Franceschetti and R. Meester, *Random Networks For Communication*. 2008.

[16] R. Ganti and M. Haenggi, "Interference and outage in clustered wireless ad hoc networks," *arXiv:0706.2434v1*, 111, 2008a.

[17] R. K. Ganti and M. Haenggi, "Interference in ad hoc networks with general motion-invariant node distributions," in *Proceedings IEEE International Symposium on Information Theory (ISIT'08)*, Toronto, Canada, July 2008b.

[18] F. Ghakov, *Boundary Value Problems*. Oxford: Pergamon Press, 1966.

[19] E. N. Gilbert, "Random plane networks," *SIAM Journal*, vol. 9, pp. 533–543, iv, 1961.

[20] E. N. Gilbert, "Random subdivisions of space into crystals," *The Annals of Mathematical Statistics*, vol. 33, no. 3, pp. 958–972, 1962.

[21] G. R. Grimmett, *Percolation*. New York: Springer-Verlag, 1989.

[22] M. Haenggi, J. Andrews, F. Baccelli, O. Dousse, and M. Franceschetti, eds., *Stochastic Geometry and Random Graphs for Wireless Networks*. Special Issue of IEEE JSAC, vol. 67, 2009.

[23] P. Hall, *Introduction to the Theory of Coverage Processes*. New York: Wiley & Sons, 1988.

[24] J. Illian, A. Penttinen, H. Stoyan, and D. Stoyan, *Statistical Analysis and Modelling of Spatial Point Patterns*. Wiley, 67, 2008.

[25] S. Janson, "Random coverings in several dimensions," *Acta Mathematica*, vol. 156, pp. 83–118, 1986.

[26] G. A. Kabatiansky and V. I. Levenshtein, "Bounds for packings on a sphere and in space," *Problemy Peredachi Informatsii*, vol. 14, no. 1, pp. 3–25, in Russian, 28, 1978.

[27] O. Kallenberg, *Random Measures*. Berlin: Akademie-Verlag, 6, 8, 1983.

[28] L. Kleinrock, *Queueing Systems*. John Wiley & Sons. iv, 1975.

[29] G. Matheron, *Random Sets and Integral Geometry*. London: John Wiley & Sons, pp. 43, 45, 1975.

[30] A. Maulloo, F. P. Kelly, and D. Tan, "Rate control in communication networks: Shadow prices, proportional fairness and stability," *Journal of the Operational Research Society*, vol. 49, pp. 237–252, 1998.

[31] R. Meester and R. Roy, *Continuum Percolation*. Cambridge University Press, pp. 39, 54, 108, 1996.

[32] J. Mo and J. Walrand, "Fair end-to-end window-based congestion control," *IEEE/ACM Transactions on Networking*, vol. 9, no. 5, pp. 556–567, 123, 2000.

[33] J. Møller, *Lectures on Random Voronoi Tessellations*. vol. 87 of *Lecture Notes in Statistics*, New York: Springer-Verlag, pp. 60, 67, 1994.

[34] A. Okabe, B. Boots, K. Sugihara, and S. N. Chiu, *Spatial Tessellations — Concepts and Applications of Voronoi Diagrams*. (second ed.), Wiley, p. 67, 2000.

[35] M. Penrose, "The longest edge of the random minimal spanning tree," *The Annals of Applied Probability*, vol. 7, no. 2, pp. 340–361, 132, 133, 1997.

[36] M. D. Penrose and J. E. Yukich, "Limit theory for random sequential packing and deposition," *The Annals of Applied Probablity*, vol. 12, no. 1, pp. 272–301, 29, 2002.

[37] C. Pugh and M. Shub, "Ergodic elements of ergodic actions," *Compositio Mathematica*, vol. 23, no. 1, pp. 115–122, 1971.

[38] B. Radunovic and J.-Y. Le Boudec, "A unified framework for max–min and min–max fairness with applications," in *40th Annual Allerton Conference on Communication, Control, and Computing*, p. 123, 2002.

[39] Y. Rozanov, *Markov Random Fields*. New York: Springer, p. 17, 1982.

[40] R. Senoussi, J. Chadoeuf, and D. Allard, "Weak homogenization of point processes by space deformations," *Advances in Applied Probability*, vol. 32, pp. 948–959, 12, 2000.

[41] R. Serfozo, *Introduction to Stochastic Networks*. New York: Springer, p. 4, 1999.

[42] D. Stoyan, W. Kendall, and J. Mecke, *Stochastic Geometry and its Applications*. Chichester: Wiley, pp. 67, 77, 87, 96, 1995.

[43] D. Stoyan and M. Schlather, "Random sequential adsorption: Relationship to dead leaves and characterization of variability," *Journal of Statistical Physics*, vol. 100, pp. 969–979, 29, 2000.

[44] S. Torquato, O. U. Uche, and F. H. Stillinger, "Random sequential addition of hard spheres in high euclidean dimensions," *Physical Review E*, vol. 74, no. 061308, p. 29, 2006.

[45] S. Weber, X. Yang, J. G. Andrews, and G. de Veciana, "Transmission capacity of wireless ad hoc networks with outage constraints," *IEEE Transactions on Information Theory*, vol. 51, no. 12, pp. 4091–4102, 111, 2005.

[46] S. Zuyev, *Strong Markov Property of Poisson Processes and Slivnyak Formula*, vol. 185 of *Lecture Notes in Statistics*. Springer, pp. 77–84, 67, 2006.

[47] S. Zuyev, *Stochastic Geometry and Telecommunications Networks*. Oxford University Press, New Perspectives in Stochastic Geometry, p. 67, 2009.