Stochastic Integral Equations for Walsh Semimartingales *

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Abstract

We construct a class of planar semimartingales which includes the Walsh Brownian motion as a special case, and derive stochastic integral equations and a change-of-variable formula for these so-called "Walsh semimartingales". Through the study of appropriate martingale problems, we examine uniqueness of the probability distribution for such processes in Markovian settings, and study some examples.

Keywords and Phrases: Skew and Walsh Brownian motions, spider and Walsh semimartingales, Skorokhod reflection, planar skew unfolding, Freidlin-Sheu formula, martingale problems, local time.

AMS 2000 Subject Classifications: Primary, 60G42; secondary, 60H10.

1 Introduction and Summary

We consider the following questions: *What is a two-dimensional analogue of the skew Brownian motion on the real line? If such a process exists, what is the corresponding stochastic integral equation that realizes its construction and describes its dynamics? Are there more general planar semimartingales with similar skew-unfolding-type structure?*

In order to answer the first question, WALSH (1978) introduced a singular planar diffusion with these properties. This diffusion is known now as the WALSH Brownian motion. In its description by BARLOW, PITMAN & YOR (1989a), "started at a point in the plane away from the origin, this process moves like a standard Brownian motion along the ray joining the starting point and the origin $\mathbf{0}$, until it reaches $\mathbf{0}$. Then it is kicked away from $\mathbf{0}$ by an entrance law that makes the radial part of the diffusion a reflecting Brownian motion, while randomizing the angular part". The WALSH Brownian motion has been generalized to the so-called spider martingales, and has been studied by several researchers (among them BARLOW, PITMAN

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& YOR (1989a), TSIREL'SON (1997), WATANABE (1999), EVANS & SOWERS (2003), PICARD (2005), FREIDLIN & SHEU (2000), MANSUY & YOR (2006), HAJRI (2011), FITZSIMMONS & KUTER (2014), HAJRI & TOUHAMI (2014), CHEN & FUKUSHIMA (2015)). In this paper we construct a family of planar semimartingales that includes the spider martingales and the WALSH Brownian motion as special cases.

There are several constructions of WALSH's Brownian motions in terms of resolvents, infinitesimal generators, semigroups, and excursion theory. Our approach in this paper can be thought of as a bridge between excursion theory and stochastic integral equations, via the folding and unfolding of semimartingales. It is also an attempt to study higher-dimensional analogues of the skew-TANAKA equation, and the semimartingale properties of planar processes that hit points.

Preview: We provide in Section 2 a system of stochastic equations (2.12) that these semimartingales satisfy. This is a two-dimensional analogue of the equation introduced by HARRISON & SHEPP (1981) for the skew Brownian motion, and answers the second and third questions stated above. Based on this integral equation description, we develop in Sections 3, 4 a stochastic calculus and establish a FREIDLIN-SHEU type change-of-variable formula for such WALSH semimartingales. In Section 5 we examine by the method of PROKAJ (2009) this two-dimensional HARRISON-SHEPP equation driven by a continuous semimartingale, as in ICHIBA & KARATZAS (2014). Pathwise uniqueness fails for the equation (2.12); we discuss in Sections 6 and 8 conditions, under which uniqueness in distribution does hold for this equation, based on the stochastic calculus developed in Section 4 and on appropriate martingale problems. As a special case, we show in Sections 7, 9 that the WALSH Brownian motion is a time-homogeneous strong Markov solution of our equation; whereas in Section 10 we examine some other examples, and discuss questions involving occupation times. Some auxiliary results and proofs are provided in appendices, Sections 11 and 12.

2 The Setting and Results

On a filtered probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$, $\widetilde{\mathbb{F}} = \{\widetilde{\mathcal{F}}(t)\}_{0 \le t < \infty}$ that satisfies the "usual conditions" of right-continuity and augmentation by null sets, we consider a real-valued, continuous semimartingale

$$U(t) \,=\, M(t) + V(t)\,, \qquad 0 \le t < \infty\,. \tag{2.1}$$

Here $M(\cdot)$ is a continuous local martingale and $V(\cdot)$ has finite variation on compact intervals; we assume that the initial position $U(0) \ge 0$ is a given real number. We denote by

$$S(t) \,:=\, U(t) + \Lambda(t)\,, \qquad \text{where} \quad \Lambda(t) = \max_{0 \le s \le t} \big(-U(s) \big)^+\,, \qquad 0 \le t < \infty\,, \tag{2.2}$$

the SKOROKHOD reflection (or "folding") of $U(\cdot)$; see, for instance, section 3.6 in KARATZAS & SHREVE (1991) for relevant theory. In particular, the continuous, increasing process $\Lambda(\cdot)$ is flat off the zero set

$$\mathfrak{Z} \,:=\, \big\{ 0 \le t < \infty : \, S(t) = 0 \big\}\,. \tag{2.3}$$

We shall impose the "non-stickiness" condition

$$\mathrm{Leb}(\mathfrak{Z}) \,\equiv\, \int_0^\infty \mathbf{1}_{\{S(t)=0\}}\, \mathrm{d}t \,=\, 0\,. \tag{2.4}$$

Let us recall the *(right) local time* $L^\Xi(\cdot)$ accumulated at the origin during the time-interval $[0, T]$ by a generic one-dimensional continuous semimartingale $\Xi(\cdot)$, namely

$$L^\Xi(T) \,:=\, \lim_{\varepsilon \downarrow 0} \frac{1}{2\,\varepsilon} \int_0^T \mathbf{1}_{\{0 \le \Xi(t) < \varepsilon\}}\, \mathrm{d}\langle \Xi \rangle(t)\,, \qquad 0 \le T < \infty\,. \tag{2.5}$$

From (2.2) and by analogy with Lemma 3.1.5 in PICARD (2005), we have the ITÔ-TANAKA-type equation

$$S(\cdot) \;=\; S(0) + \int_0^{\cdot} \mathbf{1}_{\{S(t)>0\}} \, \mathrm{d}U(t) + L^S(\cdot)\,, \qquad S(0) = U(0) \geq 0\,. \tag{2.6}$$

On the other hand, the theory of semimartingale local time (e.g., section 3.7 in KARATZAS & SHREVE (1991)) gives the properties

$$\int_0^{\infty} \mathbf{1}_{\{S(t)\,=\,0\}} \mathrm{d}\langle S\rangle(t) \;=\; 0\,, \qquad L^S(\cdot) \;=\; \int_0^{\cdot} \mathbf{1}_{\{S(t)\,=\,0\}} \mathrm{d}S(t)\,. \tag{2.7}$$

2.1 The Main Result

Theorem 2.1 below is the first key result of this paper. It produces a planar "skew-unfolding" $X(\cdot) = (X_1(\cdot), X_2(\cdot))'$ for the folding $S(\cdot)$ of the given continuous semimartingale $U(\cdot)$. This planar "skew-unfolded" process has radial part $||X(\cdot)|| = S(\cdot)$, and its motion away from the origin follows the one-dimensional dynamics of $S(\cdot)$ along rays emanating from the origin. Once at the origin, the process chooses the next ray for its voyage (according to the dynamics of $S(\cdot)$) independently of its past history and in a random fashion, according to a given probability measure on the collection of angles in $[0, 2\pi)$. Whenever $S(\cdot)$ is a reflecting Brownian motion or, more generally, a reflecting diffusion, these one-dimensional dynamics away from the origin are of course diffusive.

In order to describe this skew-unfolding with some detail and rigor, we shall need appropriate notation. Let us consider the unit circumference

$$\mathfrak{S} := \big\{ (z_1, z_2)' : z_1^2 + z_2^2 = 1 \big\}\,.$$

Here and throughout the paper, vectors are columns and the superscript $'$ denotes transposition. For every point $x := (x_1, x_2)' \in \mathbb{R}^2$ we introduce the mapping $\mathfrak{f} = \big(\mathfrak{f}_1, \mathfrak{f}_2\big)' : \mathbb{R}^2 \to \mathfrak{S} \cup \{\mathbf{0}\}$ via $\mathfrak{f}(\mathbf{0}) := \mathbf{0}$ and

$$\mathfrak{f}(x) \;:=\; \frac{x}{\|x\|} \;=\; \big(\cos(\arg(x))\,,\, \sin(\arg(x))\,\big)'\,; \qquad x \in E := \mathbb{R}^2 \setminus \{\mathbf{0}\} \tag{2.8}$$

with the notation $\mathbf{0} := (0,0)'$ and with $\arg(x) \in [0, 2\pi)$ denoting the *argument of the vector* $x \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$ in its polar coördinates. We fix a probability measure $\boldsymbol{\mu}$ on the collection $\mathcal{B}(\mathfrak{S})$ of Borel subsets of the unit circumference $\mathfrak{S}$, and consider also its expression

$$\boldsymbol{\nu}(\mathrm{d}\theta) \;:=\; \boldsymbol{\mu}(\mathrm{d}z)\,, \quad z = \big(\cos(\theta), \sin(\theta)\big)' \in \mathfrak{S}\,, \;\; \theta \in [0, 2\pi) \tag{2.9}$$

in polar coördinates. We introduce the real constants

$$\alpha_i^{(\pm)} \;:=\; \int_{\mathfrak{S}} \big(\mathfrak{f}_i(z)\big)^{\pm} \boldsymbol{\mu}(\mathrm{d}z)\,, \quad \gamma_i \;:=\; \alpha_i^{(+)} - \alpha_i^{(-)} \;=\; \int_{\mathfrak{S}} \mathfrak{f}_i(z)\, \boldsymbol{\mu}(\mathrm{d}z)\,, \quad i \;=\; 1, 2 \tag{2.10}$$

as well as the vector on the unit disc

$$\boldsymbol{\gamma} \;:=\; \big(\gamma_1, \gamma_2\big)' \;=\; \bigg(\int_0^{2\pi} \cos(\theta)\, \boldsymbol{\nu}(\mathrm{d}\theta)\,, \int_0^{2\pi} \sin(\theta)\, \boldsymbol{\nu}(\mathrm{d}\theta) \bigg)'. \tag{2.11}$$

Finally, we fix a vector $\mathrm{x} := (\mathrm{x}_1, \mathrm{x}_2)' \in \mathbb{R}^2$ with $\mathrm{x}_i = \mathfrak{f}_i(\mathrm{x})\, S(0)\,,\; i = 1, 2\,.$

Theorem 2.1. Construction of WALSH Semimartingales: *Consider the* SKOROKHOD *reflection* $S(\cdot)$ *of the continuous semimartingale* $U(\cdot)$ *as in (2.1) – (2.4), and fix* $\mathrm{x} = (\mathrm{x}_1, \mathrm{x}_2)' \in \mathbb{R}^2$ *as above.*

On a suitable enlargement $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{F} := \{\mathcal{F}(t)\}_{0 \leq t < \infty}$ *of the filtered probability space* $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$, $\widetilde{\mathbb{F}}$ *with a measure preserving map* $\boldsymbol{\pi} : \Omega \to \widetilde{\Omega}$, *there exists a planar continuous semimartingale* $X(\cdot) := (X_1(\cdot), X_2(\cdot))'$ *which solves the system of stochastic integral equations*

$$X_i(T) \;=\; \mathrm{x}_i + \int_0^T \mathfrak{f}_i\big(X(t)\big)\, \mathrm{d}S(t) + \Big(\alpha_i^{(+)} - \alpha_i^{(-)}\Big) L^S(T)\,, \qquad 0 \leq T < \infty \tag{2.12}$$

for $i = 1, 2$ *and whose radial part is*

$$\|X(\cdot)\| \,:=\, \sqrt{X_1^2(\cdot)+X_2^2(\cdot)} \,=\, S(\cdot)\,. \tag{2.13}$$

This continuous semimartingale $X(\cdot) := (X_1(\cdot), X_2(\cdot))'$ *has the following properties:*

(i) *With* $\mathrm{x} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$ *and* $\tau(s) := \inf\big\{t > s : X(t) = \mathbf{0}\big\}$ *the first time it reaches the origin after time* $s \geq 0$, *this process* $X(\cdot)$ *satisfies for every* $s \in (0,\infty)$, $B \in \mathcal{B}(\mathfrak{S})$ *and for Lebesgue almost every* $t \in (0,\infty)$ *the properties*

$$\mathfrak{f}(X(s)) \,=\, \mathfrak{f}(\mathrm{x})\,, \quad \mathbb{P}-\textit{a.e. on } \{\tau(0) > s\}\,, \tag{2.14}$$

$$\mathbb{P}\big(\mathfrak{f}(X(\tau(s)+t)) \in B \,\big|\, \mathcal{F}^X(\tau(s))\big) \,=\, \boldsymbol{\mu}(B)\,, \quad \mathbb{P}-\textit{a.e. on } \{\tau(s) < \infty\}\,. \tag{2.15}$$

(ii) *The local times at the origin of the component processes* $X_i(\cdot)$ *are given as*

$$L^{X_i}(\cdot) \,\equiv\, \alpha_i^{(+)}\, L^{\|X\|}(\cdot) \tag{2.16}$$

and are thus flat off the random set $\mathfrak{Z}$ *in (2.3) which has zero* LEBESGUE *measure by (2.4); in particular,*

$$\int_0^\infty \mathbf{1}_{\{X(t)=\mathbf{0}\}}\,\mathrm{d}t \,\equiv\, 0\,. \tag{2.17}$$

(iii) *Finally, for every* $A \in \mathcal{B}([0,2\pi))$, *the semimartingale local time at the origin of the "thinned" process* $R^A(\cdot) := \|X(\cdot)\| \cdot \mathbf{1}_A\big(\mathit{arg}\big(X(\cdot)\big)\big)$ *is given by*

$$L^{R^A}(\cdot) \,\equiv\, \boldsymbol{\nu}(A)\, L^{\|X\|}(\cdot)\,. \tag{2.18}$$

Terminology 2.1. We shall call the process $X(\cdot)$ constructed in the above Theorem a WALSH *semimartingale with "driver"* $U(\cdot)$ (*and "folded driver"* $S(\cdot)$). This process $X(\cdot)$ can be thought of as a *planar skew-unfolding* of the SKOROKHOD reflection $S(\cdot)$ of the driving continuous semimartingale $U(\cdot)$.

With the family of increasing processes $L^{R^A}(\cdot)$, $A \in \mathcal{B}\big([0,2\pi)\big)$ in (2.18), we shall find it convenient to associate a random measure $\boldsymbol{\Lambda}^X(\mathrm{d}t, \mathrm{d}\theta)$ on $[0,\infty) \times [0,2\pi)$ via

$$\boldsymbol{\Lambda}^X\big([0,t) \times A\big) \,:=\, L^{R^A}(t) \,=\, \boldsymbol{\nu}(A)\, L^{\|X\|}(t)\,; \qquad 0 \leq t < \infty\,, \quad A \in \mathcal{B}\big([0,2\pi)\big)\,. \tag{2.19}$$

3 Discussion and Ramifications

An intuitive interpretation of the stochastic integral equations (2.12) with the property (2.13) is as follows: We first "fold" the driving semimartingale $U(\cdot)$ to get its SKOROKHOD reflection $S(\cdot)$ as in (2.2) and then, starting from the point $\mathrm{x} = (\mathrm{x}_1, \mathrm{x}_2)' \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$ with $\mathrm{x}_i = \mathfrak{f}_i(\mathrm{x})\, S(0)$, $i = 1,2$ and up until the time $\tau(0)$ of Theorem 2.1(i), we run the planar process $X(\cdot) = (X_1(\cdot),\, X_2(\cdot))'$ according to the integral equation

$$X_i(\cdot) \,=\, \mathrm{x}_i + \int_0^\cdot \mathfrak{f}_i\big(X(t)\big)\,\mathrm{d}S(t)\,, \quad \text{for} \quad i = 1,2 \tag{3.1}$$

on $[0,\tau(0))$. This is the equation to which (2.12) reduces on the interval $[0,\tau(0))$. By the definition of the function $\mathfrak{f} = (\mathfrak{f}_1, \mathfrak{f}_2)'$ of (2.8), *the motion of the two-dimensional process* $X(\cdot) = (X_1(\cdot), X_2(\cdot))'$ *during the time-interval* $[0,\tau(0))$ *is along the ray that connects the origin* **0** *to the starting point* x.

Here is an argument for this claim, which proceeds by applying ITÔ's rule to the stochastic integral equation (2.12) in conjunction with (2.13): Given $\varepsilon \in (0, \|\mathrm{x}\|)$, let us define the stopping time $\sigma_\varepsilon := \inf\{t > 0 : S(t) = \|X(t)\| \leq \varepsilon\}$. Let us recall that the local time $L^S(\cdot)$ is flat off the random set $\mathfrak{Z}$ in (2.3); thus (2.12) reduces to (3.1) on $[0,\sigma_\varepsilon]$. Applying ITÔ's rule, we observe

$$\frac{X_i(t\wedge\sigma_\varepsilon)}{\|X(t\wedge\sigma_\varepsilon)\|} \,=\, \frac{\mathrm{x}_i}{\|\mathrm{x}\|} + \int_0^{t\wedge\sigma_\varepsilon} \frac{\mathrm{d}X_i(s)}{\|X(s)\|} - \int_0^{t\wedge\sigma_\varepsilon} \frac{X_i(s)}{\|X(s)\|^2}\,\mathrm{d}\|X(s)\|$$

$$+\int_0^{t\wedge\sigma_\varepsilon} \frac{X_i(s)}{\|X(s)\|^3}\,\mathrm{d}\langle\|X\|\rangle(s) - \int_0^{t\wedge\sigma_\varepsilon} \frac{1}{\|X(s)\|^2}\,\mathrm{d}\langle X_i, \|X\|\rangle(s) \tag{3.2}$$

for $t \geq 0$ and $i = 1, 2$. Because of the definition of $\mathfrak{f}(\cdot)$ and (2.13), we obtain

$$X_i(t) = \|X(t)\|\,\mathfrak{f}_i(X(t))\,, \qquad \mathrm{d}X_i(t) = \mathfrak{f}_i(X(t))\mathrm{d}S(t) = \frac{X_i(t)}{\|X(t)\|}\,\mathrm{d}\|X(t)\|\,,$$

$$\langle X_i, \|X\|\rangle(t) = \int_0^t \mathfrak{f}_i(X(s))\,\mathrm{d}\langle\|X\|\rangle(s) = \int_0^t \frac{X_i(s)}{\|X(s)\|}\,\mathrm{d}\langle\|X\|\rangle(s)$$

on $[0, \sigma_\varepsilon]$, for $i = 1, 2$. Substituting these relations into (3.2), we deduce

$$\mathfrak{f}_i(X(t)) = \frac{X_i(t)}{\|X(t)\|} = \frac{\mathrm{x}_i}{\|\mathrm{x}\|} = \mathfrak{f}_i(\mathrm{x}) \quad \text{on} \quad [0, \sigma_\varepsilon]\,,\ i = 1, 2\,.$$

Since $\varepsilon > 0$ is arbitrary, this concludes the proof of the above claim, in accordance with (2.14).

Now, every time the planar process $X(\cdot)$ visits the origin, the direction of the next ray for its $S(\cdot)$-governed motion is instantaneously chosen at random according to the probability distribution $\boldsymbol{\mu}$, the "spinning measure" of the process $X(\cdot)$, in a manner described in more detail later. If the origin is visited infinitely often during a time-interval of finite length, it is not surprising that this random choice should lead to the accumulation of local time at the origin, as indicated in the equations (2.12). It follows from (2.17) that set of times spent by $X(\cdot)$ at the origin has zero Lebesgue measure. The process continues to move then along the newly chosen ray, its motion governed by the stochastic integral equations of (3.1) just described, as long as it stays away from the origin. The path $t \mapsto X(t)$ is, with probability one, continuous in the topology induced by the tree-metric (French railway metric) on the plane, namely

$$\varrho(x, y) := (r_1 + r_2)\,\mathbf{1}_{\{\theta_1 \neq \theta_2\}} + |r_1 - r_2|\,\mathbf{1}_{\{\theta_1 = \theta_2\}}\,, \qquad x = (r_1, \theta_1)\,,\ y = (r_2, \theta_2)\,. \tag{3.3}$$

The reader may find it useful at this juncture to think of a roundhouse at the origin, of the spokes of a bicycle wheel – or of the Aeolian winds of Homeric lore, that blow the raft of Odysseus in all directions at once.

3.1 Spider Semimartingales

Suppose that the measure $\boldsymbol{\mu}$ charges only a finite number m of points on the unit circumference (equivalently, rays passing through the origin). We can think then of the planar process $X(\cdot)$ constructed in Theorem 2.1 as a *Spider Semimartingale,* whose radial part $\|X(\cdot)\| = S(\cdot)$ is the SKOROKHOD reflection of the driver $U(\cdot)$ according to (2.13).

When the driving semimartingale $U(\cdot)$ is Brownian motion, the process $X(\cdot)$ of Theorem 2.1 becomes the original WALSH *Brownian Motion* (as constructed, for instance, in BARLOW, PITMAN & YOR (1989a)) with roundhouse singularity in a multipole field; this will be shown in Proposition 7.2 below. When $m = 2$ and $\boldsymbol{\nu}(\{0\}) = \alpha \in (0, 1)$, $\boldsymbol{\nu}(\{\pi\}) = 1 - \alpha$, this construction recovers the familiar *Skew Brownian Motion,* introduced in ITÔ & MCKEAN (1963) and studied by WALSH (1978) and by HARRISON & SHEPP (1981).

3.2 Generalized HARRISON-SHEPP and Skew-TANAKA Equations

In the context of Theorem 2.1 (in particular, with the property (2.13)), the equations of (2.12) can be cast in equivalent forms, now driven by the original semimartingale $U(\cdot)$, as follows:

$$X_i(\cdot) = \mathrm{x}_i + \int_0^{\cdot} \mathfrak{f}_i\big(X(t)\big)\,\mathrm{d}U(t) + \gamma_i\, L^{\|X\|}(\cdot)\,, \qquad i = 1, 2\,, \tag{3.4}$$

$$X_i(\cdot) \,=\, \mathrm{x}_i + \int_0^{\cdot} \mathfrak{f}_i\big(X(t)\big)\, \mathrm{d}U(t) + \left(1 - \frac{\alpha_i^{(-)}}{\alpha_i^{(+)}}\right) L^{X_i}(\cdot)\,, \quad i = 1,2 \tag{3.5}$$

(the latter when $\alpha_i^{(+)} > 0$). This last system (3.5) can be thought of as a planar semimartingale version of the HARRISON & SHEPP (1981) equation for the skew Brownian motion; it is also a two-dimensional version of the *skew*-TANAKA *equation* studied by ICHIBA & KARATZAS (2014).

The system of equations (3.4), on the other hand, can be thought of as a planar analogue of the equation (2.6). For two fixed real constants γ_1, γ_2, and a folded driver $S(\cdot)$ that satisfies the condition

$$\mathbb{P}\big(L^S(\infty) > 0\big) \,>\, 0 \tag{3.6}$$

(e.g., reflecting Brownian motion), we have the following necessary and sufficient condition for the solvability of the system (3.4), subject to the condition (2.13). Its proof is given in section 5.

Proposition 3.1. *Consider a continuous semimartingale $U(\cdot)$ along with its* SKOROKHOD *reflection $S(\cdot)$ as in section 2.1, two real numbers γ_1, γ_2, and a vector $\mathrm{x} := (\mathrm{x}_1, \mathrm{x}_2)' \in \mathbb{R}^2$ with $\mathrm{x}_i = \mathfrak{f}_i(\mathrm{x})\, S(0)\,,\ i = 1,2$.*

(i) Suppose that the real numbers γ_1, γ_2 satisfy $\gamma_1^2 + \gamma_2^2 \le 1$. Then there exists a continuous planar semimartingale $X(\cdot) = (X_1(\cdot), X_2(\cdot))'$ that satisfies the system (3.4), as well as the condition (2.13).

(ii) Conversely, suppose that (3.6) holds, and that there exists a continuous planar semimartingale $X(\cdot) = (X_1(\cdot), X_2(\cdot))'$ that satisfies the system (3.4) and the condition (2.13). Then we have $\gamma_1^2 + \gamma_2^2 \le 1$.

3.3 Open Questions

• It would be of considerable interest to extend the methodology of this paper to a situation with an entire family $U(\cdot\,;\mathbf{z})\,,\ \mathbf{z} \in \mathfrak{S}$ of semimartigales so that, when the point $\mathbf{z}$ is selected on the unit circumference by the spinning measure $\boldsymbol{\mu}$, the motion along the corresponding ray is according to the SKOROKHOD reflection $S(\cdot\,;\mathbf{z})$ of this semimartingale $U(\cdot\,;\mathbf{z})$. Some results on this issue are obtained in section 8, in the context of the diffusion case and by the method of scale function and time-change.

• What are the descriptive statistics of the WALSH semimartingale? For example, what is the area of the convex hull of its path $\{X(s),\ 0 \le s \le T\}$ for some time T? In the spirit of this question, we discuss occupations times for WALSH's Brownian motion in Example 10.6.

4 A FREIDLIN-SHEU-type Formula

Let us consider now a twice continuously differentiable function $g : \mathbb{R}^2 \to \mathbb{R}$. If $X(\cdot)$ is a continuous planar semimartingale that satisfies the system of equations (2.12) with the property (2.13), an application of ITÔ's rule with the notation of (2.8) gives

$$g(X(T)) \,=\, g(\mathrm{x}) \,+ \int_0^T \left(\sum_{i=1}^2 D_i g(X(t))\, \mathfrak{f}_i(X(t))\right) \mathrm{d}S(t) \,+\, \sum_{i=1}^2 D_i g(\mathbf{0})\, \gamma_i \cdot L^S(T)$$

$$+\, \frac{1}{2} \int_0^T \left(\sum_{i=1}^2 \sum_{j=1}^2 D^2_{ij} g(X(t))\, \mathfrak{f}_i(X(t))\, \mathfrak{f}_j(X(t))\right) \mathrm{d}\langle S\rangle(t)\,, \quad 0 \le T < \infty\,.$$

We define now the functions

$$G'(x) := \sum_{i=1}^2 D_i g(x)\, \mathfrak{f}_i(x)\,, \qquad G''(x) := \sum_{i=1}^2 \sum_{j=1}^2 D^2_{ij} g(x)\, \mathfrak{f}_i(x)\, \mathfrak{f}_j(x) \tag{4.1}$$

on the punctured plane $E = \mathbb{R}^2 \setminus \{\mathbf{0}\}$, and consider them as the first and second derivatives, respectively, of the function $g(\cdot)$ in its radial argument $r = \sqrt{x_1^2 + x_2^2}$. With this notation and that of (2.10), (2.11), the above decomposition can be written in the FREIDLIN-SHEU (2000) form

$$g(X(\cdot)) = g(\mathrm{x}) + \int_0^{\cdot} \mathbf{1}_{\{X(t) \neq \mathbf{0}\}} \left(G'(X(t))\, \mathrm{d}S(t) + \frac{1}{2}\, G''(X(t))\, \mathrm{d}\langle S \rangle(t) \right) + \sum_{i=1}^{2} \gamma_i\, D_i g(\mathbf{0}) \cdot L^S(\cdot)\,. \tag{4.2}$$

We note that the real constant $\sum_{i=1}^{2} \gamma_i\, D_i g(\mathbf{0})$ which multiplies the local time term in (4.2), can be cast as

$$\sum_{i=1}^{2} \gamma_i\, D_i g(\mathbf{0}) \;=\; \int_0^{2\pi} h(\theta) \boldsymbol{\nu}(\mathrm{d}\theta)\,, \qquad \text{the integral of} \qquad h(\theta) \;:=\; \lim_{\|x\| \downarrow 0} G'(x) \Big|_{\arg(x) = \theta} \tag{4.3}$$

with respect to the spinning measure expressed here in polar coördinates, as in HAJRI & TOUHAMI (2014).

The following result is now an immediate corollary of (4.2), (2.2) and of the fact that the finite-variation process $S(\cdot) - U(\cdot) = \Lambda(\cdot)$ in (2.2) is flat off the zero set in (2.3). The planar process $X(\cdot)$ constructed in Theorem 2.1 satisfies its requirements.

Proposition 4.1. *Suppose that the semimartingale $U(\cdot)$ in (2.1) is a continuous local martingale, define its SKOROKHOD reflection $S(\cdot)$ as in (2.2), and consider any planar continuous semimartingale $X(\cdot)$ which solves the system of equations (2.12) and satisfies the property (2.13).*

Consider also a twice continuously differentiable function $g : \mathbb{R}^2 \to \mathbb{R}$ which satisfies the "slope-averaging" condition $\sum_{i=1}^{2} \gamma_i\, D_i g(\mathbf{0}) = 0$. Then the process

$$g(X(\cdot)) - g(X(0)) - \frac{1}{2} \int_0^{\cdot} G''(X(t))\, \mathbf{1}_{\{X(t) \neq \mathbf{0}\}}\, \mathrm{d}\langle U \rangle(t) \;=\; \int_0^{\cdot} G'(X(t))\, \mathbf{1}_{\{X(t) \neq \mathbf{0}\}}\, \mathrm{d}U(t) \tag{4.4}$$

is also a continuous local martingale.

4.1 A Generalization of the Change-of-Variable Formula (4.2)

Let us try to refine somewhat the considerations of the previous subsection. It is clear that, along the paths of the process $X(\cdot)$ constructed in Theorem 2.1, only derivatives of the form indicated in (4.1) – i.e., radial – appear in the FREIDLIN-SHEU-like formula (4.2). This suggests that the smoothness assumption in (4.2) and in Proposition 4.1 can be relaxed, as follows.

Definition 4.1. We consider the class $\mathfrak{D}$ of BOREL-measurable functions $g : \mathbb{R}^2 \to \mathbb{R}$ with the following properties:

(i) they are continuous in the topology induced by the tree-metric (3.3) on the plane;

(ii) for every $\theta \in [0, 2\pi)$, the function $r \longmapsto g_\theta(r) := g(r, \theta)$ is twice continuously differentiable on $(0, \infty)$ and has finite first and second right-derivatives at the origin;

(iii) the resulting functions $(r, \theta) \mapsto g'_\theta(r)$ and $(r, \theta) \mapsto g''_\theta(r)$ are BOREL measurable; and

(iv) $\sup_{\substack{0<r<K \\ \theta \in [0,2\pi)}} \big(|g'_\theta(r)| + |g''_\theta(r)| \big) < \infty$ holds for all $K \in (0, \infty)$.

Here we consider BOREL sets with respect to the Euclidean topology.

We introduce also the subclasses

$$\mathfrak{D}^{\boldsymbol{\mu}} \;:=\; \left\{ g \in \mathfrak{D} \;:\; \int_0^{2\pi} g'_\theta(0+)\, \boldsymbol{\nu}(\mathrm{d}\theta) \;=\; 0 \right\}, \quad \mathfrak{D}^{\boldsymbol{\mu}}_{+} \;:=\; \left\{ g \in \mathfrak{D} \;:\; \int_0^{2\pi} g'_\theta(0+)\, \boldsymbol{\nu}(\mathrm{d}\theta) \;\geq\; 0 \right\}. \tag{4.5}$$

Definition 4.2. For every given function $g : \mathbb{R}^2 \to \mathbb{R}$ in $\mathfrak{D}$ we set by analogy with (4.1):

$$G'(x) \;:=\; g'_\theta(r)\,, \quad G''(x) \;:=\; g''_\theta(r) \quad \text{for } x = (r, \theta) \quad \text{with } r > 0\,.$$

With this notation in place, we can formulate our second major result.

Theorem 4.1. A Generalized FREIDLIN-SHEU Formula: *With the above notation, every continuous semimartingale* $X(\cdot) = (X_1(\cdot), X_2(\cdot))'$ *which solves the system of equations (2.12) and satisfies the properties (2.13) and (2.18), also satisfies the generalized* FREIDLIN-SHEU *identity*

$$g(X(\cdot)) = g(\mathrm{x}) + \int_0^{\cdot} \mathbf{1}_{\{X(t)\neq \mathbf{0}\}} \Big(G'(X(t))\,\mathrm{d}U(t) + \frac{1}{2}\, G''(X(t))\,\mathrm{d}\langle U\rangle(t) \Big) + \Big(\int_0^{2\pi} g'_\theta(0+)\,\boldsymbol{\nu}(\mathrm{d}\theta) \Big) L^S(\cdot) \tag{4.6}$$

for every $g \in \mathfrak{D}$; *or equivalently, with the random measure* $\boldsymbol{\Lambda}^X(\mathrm{d}t, \mathrm{d}\theta)$ *as in (2.19), the identity*

$$g(X(\cdot)) = g(\mathrm{x}) + \int_0^{\cdot} \mathbf{1}_{\{X(t)\neq \mathbf{0}\}} \Big(G'(X(t))\,\mathrm{d}\|X(t)\| + \frac{1}{2}\, G''(X(t))\,\mathrm{d}\langle \|X\|\rangle(t) \Big) + \int_0^{\cdot} \int_0^{2\pi} g'_\theta(0+)\,\boldsymbol{\Lambda}^X(\mathrm{d}t, \mathrm{d}\theta) \tag{4.7}$$

In particular, the continuous semimartingale $X(\cdot)$ of Theorem 2.1 satisfies (4.6), (4.7).

4.2 Slope-Averaging Martingales

For any given bounded, measurable $\varphi : [0, 2\pi) \to \mathbb{R}$, let us define the functions

$$h_{(\varphi)}(x) \;:=\; \big(\varphi(\arg(x)) - \mathbb{E}[\varphi(\arg(\boldsymbol{\xi}_1))]\big)\cdot \mathbf{1}_{\{x\neq \mathbf{0}\}}\,, \qquad g_{(\varphi)}(x) \;:=\; \|x\| \cdot h_{(\varphi)}(x) \tag{4.8}$$

for $x \in \mathbb{R}^2$, where $\boldsymbol{\xi}_1$ is an $\mathfrak{S}$-valued random variable with distribution $\boldsymbol{\mu}$ as in (5.1). Such functions were first introduced by BARLOW, PITMAN & YOR (1989a), in their study of the WALSH Brownian motion. Using polar coördinates, we observe that $g_{(\varphi)}(x) \equiv g_{(\varphi)}(r, \theta)$ belongs to the class $\mathfrak{D}$ and satisfies

$$G'_{(\varphi)}(x) \;\equiv\; (g_{(\varphi)})'_\theta(r) \;=\; h_{(\varphi)}(r,\theta)\,, \qquad G''_{(\varphi)}(x) \;\equiv\; (g_{(\varphi)})''_\theta(r) \;=\; 0\,, \qquad \int_0^{2\pi} h_{(\varphi)}(r,\theta)\,\boldsymbol{\nu}(\mathrm{d}\theta) \;=\; 0\,.$$

Here $'$ denotes differentiation with respect to $r \in (0, \infty)$.

Direct application of Theorem 4.1 gives the following result.

Proposition 4.2. *Assume that* $U(\cdot)$ *in (2.1) is a continuous local martingale, and construct its* SKOROKHOD *reflection* $S(\cdot)$ *as in (2.2). Consider any continuous semimartingale* $X(\cdot) := (X_1(\cdot), X_2(\cdot))'$ *which satisfies the system of equations (2.12), along with the properties (2.13) and (2.18).*
(i) For any $g : \mathbb{R}^2 \to \mathbb{R}$ *in the class* $\mathfrak{D}$ *with the slope-averaging* $\int_0^{2\pi} g'_\theta(0+)\,\boldsymbol{\nu}(\mathrm{d}\theta) = 0$, *the process*

$$g(X(\cdot)) - g(X(0)) - \frac{1}{2}\int_0^{\cdot} G''(X(t))\,\mathbf{1}_{\{X(t)\neq \mathbf{0}\}}\,\mathrm{d}\langle U\rangle(t)$$

is a continuous local martingale with quadratic variation process $\int_0^{\cdot} \big(G'(X(t))\big)^2\,\mathbf{1}_{\{X(t)\neq \mathbf{0}\}}\,\mathrm{d}\langle U\rangle(t)$.
(ii) For any given bounded, measurable function $\varphi : [0, 2\pi) \to \mathbb{R}$, *the process*

$$g_{(\varphi)}\big(X(\cdot)\big) \;=\; \big|\big|X(\cdot)\big|\big|\, h_{(\varphi)}(X(\cdot)) \;=\; g_{(\varphi)}(\mathrm{x}) + \int_0^{\cdot} h_{(\varphi)}(X(t))\,\mathrm{d}U(t)\,,$$

with the notation of (4.8), is a continuous local martingale.

5 The Proofs of Theorems 2.1, 4.1, and of Proposition 3.1

The way we construct a process $X(\cdot)$ which satisfies the equation (2.12) is via "folding and unfolding of semimartingales", with additional randomness coming from a sequence $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots$ of $\mathfrak{S}-$valued, I.I.D. random variables. These have common probability distribution $\boldsymbol{\mu}$ on $\mathfrak{S}$, such that the components of the random vector $\boldsymbol{\xi}_1 \;:=\; \big(\xi_{1,1}, \xi_{1,2}\big)'$ have expectations that are matched with the parameter vector $\big(\alpha_1^{(+)}, \alpha_1^{(-)}, \alpha_2^{(+)}, \alpha_2^{(-)}\big) \in [0,1]^4$ in (2.10), (2.12) as

$$\mathbb{E}\big(\xi_{1,i}^{\pm}\big) \,=\, \alpha_i^{(\pm)}\,, \quad \mathbb{E}\big(\xi_{1,i}\big) \,=\, \alpha_i^{(+)} - \alpha_i^{(-)} \,=\, \gamma_i\,, \quad \mathbb{E}\big(|\xi_{1,i}|\big) \,=\, \alpha_i^{(+)} + \alpha_i^{(-)}\,; \quad i \,=\, 1,2\,. \tag{5.1}$$

Proof of Theorem 2.1: For simplicity, we consider the case $\mathrm{x}_1 = \mathrm{x}_2 = 0$ first. Following PROKAJ (2009) and ICHIBA & KARATZAS (2014), we enlarge the original probability space by means of the above sequence $\{\boldsymbol{\xi}_k\}_{k\in\mathbb{N}}$ of $\mathfrak{S}$-valued, I.I.D. random variables. These are independent of the $\sigma-$algebra $\widetilde{\mathcal{F}}(\infty) := \bigvee_{0\le t<\infty}\widetilde{\mathcal{F}}(t)$ and have expectation $\mathbb{E}(\boldsymbol{\xi}_1) = \boldsymbol{\gamma}$ as in (5.1), (2.11).

• Let us decompose the nonnegative half-line into the zero set $\mathfrak{Z}$ of $S(\cdot)$ as in (2.3) on the one hand, and the countable collection $\{\mathcal{C}_k\}_{k\in\mathbb{N}}$ of open disjoint components of $[0,\infty)\setminus\mathfrak{Z}$ on the other. Each of these components represents an excursion interval away from the origin for the SKOROKHOD reflection process $S(\cdot)$ in (2.2). Here we enumerate these countably-many excursion intervals $\{\mathcal{C}_k\}_{k\in\mathbb{N}}$ in a measurable manner, so that $\{t\in\mathcal{C}_k\}\in\widetilde{\mathcal{F}}(\infty)$ holds for all $t\ge 0$, $k\in\mathbb{N}$. For notational simplicity, we declare

$$\mathcal{C}_0 \,:=\, \mathfrak{Z}\,, \qquad \boldsymbol{\xi}_0 \,:=\, \mathbf{0}\,.$$

We shall denote

$$Z(t) \,:=\, \sum_{k\in\mathbb{N}_0}\boldsymbol{\xi}_k\cdot\mathbf{1}_{\mathcal{C}_k}(t)\,, \qquad X(t) \,:=\, Z(t)S(t)\,, \qquad \mathcal{F}^Z(t) \,:=\, \sigma(Z(s), 0\le s\le t) \tag{5.2}$$

for $0\le t<\infty$ and introduce the enlarged filtration $\mathbb{F} := \{\mathcal{F}(t)\}_{0\le t<\infty}$ via $\mathcal{F}(t) := \widetilde{\mathcal{F}}(t)\vee\mathcal{F}^Z(t)$.

This procedure corresponds exactly to the program outlined by J.B. WALSH in the appendix to his 1978 paper, as follows: *"The idea is to take each excursion of (reflecting Brownian motion) and, instead of giving a random sign, to assign it a random variable with a given distribution in* $[0,2\pi)$*, and to do so independently for each excursion".* Of course the process $S(\cdot)$ is one-dimensional, while

$$Z(\cdot) = \big(Z_1(\cdot), Z_2(\cdot)\big)' \qquad \text{and} \qquad X(\cdot) = \big(X_1(\cdot), X_2(\cdot)\big)'$$

are two-dimensional processes with $\mathfrak{f}(X(\cdot)) = \big(\mathfrak{f}_1(X(\cdot)),\ \mathfrak{f}_2(X(\cdot))\big)' = Z(\cdot)$, i.e.,

$$\mathfrak{f}_i(X(\cdot)) \,=\, Z_i(\cdot)\,; \quad i \,=\, 1,2\,. \tag{5.3}$$

Here the functions $\mathfrak{f}_i(\cdot)$ are as defined in (2.8). In particular, the zero set of (2.3) is

$$\mathfrak{Z} \,=\, \big\{t\ge 0 : S(t)=0\big\} = \big\{t\ge 0 : Z(t)=\mathbf{0}\big\} = \big\{t\ge 0 : X(t)=\mathbf{0}\big\}\,. \tag{5.4}$$

We can also think of the vector process $X(\cdot)$ of (5.2) as expressed in its polar coördinates

$$S(t) = \sqrt{X_1^2(t)+X_2^2(t)} \qquad \text{and} \qquad \Theta(t) = \arg\big(Z(t)\big) = \sum_{k\in\mathbb{N}}\arg(\boldsymbol{\xi}_k)\cdot\mathbf{1}_{\mathcal{C}_k}(t) \tag{5.5}$$

from (5.2). We shall see presently that this process $X(\cdot)$ satisfies the system of equations (2.12).

• We claim that, because of independence and of the way the probability space was enlarged, *both processes* $U(\cdot)$ *and* $S(\cdot)$ *are continuous* $\mathbb{F}-$*semimartingales*. This claim can be established as in the proof of Proposition 2 in PROKAJ (2009); see also Proposition 3.1 in ICHIBA & KARATZAS (2014).

Let us go briefly over the argument. By localization if necessary, we may assume that the $\widetilde{\mathbb{F}}-$local martingale $M(\cdot)$ in (2.1) is actually an $\widetilde{\mathbb{F}}-$martingale; then show that it is also an $\mathbb{F}-$martingale, i.e.,

$$\mathbb{E}\,\Big[\,\big(M(t)-M(s)\big)\cdot\mathbf{1}_A\,\Big] \,=\, 0\,; \quad A\in\mathcal{F}(s)\,, \quad 0\le s<t<\infty\,. \tag{5.6}$$

In order to do this, let us fix such s and t as above, as well as disjoint BOREL subsets $\mathfrak{S}_1,\ldots,\mathfrak{S}_{\ell_0}$ of the unit circumference $\mathfrak{S}$ with $\bigcup_{1\le\ell\le\ell_0}\mathfrak{S}_\ell = \mathfrak{S}$ and $\boldsymbol{\mu}(\mathfrak{S}_\ell)>0$, $\ell=1,\ldots,\ell_0$ for some $\ell_0\in\mathbb{N}$. For any given $n\in\mathbb{N}$, $0\le s_1<s_2<\cdots<s_n\le s<t$ and $E_j\in\{\mathfrak{S}_1,\ldots,\mathfrak{S}_{\ell_0}\}$, $j=1,\ldots,n$, we consider sets of the form

$$D := \bigcap_{j=1}^{n} \{Z(s_j) \in E_j\} = \bigcap_{j=1}^{n} \{\boldsymbol{\xi}_{\kappa(s_j)} \in E_j\} ; \tag{5.7}$$

here $\kappa(u)$ denotes the (random) index of the excursion interval $\mathcal{C}_k$ to which a given time $u \in [0, \infty)$ belongs. Note that if the probability $\mathbb{P}(\boldsymbol{\xi}_{k_1} \in E_1, \ldots, \boldsymbol{\xi}_{k_n} \in E_n)$ is strictly positive for some (nonrandom) collections of indices $\{k_1, \ldots, k_n\} \subset \mathbb{N}_0$ and subsets $\{E_1, \ldots, E_n\}$, then

$$\mathbb{P}\big(\boldsymbol{\xi}_{k_1} \in E_1, \ldots, \boldsymbol{\xi}_{k_n} \in E_n\big) = \big[\boldsymbol{\mu}(\mathfrak{S}_1)\big]^{\lambda_1} \cdot \big[\boldsymbol{\mu}(\mathfrak{S}_2)\big]^{\lambda_2} \cdot \cdots \cdot \big[\boldsymbol{\mu}(\mathfrak{S}_{\ell_0})\big]^{\lambda_{\ell_0}} ;$$

we have denoted here by λ_ℓ the number from among those distinct indices of the corresponding E_j's that are equal to $\mathfrak{S}_\ell$, for each $\ell = 1, \ldots, \ell_0$. This probability is zero, of course, whenever the sets $\{E_1, \ldots, E_n\}$ contradict the indices $\{k_1, \ldots, k_n\}$ and hence the set $\bigcap_{1 \le j \le n} \{\boldsymbol{\xi}_{k_j} \in E_j\}$ is empty.

Given the condition $k_1 = \kappa(s_1), \cdots, k_n = \kappa(s_n)$, from the trajectory $S(u),\, 0 \le u \le s$ of $S(\cdot)$ up to time s we may determine the numbers $(\lambda_1, \ldots, \lambda_{\ell_0})$, and also determine whether the set $\bigcap_{1 \le j \le n} \{\boldsymbol{\xi}_{k_j} \in E_j\}$ is empty or not. Thus the conditional probability

$$\mathbb{P}\big(D \,|\, \widetilde{\mathcal{F}}(\infty)\big) = \mathbb{P}\big(\boldsymbol{\xi}_{k_1} \in E_1, \ldots, \boldsymbol{\xi}_{k_n} \in E_n\big)\Big|_{k_1 = \kappa(s_1), \cdots, k_n = \kappa(s_n)}$$

is an $\widetilde{\mathcal{F}}(s)-$measurable random variable. Then the martingale property of $M(\cdot)$ with respect to $\widetilde{\mathbb{F}}$ gives

$$\mathbb{E}\big[(M(t) - M(s)) \cdot \mathbf{1}_{B \cap D}\big] = \mathbb{E}\big[(M(t) - M(s)) \cdot \mathbf{1}_B \cdot \mathbb{P}\big(D \,|\, \widetilde{\mathcal{F}}(\infty)\big)\big] = 0$$

for every set $B \in \widetilde{\mathcal{F}}(s)$, $0 \le s < t < \infty$ and every set D of the form in (5.7). This implies (5.6), and hence that both $U(\cdot)$ and $S(\cdot)$ are continuous $\mathbb{F}-$semimartingales.

• In order to describe the dynamics of the process $X(\cdot)$ defined in (5.2), we approximate the process $Z(\cdot)$ also defined there, by a family of processes $Z^\varepsilon(\cdot)$ with finite first variation over compact intervals indexed by $\varepsilon \in (0, 1)$, as follows. We define the sequence of stopping times $\tau_0^\varepsilon := \inf\big\{t \ge 0 : \|X(t)\| = 0\big\}$ and

$$\tau_{2\ell+1}^\varepsilon := \inf\big\{t > \tau_{2\ell}^\varepsilon : \|X(t)\| \ge \varepsilon\big\}, \quad \tau_{2\ell+2}^\varepsilon := \inf\big\{t > \tau_{2\ell+1}^\varepsilon : \|X(t)\| = 0\big\}; \quad \ell \in \mathbb{N}_0 \tag{5.8}$$

recursively. We also introduce a piecewise-constant process $Z^\varepsilon(\cdot) := \big(Z_1^\varepsilon(\cdot), Z_2^\varepsilon(\cdot)\big)'$ with

$$Z^\varepsilon(t) := \sum_{\ell \in \mathbb{N}_0} Z(t)\, \mathbf{1}_{[\tau_{2\ell+1}^\varepsilon, \tau_{2\ell+2}^\varepsilon)}(t) = \sum_{(k,\ell) \in \mathbb{N}_0^2} \boldsymbol{\xi}_k\, \mathbf{1}_{\mathcal{C}_k \cap [\tau_{2\ell+1}^\varepsilon, \tau_{2\ell+2}^\varepsilon)}(t), \qquad 0 \le t < \infty, \tag{5.9}$$

i.e., constant on each of the "downcrossing intervals" $[\tau_{2\ell+1}^\varepsilon, \tau_{2\ell+2}^\varepsilon)$. For this process, the product rule gives

$$X^\varepsilon(T) := Z^\varepsilon(T) S(T) = \int_0^T Z^\varepsilon(t) \mathrm{d}S(t) + \int_0^T S(t) \mathrm{d}Z^\varepsilon(t), \qquad 0 \le T < \infty. \tag{5.10}$$

Passing to the limit as $\varepsilon \downarrow 0$ and using (5.1)-(5.3), as well as the characterization of the local time $L^S(\cdot)$ of the semimartingale $S(\cdot)$ in terms of the number of its downcrossings, we obtain the decomposition

$$X(T) = Z(T) S(T) = \int_0^T Z(t)\, \mathrm{d}S(t) + \mathbb{E}[\boldsymbol{\xi}_1]\, L^S(T) = \int_0^T \mathfrak{f}(X(t))\, \mathrm{d}S(t) + \boldsymbol{\gamma}\, L^S(T) \tag{5.11}$$

in the notation of (2.11). Indeed, the second term on the right-hand side of (5.10) can be estimated by the strong law of large numbers and Theorem VI.1.10 in Revuz & Yor (1999): namely, we have the convergence

$$\int_0^T S(t)\mathrm{d}Z^\varepsilon(t) \,=\, \sum_{\{\ell:\,\tau^\varepsilon_{2\ell+1}<T\}} S(\tau^\varepsilon_{2\ell+1})Z^\varepsilon(\tau^\varepsilon_{2\ell+1}) \,=\, \varepsilon \sum_{j=1}^{N(T,\varepsilon)} \boldsymbol{\xi}_{\ell_j} + O(\varepsilon)$$
$$= \varepsilon\, N(T,\varepsilon)\cdot \frac{1}{N(T,\varepsilon)} \sum_{j=1}^{N(T,\varepsilon)} \boldsymbol{\xi}_{\ell_j} + O(\varepsilon) \underset{\varepsilon\downarrow 0}{\longrightarrow} L^S(T)\cdot \mathbb{E}[\boldsymbol{\xi}_1] \tag{5.12}$$

in probability. Here $\{\boldsymbol{\xi}_{\ell_j}\}_{j=1}^{N(T,\varepsilon)}$ is an enumeration of $Z^\varepsilon(\tau^\varepsilon_{2\ell+1})$, and $N(T,\varepsilon) := \sharp\{\ell\in\mathbb{N} : \tau^\varepsilon_{2\ell} < T\}$ the number of downcrossings of the interval $(0,\varepsilon)$ that the process $S(\cdot)$ has completed during $[0,T)$. We deduce from (5.11), in particular, that the process $X(\cdot)$ is a continuous planar $\mathbb{F}-$semimartingale.

By analogy with (5.11), we can approximate the process $|Z_i(\cdot)|$ by $|Z_i^\varepsilon(\cdot)|$, the absolute value of each of the components $Z_i^\varepsilon(\cdot)$ of the piecewise-constant process in (5.9); passing to the limit as $\varepsilon\downarrow 0$, we obtain

$$\big|X_i(T)\big| \,=\, \big|Z_i(T)\big|S(T) \,=\, \int_0^T \big|Z_i(t)\big|\,\mathrm{d}S(t) + \mathbb{E}\big(|\xi_{1,i}|\big)L^S(T)\,, \qquad 0\le T<\infty \tag{5.13}$$

for $i=1,2$. We appeal now to Exercise VI (1.16) 3^o) of REVUZ & YOR (1999); recalling the form of $S(\cdot)$ in (2.2) along with (5.4) we deduce that, with the normalization of (2.5), the continuous, nonnegative semimartingale $|X_i(\cdot)|$, $i=1,2$ with the decomposition (5.13) has local time at the origin

$$L^{|X_i|}(\cdot) \,=\, \int_0^\cdot \mathbf{1}_{\{X_i(t)=0\}}\,\big[\,\big|Z_i(t)\big|\,\mathrm{d}S(t) + \big(\alpha_i^{(+)}+\alpha_i^{(-)}\big)\,\mathrm{d}L^S(t)\,\big] \,=\, \big(\alpha_i^{(+)}+\alpha_i^{(-)}\big)\,L^S(\cdot)\,. \tag{5.14}$$

• Next, we need to identify the local times $L^{X_i}(\cdot)$ of each component $X_i(\cdot)$ in terms of the local time $L^S(\cdot)$. Since $X_i(\cdot) = Z_i(\cdot)S(\cdot)$ is a continuous semimartingale for $i=1,2$, we recall the decomposition (5.11) and properties of semimartingale local time, to obtain the string of identities

$$2\,L^{X_i}(\cdot) - L^{|X_i|}(\cdot) \,=\, \int_0^\cdot \mathbf{1}_{\{X_i(t)=0\}}\,\mathrm{d}X_i(t) \,=\, \int_0^\cdot \mathbf{1}_{\{X_i(t)=0\}}\big[Z_i(t)\mathrm{d}S(t) + \mathbb{E}(\xi_{1,i})\mathrm{d}L^S(t)\big]$$
$$= \mathbb{E}(\xi_{1,i})\,L^S(\cdot) \,=\, \big(\alpha_i^{(+)}-\alpha_i^{(-)}\big)\,L^S(\cdot) \tag{5.15}$$

(cf. subsection 2.1 in ICHIBA, KARATZAS & PROKAJ (2013)). Thus, combining with (5.14), we deduce

$$2\,L^{X_i}(\cdot) \,=\, \big(\mathbb{E}(|\xi_{1,i}|)+\mathbb{E}(\xi_{1,i})\big)L^S(\cdot) \,=\, 2\,\mathbb{E}(\xi^+_{1,i})\,L^S(\cdot) \,=\, 2\,\alpha_i^{(+)}L^S(\cdot)\,, \quad i=1,2\,, \tag{5.16}$$

i.e., property (2.16). The equations (2.12) and the properties of (2.13), (2.17) follow now from (2.4), (2.10), (5.5) and (5.11).

• The property (2.18) can be shown by an approximation similar in spirit and manner to that just carried out in the proof of (2.16). We take now throughout the "thinned" sequence $\boldsymbol{\xi}_k^A := \mathbf{1}_A(\arg(\boldsymbol{\xi}_k))$, $k\in\mathbb{N}_0$ in place of $\boldsymbol{\xi}_k$, $k\in\mathbb{N}_0$; and in lieu of $Z(\cdot)$ and $S(\cdot)$ in (5.2), respectively, the processes

$$Z^{(A)}(\cdot) := \sum_{k\in\mathbb{N}_0} \boldsymbol{\xi}_k^A\cdot \mathbf{1}_{\mathcal{C}_k}(\cdot) \qquad \text{and} \qquad R^A(\cdot) \,=\, \big\|X(\cdot)\big\|\cdot\mathbf{1}_A\big(\arg(X(\cdot)\big) \,=\, S(\cdot)\,Z^{(A)}(\cdot)\,.$$

• When the initial value $\mathrm{x} = (\mathrm{x}_1,\mathrm{x}_2)'$ is not the origin, we define

$$Z(0) := \mathfrak{f}(X(0)) = \mathfrak{f}(\mathrm{x}) \qquad \text{and} \qquad X(t) := Z(0)S(t)\,, \quad \text{for } 0\le t<\tau(0)\,,$$

i.e., until the process $X(\cdot)$ first attains the origin, very much in accordance with (3.1). Here $\tau(0)$ is defined as in Theorem 2.1(i). The so-constructed process $X(\cdot)$ satisfies the stochastic differential equation (3.1), to which (2.12) reduces on the interval $[0,\tau(0))$ as in the discussion at the start of section 3. On the interval $[\tau(0),\infty)$ we use the recipe (5.2) above, to construct $X(\cdot)$ starting from the origin.

With these considerations we obtain $\{\mathfrak{f}(X(s)) = \mathfrak{f}(\mathrm{x}),\ s < \tau(0)\} = \{s < \tau(0)\}$, mod. $\mathbb{P}$, and hence we verify (2.14). Moreover, for every $(s,t) \in (0,\infty)^2$, there exists by construction an $\widetilde{\mathcal{F}}(\infty)-$measurable random index $\kappa_0(s,t) : \Omega \to \mathbb{N}$ such that we have, either $\tau(s) + t \in \mathcal{C}_{\kappa_0(s,t)}$, or $\tau(s) + t \in \mathfrak{Z}$ on $\{\tau(s) < \infty\}$. If $\tau(s) + t \in \mathfrak{Z}$ and $\tau(s) < \infty$, then $\mathfrak{f}(X(\tau(s) + t)) = \mathbf{0}$. By (2.4) and the construction of $X(\cdot)$ we obtain $\mathbb{P}(\mathfrak{f}(X(\tau(s) + t)) = \mathbf{0}) = \mathbb{P}(S(\tau(s) + t) = 0) = 0$ for a.e. $t \in (0,\infty)$. Therefore,

$$\big\{\mathfrak{f}(X(\tau(s){+}t)) \in B,\ \tau(s) < \infty\big\} = \Big\{\sum_{k\in\mathbb{N}_0} \boldsymbol{\xi}_k\, \mathbf{1}_{\mathcal{C}_k}(\tau(s){+}t) \in B\,,\ \tau(s) < \infty\Big\} = \big\{\boldsymbol{\xi}_{\kappa_0(s,t)} \in B\,,\ \tau(s) < \infty\big\}$$

holds mod. $\mathbb{P}$ for every $B \in \mathcal{B}(\mathfrak{S})$ and almost every $t \in (0,\infty)$. We conclude that (2.15) holds, namely

$$\mathbb{P}\big(\,\mathfrak{f}(X(\tau(s) + t)) \in B \,\big|\, \mathcal{F}^X(\tau(s))\big) \;=\; \mathbb{P}\big(\boldsymbol{\xi}_{\kappa_0(s,t)} \in B \,\big|\, \mathcal{F}^X(\tau(s))\big)$$

$$=\; \mathbb{E}\Big[\mathbb{P}\big[\boldsymbol{\xi}_{\kappa_0(s,t)} \in B \,\big|\, \widetilde{\mathcal{F}}(\infty) \vee \mathcal{F}^Z(\tau(s))\big] \,\big|\, \mathcal{F}^X(\tau(s))\Big] \;=\; \mathbb{E}\big[\,\mathbb{P}(\boldsymbol{\xi}_1 \in B) \,\big|\, \mathcal{F}^X(\tau(s))\,\big] \;=\; \boldsymbol{\mu}(B)\,,$$

for every $s \in (0,\infty)$, $B \in \mathcal{B}(\mathfrak{S})$ and almost every $t \in (0,\infty)$. We have used here the definitions of $\mathcal{F}^X(\cdot) \subseteq \widetilde{\mathcal{F}}(\cdot) \vee \mathcal{F}^Z(\cdot)$ and $Z(\cdot)$ in (5.2), the $\widetilde{\mathcal{F}}(\infty)-$measurability of the stopping time $\tau(s)$ and of the random index $\kappa_0(s,t)$, as well as the independence between $\widetilde{\mathcal{F}}(\infty)$ and the sequence $\{\boldsymbol{\xi}_k\}_{k\in\mathbb{N}}$ of I.I.D. random variables.

This completes the proof of Theorem 2.1. □

Proof of Theorem 4.1: Let us fix a function $g : \mathbb{R}^2 \to \mathbb{R}$ in the class $\mathfrak{D}$ as in the statement of the theorem, and recall the notation established in Definitions 4.1, 4.2. Consider also a continuous planar semimartingale $X(\cdot)$ satisfying the equations of (2.12) along with the properties (2.13) and (2.18). With $\{\tau_k^\varepsilon\}_{k\in\mathbb{N}_0}$ defined as in (5.8), and with $\tau_{-1}^\varepsilon \equiv 0$ and $\mathbb{N}_{-1} := \mathbb{N}_0 \cup \{-1\}$, the value $g(X(T))$ is decomposed into

$$g(X(T)) = g(\mathrm{x}) + \sum_{\ell\in\mathbb{N}_{-1}} \big(g(X(T\wedge\tau^\varepsilon_{2\ell+2})) - g(X(T\wedge\tau^\varepsilon_{2\ell+1}))\big) + \sum_{\ell\in\mathbb{N}_0} \big(g(X(T\wedge\tau^\varepsilon_{2\ell+1})) - g(X(T\wedge\tau^\varepsilon_{2\ell}))\big)\,. \tag{5.17}$$

• We recall from the discussion at the beginning of section 3, that the process $X(\cdot)$ moves along the ray that connects $\mathbf{0}$ to the starting point $\mathrm{x} \neq \mathbf{0}$, during the time-interval $[0, \tau(0)) = [\tau^\varepsilon_{-1}, \tau^\varepsilon_0)$. In a similar manner, the processes $\mathfrak{f}_i(X(\cdot))$ are constant on every interval $[\tau^\varepsilon_{2\ell+1}, \tau^\varepsilon_{2\ell+2})$ for $\ell \in \mathbb{N}_0$, $i = 1, 2$.

The first summation in (5.17) can thus be rewritten as

$$\sum_{\ell\in\mathbb{N}_{-1}} \big(g(X(T\wedge\tau^\varepsilon_{2\ell+2})) - g(X(T\wedge\tau^\varepsilon_{2\ell+1}))\big) \;=\; \sum_{\ell\in\mathbb{N}_{-1}} \Big(g_\theta(S(T\wedge\tau^\varepsilon_{2\ell+2})) - g_\theta(S(T\wedge\tau^\varepsilon_{2\ell+1}))\Big)\Big|_{\theta\,=\,\Theta(T\wedge\tau^\varepsilon_{2\ell+1})}$$

$$= \sum_{\ell\in\mathbb{N}_{-1}} \int_{T\wedge\tau^\varepsilon_{2\ell+1}}^{T\wedge\tau^\varepsilon_{2\ell+2}} \Big(g'_\theta(S(t))\,\mathrm{d}S(t) + \frac{1}{2}\,g''_\theta(S(t))\,\mathrm{d}\langle S\rangle(t)\Big)\Big|_{\theta\,=\,\Theta(t)}$$

$$= \int_0^T \Big(\sum_{\ell\in\mathbb{N}_{-1}} \mathbf{1}_{(\tau^\varepsilon_{2\ell+1},\, \tau^\varepsilon_{2\ell+2})}(t)\Big)\Big(G'(X(t))\,\mathrm{d}S(t) + \frac{1}{2}\,G''(X(t))\,\mathrm{d}\langle S\rangle(t)\Big).$$

We have set here $\Theta(\cdot) := \arg(X(\cdot))$, and applied ITÔ's rule (Problem 3.7.3 in KARATZAS & SHREVE (1991)) to the process $g_\theta(S(\cdot))$. Letting $\varepsilon \downarrow 0$, we obtain in the limit

$$\int_0^T \mathbf{1}_{\{X(t)\neq\mathbf{0}\}}\Big(G'(X(t))\,\mathrm{d}S(t) + \frac{1}{2}\,G''(X(t))\,\mathrm{d}\langle S\rangle(t)\Big) = \int_0^T \mathbf{1}_{\{X(t)\neq\mathbf{0}\}}\Big(G'(X(t))\,\mathrm{d}U(t) + \frac{1}{2}\,G''(X(t))\,\mathrm{d}\langle U\rangle(t)\Big)\,. \tag{5.18}$$

• For the second summation in (5.17), we observe $g(\mathbf{0}) = g_\theta(0)$ by definition and hence

$$\sum_{\ell\in\mathbb{N}_0}\big(g(X(T\wedge\tau^\varepsilon_{2\ell+1}))-g(X(T\wedge\tau^\varepsilon_{2\ell}))\big) \tag{5.19}$$

$$=\sum_{\{\ell\,:\,\tau^\varepsilon_{2\ell+1}<T\}}\big(g_\theta(S(\tau^\varepsilon_{2\ell+1}))-g_\theta(0)\big)\Big|_{\theta=\Theta(\tau^\varepsilon_{2\ell+1})}+O(\varepsilon)\;=\sum_{\{\ell\,:\,\tau^\varepsilon_{2\ell+1}<T\}}\big(g_\theta(\varepsilon)-g_\theta(0)\big)\Big|_{\theta=\Theta(\tau^\varepsilon_{2\ell+1})}+O(\varepsilon)$$

$$=\sum_{\{\ell\,:\,\tau^\varepsilon_{2\ell+1}<T\}}\Big(\varepsilon g'_\theta(0+)+\int_0^\varepsilon(\varepsilon-u)g''_\theta(u)\mathrm{d}u\Big)\Big|_{\theta=\Theta(\tau^\varepsilon_{2\ell+1})}+O(\varepsilon)\;\underset{\varepsilon\downarrow 0}{\longrightarrow}\;L^S(T)\int_0^{2\pi}g'_\theta(0+)\boldsymbol{\nu}(\mathrm{d}\theta)$$

in probability. Indeed, by analogy with (5.12) we can verify

$$\Bigg|\sum_{\{\ell\,:\,\tau^\varepsilon_{2\ell+1}<T\}}\Big(\int_0^\varepsilon(\varepsilon-u)\,g''_\theta(u)\mathrm{d}u\Big)\Big|_{\theta=\Theta(\tau^\varepsilon_{2\ell+1})}\Bigg|\;\le\;c\sum_{\{\ell\,:\,\tau^\varepsilon_{2\ell+1}<T\}}\varepsilon^2\;=\;c\,\varepsilon\cdot\big(\varepsilon\,N(T,\varepsilon)+O(\varepsilon)\big)\;\underset{\varepsilon\downarrow 0}{\longrightarrow}\;0$$

in probability, where $c\;:=\;\sup_{\theta\in\text{supp}(\boldsymbol{\mu})}\max_{0\le u\le 1}\big(g''_\theta(u)\,/\,2\big)<+\infty$ by assumption.

• We also check that for every $A\in\mathcal{B}([0,2\pi))$ we have, on account of the property (2.18) for the process $R^A(\cdot)\;=\;\|X(\cdot)\|\,\mathbf{1}_A(\Theta(\cdot))$, the convergence

$$\sum_{\{\ell\,:\,\tau^\varepsilon_{2\ell+1}<T\}}\varepsilon\,\mathbf{1}_{\{\Theta(\tau^\varepsilon_{2\ell+1})\in A\}}\;=\sum_{\{\ell\,:\,\tau^\varepsilon_{2\ell+1}<T\}}S(\tau^\varepsilon_{2\ell+1})\,\mathbf{1}_{\{\Theta(\tau^\varepsilon_{2\ell+1})\in A\}}\;=\sum_{\{\ell\,:\,\tau^\varepsilon_{2\ell+1}<T\}}\|X(\tau^\varepsilon_{2\ell+1})\|\,\mathbf{1}_{\{\Theta(\tau^\varepsilon_{2\ell+1})\in A\}}$$

$$=\sum_{\{\ell\,:\,\widetilde{\tau}^\varepsilon_{2\ell+1}<T\}}R^A(\widetilde{\tau}^\varepsilon_{2\ell+1})\;=\;\varepsilon\,\widetilde{N}(T,\varepsilon)+O(\varepsilon)\;\underset{\varepsilon\downarrow 0}{\longrightarrow}\;L^{R^A}(T)\;=\;\boldsymbol{\nu}(A)\,L^S(T)$$

in probability. Here we define $\widetilde{\tau}^\varepsilon_0\;:=\;\inf\big\{t\ge 0\;:\;R^A(t)\;=\;0\big\}$, and recursively

$$\widetilde{\tau}^\varepsilon_{2\ell+1}\;:=\;\inf\big\{t>\widetilde{\tau}^\varepsilon_{2\ell}\;:\;R^A(t)\ge\varepsilon\big\}\,,\quad\widetilde{\tau}^\varepsilon_{2\ell+2}\;:=\;\inf\big\{t>\widetilde{\tau}^\varepsilon_{2\ell+1}\;:\;R^A(t)\;=\;0\big\}$$

for $\ell\in\mathbb{N}_0$, and denote by $\widetilde{N}(\varepsilon,T)$ the number of downcrossings of the interval $(0,\varepsilon)$ that the process $R^A(\cdot)$ has completed during the interval $[0,T)$ (please note that we count here the number of downcrossings corresponding to the rays in the directions in the subset A of $[0,2\pi)$). Thus, approximating the function $\theta\mapsto g'_\theta(0+)$ by indicators $\theta\mapsto\mathbf{1}_A(\theta)$, $A\in\mathcal{B}([0,2\pi))$, we verify the convergence in probability

$$\sum_{\{\ell\,:\,\tau^\varepsilon_{2\ell+1}<T\}}\varepsilon\,g'_{\Theta(\tau^\varepsilon_{2\ell+1})}(0+)\;\underset{\varepsilon\downarrow 0}{\longrightarrow}\;L^S(T)\int_0^{2\pi}g'_\theta(0+)\boldsymbol{\nu}(\mathrm{d}\theta)\,.$$

• Therefore, the limit of the expression in (5.17) is the sum of the limits of the expressions in (5.18) and (5.19), and we conclude

$$g(X(T))=g(\mathrm{x})+\Big(\int_0^{2\pi}g'_\theta(0+)\boldsymbol{\nu}(\mathrm{d}\theta)\Big)L^S(T)+\int_0^T\mathbf{1}_{\{X(t)\neq\mathbf{0}\}}\Big(G'(X(t))\mathrm{d}S(t)+\frac{1}{2}G''(X(t))\mathrm{d}\langle S\rangle(t)\Big).$$

This establishes (4.6), and completes the proof of the first claim in Theorem 4.1; the second and third claims follow then in a fairly direct manner. □

Proof of Proposition 3.1: (i) Assume $\gamma_1^2+\gamma_2^2\le 1$, and consider the vector $\boldsymbol{\gamma}:=(\gamma_1,\gamma_2)'\in\mathbb{R}^2$. Then we define the probability measure $\boldsymbol{\mu}:=\big((1+\beta)/2\big)\,\delta_{z_0}+\big((1-\beta)/2\big)\,\delta_{-z_0}$ on $(\mathfrak{S},\mathcal{B}(\mathfrak{S}))$ with $\beta:=\|\boldsymbol{\gamma}\|\le 1$ and $z_0:=\boldsymbol{\gamma}/\beta\in\mathfrak{S}$ provided that $\beta\neq 0$ (if $\beta=0$, we simply pick up an arbitrary $z_0\in\mathfrak{S}$), and note

$$\int_{\mathfrak{S}}\mathfrak{f}(z)\,\boldsymbol{\mu}(\mathrm{d}z)\;=\;\int_{\mathfrak{S}}z\,\boldsymbol{\mu}(\mathrm{d}z)\;=\;\frac{1+\beta}{2}\,z_0+\frac{1-\beta}{2}\,(-z_0)\;=\;\beta z_0\;=\;\boldsymbol{\gamma}\,.$$

Thus, if we take the process $S(\cdot)$ in section 2 as the “folded driver” and $\boldsymbol{\mu}$ as the “spinning measure”, Theorem 2.1 constructs a continuous planar semimartingale $X(\cdot)=(X_1(\cdot),X_2(\cdot))'$ that satisfies the condition (2.13) and the system of equations (2.12) – thus also the system (3.4).

(ii) Suppose now that (3.6) holds, and that there exists a continuous semimartingale $X(\cdot) = (X_1(\cdot), X_2(\cdot))'$ that satisfies (2.13) and the system of equations (3.4), thus also of (2.12). For every $\varepsilon > 0$, we define $\tau_{-1}^\varepsilon \equiv 0$ and $\{\tau_m^\varepsilon\}_{m \in \mathbb{N}_0}$ as in (5.8). Following the idea in the proof of Theorem 4.1, we write

$$X(T) = \mathrm{x} + \sum_{\ell \in \mathbb{N}_{-1}} \big(X(T \wedge \tau_{2\ell+2}^\varepsilon) - X(T \wedge \tau_{2\ell+1}^\varepsilon) \big) + \sum_{\ell \in \mathbb{N}_0} \big(X(T \wedge \tau_{2\ell+1}^\varepsilon) - X(T \wedge \tau_{2\ell}^\varepsilon) \big) \,.$$

Then as $\varepsilon \downarrow 0$, on account of (2.12) and in the same manner as in the proof of Theorem 4.1, the first summation in the above expression converges in probability to $\int_0^T \mathfrak{f}\big(X(t)\big)\, \mathrm{d}U(t)$. Thus, the second summation converges in probability to $\boldsymbol{\gamma}\, L^{\|X\|}(T)$, thanks to (3.4). This implies the convergence in probability

$$\sum_{\ell \in \mathbb{N}_0} \big(X(T \wedge \tau_{2\ell+1}^\varepsilon) - X(T \wedge \tau_{2\ell}^\varepsilon) \big) \;=\; \sum_{\ell=0}^{N(T,\varepsilon)-1} \varepsilon\, \mathfrak{f}(X(\tau_{2\ell+1}^\varepsilon)) + O(\varepsilon) \;\underset{\varepsilon \downarrow 0}{\longrightarrow}\; \boldsymbol{\gamma} L^{\|X\|}(T) \,.$$

We also have the convergence in probability $\varepsilon\, N(T, \varepsilon) \to L^{\|X\|}(T)$ as $\varepsilon \downarrow 0$ by Theorem VI.1.10 in REVUZ & YOR (1999), where $N(T, \varepsilon) := \sharp \big\{ \ell \in \mathbb{N} : \tau_{2\ell}^\varepsilon < T \big\}$. Therefore, on the event $\big\{ L^{\|X\|}(T) > 0 \big\}$ for some $T \in (0, \infty)$ sufficiently large (such a T can indeed be selected, by (3.6) and (2.13)), we have

$$\frac{1}{N(T,\varepsilon)} \sum_{\ell=0}^{N(T,\varepsilon)-1} \mathfrak{f}\big(X(\tau_{2\ell+1}^\varepsilon)\big) \;\underset{\varepsilon \downarrow 0}{\longrightarrow}\; \boldsymbol{\gamma} \qquad \text{in probability.}$$

Now $\|\boldsymbol{\gamma}\| \le 1$ follows from $\|\mathfrak{f}(\cdot)\| \le 1$. □

6 Connection to Martingale Problems

We cannot expect pathwise uniqueness, therefore neither can we expect strength, to hold for the equations of (2.12) or (3.5). Any such lingering hope is dashed by the realization that, when $U(\cdot)$ is standard Brownian motion, thus $S(\cdot)$ a reflecting Brownian motion, the process $X(\cdot)$ constructed in Theorem 2.1 is the WALSH Brownian motion – a process whose filtration cannot be generated by *any* Brownian motion of *any* dimension. For this result see Proposition 7.2 below and the celebrated paper by TSIREL'SON (1997), as well as MANSUY & YOR (2006), pages 103-116.

In light of these observations, it is natural to ask whether the next best thing, that is, *uniqueness in distribution*, might hold for these equations under appropriate conditions. We try in this section to provide some affirmative answers to this question, when the folded driving semimartingale $S(\cdot)$ is a reflected diffusion; the main results appear in Proposition 6.2 and Corollary 6.1.

6.1 The Folded Driving Semimartingale as a Reflected Diffusion

Let us start by considering the canonical space $\Omega_1 := C([0,\infty);[0,\infty))$ of nonnegative, continuous functions on $[0,\infty)$. We endow this space with the usual topology of uniform convergence over compact intervals and with the $\sigma-$algebra $\mathcal{F}_1 := \mathcal{B}(\Omega_1)$ of its BOREL sets. We consider also the filtration $\mathbb{F}_1 := \{\mathcal{F}_1(t)\}_{0 \le t < \infty}$ generated by its coördinate mapping, i.e., $\mathcal{F}_1(t) = \sigma\big(\omega_1(s),\ 0 \le s \le t\big)$.

Given BOREL-measurable coëfficients $\boldsymbol{b} : [0,\infty) \to \mathbb{R}$ and $\boldsymbol{\sigma} : [0,\infty) \to \mathbb{R} \setminus \{0\}$ and setting $\boldsymbol{a}(\cdot) := \boldsymbol{\sigma}^2(\cdot)$, we define the process

$$K^\psi(\cdot\,;\omega_1) \;:=\; \psi(\omega_1(\cdot)) - \psi(\omega_1(0)) - \int_0^\cdot \mathcal{G}\psi(\omega_1(t)) \cdot \mathbf{1}_{\{\,\omega_1(t)>0\}}\, \mathrm{d}t \,, \tag{6.1}$$

where

$$\mathcal{G}\psi(r) \;:=\; \boldsymbol{b}(r)\, \psi'(r) + \frac{1}{2}\, \boldsymbol{a}(r)\, \psi''(r)\,; \quad r \in [0,\infty)\,, \quad \psi \in C_0^2\big([0,\infty);\mathbb{R}\big)\,.$$

6.1.1 Local Submartingale Problem for a Reflected Diffusion

In the manner of STROOCK & VARADHAN (1971), we formulate the **Local Submartingale Problem associated with the pair $(\boldsymbol{\sigma}, \boldsymbol{b})$** as follows.

For every given $x \in [0, \infty)$, to find a probability measure $\mathbb{Q}^\bullet$ on the space $(\Omega_1, \mathcal{F}_1)$, under which:
(i) $\omega_1(0) = x$ and $\int_0^\infty \mathbf{1}_{\{\omega_1(t)=0\}} \, \mathrm{d}t = 0$ hold $\mathbb{Q}^\bullet-$ a.e.; and moreover,
(ii) for every function $\psi \in C^2([0,\infty); \mathbb{R})$ with $\psi'(0+) \geq 0$, the process $K^\psi(\cdot)$ is a continuous local submartingale with respect to the filtration

$$\mathbb{F}_1^\bullet = \{\mathcal{F}_1^\bullet(t)\}_{0 \leq t < \infty} \qquad \textit{with} \qquad \mathcal{F}_1^\bullet(t) := \mathcal{F}_1^\circ(t+),$$

and is a continuous $\mathbb{F}_1^\bullet-$local-martingale whenever $\psi'(0+) = 0$.

Here we have denoted by $\mathbb{F}_1^\circ := \{\mathcal{F}_1^\circ(t), \, 0 \leq t < \infty\}$ the augmentation of $\mathbb{F}_1$ under $\mathbb{Q}^\bullet$. As usual, we shall say that this problem is *well-posed,* if it admits exactly one solution. For a recent study of the well-posedness of submartingale problems for obliquely reflected diffusions, in domains with piecewise smooth boundaries, see KANG & RAMANAN (2014).

6.2 A Local Martingale Problem for the Planar Diffusion

Next, we consider the canonical space $\Omega_2 := C([0,\infty); \mathbb{R}^2)$ of $\mathbb{R}^2-$valued continuous functions on $[0, \infty)$ endowed with the $\sigma-$algebra $\mathcal{F}_2 := \mathcal{B}(\Omega_2)$ of its BOREL sets. We consider also its coördinate mapping and the natural filtration $\mathbb{F}_2 := \{\mathcal{F}_2(t)\}_{0 \leq t < \infty}$ with $\mathcal{F}_2(t) = \sigma\big(\omega_2(s), \ 0 \leq s \leq t\big)$. We recall, here and in what follows, the Definitions 4.1 and 4.2.

Given a probability measure $\boldsymbol{\mu}$ on the BOREL subsets of the unit circumference $\mathfrak{S}$, and BOREL-measurable functions $\boldsymbol{b} : [0,\infty) \to \mathbb{R}$, $\boldsymbol{\sigma} : [0,\infty) \to \mathbb{R} \setminus \{0\}$ as in subsection 6.1, we define for every function $g \in \mathfrak{D}$ the process

$$M^g(\cdot\,; \omega_2) := g(\omega_2(\cdot)) - g(\omega_2(0)) - \int_0^{\cdot} \mathcal{L}\, g(\omega_2(t)) \cdot \mathbf{1}_{\{\, \|\omega_2(t)\| > 0\}} \, \mathrm{d}t, \tag{6.2}$$

where

$$\mathcal{L}\, g(x) := \boldsymbol{b}(\|x\|)\, G'(x) + \frac{1}{2}\, \boldsymbol{a}(\|x\|)\, G''(x); \qquad x \in \mathbb{R}^2.$$

6.2.1 The Local Martingale Problem

Motivated by the generalized FREIDLIN-SHEU formula (4.6) in Theorem 4.1, we formulate now the **Local Martingale Problem associated with the triple $(\boldsymbol{\sigma}, \boldsymbol{b}, \boldsymbol{\mu})$** as follows.

For every fixed $\mathrm{x} \in \mathbb{R}^2$, to find a probability measure $\mathbb{Q}$ on the canonical space $(\Omega_2, \mathcal{F}_2)$, such that:
(i) $\omega_2(0) = \mathrm{x}$ holds $\mathbb{Q}-$ a.e.;
(ii) the analogue of the property (2.17) holds, namely

$$\int_0^\infty \mathbf{1}_{\{\omega_2(t) = \mathbf{0}\}} \, \mathrm{d}t = 0, \qquad \mathbb{Q} - a.e.; \tag{6.3}$$

(iii) for every function g in $\mathfrak{D}_+^{\boldsymbol{\mu}}$ (respectively, $\mathfrak{D}^{\boldsymbol{\mu}}$) as in (4.5), the process $M_2^g(\cdot\,; \omega_2)$ of (6.2) is a continuous local submartingale (resp., martingale) with respect to the filtration $\mathbb{F}_2^\bullet := \{\mathcal{F}_2^\bullet(t)\}_{0 \leq t < \infty}$.

Here we have set $\mathcal{F}_2^\bullet(t) := \mathcal{F}_2^\circ(t+)$, and have denoted by $\mathbb{F}_2^\circ = \{\mathcal{F}_2^\circ(t)\}_{0 \leq t < \infty}$ the $\mathbb{Q}-$augmentation of the filtration $\mathbb{F}_2$. Again, this problem is called "well-posed" if it admits exactly one solution.

• The theory of the STROOCK & VARADHAN martingale problem is extended in Proposition 6.1 below, for a continuous planar semimartingale $X(\cdot)$ that satisfies the properties (2.17), (2.18) and, with coëfficients γ_i, $i = 1, 2$ given through (2.11), (2.10), the system of stochastic equations

$$X_i(\cdot) \;=\; X_i(0) + \int_0^{\cdot} \mathfrak{f}_i(X(t)) \Big[\boldsymbol{b}\big(\|X(t)\|\big)\mathrm{d}t \;+\; \boldsymbol{\sigma}\big(\|X(t)\|\big)\mathrm{d}W(t) \Big] + \gamma_i \, L^{\|X\|}(\cdot)\,. \tag{6.4}$$

Proposition 6.1. *(a) For every weak solution* (X, W), $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{F} = \{\mathcal{F}(t)\}_{0 \le t < \infty}$ *to the system of stochastic equations (6.4), we have*

$$\|X(\cdot)\| \;=\; \|X(0)\| + \int_0^{\cdot} \mathbf{1}_{\{\|X(t)\|>0\}} \Big(\boldsymbol{b}(\|X(t)\|)\mathrm{d}t + \boldsymbol{\sigma}(\|X(t)\|)\mathrm{d}W(t) \Big) + L^{\|X\|}(\cdot)\,; \tag{6.5}$$

and if this weak solution also satisfies (2.17)-(2.18), then it induces a solution to the local martingale problem associated with the triple $(\boldsymbol{\sigma}, \boldsymbol{b}, \boldsymbol{\mu})$ *.*

(b) Conversely, every solution to the local martingale problem associated with the triple $(\boldsymbol{\sigma}, \boldsymbol{b}, \boldsymbol{\mu})$ *induces a weak solution to the system (6.4) which satisfies the properties (2.17), (2.18). The state process* $X(\cdot)$ *in this weak solution solves also the system of stochastic equations (2.12) with "folded driver"* $S(\cdot) = \|X(\cdot)\|$ *.*

(c) Uniqueness holds for the local martingale problem associated with $(\boldsymbol{\sigma}, \boldsymbol{b}, \boldsymbol{\mu})$*, if and only if uniqueness in distribution holds for the system of (6.4) subject to the conditions (2.17), (2.18).*

Proof of Part (a): We first validate (6.5) for any weak solution to (6.4). From (6.4) we see

$$\int_0^T \big(|\mathfrak{f}_i(X(t))\boldsymbol{b}(\|X(t)\|)| + \mathfrak{f}_i^2(X(t))\boldsymbol{a}(\|X(t)\|) \big)\mathrm{d}t < \infty\,, \quad i = 1, 2\,, \quad 0 \le T < \infty\,.$$

Since $\mathfrak{f}_1^2(\mathrm{x}) + \mathfrak{f}_2^2(\mathrm{x}) = 1$ and $|\mathfrak{f}_1(\mathrm{x})| + |\mathfrak{f}_2(\mathrm{x})| \ge 1$ hold for any $\mathrm{x} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$, we obtain then

$$\int_0^T \mathbf{1}_{\{||X(t)||>0\}} \big(|\boldsymbol{b}(\|X(t)\|)| + \boldsymbol{a}(\|X(t)\|) \big)\mathrm{d}t < \infty\,, \qquad 0 \le T < \infty\,. \tag{6.6}$$

Let us recall the stopping time $\sigma_\varepsilon = \inf\{t > 0 : \|X(t)\| \le \varepsilon\}$ for every $\varepsilon > 0$. Since the function $x \mapsto \|x\| = \sqrt{x_1^2 + x_2^2}$ is smooth on $\mathbb{R}^2 \setminus \{\mathbf{0}\}$, we get the following from (6.4) by ITÔ's formula:

$$\|X(\cdot \wedge \sigma_\varepsilon)\| \;= \|X(0)\| + \int_0^{\cdot \wedge \sigma_\varepsilon} \big(\boldsymbol{b}(\|X(t)\|)\mathrm{d}t + \boldsymbol{\sigma}(\|X(t)\|)\mathrm{d}W(t) \big)\,. \tag{6.7}$$

With $\tau(0) := \inf\{t \ge 0 : \|X(t)\| = 0\} = \lim_{\varepsilon \downarrow 0} \sigma_\varepsilon$, we let $\varepsilon \downarrow 0$ in (6.7) and obtain (from (6.6))

$$\|X(\cdot \wedge \tau(0))\| \;= \|X(0)\| + \int_0^{\cdot} \mathbf{1}_{(0,\,\tau(0))}(t) \, \big(\boldsymbol{b}(\|X(t)\|)\mathrm{d}t + \boldsymbol{\sigma}(\|X(t)\|)\mathrm{d}W(t) \big)\,.$$

Recall now the stopping times $\big\{\tau_m^\varepsilon\,,\, m \in \mathbb{N}_0\big\}$ defined in (5.8). In the same manner as above we obtain

$$\|X(\cdot \wedge \tau_{2\ell+2}^\varepsilon)\| - \|X(\cdot \wedge \tau_{2\ell+1}^\varepsilon)\| = \int_0^{\cdot} \mathbf{1}_{(\tau_{2\ell+1}^\varepsilon,\, \tau_{2\ell+2}^\varepsilon)}(t) \, \big(\boldsymbol{b}(\|X(t)\|)\mathrm{d}t + \boldsymbol{\sigma}(\|X(t)\|)\mathrm{d}W(t) \big)$$

for $\ell \in \mathbb{N}_{-1} = \mathbb{N}_0 \cup \{-1\}$ with $\tau_{-1}^\varepsilon \equiv 0$. We decompose $\|X(T)\|$ as in the proof of Theorem 4.1:

$$\|X(T)\| = \|X(0)\| + \sum_{\ell \in \mathbb{N}_{-1}} \big(\|X(T \wedge \tau_{2\ell+2}^\varepsilon)\| - \|X(T \wedge \tau_{2\ell+1}^\varepsilon)\| \big) + \sum_{\ell \in \mathbb{N}_0} \big(\|X(T \wedge \tau_{2\ell+1}^\varepsilon)\| - \|X(T \wedge \tau_{2\ell}^\varepsilon)\| \big)\,. \tag{6.8}$$

With the previous considerations, we can write the first summation of (6.8) as

$$\sum_{\ell \in \mathbb{N}_{-1}} \big(\|X(T \wedge \tau_{2\ell+2}^\varepsilon)\| - \|X(T \wedge \tau_{2\ell+1}^\varepsilon)\| \big) = \sum_{\ell \in \mathbb{N}_{-1}} \int_0^T \mathbf{1}_{(\tau_{2\ell+1}^\varepsilon,\, \tau_{2\ell+2}^\varepsilon)}(t)\big) \big(\boldsymbol{b}(\|X(t)\|)\mathrm{d}t + \boldsymbol{\sigma}(\|X(t)\|)\mathrm{d}W(t). \tag{6.9}$$

As $\varepsilon \downarrow 0$, the right-hand side of (6.9) converges to $\int_0^T \mathbf{1}_{\{\|X(t)\|>0\}} \big(\boldsymbol{b}(\|X(t)\|)\mathrm{d}t + \boldsymbol{\sigma}(\|X(t)\|)\mathrm{d}W(t) \big)$ in probability, thanks to (6.6). On the other hand, with $N(T, \varepsilon) \;:=\; \sharp\big\{\ell \in \mathbb{N} : \tau_{2\ell}^\varepsilon < T\big\}$ we have for the second summation of (6.8) the convergence

$$\sum_{\ell \in \mathbb{N}_0} \big(\| X(T \wedge \tau^\varepsilon_{2\ell+1}) \| - \| X(T \wedge \tau^\varepsilon_{2\ell}) \| \big) \,=\, \varepsilon\, N(T, \varepsilon) \,+ O(\varepsilon) \,\underset{\varepsilon \downarrow 0}{\longrightarrow}\, L^{\|X\|}(T)$$

in probability, by Theorem VI.1.10 in REVUZ & YOR (1999). Therefore, letting $\varepsilon \downarrow 0$ in (6.8), we obtain the equation (6.5) for the radial process $\| X(\cdot) \|$. The continuous semimartingale $X(\cdot)$ thus solves also the system (2.12) with the "folded driver" $S(\cdot) = \| X(\cdot) \|$.

Suppose now that the properties (2.17)-(2.18) are also satisfied by the weak solution we have posited. Thanks to Theorem 4.1, for every given function $g \in \mathfrak{D}^{\boldsymbol{\mu}}_+$ (resp., $g \in \mathfrak{D}^{\boldsymbol{\mu}}$), the process $M^g(\cdot\,; X)$ as in (6.2) is then a local submartingale (resp., martingale). The property (6.3) comes from (2.17). Consequently, a solution $\mathbb{Q}$ to the local martingale problem associated with the triple $(\boldsymbol{\sigma}, \boldsymbol{b}, \boldsymbol{\mu})$ is given by the probability measure $\mathbb{Q} = \mathbb{P} X^{-1}$ induced by the process $X(\cdot)$ on the canonical space $(\Omega_2, \mathcal{F}_2)$. □

Proof of Part (b): Conversely, suppose that the local martingale problem associated with the triple $(\boldsymbol{\sigma}, \boldsymbol{b}, \boldsymbol{\mu})$ has a solution $\mathbb{Q}$. We recall the notation in (2.11) and define on the canonical space the processes

$$X(\cdot) \,\equiv\, \big(X_1(\cdot), X_2(\cdot) \big)' \,:=\, \big(\, \|\omega_2(\cdot)\| \, \mathfrak{f}_1(\omega_2(\cdot)) \,,\; \|\omega_2(\cdot)\| \, \mathfrak{f}_2(\omega_2(\cdot)) \, \big)', \tag{6.10}$$

$$M_i(\cdot) \,:=\, X_i(\cdot) - X_i(0) - \int_0^\cdot \boldsymbol{b}(\|X(t)\|) \mathfrak{f}_i(X(t)) \mathrm{d}t - \gamma_i \Big(\|X(\cdot)\| - \|X(0)\| - \int_0^\cdot \boldsymbol{b}(\|X(t)\|) \, \mathbf{1}_{\{\|X(t)\|>0\}} \mathrm{d}t \Big), \tag{6.11}$$

$$M_{i,k}(\cdot) \,:=\, g_{i,k}(X(\cdot)) - g_{i,k}(X(0)) - 2 \int_0^\cdot \|X(t)\| \, \boldsymbol{b}(\|X(t)\|) \, \big(\mathfrak{f}_i(X(t)) - \gamma_i \big) \big(\mathfrak{f}_k(X(t)) - \gamma_k \big) \, \mathbf{1}_{\{\|X(t)\|>0\}} \mathrm{d}t$$
$$- \int_0^\cdot \boldsymbol{a}(\|X(t)\|) \big(\mathfrak{f}_i(X(t)) - \gamma_i \big) \big(\mathfrak{f}_k(X(t)) - \gamma_k \big) \, \mathbf{1}_{\{\|X(t)\|>0\}} \mathrm{d}t$$

for $1 \le i, k \le 2$, as well as

$$M^\circ_{i,i}(\cdot) \,:=\, X_i^2(\cdot) - X_i^2(0) - \int_0^\cdot \mathfrak{f}_i^2(X(t)) \Big[2 \, \|X(t)\| \, \boldsymbol{b}(\|X(t)\|) + \boldsymbol{a}(\|X(t)\|) \Big] \, \mathrm{d}t \,. \tag{6.12}$$

- Here, as in Proposition 4.2, we consider the following functions in the family $\mathfrak{D}^{\boldsymbol{\mu}}$ of (4.5):

$$g_1(x) \,:=\, r \big(\cos(\theta) - \gamma_1 \big) \,, \quad g_2(x) \,:=\, r \big(\sin(\theta) - \gamma_2 \big) \,, \quad g_{i,k}(x) \,:=\, g_i(x) \, g_k(x) \tag{6.13}$$

for $x = (r, \theta) \in \mathbb{R}^2$, $1 \le i, k \le 2$. We consider also the functions $g^\circ_{i,i} \in \mathfrak{D}^{\boldsymbol{\mu}}$ and $g_3 \in \mathfrak{D}^{\boldsymbol{\mu}}_+$ defined by

$$g^\circ_{1,1}(x) \,:=\, r^2 \cos^2(\theta) \,, \quad g^\circ_{2,2}(x) \,:=\, r^2 \sin^2(\theta) \,, \quad g_3(x) := r \,; \qquad x \in \mathbb{R}^2 \,. \tag{6.14}$$

- We deduce then from (6.2) that the processes

$$M_i(\cdot) \equiv M^{g_i}(\cdot\,; X) \,, \quad M_{i,k}(\cdot) \equiv M^{g_{i,k}}(\cdot\,; X) \qquad \text{as well as} \qquad M^\circ_{i,i}(\cdot) \equiv M^{g^\circ_{i,i}}(\cdot\,; X) \,,$$

are continuous local martingales for $1 \le i, k \le 2$; and that so are the processes

$$M_{i,k}(\cdot) - g_i(X(0)) M_k(\cdot) - g_k(X(0)) M_i(\cdot) - \int_0^\cdot \Big(\int_0^t \big(\mathfrak{f}_k(X(s)) - \gamma_k \big) \boldsymbol{b}(\|X(s)\|) \, \mathbf{1}_{\{\|X(s)\|>0\}} \mathrm{d}s \Big) \mathrm{d}M_i(t)$$
$$- \int_0^\cdot \Big(\int_0^t \big(\mathfrak{f}_i(X(s)) - \gamma_i \big) \boldsymbol{b}(\|X(s)\|) \, \mathbf{1}_{\{\|X(s)\|>0\}} \mathrm{d}s \Big) \mathrm{d}M_k(t) \,=\, M_i(\cdot) M_k(\cdot) - \int_0^\cdot r_{i,k}(t) \, \mathrm{d}t \,.$$

This way, we identify for $1 \le i, k \le 2$ the cross-variation structure

$$\langle M_i, M_k \rangle(\cdot) = \int_0^\cdot r_{i,k}(t) \mathrm{d}t \,, \quad r_{i,k}(t) := \boldsymbol{a}(\|X(t)\|) \, \big(\mathfrak{f}_i(X(t)) - \gamma_i \big) \big(\mathfrak{f}_k(X(t)) - \gamma_k \big) \, \mathbf{1}_{\{\|X(t)\|>0\}} \,. \tag{6.15}$$

- We also observe that the continuous process

$$N(\cdot) \,:=\, M^{g_3}(\cdot\,; X) \,=\, \|X(\cdot)\| - \|X(0)\| - \int_0^\cdot \boldsymbol{b}(\|X(t)\|) \, \mathbf{1}_{\{\|X(t)\|>0\}} \, \mathrm{d}t \tag{6.16}$$

is a local submartingale; this way we obtain the semimartingale property of the radial process $||X(\cdot)||$.

By the DOOB-MEYER decomposition (e.g., KARATZAS & SHREVE (1991), Theorem 1.4.10), there exists then an adapted, continuous and increasing process $A(\cdot)$ such that

$$M_3(\cdot) \,:=\, N(\cdot) - A(\cdot) \,=\, \|X(\cdot)\| - \|X(0)\| - \int_0^{\cdot} \boldsymbol{b}(\|X(t)\|)\, \mathbf{1}_{\{\|X(t)\|>0\}}\, \mathrm{d}t - A(\cdot) \tag{6.17}$$

is a continuous local martingale. *We claim that this increasing process is* $A(\cdot) = L^{||X||}(\cdot)$*, the local time at the origin of the continuous, nonnegative semimartingale* $||X(\cdot)||$.

• In order to substantiate this claim, let us fix two arbitrary positive constants c_1, c_2 with $c_1 < c_2$ and define a sequence of stopping times inductively, via $\sigma_0 := \inf\{t \ge 0 : \|X(t)\| = c_2\}$ if $\|X(0)\| < c_2$ and $\sigma_0 := 0$ otherwise; as well as

$$\sigma_{2n+1} \,:=\, \inf\{t \ge \sigma_{2n} : \|X(t)\| \,=\, c_1\}\,, \quad \sigma_{2n+2} \,:=\, \inf\{t \ge \sigma_{2n+1} : \|X(t)\| \,=\, c_2\}\,; \quad n \in \mathbb{N}_0\,.$$

We note that $\|X(t)\| \ge c_1$ holds for $t \in (\sigma_{2n}, \sigma_{2n+1})$; and conversely, that $\|X(t)\| > c_2$ implies $t \in (\sigma_{2n}, \sigma_{2n+1})$ for some $n \in \mathbb{N}_0$. Thus, by taking an appropriate smooth function $g_4 \in \mathfrak{D}^{\boldsymbol{\mu}}$ of the form $g_4(r,\theta) = \psi(r)$ where $\psi : [0,\infty) \to [0,\infty)$ is smooth with $\psi(r) = r$ for $r \ge c_1$, one can show that $N(\cdot \wedge \sigma_{2n+1}) - N(\cdot \wedge \sigma_{2n})$ is a continuous local martingale.

Then, since both processes $N(\cdot \wedge \sigma_{2n+1}) - A(\cdot \wedge \sigma_{2n+1})$ and $N(\cdot \wedge \sigma_{2n}) - A(\cdot \wedge \sigma_{2n})$ are continuous local martingales, so is $A(\cdot \wedge \sigma_{2n+1}) - A(\cdot \wedge \sigma_{2n})$. But this last process is of bounded variation, so $A(\cdot \wedge \sigma_{2n+1}) \equiv A(\cdot \wedge \sigma_{2n})$ for every $n \in \mathbb{N}_0$. In other words, the process $A(\cdot)$ is flat on $[\sigma_{2n}, \sigma_{2n+1}]$ for every $n.$ Therefore we have $\int_0^\infty \mathbf{1}_{\{\|X(t)\| \in (c_2,\infty)\}}\, \mathrm{d}A(t) \,\equiv\, 0$, because $\|X(t)\| \in (c_2,\infty)$ implies $t \in (\sigma_{2n}, \sigma_{2n+1})$ for some $n \in \mathbb{N}_0$. Since $c_2 > 0$ can be chosen arbitrarily small, we obtain

$$A(\cdot) \,=\, \int_0^{\cdot} \mathbf{1}_{\{\|X(t)\| = 0\}}\, \mathrm{d}A(t)\,, \quad \text{and} \quad \int_0^{\cdot} \|X(t)\| \mathrm{d}A(t) \,=\, 0\,. \tag{6.18}$$

In conjunction with (6.16)–(6.18), the characterization $L^{\|X\|}(\cdot) = \int_0^{\cdot} \mathbf{1}_{\{\|X(t)\|=0\}} \mathrm{d}\|X(t)\|$ for the local time of a continuous, nonnegative semimartingale such as $||X(\cdot)||$, establishes then the claim, since

$$L^{||X||}(\cdot) \,=\, \int_0^{\cdot} \mathbf{1}_{\{\|X(t)\|=0\}} \Big(\boldsymbol{b}(\|X(t)\|)\, \mathbf{1}_{\{\|X(t)\|>0\}}\, \mathrm{d}t + \mathrm{d}A(t) \Big) \,=\, A(\cdot)\,.$$

• We return now to the computation of the cross-variations $\langle M_i, M_k \rangle(\cdot)$ for $1 \le i,k \le 3$. Recalling (6.17), (6.18) and $\langle \|X\| \rangle(\cdot) = \langle M_3 \rangle(\cdot) = \langle N \rangle(\cdot)$, an application of ITÔ's rule to $\|X(\cdot)\|^2$ gives

$$\|X(\cdot)\|^2 - \|X(0)\|^2 - 2\int_0^{\cdot} \|X(t)\| \boldsymbol{b}(\|X(t)\|)\, \mathbf{1}_{\{\|X(t)\|>0\}} \mathrm{d}t - \langle N \rangle(\cdot) \,=\, 2\int_0^{\cdot} \|X(t)\|\, \mathrm{d}M_3(t)\,.$$

Combining the last identity with (6.12), we observe that

$$M^\circ_{1,1}(\cdot) + M^\circ_{2,2}(\cdot) - 2\int_0^{\cdot} \|X(t)\| \mathrm{d}M_3(t) \,=\, \langle N \rangle(\cdot) - \int_0^{\cdot} \boldsymbol{a}(\|X(t)\|)\, \mathbf{1}_{\{\|X(t)\|>0\}}\, \mathrm{d}t \tag{6.19}$$

is both a local martingale and a continuous process of bounded variation; thus we identify

$$\langle \|X\| \rangle(\cdot) = \langle N \rangle(\cdot) = \langle M_3 \rangle(\cdot) = \int_0^{\cdot} r_{3,3}(t) \mathrm{d}t \quad \text{where} \quad r_{3,3}(t) := \boldsymbol{a}(\|X(t)\|)\, \mathbf{1}_{\{\|X(t)\|>0\}}\,. \tag{6.20}$$

By analogy with the derivation of (6.19), and taking (6.11) into account, we observe that

$$M^\circ_{i,i}(\cdot) - 2\int_0^{\cdot} X_i(t) \mathrm{d}M_i(t) - 2\,\gamma_i \int_0^{\cdot} X_i(t) \mathrm{d}M_3(t) \,=\, \langle X_i \rangle(\cdot) - \int_0^{\cdot} \boldsymbol{a}(\|X(t)\|)\, \big[\mathfrak{f}_i(X(t))\big]^2 \mathrm{d}t$$

is both a local martingale and a continuous process of bounded variation for $i = 1,2$; thus we identify

$$\langle X_i \rangle(\cdot) \;=\; \int_0^{\cdot} \boldsymbol{a}(\|X(t)\|)\,\big[\,\mathfrak{f}_i(X(t))\big]^2\,\mathrm{d}t\,, \quad i=1,2\,. \tag{6.21}$$

It follows now from (6.10) that $\langle M_i \rangle(\cdot) = \langle X_i \rangle(\cdot) + \gamma_i^2 \langle \|X\| \rangle(\cdot) - 2\gamma_i \langle X_i, \|X\| \rangle(\cdot)$; and in conjunction with (6.20), (6.21), (6.15), this gives $\langle X_i, \|X\| \rangle(\cdot) \;=\; \int_0^{\cdot} \mathbf{1}_{\{X(t)\neq \mathbf{0}\}} \boldsymbol{a}(\|X(t)\|)\,\mathfrak{f}_i(X(t))\,\mathrm{d}t\,.$

Hence, with $r_{i,3}(t) \;\equiv\; r_{3,i}(t) \;:=\; \boldsymbol{a}(\|X(t)\|)\,\big(\mathfrak{f}_i(X(t)) - \gamma_i\big)\,\mathbf{1}_{\{X(t)\neq \mathbf{0}\}}$ for $i \;=\; 1,2$, we obtain

$$\langle M_i, M_3 \rangle(\cdot) \;\equiv\; \langle M_3, M_i \rangle(\cdot) \;=\; \langle X_i, \|X\| \rangle(\cdot) - \gamma_i \langle \|X\| \rangle(\cdot) \;=\; \int_0^{\cdot} r_{i,3}(t)\mathrm{d}t\,.$$

• We have now computed all elements of the (3×3) matrix $(\mathrm{d}\langle M_i, M_k\rangle(t)\,/\,\mathrm{d}t)_{1\le i,k\le 3} \;=\; (r_{i,k}(t))_{1\le i,k\le 3}$; we observe also that this matrix is of rank 1, on $\{t \ge 0 : X(t) \neq \mathbf{0}\}$. By Theorem 3.4.2 and Proposition 5.4.6 in KARATZAS & SHREVE (1991), there exists an extension of the original probability space, and on it
(i) a three-dimensional standard Brownian motion $\widetilde{W}(\cdot) \;=\; \big(\widetilde{W}_1(\cdot), \widetilde{W}_2(\cdot), \widetilde{W}_3(\cdot)\big)'$,
(ii) a one-dimensional standard Brownian motion $W(\cdot)$, and
(iii) measurable, adapted, matrix-valued processes $(\rho_{i,k}(\cdot))_{1\le i,k\le 3}$ with $\int_0^T [\rho_{i,k}(t)]^2 \mathrm{d}t < \infty$,
such that we have the representations

$$M_i(\cdot) \;=\; \sum_{k=1}^{3} \int_0^{\cdot} \rho_{i,k}(t)\,\mathrm{d}\widetilde{W}_k(t) \;=\; \int_0^{\cdot} \boldsymbol{\sigma}(\|X(t)\|)\,\big(\mathfrak{f}_i(X(t)) - \gamma_i\big)\,\mathbf{1}_{\{X(t)\neq \mathbf{0}\}}\,\mathrm{d}W(t)\,, \qquad i=1,2 \tag{6.22}$$

and $M_3(\cdot) \;=\; \int_0^{\cdot} \boldsymbol{\sigma}(\|X(t)\|)\,\mathbf{1}_{\{X(t)\neq \mathbf{0}\}}\,\mathrm{d}W(t)$. Substituting this into the decomposition $N(\cdot) = M_3(\cdot) + L^{||X||}(\cdot)$ and then into (6.16), we obtain the stochastic equation (6.5) for the radial process $||X(\cdot)||$. Substituting the expressions of (6.22), (6.5) into $M_i(\cdot)$ in (6.10) for $i = 1,2$, we observe that the process $X(\cdot)$ defined in (6.10) satisfies the system of (6.4). It follows from (6.3) that $X(\cdot)$ satisfies the property (2.17) .

• Finally, for every set $A \in \mathcal{B}([0, 2\pi))$, we consider the functions

$$g_5(x) \;:=\; g_5(r,\theta) \;=\; r\big(\mathbf{1}_A(\theta) - \boldsymbol{\nu}(A)\big) \qquad \text{and} \qquad g_6(x) \;:=\; g_6(r,\theta) \;=\; r\,\mathbf{1}_A(\theta) \tag{6.23}$$

in polar coördinates. Since $g_5 \in \mathfrak{D}^{\boldsymbol{\mu}}$ and $g_6 \in \mathfrak{D}^{\boldsymbol{\mu}}_+$ we obtain that the process

$$g_5(X(\cdot)) - g_5(X(0)) - \int_0^{\cdot} \boldsymbol{b}(\|X(t)\|)\big(\mathbf{1}_{\{\arg(X(t))\,\in\, A\}} - \boldsymbol{\nu}(A)\big)\mathbf{1}_{\{\|X(t)\|>0\}}\mathrm{d}t \tag{6.24}$$

is a continuous local martingale, and that the process

$$g_6(X(\cdot)) - g_6(X(0)) - \int_0^{\cdot} \boldsymbol{b}(\|X(t)\|)\mathbf{1}_{\{\arg(X(t))\,\in\, A\}\cup\{\|X(t)\|>0\}}\mathrm{d}t$$

is a continuous local submartingale.

Repeating an argument similar to the one deployed above, we identify $\boldsymbol{\nu}(A)L^{\|X\|}(\cdot)$ as the local time $L^{R_A}(\cdot)$ at the origin for the continuous, non-negative semimartingale $R^A(\cdot) \;:=\; g_6(X(\cdot))$. Indeed,

$$R^A(\cdot) - R^A(0) - \int_0^{\cdot} \boldsymbol{b}(\|X(t)\|)\mathbf{1}_{\{\arg(X(t))\in A\}\cup\{\|X(t)\|>0\}}\mathrm{d}t - L^{R^A}(\cdot) \tag{6.25}$$

is a continuous local martingale. Moreover, on account of (6.17), we see that

$$\boldsymbol{\nu}(A)\Big(\|X(\cdot)\| - \|X(0)\| - \int_0^{\cdot} \boldsymbol{b}(\|X(t)\|)\mathbf{1}_{\{\|X(t)\|>0\}}\mathrm{d}t - L^{\|X\|}(\cdot)\Big) \tag{6.26}$$

is also a continuous local martingale. Subtracting (6.25) from (6.24) and adding (6.26), we deduce that the finite variation process $L^{R^A}(\cdot) - \boldsymbol{\nu}(A)L^{\|X\|}(\cdot)$ is a continuous local martingale, and hence identically zero, i.e., $L^{R^A}(\cdot) \equiv \boldsymbol{\nu}(A)\,L^{\|X\|}(\cdot)$ as in (2.18).

We conclude from this analysis, that the system of equations (6.4) admits a weak solution with the properties (2.17) and (2.18). This proves Part (b); Part (c) is now evident. $\square$

Remark 6.1. Looking back to the definition of the above local martingale problem for the planar diffusion, we remark the following statements (i)-(ii) are equivalent:
(i) For every $g \in \mathfrak{D}^{\boldsymbol{\mu}}_{+}$, the process $M^g(\cdot\,;\omega_2)$ is a continuous local submartingale;
(ii) For every $g \in \mathfrak{D}^{\boldsymbol{\mu}}$, the process $M^g(\cdot\,;\omega_2)$ is a continuous local martingale, and the process $M^{g_3}(\cdot\,;\omega_2)$ is a continuous local submartingale, where $g_3(x) = \|x\| = r$ is defined in (6.14).

In fact, let us assume (i) first. Since $g_3(x) = \|x\|$ belongs to $\mathfrak{D}^{\mu}_{+}$, $M^{g_3}(\cdot\,;\omega_2)$ is a continuous local submartingale. For every $g \in \mathfrak{D}^{\boldsymbol{\mu}}$ we observe $g \in \mathfrak{D}^{\boldsymbol{\mu}}_{+}$ and $-g \in \mathfrak{D}^{\mu}_{+}$, and hence both $M^g(\cdot\,;\omega_2)$ and $M^{-g}(\cdot\,;\omega_2) = -M^g(\cdot\,;\omega_2)$ are continuous local submartingales. Thus $M^g(\cdot\,;\omega_2)$ is a continous local martingale, and (ii) follows.

Next, let us assume (ii). Then every $g \in \mathfrak{D}^{\boldsymbol{\mu}}_{+}$ can be decomposed as $g := g_{(1)} + g_{(2)}$, where $g_{(1)}(x) = c\|x\|$ with $c := \int_0^{2\pi} g'_\theta(0+)\boldsymbol{\nu}(\mathrm{d}\theta) \ge 0$ and $g_{(2)}(\cdot) := g(\cdot) - g_{(1)}(\cdot) \in \mathfrak{D}^{\boldsymbol{\mu}}$. Thus the above condition (ii) implies that $M^{g_{(1)}}(\cdot\,;\omega_2) = c\|\omega_2(\cdot)\|$ is a local submartingale and $M^{g_{(2)}}(\cdot\,;\omega_2)$ is a local martingale, and hence $M^g(\cdot\,;\omega_2) = M^{g_{(1)}}(\cdot\,;\omega_2) + M^{g_{(2)}}(\cdot;\omega_2)$ is a local submartingale, and (i) follows.

6.3 Well-Posedness

We conjecture that, if the local submartingale problem associated with the pair $(\boldsymbol{\sigma}, \boldsymbol{b})$ is well-posed, then the same is true for the local martingale problem associated with the triple $(\boldsymbol{\sigma}, \boldsymbol{b}, \boldsymbol{\mu})$.

The result that follows settles this conjecture in the affirmative for the driftless case $\boldsymbol{b} \equiv \boldsymbol{0}$. Corollary 6.1 then deals with the case of a drift $\boldsymbol{b} = \boldsymbol{\sigma c}$ with $\boldsymbol{c} : \mathbb{R}_+ \to \mathbb{R}$ bounded and measurable.

Proposition 6.2. *Suppose that*
(i) the drift $\boldsymbol{b}$ *is identically equal to zero; and that*
(ii) the reciprocal of the dispersion coëfficient $\boldsymbol{\sigma} : [0,\infty) \to \mathbb{R} \setminus \{0\}$ *is locally square-integrable; i.e.,*

$$\int_K \frac{\mathrm{d}y}{\boldsymbol{\sigma}^2(y)} \;<\; \infty\,, \qquad \textit{for every compact set } K \subset [0,\infty)\,. \tag{6.27}$$

Then the local submartingale problem of subsection 6.1, associated with the pair $(\boldsymbol{\sigma}, \boldsymbol{0})$*, is well-posed. Moreover, the local martingale problem of subsection 6.2 associated with the triple* $(\boldsymbol{\sigma}, \boldsymbol{0}, \boldsymbol{\mu})$ *is also well-posed; and uniqueness in distribution holds, subject to the properties in (2.17) and (2.18), for the corresponding system of stochastic integral equations in (6.4) with* $\boldsymbol{b} \equiv 0$*, namely,*

$$X_i(\cdot) \;=\; X_i(0) + \int_0^{\cdot} \mathfrak{f}_i(X(t))\, \boldsymbol{\sigma}\big(\|X(t)\|\big)\, \mathrm{d}W(t) + \gamma_i\, L^{\|X\|}(\cdot)\,, \qquad i = 1, 2\,. \tag{6.28}$$

Proof of Existence: Let us consider the stochastic integral equation

$$S(\cdot) \;=\; r + \int_0^{\cdot} \boldsymbol{\sigma}(S(t))\, \mathrm{d}W(t) + L^S(\cdot) \tag{6.29}$$

driven by one-dimensional Brownian motion $W(\cdot)$. It is shown in SCHMIDT (1989) that, under (6.27), this equation (6.29) has a non-negative, unique-in-distribution weak solution; equivalently, the STROOCK & VARADHAN (1971) local submartingale problem associated with $(\boldsymbol{\sigma}, \boldsymbol{0})$ for $K^{\psi}(\cdot)$ in (6.1) is well-posed.

Let us also verify the property (2.4). From Exercise 3.7.10 in KARATZAS & SHREVE (1991),

$$0 \;=\; \int_0^{\infty} \mathbf{1}_{\{S(t)=0\}}\, \mathrm{d}\langle S\rangle(t) \;=\; \int_0^{\infty} \mathbf{1}_{\{S(t)=0\}}\, \boldsymbol{\sigma}^2(S(t))\, \mathrm{d}t$$

holds, and $\int_0^{\infty} \mathbf{1}_{\{S(t)=0\}}\, \mathrm{d}t = 0$ follows because $\boldsymbol{\sigma}(\cdot)$ never vanishes.

It follows then from Theorem 2.1 that, on a suitably enlarged probability space, we may construct from this reflected diffusion $S(\cdot)$ a continuous, planar semimartingale $X(\cdot)$ which satisfies $||X(\cdot)|| = S(\cdot)$, the system of stochastic integral equations (6.28), as well as the properties (2.14)–(2.18). On the strength of Proposition 6.1(a), the local martingale problem associated with the triple $(\boldsymbol{\sigma}, \mathbf{0}, \boldsymbol{\mu})$ admits a solution. $\square$

Proof of Uniqueness: We adopt the idea of proof in Theorem 3.2 of BARLOW, PITMAN & YOR (1989a). Suppose now that there are *two* solutions $\mathbb{Q}_j$, $j = 1, 2$ to this local martingale problem associated with the triple $(\boldsymbol{\sigma}, \mathbf{0}, \boldsymbol{\mu})$. Let us take an arbitrary set $A \in \mathcal{B}([0, 2\pi))$, and the functions $h_A(\cdot)$ and $g_A(\cdot)$ defined as in (4.8) for the indicator $\varphi = \mathbf{1}_A$, namely

$$h_A(x) := \big(\mathbf{1}_{\{\arg(x) \in A\}} - \boldsymbol{\nu}(A)\big) \cdot \mathbf{1}_{\{\|x\|>0\}} \tag{6.30}$$
$$= \big(\boldsymbol{\nu}(A^c)\, \mathbf{1}_{\{\arg(x) \in A\}} - \boldsymbol{\nu}(A)\, \mathbf{1}_{\{\arg(x) \in A^c\}}\big) \cdot \mathbf{1}_{\{\|x\|>0\}}$$

and

$$g_A(x) := \|x\|\, h_A(x), \qquad x \in \mathbb{R}^2. \tag{6.31}$$

The above function $g_A(\cdot)$ belongs to the family $\mathfrak{D}^{\boldsymbol{\mu}}$ in (4.5), as does the function $[g_A(\cdot)]^2$. By assumption and Proposition 4.2, the process $M_A(\cdot) := g_A(\omega_2(\cdot))$ is then a $\mathbb{Q}_j-$local martingale, with

$$\langle M_A \rangle(T) = \big\langle g_A(\omega_2(\cdot))\big\rangle(T) = \int_0^T \big[h_A(\omega_2(t))\big]^2 \boldsymbol{a}(\|\omega_2(t)\|)\, \mathrm{d}t\,; \quad 0 \le T < \infty\,, \quad j = 1, 2\,.$$

Note that $\omega_2(\cdot)$ and $\|\omega_2(\cdot)\|$ solve in the weak sense the equations (6.28) and (6.29), respectively. Thus the argument at the beginning of Section 3 implies that $\omega_2(\cdot)$ stays on the same ray on each of its excursion away from the origin. Moreover, $\|\omega_2(\cdot)\|$ is strongly Markovian with respect to the filtration $\mathbb{F}_2$, and its distribution is uniquely determined.

• Let us assume $0 < \boldsymbol{\nu}(A) < 1$ first. We note that $g_A(x) > 0$, if $\arg(x) \in A$; $g_A(x) < 0$ if $\arg(x) \in A^c$; and $g_A(x) = 0$ if $x = \mathbf{0}$. It is also easy to verify that the process

$$U_A(\cdot) := \int_0^\cdot \Big(\frac{1}{\boldsymbol{\nu}(A^c)} \cdot \mathbf{1}_{\{g_A(\omega_2(t))>0\}} + \frac{1}{\boldsymbol{\nu}(A)} \cdot \mathbf{1}_{\{g_A(\omega_2(t))\le 0\}}\Big) \cdot \frac{\mathrm{d}M_A(t)}{\boldsymbol{\sigma}(\|\omega_2(t)\|)} \tag{6.32}$$

is a continuous $\mathbb{Q}_j-$local martingale with $\langle U_A \rangle(t) = t$ for $t \ge 0$; i.e., a $\mathbb{Q}_j-$Brownian motion for $j = 1, 2$. The probability distribution of the process $M_A(\cdot) = g_A(\omega_2(\cdot))$ is then determined uniquely and *independently of the solution* $\mathbb{Q}_j$, $j = 1, 2$ *to the local martingale problem.* This is because, under the assumption (6.27) on the dispersion coëfficient and thanks to the theory of ENGELBERT & SCHMIDT (1984), the stochastic differential equation driven by the Brownian motion $U_A(\cdot)$ and derived from (6.32),

$$\mathrm{d}M_A(t) = \boldsymbol{\varrho}\big(M_A(t)\big)\, \mathrm{d}U_A(t)\,, \quad 0 \le t < \infty \tag{6.33}$$

with $c_0 := \boldsymbol{\nu}(A^c)$, $c_1 := \boldsymbol{\nu}(A)$ and the new dispersion function

$$\boldsymbol{\varrho}(x) := c_0 \cdot \boldsymbol{\sigma}\Big(\frac{x}{c_0}\Big) \cdot \mathbf{1}_{\{x>0\}} + c_1 \cdot \boldsymbol{\sigma}\Big(-\frac{x}{c_1}\Big) \cdot \mathbf{1}_{\{x\le 0\}}\,; \quad x \in \mathbb{R}\,, \tag{6.34}$$

admits a weak solution, which is unique in the sense of the probability distribution. This follows from Theorem 5.5.7 in KARATZAS & SHREVE (1991), and from the fact that the reciprocal of the function $\boldsymbol{\varrho}(\cdot)$ inherits the local square-integrability property (6.27) of the reciprocal of $\boldsymbol{\sigma}(\cdot)$. Moreover, $M_A(\cdot) = g_A(\omega_2(\cdot))$ is strongly Markovian with respect to the filtration $\mathbb{F}_2$ (cf. the proof of Lemma 9.2 in Section 9). Therefore, for an arbitrary $C \in \mathcal{B}((0, \infty))$, recalling (6.30) and (6.31), we have

$$\mathbb{Q}_j\big(\,||\omega_2(t)|| \in C\,,\ \arg(\omega_2(t)) \in A \,\big|\, \mathcal{F}_2(s)\,\big) = \mathbb{Q}_j\big(\,g_A(\omega_2(t)) \in \boldsymbol{\nu}(A^c)\, C \,\big|\, \mathcal{F}_2(s)\,\big)$$

$$= \mathbb{Q}_j\big(g_A(\omega_2(t)) \in \boldsymbol{\nu}(A^c)\, C \,\big|\, g_A(\omega_2(s))\big), \qquad\qquad 0 \le s < t < \infty,\ \ j = 1, 2.$$

Since the distribution of the process $g_A(\omega_2(\cdot))$ is uniquely determined, the above probability does not depend on $j = 1, 2$. We conclude then that $\mathbb{Q}_j\big(||\omega_2(t)|| \in C\,,\ \arg(\omega_2(t)) \in A \,\big|\, \mathcal{F}_2(s)\big)$ does not depend on $j = 1, 2$, if $0 < \boldsymbol{\nu}(A) < 1$.

- For the resulting diffusion process in natural scale, we shall denote by

$$p_A\big(s,t;y,B\big) := \mathbb{Q}_j\big(g_A(\omega_2(t)) \in B \,\big|\, g_A(\omega_2(s)) = y\big)$$
$$p^*_A\big(s,t;y,B\big) := \mathbb{Q}_j\big(g_A(\omega_2(t)) \in B\,,\ \tau_s(\omega_2) > t \,\big|\, g_A(\omega_2(s)) = y\big)$$
$$q_A\big(s,t;y\big) := p^*_A\big(s,t;y,\mathbb{R}\big) := \mathbb{Q}_j\big(\tau_s(\omega_2) > t \,\big|\, g_A(\omega_2(s)) = y\big)$$

its transition, taboo-transition, and survival probabilities (for both $j = 1, 2$ on the strength of uniqueness in distribution for (6.33)). Here $0 \le s < t < \infty$, $y \in \mathbb{R}$ and $B \in \mathcal{B}$ are arbitrary, and we have denoted the first hitting time of the origin by

$$\tau_s(\omega_2) := \inf\big\{u \ge s \,:\, ||\omega_2(u)|| = 0\big\} = \inf\big\{u \ge s \,:\, g_A(\omega_2(u)) = 0\big\}$$

For an arbitrary $C \in \mathcal{B}((0,\infty))$ and recalling (6.30), (6.31), we have then the expression

$$\mathbb{Q}_j\big(||\omega_2(t)|| \in C\,,\ \arg(\omega_2(t)) \in A\,,\ \tau_s(\omega_2) > t \,\big|\, \omega_2(s) = x\big) = p^*_A\big(s,t;\boldsymbol{\nu}(A^c)\,||x||, \boldsymbol{\nu}(A^c)\, C\big)\, \mathbf{1}_A(\arg(x))\,,$$

whose right-hand side does not depend on $j = 1, 2$. Similarly, we observe that the transition probability

$$\mathbb{Q}_j\big(||\omega_2(t)|| \in C\,,\ \arg(\omega_2(t)) \in A\,,\ \tau_s(\omega_2) < t \,\big|\, \omega_2(s) = x\big)$$

is given, with $m := \boldsymbol{\nu}(A^c)\, \mathbf{1}_A(\arg(x)) - \boldsymbol{\nu}(A)\, \mathbf{1}_{A^c}(\arg(x))$, by the expression

$$\int_s^t \mathbb{Q}_j\big(||\omega_2(t)|| \in C\,,\ \arg(\omega_2(t)) \in A \,\big|\, \tau_s(\omega_2) = \theta,\, \omega_2(s) = x\big) \cdot \mathbb{Q}_j\big(\tau_s(\omega_2) \in \mathrm{d}\theta \,\big|\, \omega_2(s) = x\big)$$
$$= \int_s^t p_A\big(\theta, t; 0, \boldsymbol{\nu}(A^c) C\big) \cdot \Big(- \mathrm{d}_\theta\, q_A\big(s, \theta\,;\, m\, ||x||\big)\Big).$$

Once again, the expression on the right-hand side does not depend on $j = 1, 2$.

- Next, we consider the case $\boldsymbol{\nu}(A) \in \{0,\, 1\}$. Let $\boldsymbol{\nu}(A) = 0$ first; then $g_A(x) = \|x\|\, \mathbf{1}_{\{\arg(x) \in A,\, \|x\|>0\}}$, and the process $M_A(\cdot) = g_A(\omega_2(\cdot))$ is a nonnegative, continuous $\mathbb{Q}_j-$local martingale, thus also a supermartingale – so it stays at the origin $\mathbf{0}$ after hitting it for the first time. It follows that with $\mathbb{Q}_j-$probability one, the angular part $\arg(\omega_2(\cdot))$ never again visits the set A, after the radial part $\|\omega_2(\cdot)\|$ first becomes zero. Thus for an arbitrary $C \in \mathcal{B}((0,\infty))$ and for every $0 \le s < t < \infty$, $x \in \mathbb{R}^2$, $j = 1, 2$ we have

$$\mathbb{Q}_j\big(||\omega_2(t)|| \in C\,,\ \arg(\omega_2(t)) \in A\,,\ \tau_s(\omega_2) < t \,\big|\, \mathcal{F}_2(s)\big) = 0\,.$$

If, on the other hand, $\boldsymbol{\nu}(A) = 1$ holds, then $\boldsymbol{\nu}(A^c) = 0$ and therefore

$$\mathbb{Q}_j\big(||\omega_2(t)|| \in C\,,\ \arg(\omega_2(t)) \in A^c\,,\ \tau_s(\omega_2) < t \,\big|\, \mathcal{F}_2(s)\big) = 0\,,$$

which implies

$$\mathbb{Q}_j\big(||\omega_2(t)|| \in C,\ \arg(\omega_2(t)) \in A\,,\ \tau_s(\omega_2) < t \,\big|\, \mathcal{F}_2(s)\big) = \mathbb{Q}_j\big(||\omega_2(t)|| \in C,\ \tau_s(\omega_2) < t \,\big|\, \mathcal{F}_2(s)\big)$$
$$= \mathbb{Q}_j\big(||\omega_2(t)|| \in C,\ \tau_s(\omega_2) < t \,\big|\, ||\omega_2(s)||\big).$$

We have also the following in both cases:

$$\mathbb{Q}_j\big(||\omega_2(t)|| \in C\,,\ \arg(\omega_2(t)) \in A\,,\ \tau_s(\omega_2) > t \,\big|\, \mathcal{F}_2(s)\big)$$
$$= \mathbb{Q}_j\big(||\omega_2(t)|| \in C\,,\ \tau_s(\omega_2) > t \,\big|\, ||\omega_2(s)||\big)\, \mathbf{1}_A(\arg(\omega_2(s)))\,.$$

Since the distribution of $\|\omega_2(\cdot)\|$ is uniquely determined and independent of $j = 1, 2$, we conclude that $\mathbb{Q}_j\big(||\omega_2(t)|| \in C\,,\ \arg(\omega_2(t)) \in A \,\big|\, \mathcal{F}_2(s)\big)$ does not depend on $j = 1, 2$, if $\boldsymbol{\nu}(A) = 0$ or 1.

• Finally, we note that $\mathbb{Q}_j\big(\omega_2(t) = 0 \,\big|\, \omega_2(s) = x\big) = \mathbb{Q}_j\big(||\omega_2(t)|| = 0 \,\big|\, ||\omega_2(s)|| = ||x||\big)$, where the right-hand side is also uniquely determined.

• Now it is clear that the conditional distribution of $\omega_2(t)$ given $\mathcal{F}_2(s)$ is uniquely determined for $0 \le s < t < \infty$. Standard arguments show then, that the finite-dimensional distributions of $\omega_2(\cdot)$ are uniquely determined. Therefore, the local martingale problem of the Proposition is well-posed. □

Corollary 6.1. *Under the setting of Proposition 6.2, and in addition to the assumptions imposed there, let us consider another function* $\boldsymbol{c} : \mathbb{R}_+ \to \mathbb{R}$ *which is bounded and measurable. We denote by* $\mathbb{Q}^{(\mathbf{0})}$ *the solution to the local martingale problem of subsection 6.2 associated with the triple* $(\boldsymbol{\sigma}, \mathbf{0}, \boldsymbol{\mu})$.

Then, for every $T \in (0, \infty)$, *the local martingale problem associated with the triple* $(\boldsymbol{\sigma}, \boldsymbol{\sigma c}, \boldsymbol{\mu})$ *for* $M^g(t)$, $0 \le t \le T$ *in (6.2), is well posed, and its solution is given by the probability measure* $\mathbb{Q}^{(\boldsymbol{c})}$ *with*

$$\frac{\mathrm{d}\mathbb{Q}^{(\boldsymbol{c})}}{\mathrm{d}\mathbb{Q}^{(\mathbf{0})}}\bigg|_{\mathcal{F}_2^\bullet(t)} := \exp\Big(\int_0^t \boldsymbol{c}(\|\omega_2(u)\|)\, \mathrm{d}W(u) - \frac{1}{2}\int_0^t \boldsymbol{c}^2(\|\omega_2(u))\|)\mathrm{d}u\Big); \quad 0 \le t \le T.$$

Proof. This is a direct consequence of Propositions 6.1, 6.2 and GIRSANOV's change of measure. Indeed, it follows from Proposition 6.1 that, under $\mathbb{Q}^{(\mathbf{0})}$, the coördinate process $\omega_2(\cdot)$ satisfies the system of stochastic integral equations (6.28), subject to (2.17) and (2.18). Because of the boundedness of the function $\boldsymbol{c}(\cdot)$, the measure $\mathbb{Q}^{(\boldsymbol{c})}$ just introduced is a probability. By GIRSANOV's theorem (e.g., KARATZAS & SHREVE (1991), Theorem 3.5.1) we see that for every fixed $T \in (0, \infty)$, the process

$$W^{(\boldsymbol{c})}(u) := W(u) - \int_0^u \boldsymbol{c}\big(S(t)\big)\,\mathrm{d}t; \quad 0 \le u \le T$$

is standard Brownian motion under this probability measure $\mathbb{Q}^{(\boldsymbol{c})}$, and thus the coördinate process $\omega_2(\cdot)$ satisfies on the time-horizon $[0, T]$ the system of stochastic integral equations

$$X_i(\cdot) = \mathrm{x} + \int_0^{\cdot} \mathfrak{f}_i(X(t))\,\boldsymbol{\sigma}\big(\|X(t)\|\big)\Big[\mathrm{d}W^{(\boldsymbol{c})}(t) + \boldsymbol{c}\big(\|X(t)\|\big)\,\mathrm{d}t\Big] + \gamma_i\, L^{\|X\|}(\cdot), \quad i = 1, 2.$$

Moreover, since the probability measure $\mathbb{Q}^{(\boldsymbol{c})}$ is absolutely continuous with respect to $\mathbb{Q}^{(\mathbf{0})}$, we obtain (2.17) and (2.18) with $X(\cdot)$ replaced by $\omega_2(\cdot)$, a.e. under $\mathbb{Q}^{(\boldsymbol{c})}$. Thanks to Proposition 6.1 again, $\mathbb{Q}^{(\boldsymbol{c})}$ solves the local martingale problem of subsection 6.2 associated with the triple $(\boldsymbol{\sigma}, \boldsymbol{\sigma c}, \boldsymbol{\mu})$.

Conversely, for any solution $\mathbb{Q}^{(\boldsymbol{c})}$ to the local martingale problem associated with $(\boldsymbol{\sigma}, \boldsymbol{\sigma c}, \boldsymbol{\mu})$ for $M^g(t)$, $0 \le t \le T$ as in (6.2), the probability measure $\mathbb{Q}^{(\mathbf{0})}$ defined via

$$\frac{\mathrm{d}\mathbb{Q}^{(\mathbf{0})}}{\mathrm{d}\mathbb{Q}^{(\boldsymbol{c})}}\bigg|_{\mathcal{F}_2^\bullet(t)} := \exp\Big(-\int_0^t \boldsymbol{c}(\|\omega_2(u)\|)\, \mathrm{d}W^{(\boldsymbol{c})}(u) - \frac{1}{2}\int_0^t \boldsymbol{c}^2(\|\omega_2(u))\|)\mathrm{d}u\Big); \quad 0 \le t \le T$$

is seen to solve the local martingale problem of subsection 6.2 associated with the triple $(\boldsymbol{\sigma}, \mathbf{0}, \boldsymbol{\mu})$. Since this problem is well-posed, the same holds for the local martingale problem associated with $(\boldsymbol{\sigma}, \boldsymbol{\sigma c}, \boldsymbol{\mu})$. □

7 Martingale Characterization of the WALSH Brownian Motion

We still have to show that, when $U(\cdot) \equiv B(\cdot)$ is standard Brownian motion, the construction of Theorem 2.1 leads to the WALSH Brownian motion as defined, for instance, in BARLOW, PITMAN & YOR (1989a). The purpose of the present section is to establish this connection; cf. Proposition 7.2.

Let us recall a few basic facts regarding the WALSH Brownian motion with spinning measure $\boldsymbol{\mu}$. Following FITZSIMMONS & KUTER (2014) (see also BARLOW, PITMAN & YOR (1989a)), we may characterize the WALSH Brownian motion $\boldsymbol{W}(\cdot)$ in terms of its FELLER semigroup $\{\mathcal{P}_t\,,\, t \ge 0\}$ defined as

$$
\begin{aligned}
\big[\mathcal{P}_t f\big](0,\theta) \,&:=\, T^+_t \overline{f}(0)\,,\\
\big[\mathcal{P}_t f\big](r,\theta) \,&:=\, T^+_t \overline{f}(r) + \big[T^0_t\big(f_\theta - \overline{f}\,\big)\big](r)\,; \quad r>0\,,\ \theta\in[0,2\pi)\,,
\end{aligned} \tag{7.1}
$$

for $f \in C_0(\overline{E})$. Here $\{T^+_t,\, 0 \le t < \infty\}$ is the semigroup of the reflected Brownian motion on $[0,\infty)$, whereas $\{T^0_t,\, 0 \le t < \infty\}$ is the semigroup of Brownian motion on $[0,\infty)$ killed upon reaching the origin. For the sake of simplicity, we use polar coördinates in the punctured plane E of (2.8). Abusing notation slightly, we define also

$$
\overline{f}(r) \,:=\, \int_{[0,2\pi)} f(r,\theta)\,\boldsymbol{\nu}(\mathrm{d}\theta)\,, \qquad f_\theta(r) \,:=\, f(r,\theta)\,; \qquad (r,\theta)\in\overline{E}\,, \tag{7.2}
$$

for $f \in C(\overline{E})$, as in (2.9). Let us assume that $\boldsymbol{W}(0) = \mathrm{x} \in \mathbb{R}^2$. BARLOW, PITMAN & YOR (1989a) show that there is a FELLER and strong MARKOV process $\boldsymbol{W}(\cdot)$ with values in $\mathbb{R}^2$, continuous paths, and $\{\mathcal{P}_t\,,\, 0 \le t < \infty\}$ as its semigroup. This is the process these authors call "WALSH Brownian motion". They show that the radial part $||\boldsymbol{W}(\cdot)||$ is one-dimensional reflecting Brownian motion. For this planar process $\boldsymbol{W}(\cdot)$, HAJRI & TOUHAMI (2014) derive a version of the FREIDLIN-SHEU formula, that involves the standard, one-dimensional Brownian motion of the filtration $\mathbb{F}^{\boldsymbol{W}} = \big\{\mathcal{F}^{\boldsymbol{W}}(t)\big\}_{0\le t<\infty}$, given by

$$
\boldsymbol{\beta}^{\boldsymbol{W}}(\cdot) \,:=\, ||\boldsymbol{W}(\cdot)|| - ||\mathrm{x}|| - L^{||\boldsymbol{W}||}(\cdot)\,. \tag{7.3}
$$

Here is an extension of Proposition 3.1 in BARLOW, PITMAN & YOR (1989a); we recall the Definitions 4.1 and 4.2, as well as the notation of (4.8).

Proposition 7.1. *Let $\boldsymbol{W}(\cdot)$ be the* WALSH *Brownian motion defined via the semigroup (7.1) and with spinning measure $\boldsymbol{\mu}$. Then:*
(i) The process $\|\boldsymbol{W}(\cdot)\|$ is reflecting Brownian motion; and $\boldsymbol{W}(\cdot)$ satisfies the properties in (2.14)–(2.15).
(ii) For any function $g : \mathbb{R}^2 \to \mathbb{R}$ in the class $\mathfrak{D}^{\boldsymbol{\mu}}$ of (4.5), the continuous process

$$
g(\boldsymbol{W}(\cdot)) - g(\mathrm{x}) - \frac{1}{2}\int_0^{\cdot} G''(\boldsymbol{W}(t))\,\mathbf{1}_{\{\boldsymbol{W}(t)\neq\mathbf{0}\}}\,\mathrm{d}t \,=\, \int_0^{\cdot} G'(\boldsymbol{W}(t))\,\mathbf{1}_{\{\boldsymbol{W}(t)\neq\mathbf{0}\}}\,\mathrm{d}\boldsymbol{\beta}^{\boldsymbol{W}}(t)
$$

is a local martingale.

Proof: The claims of (i) are proved in BARLOW, PITMAN & YOR (1989a). Claim (ii) follows by applying the FREIDLIN-SHEU formula of Theorem 1.2 in HAJRI & TOUHAMI (2014) to the process $g(\boldsymbol{W}(\cdot))$. We also note that, with the notation of (4.8), both processes below are continuous martingales:

$$
\mathcal{M}^{\boldsymbol{W}}_{(\varphi)}(\cdot) = g_{(\varphi)}\big(\boldsymbol{W}(\cdot)\big) - g_{(\varphi)}(\mathrm{x}) = \int_0^{\cdot} h_{(\varphi)}\big(\boldsymbol{W}(t)\big)\,\mathrm{d}\boldsymbol{\beta}^{\boldsymbol{W}}(t)\,, \quad \mathcal{N}^{\boldsymbol{W}}_{(\varphi)}(\cdot) = \big(\mathcal{M}^{\boldsymbol{W}}_{(\varphi)}(\cdot)\big)^2 - \langle \mathcal{M}^{\boldsymbol{W}}_{(\varphi)}\rangle(\cdot). \quad \square
$$

We deduce from Proposition 7.1 that the WALSH Brownian motion with spinning measure $\boldsymbol{\mu}$, defined via the semigroup (7.1), generates a solution to the local martingale problem associated with the triple $(\mathbf{1}, \mathbf{0}, \boldsymbol{\mu})$ (cf. Remark 6.1).

Proposition 7.2. *Suppose that the semimartingale $U(\cdot) \equiv B(\cdot)$ of (2.1) is Brownian motion. Then the planar process $X(\cdot)$ constructed as in Theorem 2.1 has the following properties:*
(i) It is the unique-in-distribution weak solution, subject to the properties (2.17), (2.18), of the system of stochastic integral equations in (3.4), namely

$$
X_i(\cdot) \,=\, \mathrm{x}_i + \int_0^{\cdot} \mathfrak{f}_i\big(X(t)\big)\,\mathrm{d}B(t) + \gamma_i\, L^{\,||X||}(\cdot)\,, \qquad i=1,2\,.
$$

(ii) It is a WALSH *Brownian motion.*
(iii) Every $\mathbb{F}^X-$local martingale $M(\cdot)$ with $M(0)=0$ has an integral representation $M(\cdot) = \int_0^{\cdot} H(t)\,\mathrm{d}B(t)$, for some $\mathbb{F}^X-$progressively measurable and locally square-integrable process $H(\cdot)$.

Proof. The first claim follows from Propositions 6.1, 6.2 with $\boldsymbol{\sigma}(\cdot) \equiv 1$; the second claim, that $X(\cdot)$ is WALSH Brownian motion, is a consequence of Propositions 6.1, 6.2 and 7.1. With $U(\cdot) \equiv B(\cdot)$ a standard Brownian motion, Proposition 4.2 shows that both processes below are continuous local martingales

$$M_{(\varphi)}(\cdot) = g_{(\varphi)}(X(\cdot)) - g_{(\varphi)}(\mathrm{x}) = \int_0^{\cdot} h_{(\varphi)}(X(t))\, \mathrm{d}B(t)\,, \qquad N_{(\varphi)}(\cdot) = \big[M_{(\varphi)}(\cdot)\big]^2 - \int_0^{\cdot} \big[h_{(\varphi)}(X(t))\big]^2 \mathrm{d}t$$

(cf. Theorem 3.1 of BARLOW, PITMAN & YOR (1989a)). The third claim is proved in Theorem 4.1 of BARLOW, PITMAN & YOR (1989a), for spinning measures $\boldsymbol{\mu}$ supported on a finite number of points on the unit circumference. The same argument works in our present generality, now that we have established the well-posedness of the local martingale problem corresponding to the triple $(\mathbf{1}, \mathbf{0}, \boldsymbol{\mu})$ in Proposition 6.2. $\square$

In the terminology adopted by MANSUY & YOR (2006), and for a spinning measure $\boldsymbol{\mu}$ that does not concentrate on one or two points in $\mathfrak{S}$, this last property says that the natural filtration $\mathbb{F}^X$ of the WALSH Brownian motion is a *weak Brownian filtration* (has the martingale representation property with respect to U) but *not a strong Brownian filtration* (cannot be generated by a Brownian motion of *any* dimension).

8 Angular Dependence

Let us admit now bounded, BOREL-measurable coëfficients $\boldsymbol{b} : \mathbb{R} \times [0, 2\pi) \to \mathbb{R}$ and $\boldsymbol{a} : \mathbb{R} \times [0, 2\pi) \to (0, \infty)$ which may depend on the angular variable $\theta \in [0, 2\pi)$ in (6.2). We assume also that $\boldsymbol{a}$ is bounded away from zero, and consider the local martingale problem of subsection 6.2 but now with the infinitesimal generator re-defined as

$$\mathcal{L}^* g(x) \;:=\; \boldsymbol{b}\big(\|x\|, \arg(x)\big)\, G'(x) + \frac{1}{2}\, \boldsymbol{a}\big(\|x\|, \arg(x)\big)\, G''(x)\,; \qquad x \in \mathbb{R}^2\,, \quad g \in \mathfrak{D}\,. \tag{8.1}$$

For every given, fixed $\theta \in [0, 2\pi)$, we set $\boldsymbol{\sigma}_\theta(r) := \boldsymbol{\sigma}(r, \theta)$ as well as $\boldsymbol{a}(r, \theta) = [\boldsymbol{\sigma}(r, \theta)]^2$, and define the scale function $\boldsymbol{p}_\theta(\cdot)$ by

$$\boldsymbol{p}_\theta(r) \;=\; \boldsymbol{p}(r, \theta) \;:=\; \int_0^r \exp\Big(-2 \int_0^\xi \frac{\boldsymbol{b}(\zeta, \theta)}{\boldsymbol{a}(\zeta, \theta)} \mathrm{d}\zeta \Big) \mathrm{d}\xi\,, \qquad r \in [0, \infty)\,,$$

as well as its inverse $\boldsymbol{q}_\theta(r) = \boldsymbol{q}(r, \theta)$ in the radial component with $\boldsymbol{q}_\theta(\boldsymbol{p}_\theta(r)) = r$. We note that these functions satisfy $\boldsymbol{p}_\theta(0) = 0 = \boldsymbol{q}_\theta(0)$ and $\boldsymbol{p}'_\theta(0+) = 1 = \boldsymbol{q}'_\theta(0+)$; that $\boldsymbol{p}_\theta(\cdot)$ has an absolutely continuous, strictly positive derivative; that the second derivative $\boldsymbol{p}''_\theta(\cdot)$ exists almost everywhere; and that both of these derivatives are bounded. Therefore, by the generalized ITÔ rule, we see that Theorem 4.1 holds also for the function $\boldsymbol{p}_\theta(\cdot)$, which may not be in the class $\mathfrak{D}$; the same is true for the function $\boldsymbol{q}_\theta(\cdot)$.

Let us consider an auxiliary diffusion coëfficient

$$\widetilde{\boldsymbol{\sigma}}_\theta(r) \;\equiv\; \widetilde{\boldsymbol{\sigma}}(r, \theta) \;:=\; \boldsymbol{p}'_\theta(\boldsymbol{q}_\theta(r))\, \boldsymbol{\sigma}_\theta(\boldsymbol{q}_\theta(r))\,, \qquad 0 < r < \infty \tag{8.2}$$

and $\theta \in [0, 2\pi)$, and write $\widetilde{\boldsymbol{\sigma}}(y) \equiv \widetilde{\boldsymbol{\sigma}}(r, \theta)$ for $y = (r, \theta) \in \mathbb{R}^2$. We introduce also the stochastic clock

$$\mathcal{Q}(\cdot) := \int_0^{\cdot} \frac{\mathrm{d}u}{\big[\widetilde{\boldsymbol{\sigma}}\big(\|X(u)\|, \Theta(u)\big)\big]^2} \qquad \text{and its inverse} \quad \mathcal{T}(t) := \inf\big\{ v \geq 0 : \mathcal{Q}(v) > t \big\}\,; \quad 0 \leq t < \infty\,.$$

Here $X(\cdot) = Z(\cdot) S(\cdot)$ is a WALSH semimartingale as constructed as in (5.2), starting from a one-dimensional Brownian motion $U(\cdot) = B(\cdot)$ in Proposition 7.2. In particular, $X(\cdot)$ is WALSH Brownian motion; whereas $\Theta(\cdot) = \arg(X(\cdot))$ is as in (5.5). We consider now the time-changed, rescaled version $Y(\cdot) = (Y_1(\cdot), Y_2(\cdot))'$ of this WALSH Brownian motion $X(\cdot)$, defined in polar coördinates via

$$\|Y(\cdot)\| \;:=\; \boldsymbol{q}\big(\|X(\mathcal{T}(\cdot))\|,\; \arg(X(\mathcal{T}(\cdot)))\big)\,, \qquad \arg(Y(\cdot)) \;:=\; \arg(X(\mathcal{T}(\cdot))) \;=\; \Theta(\mathcal{T}(\cdot))\,. \tag{8.3}$$

In terms of this rescaling, we have the representation

$$\mathcal{T}(\cdot) = \int_0^{\cdot} \big(\boldsymbol{p}_\theta'(r)\, \boldsymbol{\sigma}_\theta(r)\big)^2 \Big|_{\theta=\arg(Y(t)),\, r=\|Y(t)\|} \,\mathrm{d}t \tag{8.4}$$

for the inverse clock. The resulting process $Y(\cdot)$ turns out to be a WALSH semimartingale with angular dependence in its local characteristics $(\boldsymbol{\sigma}, \boldsymbol{b}, \boldsymbol{\mu})$, as follows.

Proposition 8.1. *The process $Y(\cdot)$ defined in (8.3) satisfies the integral equations*

$$Y(\cdot) \,=\, Y(0) + \int_0^{\cdot} \mathfrak{f}(Y(t)) \big[\, \boldsymbol{\sigma}\big(\|Y(t)\|, \mathit{arg}(Y(t))\big)\, \mathrm{d}W(t) + \boldsymbol{b}\big(\|Y(t)\|, \mathit{arg}(Y(t))\big) \mathrm{d}t \,\big] + \boldsymbol{\gamma}\, L^{\|Y\|}(\cdot) \tag{8.5}$$

$$\|Y(\cdot)\| \,=\, \|Y(0)\| + \int_0^{\cdot} \mathbf{1}_{\{\|Y(t)\|>0\}} \big(\boldsymbol{b}(\|Y(t)\|, \mathit{arg}(Y(t)))\mathrm{d}t + \boldsymbol{\sigma}(\|Y(t)\|, \mathit{arg}(Y(t)))\mathrm{d}W(t)\big) + L^{\|Y\|}(\cdot)$$

as well as the properties

$$\int_0^{\cdot} \mathbf{1}_{\{Y(t)=0\}} \mathrm{d}t \,\equiv\, 0 \qquad \textit{and} \qquad L^{R_*^A}(\cdot) \,\equiv\, \boldsymbol{\nu}(A)\, L^{\|Y\|}(\cdot)\,, \;\; \forall\;\, A \in \mathcal{B}([0, 2\pi)).$$

It induces a solution to the local martingale problem associated with the triple $(\boldsymbol{\sigma}, \boldsymbol{b}, \boldsymbol{\mu})$ and $\mathcal{L}^*$ in (8.1).

In the above expressions $\mathfrak{f} = (\mathfrak{f}_1, \mathfrak{f}_2)'$ is defined in (2.8), $W(\cdot)$ is one-dimensional Brownian motion, and the "thinned" process $R_*^A(\cdot) := \|Y(\cdot)\| \cdot \mathbf{1}_A(\mathit{arg}(Y(\cdot))$ is defined for $A \in \mathcal{B}([0, 2\pi))$.

Proof: Applying the FREIDLIN-SHEU formula in Theorem 4.1 to $\boldsymbol{q}(X(\cdot))$, we obtain

$$\|Y(\cdot)\| \,=\, \boldsymbol{q}(X(\mathcal{T}(\cdot))) \,=\, \boldsymbol{q}(\mathrm{x}) + \int_0^{\mathcal{T}(\cdot)} \boldsymbol{Q}'(X(u))\, \mathbf{1}_{\{X(u)\neq\mathbf{0}\}} \mathrm{d}B(u) + \Big(\int_0^{2\pi} \boldsymbol{q}_\theta'(0+)\boldsymbol{\nu}(\mathrm{d}\theta) \Big) \cdot L^{\|X\|}(\mathcal{T}(\cdot))$$

$$+ \,\frac{1}{2} \int_0^{\mathcal{T}(\cdot)} \boldsymbol{Q}''(X(u))\, \mathbf{1}_{\{X(u)\neq 0\}} \mathrm{d}u\,. \tag{8.6}$$

Here by direct calculation

$$\boldsymbol{Q}'(x) \,:=\, \boldsymbol{q}_\theta'(r) \,=\, \frac{1}{\boldsymbol{p}_\theta'\big(\boldsymbol{q}_\theta(r)\big)}\,, \qquad \boldsymbol{Q}''(x) \,:=\, \boldsymbol{q}_\theta''(r) \,=\, \frac{2\, \boldsymbol{b}(\boldsymbol{q}_\theta(r), \theta)}{\boldsymbol{a}\big(\boldsymbol{q}_\theta(r), \theta\big) \cdot \big(\boldsymbol{p}_\theta'(\boldsymbol{q}_\theta(r))\big)^2} \tag{8.7}$$

hold for every $x = (r, \theta)$, where r is not in a set of Lebesgue measure zero that depends on $\theta \in [0, 2\pi)$. Thanks to the P. LÉVY Theorem, the continuous local martingale

$$W(\cdot) \,:=\, \int_0^{\mathcal{T}(\cdot)} \frac{\mathrm{d}B(u)}{\widetilde{\boldsymbol{\sigma}}(\|X(u)\|, \arg(X(u)))}$$

is one-dimensional standard Brownian motion. Since $\mathrm{Leb}(\{t : X(t) = \mathbf{0}\}) \,=\, \mathrm{Leb}(\{t : S(t) = 0\}) = 0$ a.s. and $\boldsymbol{q}(\mathbf{0}) \,=\, 0$ from the construction, we obtain

$$\mathrm{Leb}\big(\{t : \|Y(t)\| = \boldsymbol{q}(X(\mathcal{T}(t))) = \mathbf{0}\}\big) \,=\, \mathrm{Leb}\big(\mathcal{T}^{-1}\{t : X(t) \,=\, \mathbf{0}\}\big) \,=\, 0 \;\text{ a.s.} \tag{8.8}$$

In conjunction with the definitions (8.2) and (8.3), we obtain now the representations

$$\int_0^{\mathcal{T}(\cdot)} \boldsymbol{Q}'(X(u))\, \mathbf{1}_{\{X(u)\neq\mathbf{0}\}} \mathrm{d}B(u) \,=\, \int_0^{\cdot} \boldsymbol{Q}'(X(\mathcal{T}(u)))\, \mathbf{1}_{\{X(\mathcal{T}(u))\neq\mathbf{0}\}}\, \mathrm{d}B(\mathcal{T}(u))$$

$$= \int_0^{\cdot} \mathbf{1}_{\{\|Y(u)\|>0\}}\, \boldsymbol{\sigma}\big(\|Y(u)\|, \arg(Y(u))\big)\, \mathrm{d}W(u) \tag{8.9}$$

(on the strength of Proposition 3.4.8 in KARATZAS & SHREVE (1991)), as well as

$$\int_0^{\mathcal{T}(\cdot)} \boldsymbol{Q}''(X(u))\, \mathbf{1}_{\{X(u)\neq 0\}}\, \mathrm{d}u \,=\, \int_0^{\cdot} \boldsymbol{Q}''(X(\mathcal{T}(u)))\, \frac{\mathrm{d}\mathcal{T}(u)}{\mathrm{d}u}\, \mathbf{1}_{\{X(\mathcal{T}(u))\neq\mathbf{0}\}}\, \mathrm{d}u$$

$$= 2\int_0^{\cdot} \mathbf{1}_{\{\|Y(u)\|>0\}}\, \boldsymbol{b}\big(\|Y(u)\|, \arg(Y(u)\big)\, \mathrm{d}u \tag{8.10}$$

(by time-change). From these considerations and (8.8) we also obtain the identification of local time

$$L^{\|Y\|}(\cdot) = \int_0^{\cdot} \mathbf{1}_{\{Y(u)=0\}}\, \mathrm{d}\|Y\|(u) = \Big(\int_0^{2\pi} \boldsymbol{q}_\theta'(0+)\boldsymbol{\nu}(\mathrm{d}\theta)\Big)\cdot L^{\|X\|}(\mathcal{T}(\cdot))\ , \tag{8.11}$$

thus also the dynamics for the radial part of the process $Y(\cdot)$, namely

$$\|Y(\cdot)\| = \|Y(0)\| + \int_0^{\cdot} \mathbf{1}_{\{\|Y(t)\|>0\}}\, \big(\boldsymbol{b}(\|Y(t)\|, \arg(Y(t)))\mathrm{d}t + \boldsymbol{\sigma}(\|Y(t)\|, \arg(Y(t)))\mathrm{d}W(t)\big) + L^{\|Y\|}(\cdot).$$

• Recalling (8.3), and applying the FREIDLIN-SHEU formula in Theorem 4.1 to the process $Y_i(\cdot) = \boldsymbol{q}(X(\mathcal{T}(\cdot)))\,\mathfrak{f}_i(X(\mathcal{T}(\cdot)))$, we obtain

$$Y_i(\cdot) = \mathrm{y}_i + \int_0^{\mathcal{T}(\cdot)} \boldsymbol{Q}'(X(u))\mathfrak{f}_i(X(u))\, \mathbf{1}_{\{X(u)\neq \mathbf{0}\}}\mathrm{d}B(u) + \frac{1}{2}\int_0^{\mathcal{T}(\cdot)} \boldsymbol{Q}''(X(u))\mathfrak{f}_i(X(u))\mathbf{1}_{\{X(u)\neq \mathbf{0}\}}\mathrm{d}u\,,$$

$$+\Big(\int_0^{2\pi} \boldsymbol{q}_\theta'(0+)\cos\Big(\theta - \frac{\pi}{2}(i-1)\Big)\boldsymbol{\nu}(\mathrm{d}\theta)\Big)L^{\|X\|}(\mathcal{T}(\cdot))\,; \quad i = 1, 2\,.$$

Hence, combining this with (8.9)-(8.11), as well as $\mathfrak{f}(\mathbf{0}) = 0$ and $\boldsymbol{q}_\theta'(0+) = 1$, we obtain the desired dynamics (8.5).

• Furthermore, for every $g \in \mathfrak{D}$ by another application of the FREIDLIN-SHEU formula in Theorem 4.1 to $g(Y(\cdot)) = g(\boldsymbol{q}(r,\theta),\theta)\big|_{r=\|X(\mathcal{T}(\cdot)\|\,,\,\theta=\arg(X(\mathcal{T}(\cdot)))}$ with $\boldsymbol{q}_\theta(0+) = 0$, we derive

$$g(Y(T)) = g(\mathrm{y}) + \int_0^T \mathbf{1}_{\{Y(t)\neq\mathbf{0}\}}\, \big(\boldsymbol{b}(\|Y(t)\|, \arg(Y(t)))G'(Y(t)) + \frac{1}{2}\boldsymbol{a}(\|Y(t)\|, \arg(Y(t)))G''(Y(t))\big)\mathrm{d}t$$

$$+\int_0^T \mathbf{1}_{\{Y(t)\neq\mathbf{0}\}}\, G'(Y(t))\boldsymbol{\sigma}(\|Y(t)\|, \arg(Y(t)))\mathrm{d}W(t) + \Big(\int_0^{2\pi} g_\theta'(0+)\,\boldsymbol{\nu}(\mathrm{d}\theta)\Big)\cdot L^{\|Y\|}(T)\,, \quad 0 \le T < \infty\,, \tag{8.12}$$

in conjunction with (8.7)-(8.11) and $\boldsymbol{q}_\theta'(0+) = 1$. When $g \in \mathfrak{D}^{\boldsymbol{\mu}}$, we can apply this to $M^g(\cdot\,;Y)$ in (6.2) – now redefined with the operator $\mathcal{L}^*$ of (8.1) – to conclude that $M^g(\cdot\,;Y)$ is equal to the local martingale

$$g(Y(\cdot)) - g(\mathrm{y}) - \int_0^{\cdot} \mathcal{L}^* g(Y(t))\,\mathbf{1}_{\{Y(t)\neq\mathbf{0}\}}\mathrm{d}t = \int_0^{\cdot} G'(Y(t))\,\mathbf{1}_{\{Y(t)\neq\mathbf{0}\}}\,\boldsymbol{\sigma}\big(\|Y(t)\|, \arg(Y(t))\big)\,\mathrm{d}W(t).$$

Therefore, the local martingale problem associated with the triple $(\boldsymbol{\sigma}, \boldsymbol{b}, \boldsymbol{\mu})$ and the second-order differential operator $\mathcal{L}^*$ in (8.1), is seen to have a solution. The properties of $Y(\cdot)$ are now verified readily. $\square$

Proposition 8.2. *With the assumptions and notation of this section, the local martingale problem of subsection 6.2, associated with the triple $(\boldsymbol{\sigma}, \boldsymbol{b}, \boldsymbol{\mu})$ and the operator $\mathcal{L}^*$ in (8.1), is well-posed.*

Proof: Existence of a solution to this local martingale problem is established by Proposition 8.1.

To prove uniqueness, we can reverse the steps of the construction in Proposition 8.1, as follows. Consider *any* solution of the local martingale problem of subsection 6.2, associated with the triple and $(\boldsymbol{\sigma}, \boldsymbol{b}, \boldsymbol{\mu})$ and the operator $\mathcal{L}^*$, and the coördinate process $Y(\cdot) := \omega_2(\cdot)$ on the canonical space for that problem. We introduce the time change $\mathcal{T}(\cdot)$ as in (8.4), along with its inverse $\mathcal{Q}(\cdot)$; as well as the time-changed, rescaled version $X(\cdot) = (X_1(\cdot), X_2(\cdot))'$ of the process $Y(\cdot)$, defined in polar coördinates via

$$\|X(\cdot)\| := \boldsymbol{p}\big(\|Y(\mathcal{Q}(\cdot))\|,\ \arg(Y(\mathcal{Q}(\cdot)))\big)\,, \qquad \arg(X(\cdot)) := \arg(Y(\mathcal{Q}(\cdot)))\,. \tag{8.13}$$

Using Proposition 6.1 (rather, its obvious generalization to coëfficients with angular dependence) and Theorem 4.1, we have for the planar process $Y(\cdot)$ the appropriate FREIDLIN-SHEU-formula. With this at hand, the planar process $X(\cdot)$ is seen to be a WALSH Brownian motion with spinning measure $\boldsymbol{\mu}$, in a manner similar to that in the proof of Proposition 8.1. The path $t \mapsto X(t)$ is, with probability one, continuous in the topology induced by the tree metric (3.3), and hence so is the path $t \mapsto Y(t)$. In terms of this WALSH Brownian motion, we can express the time change $\mathcal{Q}(\cdot)$ as

$$\mathcal{Q}(\cdot) = \int_0^{\cdot} \frac{\mathrm{d}u}{\big[\widetilde{\boldsymbol{\sigma}}\big(\|X(u)\|, \arg(X(u))\big)\big]^2}\,.$$

The crucial step now, is to note that the process $Y(\cdot)$ can be written as $Y(t) = \Psi_t(X(\cdot))$. Here $\Psi_\cdot$ is a measurable mapping defined by $\Psi_t(\omega_2) = \boldsymbol{q}\big(\Pi_{\mathcal{T}(t;\omega_2)}(\omega_2)\big)$, in terms of the measurable projection mapping $\Pi_t(\omega_2) := \omega_2(t)$ and the continuous time change

$$\mathcal{T}(t;\omega_2) := \inf\left\{ v \geq 0 \,:\, \int_0^v \frac{\mathrm{d}u}{\big[\widetilde{\boldsymbol{\sigma}}(\|\omega_2(u)\|, \arg(\omega_2(u)))\big]^2} > t \right\}, \qquad 0 \leq t < \infty\,.$$

Since the distribution of the WALSH Brownian motion $X(\cdot)$ is uniquely determined (see section 7), the distribution of $Y(\cdot)$ is also determined uniquely from these considerations.

We conclude that the local martingale problem associated with the triple $(\boldsymbol{\sigma}, \boldsymbol{b}, \boldsymbol{\mu})$ is well-posed. □

9 The Time-Homogeneous Strong MARKOV Property

From section 7, we know that the unique solution to the well-posed local martingale problem associated with the triple $(\mathbf{1}, \mathbf{0}, \boldsymbol{\mu})$ induces a WALSH Brownian motion, which is a time-homogeneous strong MARKOV process as shown in BARLOW, PITMAN & YOR (1989a). We generalize this result in subsection 9.1, by showing that every solution to a well-posed local martingale problem as in subsection 6.2, associated with a triple $(\boldsymbol{\sigma}, \boldsymbol{b}, \boldsymbol{\mu})$, induces a time-homogeneous strong MARKOV process.

Next, we try to pick up the thread of Part (a) in Proposition 6.1, and see what we can say about solutions to *the system of stochastic equations (6.4) for given* $(\gamma_1, \gamma_2) \in \mathbb{R}^2$*, subject only to the condition (2.17).* We find that for some such solutions there is no "spinning measure" $\boldsymbol{\mu}$ such that (2.18) is satisfied. We show that the time-homogeneous strong MARKOV property can be used to rule out these solutions. Then for every solution with an appropriate version of this property, we prove the existence of a "spinning measure" $\boldsymbol{\mu}$ for which the local martingale problem associated with the triple $(\boldsymbol{\sigma}, \boldsymbol{b}, \boldsymbol{\mu})$ is solved by the distribution of the state process $X(\cdot)$ in the solution. In this spirit we obtain in subsection 9.2 a similar conclusion as in Part (a) of Proposition 6.1, but with the notable difference that here $\boldsymbol{\mu}$ is not given in advance; its existence is established in the proof of Theorem 9.1, the third major result of this work. As a corollary of this result, we show in subsection 9.3 that with $\boldsymbol{b} = \mathbf{0}$, $\boldsymbol{\sigma} = \mathbf{1}$ the equations (6.4), subject to (2.17) and to the time-homogeneous strong MARKOV property, characterize WALSH Brownian motions.

Throughout this section, we shall always refer to subsection 6.1 for local submartingale problems associated with pairs $(\boldsymbol{\sigma}, \boldsymbol{b})$ (corresponding to one-dimensional reflected diffusions), and to subsection 6.2 for local martingale problems associated with triples $(\boldsymbol{\sigma}, \boldsymbol{b}, \boldsymbol{\mu})$ (corresponding to planar diffusions).

9.1 On Well-posed Local Martingale Problems

Definition 9.1. Given a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{F} = \big\{\mathcal{F}(t)\big\}_{0 \leq t < \infty}$, we say that a progressively measurable process $X(\cdot)$ with values in some Euclidean space $\mathbb{R}^d$ is *time-homogeneous strongly Markovian* with respect to it if, for every stopping time T of $\mathbb{F}$, real number $t \geq 0$, and set $\Gamma \in \mathcal{B}(\mathbb{R}^d)$,

$$\mathbb{P}\big(X(T+t)\in\Gamma\,|\,\mathcal{F}(T)\big) \;=\; \mathbb{P}\big(X(T+t)\in\Gamma\,|\,X(T)\big) \;=\; \mathfrak{g}\big(X(T)\big) \quad \text{holds } \mathbb{P}-\text{a.e. on } \{T<\infty\}\,.$$

Here $\mathfrak{g}:\mathbb{R}^d\to\mathbb{R}$ is a bounded measurable function that depends on t and Γ, but not on T.

It is clear that every strong MARKOV process with a one-parameter transition semigroup is time-homogeneous strongly Markovian. Also, a diffusion, or a strong MARKOV family, is time-homogeneous strongly Markovian under every probability measure in the family (see Definition 5.1, Chapter IV of IKEDA & WATANABE (1989), and Definition 2.6.3 in KARATZAS & SHREVE (1991)). We show here that every solution to a well-posed local martingale problem associated with the triple $(\boldsymbol{\sigma},\boldsymbol{b},\boldsymbol{\mu})$ induces a time-homogeneous strongly Markovian process. This is an extension of Theorem 5.4.20 in KARATZAS & SHREVE (1991) in the context of subsection 6.2. Its proof given here is in the same context.

Proposition 9.1. *Suppose that the local martingale problem associated with the triple $(\boldsymbol{\sigma},\boldsymbol{b},\boldsymbol{\mu})$ is well-posed, and let $\mathbb{Q}^{\mathrm{x}}$ be its solution with $\omega_2(0)=\mathrm{x}$, $\mathbb{Q}^{\mathrm{x}}-$a.e.*

Then for every stopping time T of $\mathbb{F}_2$, $C\in\mathcal{F}_2$, and $\mathrm{x}\in\mathbb{R}^2$, the process $\omega_2(\cdot)$ satisfies the property

$$\mathbb{Q}^{\mathrm{x}}\big(\theta_T^{-1}C\,\big|\,\mathcal{F}_2(T)\big)(\omega_2)=\mathbb{Q}^{\omega_2(T)}\big(C\big)\,,\qquad \mathbb{Q}^{\mathrm{x}}-\text{a.e. on } \{T<\infty\},$$

where θ_T is the shift operator $(\theta_T(\omega_2))(\cdot)=\omega_2(T(\omega_2)+\cdot)$. In particular, $\omega_2(\cdot)$ is time-homogeneous strongly Markovian with respect to $(\Omega_2,\mathcal{F}_2,\mathbb{Q}^{\mathrm{x}})$ and the filtration $\mathbb{F}_2$, for every $\mathrm{x}\in\mathbb{R}^2$.

We shall need a countable determining class for our local martingale problem, so we introduce it next. A crucial result in this regard, Lemma 9.1 below, is proved in an Appendix, section 11.

Definition 9.2. We shall denote by $\mathfrak{E}\subseteq\mathfrak{D}^{\boldsymbol{\mu}}_+$ the collection that consists of
(i) the functions $g_A(x):=\|x\|\,\big(\mathbf{1}_A(\arg(x))-\boldsymbol{\nu}(A)\big)$ as in (6.23), where $A\subset[0,2\pi)$ is of the form $[a,b)$ and a,b are rational numbers; and of
(ii) the following functions in $\mathfrak{D}^{\boldsymbol{\mu}}_+$ used in the proof of Part (b) of Proposition 6.1: namely, g_1, g_2, $g_{i,k}$, $1\le i,k\le 2$ in (6.13); $g^{\circ}_{1,1}$, $g^{\circ}_{2,2}$, g_3 in (6.14); as well as, for every rational $c_1>0$, a function $g_4\in\mathfrak{D}^{\boldsymbol{\mu}}$ of the form $g_4(r,\theta)=\psi(r)$ where $\psi:[0,\infty)\to[0,\infty)$ is smooth with $\psi(r)=r$ for $r\ge c_1$.

This way, we ensure that $\mathfrak{E}$ is a *countable collection.* □

Lemma 9.1. *Suppose $\mathbb{Q}$ is a probability measure on $(\Omega_2,\mathcal{F}_2)$ with $\omega_2(0)=\mathrm{x}$, $\mathbb{Q}-$a.e., under which $M^g(\cdot\,;\omega_2)$ is a continuous local martingale (resp., submartingale) of the filtration $\mathbb{F}_2$ for every function $g\in\mathfrak{D}^{\boldsymbol{\mu}}\cap\mathfrak{E}$ (resp., $\mathfrak{E}$). Then this is also true for every function $g\in\mathfrak{D}^{\boldsymbol{\mu}}$ (resp., $\mathfrak{D}^{\boldsymbol{\mu}}_+$).*

Proof of Proposition 9.1: We proceed as in the proof of Theorem 5.4.20, including Lemma 5.4.18 and Lemma 5.4.19, in KARATZAS & SHREVE (1991). It is easy to check that all the arguments there apply to our context (with some standard localization and application of optional sampling to submartingales), except for the final step of the proof of Lemma 5.4.19. To get through it, we only need to find a countable collection $\mathfrak{E}\subset\mathfrak{D}^{\boldsymbol{\mu}}_+$ with the property that, in order to show that $M^g(\cdot\,;\omega_2)$ is a continuous local martingale (resp., submartingale) for every function $g\in\mathfrak{D}^{\boldsymbol{\mu}}$ (resp., $\mathfrak{D}^{\boldsymbol{\mu}}_+$), it suffices to have these properties for all functions in $\mathfrak{E}$. We appeal now to Lemma 9.1, and the proof of Proposition 9.1 follows. □

9.2 Time-homogeneous Strongly Markovian Solutions to (6.4), under only (2.17)

Let us recall Part (a) of Proposition 6.1. Suppose that we do not specify a measure $\boldsymbol{\mu}$ in advance, and that condition (2.18) is not imposed. In particular, *with given* BOREL*-measurable functions $\boldsymbol{b}:[0,\infty)\to\mathbb{R}$, $\boldsymbol{\sigma}:[0,\infty)\to\mathbb{R}\setminus\{0\}$ and real numbers γ_i, $i=1,2$, we consider the system of stochastic equations (6.4) subject only to the condition (2.17).*

From Part (b) of Proposition 6.1 we know that, for a probability measure $\boldsymbol{\mu}$ on $(\mathfrak{S},\,\mathcal{B}(\mathfrak{S}))$ with

$$\gamma_i = \int_{\mathfrak{S}} \mathfrak{f}_i(z)\, \boldsymbol{\mu}(\mathrm{d}z)\,, \qquad i = 1, 2\,, \tag{9.1}$$

every solution to the local martingale problem associated with the triple $(\boldsymbol{\sigma}, \boldsymbol{b}, \boldsymbol{\mu})$ induces a solution to the system (6.4), subject to (2.17). *But can we obtain all the solutions of (6.4), (2.17) in this way?*

The answer is negative: There are usually several probability measures $\boldsymbol{\mu}$ satisfying (9.1), so we can construct a solution to (6.4) that satisfies (2.17) and features two different "spinning measures", both satisfying (9.1). Then this solution is not related to that of a local martingale problem associated with the triple $(\boldsymbol{\sigma}, \boldsymbol{b}, \boldsymbol{\mu})$, for *any* $\boldsymbol{\mu}$. The construction will be given in detail at the end of this subsection (Remark 9.3).

Interestingly, if we restrict our scope to solutions with some appropriate time-homogeneous strong MARKOV properties, then each solution to (6.4) subject to (2.17) *is* related to that of a local martingale problem associated with the triple $(\boldsymbol{\sigma}, \boldsymbol{b}, \boldsymbol{\mu})$, for some $\boldsymbol{\mu}$ that depends on this solution. This is the main result of the present subsection, Theorem 9.1 below.

Before stating this result, we note that Proposition 2.6.6 (c') in KARATZAS & SHREVE (1991), proved for strong MARKOV families, admits a version for continuous, time-homogeneous strongly Markovian processes with exactly the same proof. We state this version here; it will be used several times in what follows.

Proposition 9.2. *Suppose $X(\cdot)$ is continuous and time-homogeneous strongly Markovian, in the sense of Definition 9.1. Then for every set $B \in \mathcal{B}(C[0,\infty)^d)$ we have*

$$\mathbb{P}\big(X(T+\cdot) \in B \,|\, \mathcal{F}(T) \big) \;=\; \mathbb{P}\big(X(T+\cdot) \in B \,|\, X(T)\big) \;=\; \mathfrak{h}(X(T))\,, \quad \mathbb{P}-\textit{a.e. on } \{T<\infty\}\,.$$

for some bounded, measurable function $\mathfrak{h} : \mathbb{R}^d \to \mathbb{R}$ that depends on the set B, but not on T.

We give now the main result of this section. Its proof is given in an Appendix, section 12.

Theorem 9.1. *Let us consider a weak solution $(X(\cdot), W(\cdot))$, $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{F} = \big\{\mathcal{F}(t)\big\}_{0 \le t < \infty}$ to the system of equations of (6.4) for some given real numbers γ_1, γ_2, namely*

$$X_i(\cdot) \;=\; X_i(0) + \int_0^{\cdot} \mathfrak{f}_i(X(t)) \Big[\boldsymbol{b}(\|X(t)\|)\mathrm{d}t \;+\; \boldsymbol{\sigma}(\|X(t)\|)\mathrm{d}W(t) \Big] + \gamma_i\, L^{\|X\|}(\cdot)\,, \quad i = 1, 2\,.$$

(i) *Then the radial part $\|X(\cdot)\|$ of the state process solves the equation (6.5), namely*

$$\|X(\cdot)\| \;=\; \|X(0)\| + \int_0^{\cdot} \mathbf{1}_{\{\|X(t)\|>0\}} \Big(\boldsymbol{b}(\|X(t)\|)\mathrm{d}t + \boldsymbol{\sigma}(\|X(t)\|)\mathrm{d}W(t) \Big) + L^{\|X\|}(\cdot)\,.$$

(ii) *If both $X(\cdot)$ and $\|X(\cdot)\|$ are time-homogeneous, strongly Markovian processes with respect to $\mathbb{F}^X = \big\{\mathcal{F}^X(t)\big\}_{0 \le t < \infty}$ with the condition (2.17), then there exists a probability measure $\boldsymbol{\mu}$ on $(\mathfrak{S}, \mathcal{B}(\mathfrak{S}))$ such that $X(\cdot)$ induces a solution to the local martingale problem associated with the triple $(\boldsymbol{\sigma}, \boldsymbol{b}, \boldsymbol{\mu})$.*

(iii) *If, in addition, the state process of this weak solution satisfies the analogue*

$$\mathbb{P}\big(L^{\|X\|}(\infty) > 0\big) \;>\; 0 \tag{9.2}$$

of the condition (3.6), then the measure $\boldsymbol{\mu}$ in (ii) is uniquely determined by $X(\cdot)$ and must satisfy (9.1).

Under appropriate conditions on $(\boldsymbol{\sigma}, \boldsymbol{b})$ in (6.4), we will only need $X(\cdot)$ itself to be time-homogeneous and strongly Markovian with respect to $\mathbb{F}^X$ in Theorem 9.1 **(ii)**. The following lemma guarantees this.

Lemma 9.2. *With the setting and the assumptions of Theorem 9.1, suppose that the local submartingale problem of subsection 6.1 associated with the pair $(\boldsymbol{\sigma}, \boldsymbol{b})$ is well-posed.*

Then $\|X(\cdot)\|$ is time-homogeneous strongly Markovian with respect to $\mathbb{F}^X$.

Proof of Lemma 9.2: We obtain the equation (6.5) for the radial part $\|X(\cdot)\|$ from Theorem 9.1 **(i)**. Applying ITÔ's formula to it in the context of subsection 6.1, we see that for every function $\psi \in C^2([0,\infty);\mathbb{R})$ with $\psi'(0+) \geq 0$ (resp. $\psi'(0+) = 0$), the process $K^{\psi}(\cdot\,; \|X(\cdot)\|)$ is a continuous local submartingale (resp. martingale) with respect to the filtration $\mathbb{F}$. This process is also adapted to $\mathbb{F}^X$ and $\mathcal{F}^X(t) \subseteq \mathcal{F}(t)$ holds for all $t \geq 0$, so the statement in the last sentence still holds with $\mathbb{F}$ replaced by $\mathbb{F}^X$.

Following the idea of Lemma 5.4.18 and Lemma 5.4.19 in KARATZAS & SHREVE (1991), we denote by $\mathbb{Q}_\omega(A) = \mathbb{Q}(\omega; A) : \Omega \times \mathcal{F} \mapsto [0,1]$ the regular conditional probability for $\mathcal{F}$ given $\mathcal{F}^X(T)$, where T is a bounded stopping time of $\mathbb{F}^X$. For every $\omega \in \Omega$, define the probability measure $\mathbb{P}_\omega$ on $\big(C[0,\infty), \mathcal{B}\big(C[0,\infty)\big)\big)$ by $\mathbb{P}_\omega(F) := \mathbb{Q}_\omega\big(\|X(T+\cdot)\| \in F\big),\ \forall\, F \in \mathcal{B}(C[0,\infty))$.

With this notation and the conclusion in the first paragraph of this proof, we can follow the arguments in the aforementioned two lemmas to show that for a.e. $\omega \in \Omega$, the probability measure $\mathbb{P}_\omega$ solves the local submartingale problem associated with the pair $(\boldsymbol{\sigma}, \boldsymbol{b})$, starting at $\|X(T,\omega)\|$. Combining this with the well-posedness of the local submartingale problem, we prove Lemma 9.2 by applying the proof of Theorem 5.4.20 in KARATZAS & SHREVE (1991). □

Remark 9.1. Just as in the proof of Proposition 9.1, the above argument needs a "countable representatives" result like Lemma 9.1. Here it suffices to take functions of the form $f(x) = x$, $g(x) = x^2$, and for every $n \in \mathbb{N}$ a function $f_n(\cdot)$ such that $f_n'(0+) = 0$ and $f_n(x) = x$ for $x \geq (1/n)$.

In conjunction with Lemma 9.2, Theorem 9.1 has the following corollary.

Corollary 9.1. *Suppose that the conditions (2.17), (9.2) are satisfied by a weak solution* $(X(\cdot), W(\cdot))$, $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{F} = \big\{\mathcal{F}(t)\big\}_{0 \leq t < \infty}$ *of the system of equations (6.4) for some given real numbers* γ_1, γ_2, *and that the local submartingale problem associated with the pair* $(\boldsymbol{\sigma}, \boldsymbol{b})$ *is well-posed.*

If $X(\cdot)$ *is time-homogeneous and strongly Markovian with respect to* $\mathbb{F}^X$, *then it determines a probability measure* $\boldsymbol{\mu}$ *on* $(\mathfrak{S}, \mathcal{B}(\mathfrak{S}))$ *which satisfies (9.1), and such that* $X(\cdot)$ *induces a solution to the local martingale problem associated with the triple* $(\boldsymbol{\sigma}, \boldsymbol{b}, \boldsymbol{\mu})$.

9.3 The case of WALSH Brownian Motion

Let us specialize the system of equations (6.4) to the case $\boldsymbol{b} = \boldsymbol{0}$, $\boldsymbol{\sigma} = \boldsymbol{1}$ as in Proposition 7.2, namely

$$X_i(\cdot) \,=\, \mathrm{x}_i + \int_0^{\cdot} \mathfrak{f}_i\big(X(t)\big)\,\mathrm{d}W(t) + \gamma_i\, L^{\|X\|}(\cdot)\,, \qquad i = 1, 2\,. \tag{9.3}$$

We shall show that when $\gamma_1^2 + \gamma_2^2 \leq 1$ this system, coupled with the time-homogeneous strong MARKOV property, characterizes WALSH Brownian motions under the "non-stickiness" condition (2.17). We note that in the statement and proof of the next proposition, neither (γ_1, γ_2) nor $\boldsymbol{\mu}$ are specified in advance. Therefore, we view (9.1) as a relationship between $\boldsymbol{\mu}$ and (γ_1, γ_2).

Proposition 9.3. *Assume that* $Z(\cdot)$ *is a continuous planar process on some filtered probability space* $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbb{P}})$, $\widetilde{\mathbb{F}} = \big\{\widetilde{\mathcal{F}}(t)\big\}_{0 \leq t < \infty}$. *Then the following two assertions are equivalent:*

(i) $Z(\cdot)$ *is a* WALSH *Brownian motion, defined via the semigroup (7.1), for some spinning measure* $\boldsymbol{\mu}$.

(ii) For a pair of real numbers (γ_1, γ_2) *with* $\gamma_1^2 + \gamma_2^2 \leq 1$, *there exists a weak solution* $(X(\cdot), W(\cdot))$, $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{F} = \big\{\mathcal{F}(t)\big\}_{0 \leq t < \infty}$ *to the system of equations (9.3), such that* $X(\cdot)$*: is time-homogeneous strongly Markovian with respect to* $\mathbb{F}^X$*; satisfies (2.17); and has the same distribution as* $Z(\cdot)$.

When these assertions hold, the measure $\boldsymbol{\mu}$ *and the coëfficients* γ_1, γ_2 *satisfy the relationship (9.1).*

Proof of Proposition 9.3: (i) ⇒ *(ii):* By Propositions 7.1 and 6.1, the process $Z(\cdot)$ induces a weak solution of (9.3) subject to (2.17), where γ_1, γ_2 are given by (9.1). Since $Z(\cdot)$ is time-homogeneous strongly Markovian with respect its own filtration, so is this solution. We also obtain $\gamma_1^2 + \gamma_2^2 \leq 1$ from (9.1).

(ii) ⇒ (i): Appealing to Proposition 6.2, we see that the local submartingale problem associated with the pair $(\mathbf{1}, \mathbf{0})$ is well-posed. By Theorem 9.1 **(i)** we have the following for the radial part of $X(\cdot)$:

$$\|X(\cdot)\| = \|X(0)\| + \int_0^{\cdot} \mathbf{1}_{\{\|X(t)\|>0\}} \mathrm{d}W(t) + L^{\|X\|}(t) = \|X(0)\| + W(\cdot) + L^{\|X\|}(\cdot) \ . \tag{9.4}$$

We also used condition (2.17) here. Therefore, $\|X(\cdot)\|$ is the SKOROKHOD reflection of the Brownian motion $\|X(0)\| + W(\cdot)$, and satisfies $\mathbb{P}\big(L^{\|X\|}(\infty) = \infty\big) = 1$, so the condition (9.2) follows.

Now from Corollary 9.1, the weak solution posited in (ii) induces a solution to the local martingale problem associated with the triple $(\mathbf{1}, \mathbf{0}, \boldsymbol{\mu})$, for some probability measure $\boldsymbol{\mu}$ that satisfies (9.1). Propositions 7.1 and 6.2 show that $X(\cdot)$ is a WALSH Brownian motion with spinning measure $\boldsymbol{\mu}$, and so is $Z(\cdot)$. □

Remark 9.2. *Similarities and Differences:* Propositions 9.3 and 3.1 show that the system of equations (9.3), with the condition $\gamma_1^2 + \gamma_2^2 \le 1$ on the coëfficients, is a two-dimensional analogue of the HARRISON & SHEPP (1981) equation for the skew Brownian motion. But with the following caveat:

The equations (9.3), (2.17) characterize WALSH Brownian motions only when we restrict attention to time-homogeneous strongly Markovian processes. If this restriction is not imposed, there will be solutions to the system (9.3) that are not WALSH Brownian motions. Such solutions are discussed in the next remark.

Furthermore, (9.3) does not describe a unique WALSH Brownian motion, but may be satisfied by many such motions with different spinning measures. This is because, given two real numbers γ_1, γ_2 with $\gamma_1^2 + \gamma_2^2 \le 1$, we cannot uniquely determine the measure $\boldsymbol{\mu}$ through (9.1). By contrast, we can read off the flipping probability from the coëfficient in the equation for the one-dimensional skew Brownian motion. The construction in Remark 9.3 is actually based on this observation.

Remark 9.3. *A solution to the system of equations (9.3) that features two different spinning measures:* Consider the system of equations (9.3) with $\gamma_1 = \gamma_2 = 0$ and $\mathrm{x} = (0, 0)$, and note that both measures

$$\boldsymbol{\mu}_1 = \frac{1}{2}\,\delta_{(1,0)} + \frac{1}{2}\,\delta_{(-1,0)} \qquad \text{and} \qquad \boldsymbol{\mu}_2 = \frac{1}{2}\,\delta_{(0,1)} + \frac{1}{2}\,\delta_{(0,-1)}$$

satisfy (9.1). Let $X(\cdot)$ be a WALSH Brownian motion that solves (9.3) with $X(0) = (0,0)$, $\gamma_1 = \gamma_2 = 0$, spinning measure $\boldsymbol{\mu}_1$ and driving Brownian motion $B(\cdot)$. Let $Y(\cdot)$ be another WALSH Brownian motion that solves (9.3) with $Y(0) = (1,0)$, $\gamma_1 = \gamma_2 = 0$, spinning measure $\boldsymbol{\mu}_2$ and driving Brownian motion $\widetilde{B}(\cdot) := B(\tau_{(1,0)} + \cdot)$. Now define $\tau_{(1,0)} := \inf\{t \ge 0 : X(t) = (1,0)\}$ as well as

$$Z(t) \,:=\, X(t), \quad 0 \,\le\, t < \tau_{(1,0)}\,, \qquad \text{and} \qquad Z(\tau_{(1,0)} + t) := Y(t), \quad \forall \ t \ge 0\,.$$

The so-defined process $Z(\cdot)$ solves (9.3) with $Z(0) = (0,0)$, $\gamma_1 = \gamma_2 = 0$ and driving Brownian motion $B(\cdot)$, but is *not* a WALSH Brownian motion: it switches from $\boldsymbol{\mu}_1$ to $\boldsymbol{\mu}_2$ after time $\tau_{(1,0)}$. It is also not time-homogeneous strongly Markovian, by virtue of either Proposition 9.3 or elementary observations.

10 Examples

Example 10.1. WALSH's Brownian Motion and Spider Martingales: When the spinning measure $\boldsymbol{\mu}$ in Theorem 2.1 is a discrete probability charging a finite number of rays that pass through the origin, and the driving semimartingale $U(\cdot)$ is Brownian motion, the process $X(\cdot)$ becomes the original WALSH Brownian motion $\boldsymbol{W}(\cdot)$ with roundhouse singularities in multipole fields as in Proposition 7.2. Given a finite number $m \ge 2$ of distinct angles $\{\theta_\ell \in [0, 2\pi),\ \ell = 1, \ldots, m\}$, let us consider m rays emanating from the origin,

$$\mathcal{I}_\ell \,:=\, \big\{x \in \mathbb{R}^2 \setminus \{\mathbf{0}\} : \arg(x) \,=\, \theta_\ell\big\}\,, \qquad \ell \,=\, 1, \ldots, m\,,$$

and assign a discrete probability measure $\boldsymbol{\mu}$ with weights $p_\ell \in (0,1)$, $\sum_{\ell=1}^m p_\ell = 1$, such that

$$\boldsymbol{\mu}\big(\{(\cos(\theta_\ell), \sin(\theta_\ell))\}\big) \,=\, \mathbb{P}\big(\arg(\boldsymbol{\xi}_1) \,=\, \theta_\ell\big) \,=\, p_\ell\,, \qquad \ell \,=\, 1, \ldots, m\,. \tag{10.1}$$

Using MARKOV semigroups and excursions, BARLOW, PITMAN & YOR (1989a) study WALSH's Brownian motion $\boldsymbol{W}(\cdot)$ on the collection of rays $\bigcup_{\ell=1}^{m} \mathcal{I}_\ell \cup \{\mathbf{0}\}$. Their approach has been generalized to "multiple spider martingales" by YOR (1997), and has been studied by TSIREL'SON (1997), BARLOW, ÉMERY, KNIGHT, SONG & YOR (1998), WATANABE (1999) and MANSUY & YOR (2006), pp. 103-116.

Example 10.2. The Case of Two Rays: Let us consider the setup of the previous example with $m = 2$ and $\theta_1 = 0$, $\theta_2 = \theta \in (0, \pi]$, as well as $\mathbb{P}\big(\arg(\boldsymbol{\xi}_1) = \theta\big) = p \in (0,1)$. The equations of (3.4), (3.5) become

$$X_1(\cdot) \;=\; \mathrm{x}_1 + \int_0^{\cdot} \cos\big(\arg\big(X(t)\big)\big)\,\mathrm{d}U(t) + \left(1 - \frac{p\cos^-(\theta)}{1-p+p\cos^+(\theta)}\right) L^{X_1}(\cdot)\,,$$

$$X_2(\cdot) \;=\; \mathrm{x}_2 + \int_0^{\cdot} \sin\big(\arg\big(X(t)\big)\big)\,\mathrm{d}U(t) + p\sin(\theta)\,L^{||X||}(\cdot)$$

with $L^{X_1}(\cdot) = \big(1-p+p\cos^+(\theta)\big)\,L^{||X||}(\cdot)\,,\; L^{X_2}(\cdot) = p\sin^+(\theta)\,L^{||X||}(\cdot)\,.$

Case I: With $\theta = \pi$, and with $\mathrm{x}_2 = 0$ for simplicity, the second of these equations has the trivial solution $X_2(\cdot) \equiv 0$, whereas the first can be cast in the form of the celebrated HARRISON-SHEPP (1981) equation

$$X_1(\cdot) \;=\; x_1 + V_1(\cdot) + \frac{1-2p}{1-p} L^{X_1}(\cdot)\,, \qquad \text{driven by} \quad V_1(\cdot) \;:=\; \int_0^{\cdot} \operatorname{sgn}(X_1(t))\,\mathrm{d}U(t)\,.$$

As these authors showed, when $U(\cdot)$ is Brownian motion the above equation has a pathwise unique, strong solution with respect to the Brownian motion $V_1(\cdot)$, and in this case $X_1(\cdot)$ is skew Brownian motion.

When written in terms of the original driver $U(\cdot)$, the above equation for $X_1(\cdot)$ is a skew version of the TANAKA equation. In particular, Proposition 2.1 of ICHIBA & KARATZAS (2014) establishes the filtration comparisons $\mathbb{F}^U(\cdot) \subsetneq \mathbb{F}^{V_1}(\cdot) = \mathbb{F}^{X_1}(\cdot) = \mathbb{F}^{(X_1,X_2)}(\cdot)$ when $U(\cdot)$ is Brownian motion.

Case II: When $\theta \in (0, \pi)$, we assume for simplicity $\arg(\mathrm{x}) \in \{0,\, \theta\}$ and consider the process

$$\Upsilon(\cdot) \;:=\; \frac{-X_2(\cdot)}{\sin(\theta)} \cdot \mathbf{1}_{\{X_2(\cdot)>0\}} + \frac{X_1(\cdot)}{\cos(0)} \cdot \mathbf{1}_{\{X_2(\cdot)=0\}}\,;$$

that is, we flatten the state space by rotating the ray. This process also satisfies a HARRISON-SHEPP-type equation, namely

$$\Upsilon(\cdot) \;=\; \Upsilon(0) + V_\bullet(\cdot) + \frac{1-2p}{1-p} L^{\Upsilon}(\cdot) \qquad \text{driven by} \quad V_\bullet(\cdot) \;:=\; \int_0^{\cdot} \operatorname{sgn}(\Upsilon(t))\,\mathrm{d}U(\cdot)\,;$$

and conversely, the coördinate processes are given in terms of $\Upsilon(\cdot)$ as

$$X_1(\cdot) \;=\; \Upsilon(\cdot)\cdot \mathbf{1}_{\{\Upsilon(\cdot)>0\}} - \Upsilon(\cdot)\cos(\theta)\cdot \mathbf{1}_{\{\Upsilon(\cdot)\le 0\}}\,, \qquad X_2(\cdot) \;=\; -\Upsilon(\cdot)\sin(\theta)\cdot \mathbf{1}_{\{\Upsilon(\cdot)\le 0\}}\,.$$

If $U(\cdot)$ is standard Brownian motion, then so is $V_\bullet(\cdot)$; in this case $\Upsilon(\cdot)$ becomes a skew Brownian motion, and we obtain as before the filtration comparisons $\mathbb{F}^U(\cdot) \subsetneq \mathbb{F}^{V_\bullet}(\cdot) = F^{\Upsilon}(\cdot) = \mathbb{F}^{(X_1,X_2)}(\cdot)$.

• We have shown that the filtration $\mathbb{F}^{(X_1,X_2)}(\cdot)$ of a WALSH Brownian motion on two rays coincides with the filtration generated by some standard Brownian motion, and is *strictly finer* than the filtration $\mathbb{F}^U(\cdot)$ generated by its driving Brownian motion.

• Suppose the driver $U(\cdot)$ is a continuous local martingale with $U(0) = 0$ and $\langle U\rangle(\infty) = \infty$, and consider its DAMBIS-DUBINS-SCHWARZ representation $U(\cdot) = \boldsymbol{\beta}(\langle U\rangle(\cdot))$ with $\boldsymbol{\beta}(\cdot)$ a standard Brownian motion.

From the above considerations and in conjunction with Proposition 2.2 in ICHIBA & KARATZAS (2014) we see that, in the case of a spinning measure $\boldsymbol{\mu}$ that charges exactly two points on the unit circumference, uniqueness in distribution holds for the system (3.5) subject to (2.13) and (2.18), provided that either

(i) $U(\cdot)$ is *pure* (i.e., $\langle U\rangle(t)$ is $\mathcal{F}^{\boldsymbol{\beta}}(\infty)-$measurable, for every $t \in [0,\infty)$); or that

(ii) the quadratic variation process $\langle U\rangle(\cdot)$ is adapted to a Brownian motion $\Gamma(\cdot) = (\Gamma_1(\cdot), \cdots, \Gamma_n(\cdot))'$ with values in some Euclidean space and independent of the Brownian motion $\boldsymbol{\beta}(\cdot)$.

Example 10.3. TSIREL'SON's triple point: When $\alpha_i^{(+)} = \alpha_i^{(-)}$ for $i = 1, 2$, the equations (2.12) and (3.4) become, respectively,

$$X_i(T) = x_i + \int_0^T \mathfrak{f}_i\big(X(t)\big)\,\mathrm{d}S(t) \qquad \text{and} \qquad X_i(T) = x_i + \int_0^T \mathfrak{f}_i\big(X(t)\big)\,\mathrm{d}U(t)\,; \quad i = 1, 2\,.$$

This is the case when the common probability distribution $\boldsymbol{\mu}$ of the I.I.D. random variables $\{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots\}$ in (5.1) has zero expectation, namely $\mathbb{E}[\boldsymbol{\xi}_1] = \mathbf{0}$. For instance, when $\boldsymbol{\mu}$ assigns equal weights of $1/3$ to three points at angles $\theta_0 + (2\pi\ell/3)$, $\ell = 0, 1, 2$ on the unit circumference $\mathfrak{S}$ that trisect it, namely,

$$\mathbb{P}\Big(\boldsymbol{\xi}_1 = \big(\cos(\theta_0 + (2\pi\ell/3))\,, \sin(\theta_0 + (2\pi\ell/3))\big)'\Big) = 1/3\,; \quad \ell = 0, 1, 2$$

for some $\theta_0 \in [0, 2\pi)$. If, in addition, $U(\cdot) = W(\cdot)$ is Brownian motion, and thus the SKOROKHOD reflection $S(\cdot) = W(\cdot) + \max_{0 \le s \le \cdot}(-W(s))^+$ in (2.2) is a reflecting Brownian motion, we deduce from subsection 3.2 that the corresponding planar process $X(\cdot) = (X_1(\cdot), X_2(\cdot))'$ is a martingale, to wit

$$X_i(T) = \mathrm{x}_i + \int_0^T \mathfrak{f}_i\big(X(t)\big)\,\mathrm{d}W(t)\,; \quad i = 1, 2\,.$$

It was conjectured by BARLOW, PITMAN & YOR (1989a), and shown in the landmark paper by TSIREL'SON (1997) (cf. YOR (1997), MANSUY & YOR (2006)), that the natural filtration of this martingale $X(\cdot)$ is not generated by *any* Brownian motion of *any* dimension.

Example 10.4. WALSH's Brownian motion with polar drifts: Let us look at the case $\boldsymbol{\sigma}(\cdot) \equiv 1$ and $\boldsymbol{c}(\cdot) \equiv -\lambda$ for some $\lambda > 0$ in Corollary 6.1. The driving one-dimensional semimartingale $U(\cdot)$ for $X(\cdot)$ is Brownian motion with negative drift $-\lambda$ and with instantaneous reflection at the origin. It follows from Theorem 2.1 the process $X(\cdot) = (X_1(\cdot), X_2(\cdot))'$ satisfies

$$X_i(T) = \mathrm{x}_i + \int_0^T \mathfrak{f}_i(X(t))\big(-\lambda\,\mathrm{d}t + \mathrm{d}W(t)\big) + \gamma_i\, L^{||X||}(T)\,, \qquad 0 \le T < \infty$$

for $i = 1, 2$, where $W(\cdot)$ is one-dimensional standard Brownian motion. Moreover, following Proposition 8.1, we may replace the constant drifts by *drifts exhibiting angular dependence*. Suppose that $\boldsymbol{a}(r,\theta) = 1$ and $\boldsymbol{b}(r,\theta) = \boldsymbol{\lambda}(\theta)$ for some measurable function $\boldsymbol{\lambda} : [0, 2\pi) \to (0, \infty)$. The resulting process $Y(\cdot)$ in Proposition 8.1 has the dynamics

$$Y(T) = \mathrm{y} + \int_0^T \mathfrak{f}(Y(t))\Big(-\boldsymbol{\lambda}(\arg\big(Y(t)\big))\mathrm{d}t + \mathrm{d}W(t)\Big) + \boldsymbol{\gamma}\, L^{\|Y\|}(T)\,, \qquad 0 \le T < \infty\,.$$

Since the driving semimartingale is positive recurrent in $\mathbb{R}_+$, the degenerate planar process $X(\cdot)$ is positive recurrent. Its stationary distribution is expressed in polar coördinates as

$$\bigg(\int_0^{2\pi} \frac{\boldsymbol{\nu}(\mathrm{d}u)}{2[\boldsymbol{\lambda}(u)]^2}\bigg)^{-1} \frac{e^{-2\boldsymbol{\lambda}(\theta)r}}{\boldsymbol{\lambda}(\theta)}\,\mathrm{d}r\,\boldsymbol{\nu}\,(\mathrm{d}\theta)\,; \qquad r > 0\,,\ \theta \in [0, 2\pi)$$

by the distribution of occupation times and the excursion theory of SALMINEN, VALLOIS & YOR (2007). If $\boldsymbol{\lambda}(\cdot) \equiv \lambda$ (constant), then the stationary distribution reduces to $\big(2\lambda e^{-2\lambda r}\mathrm{d}r\big)\,\boldsymbol{\nu}(\mathrm{d}\theta)$, $r > 0$, $\theta \in [0, 2\pi)$.

Example 10.5. WALSH semimartingale driven by BESSEL processes: Suppose that $R^2(\cdot)$ is a squared BESSEL process with dynamics

$$\mathrm{d}R^2(t) = \delta\,\mathrm{d}t + 2\sqrt{R^2(t)}\,\mathrm{d}W(t)\,,$$

where $\delta \in (1, 2)$ and $W(\cdot)$ is one-dimensional standard Brownian. We take the square root $|R(\cdot)|$ of this process as the driving semimartingale, i.e., $U(\cdot) = |R(\cdot)| = S(\cdot)$ in Theorem 2.1.

This process $S(\cdot)$ does not accumulate local time at the origin, i.e., $L^S(\cdot) \equiv 0$ holds for $\delta \in (1, 2)$, hence the resulting planar process $X(\cdot)$ of Theorem 2.1 has the dynamics

$$X_i(T) \,=\, \mathrm{x}_i + \int_0^T \mathfrak{f}_i(X(t))\Big(\frac{\delta - 1}{2\,\|X(t)\|} \cdot \mathbf{1}_{\{\|X(t)\neq 0\|\}}\,\mathrm{d}t + \mathrm{d}W(t)\Big)\,, \quad 0 \le T < \infty$$

for $i = 1, 2$. Note that when $\delta \,=\, 1$, the process $X(\cdot)$ becomes WALSH Brownian motion; when $\delta \in (0, 1)$, the semimartingale property is violated; when $\delta \ge 2$, the process $R(\cdot)$ never reaches the origin.

Furthermore, and by analogy with Example 10.4, given a measurable function $\boldsymbol{\delta} : [0, 2\pi) \to (1, 2)$ we may use the time-change technique with the dispersion $\boldsymbol{a}(r, \theta) \,=\, 4\,r$ and the drift $\boldsymbol{b}(r, \theta) \,=\, \boldsymbol{\delta}(\theta)$ and consider the WALSH semimartingale $Y(\cdot)$ driven by angular dependent, squared-BESSEL process

$$Y(T) \,=\, \mathrm{y} + \int_0^T \mathfrak{f}(Y(t))\Big(\boldsymbol{\delta}(\arg(Y(t))\,\mathrm{d}t + 2\sqrt{\|Y(t)\|}\,\mathrm{d}W(t)\Big)\,, \qquad 0 \le T < \infty.$$

Here, the process $\|Y(\cdot)\|$ does not accumulate local time at the origin. The corresponding scale function, inverse function and stochastic clock are given by $\boldsymbol{p}_\theta(r) \,=\, r^{(2-\boldsymbol{\delta}(\theta))\,/\,2}$, $\boldsymbol{q}_\theta(r) \,=\, r^{2\,/\,(2-\boldsymbol{\delta}(\theta))}$, and

$$\mathcal{T}(\cdot) \,=\, \int_0^\cdot \Big((2 - \boldsymbol{\delta}(\theta))^2\, r^{-(\boldsymbol{\delta}(\theta)-1)}\Big)\Big|_{r = \|Y(t)\|,\, \theta = \arg(Y(t))}\,\mathrm{d}t\,,$$

respectively. It can be shown that the stochastic clock does not explode (cf. Lemma 3.1 of BIANE & YOR (1987), Proposition XI.1.11 of REVUZ & YOR (1999), pages 285-289 of ROGERS & WILLIAMS (2000) and Appendix A.1 of ICHIBA ET AL. (2011)). From this process $Y(\cdot)$ we may define now the WALSH semimartingale $\Xi(\cdot) \,=\, \big(\Xi_1(\cdot), \Xi_2(\cdot)\big)'$, $\Xi_i(\cdot) := \mathfrak{f}_i(Y(\cdot))\,\|Y(\cdot)\|^{1/2}$, $i \,=\, 1, 2$ driven by a BESSEL process with angular dependence, which satisfies the vector integral equation derived from (8.12), namely,

$$\Xi(T) \,=\, \Xi(0) + \int_0^T \mathfrak{f}(\Xi(t))\Big(\frac{\boldsymbol{\delta}(\arg(\Xi(t))) - 1}{2\,\|\Xi(t)\|}\,\mathbf{1}_{\{\|\Xi(t) \neq 0\|\}}\mathrm{d}t + \mathrm{d}W(t)\Big)\,, \qquad 0 \le T < \infty\,.$$

Example 10.6. Occupation times for WALSH's Brownian motion:

When the driving semimartingale is a BESSEL process and the spinning probability measure $\boldsymbol{\mu}$ is discrete, with probability masses on a finite number of points as in (10.1) of Example 10.1, BARLOW, PITMAN & YOR (1989b) consider the joint distribution of occupation times $\mathfrak{A}_{\mathcal{I}_\ell}(t) \,:=\, \int_0^t \mathbf{1}_{\{\,X(s)\,\in\,\mathcal{I}_\ell\,\}}\mathrm{d}s$, $\ell \,=\, 1, \cdots, m$ and the local time $L^{\|X\|}(t)$ for $\|X(\cdot)\|$ accumulated at the origin for $0 \le t < \infty$, where $\mathcal{I}_1, \cdots, \mathcal{I}_m$ is the collection of rays, each of which starts at the origin and passes through the point in the support of the spinning measure $\boldsymbol{\mu}$.

When the spinning probability measure is not necessarily discrete, we may consider the random times

$$\mathfrak{A}_{C_\ell}(1) \,:=\, \int_0^1 \mathbf{1}_{\{\boldsymbol{W}(s) \in C_\ell\}}\mathrm{d}s\,; \quad \ell \,=\, 1, \ldots, m$$

occupied by WALSH's Brownian motion $\boldsymbol{W}(\cdot)$ for the disjoint, planar semi-infinite cones $C_1, \cdots, C_m$ that partition the whole $\mathbb{R}^2 \,=\, \bigcup_{\ell=1}^m C_\ell$ with $C_\ell \cap C_k \,=\, \emptyset$ through the origin. Then we have the identity

$$\Big(\mathfrak{A}_{C_1}(1), \ldots, \mathfrak{A}_{C_m}(1), \big(L^{\|X\|}(1)\big)^2\Big) \overset{(\mathcal{D})}{=} \left(\frac{\overline{p}_1^2\,\mathfrak{T}_1}{\sum_{\ell=1}^m \overline{p}_\ell^2\,\mathfrak{T}_\ell}, \ldots, \frac{\overline{p}_m^2\,\mathfrak{T}_m}{\sum_{\ell=1}^m \overline{p}_\ell^2\,\mathfrak{T}_\ell}, \frac{1}{4}\sum_{\ell=1}^m \overline{p}_\ell^2\,\mathfrak{T}_\ell\right) \tag{10.2}$$

in distribution by analogy with the result in BARLOW, PITMAN & YOR (1989b). Here we denote by $\{\mathfrak{T}_\ell\}_{\ell=1}^m$ the I.I.D. stable random variables with index $1/2$, by $\mathfrak{f}(C_\ell)$ the projection of the cone C_ℓ on the unit circumference $\mathfrak{S}$, by the definition of the function $\mathfrak{f}$ in (2.8), and set $\overline{p}_\ell := \boldsymbol{\mu}\big(\mathfrak{f}(C_\ell)\big)$, $\ell \,=\, 1, \ldots, m$.

11 Appendix: The Proof of Lemma 9.1

We denote by $\widetilde{\mathfrak{D}^{\boldsymbol{\mu}}}$ (resp. $\widetilde{\mathfrak{D}^{\boldsymbol{\mu}}_+}$) the collection of functions g in $\mathfrak{D}^{\boldsymbol{\mu}}$ (resp. $\mathfrak{D}^{\boldsymbol{\mu}}_+$) such that $M^g(\cdot\,;\omega_2)$ is a continuous local martingale (resp. submartingale) of the filtration $\mathbb{F}_2$, under $\mathbb{Q}$. Then we have $\widetilde{\mathfrak{D}^{\boldsymbol{\mu}}} \supseteq \mathfrak{D}^{\boldsymbol{\mu}} \cap \mathfrak{E}$ and $\widetilde{\mathfrak{D}^{\boldsymbol{\mu}}_+} \supseteq \mathfrak{E}$ by assumption. The goal here is to show $\widetilde{\mathfrak{D}^{\boldsymbol{\mu}}} = \mathfrak{D}^{\boldsymbol{\mu}}$ and $\widetilde{\mathfrak{D}^{\boldsymbol{\mu}}_+} = \mathfrak{D}^{\boldsymbol{\mu}}_+$.

Recalling that $\mathfrak{E}$ contains the functions in Definition 9.2(ii), we can follow the proof of Part (b) of Proposition 6.1 and show that there exists a one-dimensional standard Brownian motion $W(\cdot)$ on an extension of the filtered probability space $(\Omega_2, \mathcal{F}_2, \mathbb{Q}), \mathbb{F}_2$ such that (6.4), (6.5) hold with $X(\cdot)$ given by (6.10), or simply $X(\cdot) := \omega_2(\cdot)$. It is clear, therefore, that $\int_0^t \mathbf{1}_{\{\|\omega_2(u)\|>0\}} \big(|\boldsymbol{b}(\|\omega_2(u)\|)| + \boldsymbol{a}(\|\omega_2(u)\|)\big)\mathrm{d}u < \infty$ holds for all $0 \le t < \infty$, $\mathbb{Q}-$a.s. We make now the following two observations.

First Observation: $\widetilde{\mathfrak{D}^{\boldsymbol{\mu}}}$ *is a linear space.* This is obvious from the linearity of stochastic integrals, derivatives, and local martingales.

Second Observation: *Suppose $\{g_n\}_{n\in\mathbb{N}} \subseteq \widetilde{\mathfrak{D}^{\boldsymbol{\mu}}}$ and $g \in \mathfrak{D}^{\boldsymbol{\mu}}$ satisfy the following: as $n \uparrow \infty$, $g_n(x) \to g(x), \forall\, x \in \mathbb{R}^2$ and $G_n'(x) \to G'(x), G_n''(x) \to G''(x), \forall\, x \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$, and all these functions (g_n, g, G_n', G', G_n'', G'') are uniformly bounded on every compact subset of $\mathbb{R}^2$. Then we have $g \in \widetilde{\mathfrak{D}^{\boldsymbol{\mu}}}$.*

To see this, we define stopping times

$$T_k \,=\, \inf\left\{t : \int_0^t \mathbf{1}_{\{\|\omega_2(u)\|>0\}} \Big(\big|\boldsymbol{b}\big(\|\omega_2(u)\|\big)\big| + \boldsymbol{a}\big(\|\omega_2(u)\|\big)\Big)\mathrm{d}u \ge k \;\text{ or }\; \|\omega_2(t)\| \ge k\right\}, \quad k \in \mathbb{N},$$

and note that $M^{g_n}(\cdot \wedge T_k\,;\omega_2)$, $n \in \mathbb{N}$ are uniformly bounded local martingales, hence uniformly bounded martingales, for all $k \in \mathbb{N}$; and that $\lim_{n\to\infty} M^{g_n}(t \wedge T_k\,;\omega_2) = M^g(t \wedge T_k\,;\omega_2)$ for any $t \in [0,\infty)$. Thus $M^g(\cdot \wedge T_k\,;\omega_2)$ is also a continuous martingale, and the conclusion $g \in \widetilde{\mathfrak{D}^{\boldsymbol{\mu}}}$ follows.

• Returning to our argument, we know that for the functions of Definition 9.2, the process $M^{g_A}(\cdot\,;\omega_2)$ is a local martingale for any interval $A \subseteq [0, 2\pi)$ of the form $[a, b)$, where a, b are rationals. Thus the same is true when A is the disjoint union of such intervals, by linearity. These sets form an algebra. By the second observation and monotone class arguments, the same is also true for every BOREL subset A of $[0, 2\pi)$. Now for any two disjoint BOREL subsets A, B of $[0, 2\pi)$, we define

$$g_{A,B}(x) := \|x\| \left(\boldsymbol{\nu}(A)\, \mathbf{1}_{\{\arg(x)\in B\}} - \boldsymbol{\nu}(B)\, \mathbf{1}_{\{\arg(x)\in A\}}\right) \quad \text{and note} \quad g_{A,B}(x) = \boldsymbol{\nu}(A)\, g_B(x) - \boldsymbol{\nu}(B)\, g_A(x),$$

thus $g_{A,B} \in \widetilde{\mathfrak{D}^{\boldsymbol{\mu}}}$ by linearity. Starting from this and using linearity and induction, we show that if $h : [0, 2\pi) \to \mathbb{R}$ is simple and satisfies $\int_0^{2\pi} h(\theta)\boldsymbol{\nu}(\mathrm{d}\theta) = 0$, then the mapping $x \mapsto \|x\| \cdot h(\arg(x))$ is in $\widetilde{\mathfrak{D}^{\boldsymbol{\mu}}}$. Using approximation and the second observation, we see that this statement is still true when "simple" is replaced by "bounded and measurable".

Let us recall now that, in the second paragraph of this section, we obtained the existence of a one-dimensional standard Brownian motion $W(\cdot)$ on an extension of the filtered probability space $(\Omega_2, \mathcal{F}_2, \mathbb{Q})$, $\mathbb{F}_2$, along with (6.4) and (6.5), where $X(\cdot) := \omega_2(\cdot)$. By defining $S(\cdot) := \|X(\cdot)\|$, we can follow the proof of Theorem 4.1 to establish for any given function $g \in \mathfrak{D}^{\boldsymbol{\mu}}$ with $g_\theta'(0+) \equiv 0$ the following FREIDLIN-SHEU-type semimartingale decomposition:

$$g(\omega_2(\cdot)) \,=\, g(\mathrm{x}) + \int_0^{\cdot} \mathbf{1}_{\{\|\omega_2(t)\|>0\}} \Big(\boldsymbol{b}(\|\omega_2(t)\|)\, G'(\omega_2(t)) + \frac{1}{2}\, \boldsymbol{a}(\|\omega_2(t)\|)\, G''(\omega_2(t))\Big)\mathrm{d}t$$

$$+ \int_0^{\cdot} \mathbf{1}_{\{\|\omega_2(t)\|>0\}} \boldsymbol{\sigma}(\|\omega_2(t)\|)\, G'(\omega_2(t))\, \mathrm{d}W(t)\,.$$

The condition (2.18) is *not* needed here; and neither are terms involving local time. This is because the use of (2.18) in the proof of Theorem 4.1 comes only when proving the convergence to local time

$$\sum_{\{\ell\,:\,\tau^\varepsilon_{2\ell+1}<T\}} \varepsilon\, h(\Theta(\tau^\varepsilon_{2\ell+1})) \ \xrightarrow[\varepsilon\downarrow 0]{} \ L^S(T)\int_0^{2\pi} h(\theta)\,\boldsymbol{\nu}(\mathrm{d}\theta)\,.$$

But this property holds here trivially, courtesy of $h(\theta) := g'_\theta(0+) \equiv 0$. It follows from the above decomposition of FREIDLIN-SHEU-type that, if $g \in \mathfrak{D}^{\boldsymbol{\mu}}$ satisfies $g'_\theta(0+) \equiv 0$, then $g \in \widetilde{\mathfrak{D}^{\boldsymbol{\mu}}}$.

Finally, we observe that every $g \in \mathfrak{D}^{\boldsymbol{\mu}}$ can be decomposed as $g = g^{(1)} + g^{(2)}$, where the function $x \mapsto g^{(1)}(x) := \|x\| \cdot g'_\theta(0+)$ is in $\widetilde{\mathfrak{D}^{\boldsymbol{\mu}}}$ by the first paragraph of this bullet, and the function $g^{(2)} := g - g^{(1)} \in \mathfrak{D}^{\boldsymbol{\mu}}$ satisfies $\big(g^{(2)}_\theta\big)'(0+) \equiv 0$. With the considerations above, we see $g \in \widetilde{\mathfrak{D}^{\boldsymbol{\mu}}}$, thus $\widetilde{\mathfrak{D}^{\boldsymbol{\mu}}} = \mathfrak{D}^{\boldsymbol{\mu}}$. We decompose then every function $g \in \mathfrak{D}^{\boldsymbol{\mu}}_+$ as $g = g_{(1)} + g_{(2)}$, where $g_{(1)}(x) := c\,\|x\|$ with a constant $c := \int_0^{2\pi} g'_\theta(0+)\boldsymbol{\nu}(\mathrm{d}\theta) \ge 0$ and $g_{(2)} := g - g_{(1)} \in \mathfrak{D}^{\boldsymbol{\mu}}$ (cf. Remark 6.1). Here $M^{g_{(2)}}(\cdot\,;\omega_2)$ is a local martingale, and $M^{g_{(1)}}(\cdot\,;\omega_2) = c\,M^{g_3}(\cdot\,;\omega_2)$ is a local submartingale, since the mapping $x \mapsto g_3(x) = \|x\|$ belongs to $\mathfrak{E} \subseteq \widetilde{\mathfrak{D}^{\boldsymbol{\mu}}_+}$ (cf. Definition 9.2 (ii)). Thus $M^g(\cdot\,;\omega_2)$ is also a local submartingale and $g \in \widetilde{\mathfrak{D}^{\boldsymbol{\mu}}_+}$. We conclude then $\widetilde{\mathfrak{D}^{\boldsymbol{\mu}}_+} = \mathfrak{D}^{\boldsymbol{\mu}}_+$, and the proof of Lemma 9.1 is complete. □

12 Appendix: The Proof of Theorem 9.1

We first identify the measure $\boldsymbol{\mu}$ from $X(\cdot)$, using the time-homogeneous strong MARKOV property of this process. Then we establish a FREIDLIN-SHEU-type formula for $X(\cdot)$, so as to relate this process to a solution of the local martingale problem associated with the triple $(\boldsymbol{\sigma}, \boldsymbol{b}, \boldsymbol{\mu})$.

• By the result of Part (a) of Proposition 6.1, we obtain the equation (6.5), thus prove Part (i) of Theorem 9.1. We also know that the "direction process" $\mathfrak{f}\big(X(\cdot)\big) = \big(\mathfrak{f}_1\big(X(\cdot)\big), \mathfrak{f}_2\big(X(\cdot)\big)\big)$ is constant on every excursion interval of $\|X(t)\|$, by applying the idea in the argument at the beginning of section 3.

• For every $\varepsilon > 0$, we define the stopping times $\big\{\tau^\varepsilon_m\,,\, m \in \mathbb{N}_0\big\}$ as in (5.8).

12.1 Proof of Theorem 9.1(ii), Part A

Let us start by assuming that, with probability one, all these stopping times $\big\{\tau^\varepsilon_m\,,\, m \in \mathbb{N}_0\big\}$ are finite.

Then for every $\varepsilon > 0$, $\ell \in \mathbb{N}_0$, we define also the probability measure $\boldsymbol{\mu}^\varepsilon_\ell$ on $(\mathfrak{S}, \mathcal{B}(\mathfrak{S}))$, by

$$\boldsymbol{\mu}^\varepsilon_\ell(B) := \mathbb{P}\big(\mathfrak{f}(X(\tau^\varepsilon_{2\ell+1})) \in B\big), \quad \forall\, B \in \mathcal{B}(\mathfrak{S})\,. \tag{12.1}$$

Proposition 12.1. *The measure $\boldsymbol{\mu}^\varepsilon_\ell$ just introduced does not depend on either ε or ℓ, so we can define $\boldsymbol{\mu} := \boldsymbol{\mu}^\varepsilon_\ell$, $\forall\, \varepsilon > 0$, $\ell \in \mathbb{N}_0$. Furthermore, $\big\{\,\mathfrak{f}(X(\tau^\varepsilon_{2\ell+1}))\,\big\}_{\ell\in\mathbb{N}_0}$ is a sequence of independent random variables with common distribution $\boldsymbol{\mu}$, for every fixed $\varepsilon > 0$.*

Proof of Proposition 12.1: Step 1: We shall show in this step that $\mathfrak{f}(X(\tau^\varepsilon_{2\ell+1}))$ is independent of $\mathcal{F}^X(\tau^\varepsilon_{2\ell})$ for any $\varepsilon > 0$, $\ell \in \mathbb{N}_0$, and that the random variables $\{\mathfrak{f}(X(\tau^\varepsilon_{2\ell+1}))\}_{\ell\in\mathbb{N}_0}$ are I.I.D. for any fixed $\varepsilon > 0$.

By Proposition 9.2, we have for every $\varepsilon > 0$, $\ell \in \mathbb{N}_0$, $B \in \mathcal{B}(\mathfrak{S})$, the identity

$$\mathbb{P}\big(\mathfrak{f}(X(\tau^\varepsilon_{2\ell+1})) \in B \,|\, \mathcal{F}^X(\tau^\varepsilon_{2\ell})\big) = \mathbb{P}\big(X(\tau^\varepsilon_{2\ell} + \cdot) \in A_1 \,|\, \mathcal{F}^X(\tau^\varepsilon_{2\ell})\big) = \mathbb{P}\big(X(\tau^\varepsilon_{2\ell} + \cdot) \in A_1 \,|\, X(\tau^\varepsilon_{2\ell})\big)\,.$$

Here

$$A_1 := \big\{\omega \in C[0,\infty)^2 : \mathfrak{f}(\omega(\tau^\varepsilon_1(\omega))) \in B,\ \omega(0) = 0\big\} \in \mathcal{B}\big(C[0,\infty)^2\big)\,,$$

and the above conditional probability also equals $\mathfrak{h}_1(X(\tau^\varepsilon_{2\ell}))$, for some bounded measurable function $\mathfrak{h}_1 : \mathbb{R}^2 \to \mathbb{R}$ that depends only on A_1. Now because $X(\tau^\varepsilon_{2\ell}) \equiv 0$, this conditional probability is a constant that is irrelevant to $\tau^\varepsilon_{2\ell}$, in particular, to ℓ. We deduce that $\mathfrak{f}(X(\tau^\varepsilon_{2\ell+1}))$ is independent of $\mathcal{F}^X(\tau^\varepsilon_{2\ell})$, and its distribution does not depend on ℓ. Therefore, the random variables in $\big\{\mathfrak{f}(X(\tau^\varepsilon_{2\ell+1}))\big\}_{\ell\in\mathbb{N}_0}$ are I.I.D.

Step 2: On the strength of Step 1, we can define $\boldsymbol{\mu}^\varepsilon := \boldsymbol{\mu}^\varepsilon_\ell\,,\ \forall\ \ell \in \mathbb{N}_0$. We shall show in this step that $\boldsymbol{\mu}^\varepsilon$ does not depend on ε . Once this is done, we shall obtain Proposition 12.1 by combining the results of the two steps. Let $\varepsilon_1 > \varepsilon_2 > 0$. We shall prove the claim

$$\boldsymbol{\mu}^{\varepsilon_1}(B) \,=\, \boldsymbol{\mu}^{\varepsilon_2}(B)\,, \quad \forall\ B \in \mathcal{B}(\mathfrak{S})\,.$$

Since $\|X(\tau_1^{\varepsilon_1})\| = \varepsilon_1 > \varepsilon_2$, and $\|X(\cdot)\| \le \varepsilon_2$ on every $[\tau_{2\ell}^{\varepsilon_2}, \tau_{2\ell+1}^{\varepsilon_2}]$, we see that for a.e. $\omega \in \Omega$ there exists a unique $\ell_2 \in \mathbb{N}_0$ (depending on ω), such that $\tau_{2\ell_2+1}^{\varepsilon_2} < \tau_1^{\varepsilon_1} < \tau_{2\ell_2+2}^{\varepsilon_2}$. Then we can partition $\Omega = \bigcup_{\ell\in\mathbb{N}_0}\{\tau_{2\ell+1}^{\varepsilon_2} < \tau_1^{\varepsilon_1} < \tau_{2\ell+2}^{\varepsilon_2}\}$, where the right-hand side is a disjoint union. On the event $\{\tau_{2\ell+1}^{\varepsilon_2} < \tau_1^{\varepsilon_1} < \tau_{2\ell+2}^{\varepsilon_2}\}$, we note that $\tau_1^{\varepsilon_1}$ and $\tau_{2\ell+1}^{\varepsilon_2}$ are on the same excursion interval of $\|X(\cdot)\|$. Then from the considerations in the first bullet, we have $\mathfrak{f}(X(\tau_1^{\varepsilon_1})) = \mathfrak{f}(X(\tau_{2\ell+1}^{\varepsilon_2}))$ on the event $\{\tau_{2\ell+1}^{\varepsilon_2} < \tau_1^{\varepsilon_1} < \tau_{2\ell+2}^{\varepsilon_2}\}$.

On the strength of Lemma 12.1 below, we can write

$$\boldsymbol{\mu}^{\varepsilon_1}(B) = \mathbb{P}\big(\mathfrak{f}(X(\tau_1^{\varepsilon_1})) \in B\big) = \sum_{\ell\in\mathbb{N}_0} \mathbb{P}\Big(\{\mathfrak{f}(X(\tau_1^{\varepsilon_1})) \in B\} \bigcap \{\tau_{2\ell+1}^{\varepsilon_2} < \tau_1^{\varepsilon_1} < \tau_{2\ell+2}^{\varepsilon_2}\}\Big)$$

$$= \sum_{\ell\in\mathbb{N}_0} \mathbb{P}\Big(\{\mathfrak{f}(X(\tau_{2\ell+1}^{\varepsilon_2})) \in B\} \bigcap \{\tau_{2\ell}^{\varepsilon_2} < \tau_1^{\varepsilon_1}\} \bigcap \Big\{ \max_{\tau_{2\ell+1}^{\varepsilon_2} \le t \le \tau_{2\ell+2}^{\varepsilon_2}} \|X(t)\| \ge \varepsilon_1 \Big\}\Big)$$

$$= \boldsymbol{\mu}^{\varepsilon_2}(B) \sum_{\ell\in\mathbb{N}_0} \mathbb{P}\Big(\big\{\tau_{2\ell}^{\varepsilon_2} < \tau_1^{\varepsilon_1}\big\} \bigcap \Big\{ \max_{\tau_{2\ell+1}^{\varepsilon_2} \le t \le \tau_{2\ell+2}^{\varepsilon_2}} \|X(t)\| \ge \varepsilon_1 \Big\}\Big)$$

$$= \boldsymbol{\mu}^{\varepsilon_2}(B) \sum_{\ell\in\mathbb{N}_0} \mathbb{P}\big(\tau_{2\ell+1}^{\varepsilon_2} < \tau_1^{\varepsilon_1} < \tau_{2\ell+2}^{\varepsilon_2}\big) \,=\, \boldsymbol{\mu}^{\varepsilon_2}(B).$$

This way we complete Step 2, and Proposition 12.1 is proved. $\square$

Lemma 12.1. *(a) We have the comparisons* $\tau_{2\ell+1}^{\varepsilon_2} < \tau_1^{\varepsilon_1} < \tau_{2\ell+2}^{\varepsilon_2}$*, if and only if* $\tau_{2\ell}^{\varepsilon_2} < \tau_1^{\varepsilon_1}$ *and* $\max_{\tau_{2\ell+1}^{\varepsilon_2} \le t \le \tau_{2\ell+2}^{\varepsilon_2}} \|X(t)\| \ge \varepsilon_1$ *hold. (b)* $\forall\, B \in \mathcal{B}(\mathfrak{S})$*, the three events* $\{\mathfrak{f}(X(\tau_{2\ell+1}^{\varepsilon_2})) \in B\}$, $\{\tau_{2\ell}^{\varepsilon_2} < \tau_1^{\varepsilon_1}\}$, $\{\max_{\tau_{2\ell+1}^{\varepsilon_2} \le t \le \tau_{2\ell+2}^{\varepsilon_2}} \|X(t)\| \ge \varepsilon_1\}$ *are independent.*

Proof of Lemma 12.1: (a) It is fairly clear that, if $\tau_{2\ell+1}^{\varepsilon_2} < \tau_1^{\varepsilon_1} < \tau_{2\ell+2}^{\varepsilon_2}$, then $\tau_{2\ell}^{\varepsilon_2} < \tau_{2\ell+1}^{\varepsilon_2} < \tau_1^{\varepsilon_1}$, and $\max_{\tau_{2\ell+1}^{\varepsilon_2} \le t \le \tau_{2\ell+2}^{\varepsilon_2}} \|X(t)\| \ge \|X(\tau_1^{\varepsilon_1})\| = \varepsilon_1$.

Conversely, if $\tau_{2\ell}^{\varepsilon_2} < \tau_1^{\varepsilon_1}$, then since $\|X(t)\| \le \varepsilon_2$ for $t \in [\tau_{2\ell}^{\varepsilon_2}, \tau_{2\ell+1}^{\varepsilon_2}]$, we have $\tau_{2\ell+1}^{\varepsilon_2} < \tau_1^{\varepsilon_1}$. On the other hand, if $\max_{\tau_{2\ell+1}^{\varepsilon_2} \le t \le \tau_{2\ell+2}^{\varepsilon_2}} \|X(t)\| \ge \varepsilon_1$, then $\exists\ t \in (\tau_{2\ell+1}^{\varepsilon_2}, \tau_{2\ell+2}^{\varepsilon_2}) \subset (\tau_0^{\varepsilon_2}, \tau_{2\ell+2}^{\varepsilon_2}) = (\tau_0^{\varepsilon_1}, \tau_{2\ell+2}^{\varepsilon_2})$, such that $\|X(t)\| \ge \varepsilon_1$. Thus $\tau_1^{\varepsilon_1} < \tau_{2\ell+2}^{\varepsilon_2}$, concluding the proof of Part (a) of Lemma 12.1.

(b) By Step 1 of the proof of Proposition 12.1, $\{\mathfrak{f}(X(\tau_{2\ell+1}^{\varepsilon_2})) \in B\}$ is independent of $\mathcal{F}^X(\tau_{2\ell}^{\varepsilon_2})$. But $\{\tau_{2\ell}^{\varepsilon_2} < \tau_1^{\varepsilon_1}\} \in \mathcal{F}^X(\tau_{2\ell}^{\varepsilon_2})$, so $\{\mathfrak{f}(X(\tau_{2\ell+1}^{\varepsilon_2})) \in B\}$ and $\{\tau_{2\ell}^{\varepsilon_2} < \tau_1^{\varepsilon_1}\}$ are independent, and both belong to $\mathcal{F}^X(\tau_{2\ell+1}^{\varepsilon_2})$.

Let $A_2 := \{\omega \in C[0,\infty) : \omega(\cdot)$ hits ε_1 before hitting 0 with $\omega(0) = \varepsilon_2\} \in \mathcal{B}(C[0,\infty))$. Proposition 9.2 applied to $\|X(\cdot)\|$, gives

$$\mathbb{P}\Big(\max_{\tau_{2\ell+1}^{\varepsilon_2} \le t \le \tau_{2\ell+2}^{\varepsilon_2}} \|X(t)\| \ge \varepsilon_1 \,\Big|\, \mathcal{F}^X(\tau_{2\ell+1}^{\varepsilon_2})\Big) \,=\, \mathbb{P}\big(\|X(\tau_{2\ell+1}^{\varepsilon_2} + \cdot)\| \in A_2 \,\big|\, \mathcal{F}^X(\tau_{2\ell+1}^{\varepsilon_2})\big)$$

$$= \mathbb{P}\big(\|X(\tau_{2\ell+1}^{\varepsilon_2} + \cdot)\| \in A_2 \,\big|\, \|X(\tau_{2\ell+1}^{\varepsilon_2})\|\big)\,,$$

which is a measurable function of $\|X(\tau_{2\ell+1}^{\varepsilon_2})\|$. But we have $\|X(\tau_{2\ell+1}^{\varepsilon_2})\| \equiv \varepsilon_2$, and therefore the event $\big\{\max_{\tau_{2\ell+1}^{\varepsilon_2} \le t \le \tau_{2\ell+2}^{\varepsilon_2}} \|X(t)\| \ge \varepsilon_1\big\}$ is independent of $\mathcal{F}^X(\tau_{2\ell+1}^{\varepsilon_2})$. Combining this observation with the last paragraph, we complete the argument for Part (b). This concludes the proof of Lemma 12.1. $\square$

• With $\boldsymbol{\mu}$ defined as in Proposition 12.1, let $\boldsymbol{\nu}$ be the "angular measure" on $([0, 2\pi), \mathcal{B}([0, 2\pi)))$ induced by the "spinning measure" $\boldsymbol{\mu}$ on $(\mathfrak{S}, \mathcal{B}(\mathfrak{S}))$, through (2.9). Thus with $\Theta(\cdot) := \arg(X(\cdot))$, for every fixed $\varepsilon > 0$ the random variables $\{\Theta(\tau^{\varepsilon}_{2\ell+1})\}_{\ell \in \mathbb{N}_0}$ are I.I.D. with common distribution $\boldsymbol{\nu}$, following Proposition 12.1.

We turn now to the proof of the FREIDLIN-SHEU formula for $X(\cdot)$ in this setting: *For every function $g : \mathbb{R}^2 \to \mathbb{R}$ in the class $\mathfrak{D}$, defined as in subsection 4.1, we have*

$$g(X(\cdot)) = g(\mathrm{x}) + \int_0^{\cdot} \mathbf{1}_{\{\|X(t)\|>0\}} \Big(\boldsymbol{b}(\|X(t)\|)\, G'(X(t)) + \frac{1}{2}\, \boldsymbol{a}(\|X(t)\|)\, G''(X(t)) \Big) \mathrm{d}t$$

$$+ \int_0^{\cdot} \mathbf{1}_{\{\|X(t)\|>0\}} \boldsymbol{\sigma}(\|X(t)\|)\, G'(X(t))\, \mathrm{d}W(t) \,+ \Big(\int_0^{2\pi} g'_{\theta}(0+)\, \boldsymbol{\nu}(\mathrm{d}\theta) \Big) L^{\|X\|}(T)\,. \qquad (12.2)$$

With the considerations at the very start of this section and defining $S(\cdot) := \|X(\cdot)\|$, we can proceed exactly as the proof of Theorem 4.1, except for the step of proving

$$\sum_{\{\ell \,:\, \tau^{\varepsilon}_{2\ell+1} < T\}} \varepsilon\, g'_{\Theta(\tau^{\varepsilon}_{2\ell+1})}(0+) \;\xrightarrow[\varepsilon \downarrow 0]{}\; L^S(T) \int_0^{2\pi} h(\theta) \boldsymbol{\nu}(\mathrm{d}\theta)$$

where $h(\theta) := g'_{\theta}(0+)$, because now we do not have the help of (2.18).
We claim this convergence is still true here. Setting $N(T, \varepsilon) := \sharp\big\{ \ell \in \mathbb{N} : \tau^{\varepsilon}_{2\ell} < T \big\}$, we write

$$\sum_{\{\ell \,:\, \tau^{\varepsilon}_{2\ell+1} < T\}} \varepsilon\, g'_{\Theta(\tau^{\varepsilon}_{2\ell+1})}(0+) \;=\; \varepsilon\, N(T, \varepsilon) \,\cdot\, \frac{1}{N(T, \varepsilon)} \sum_{\ell=0}^{N(T,\varepsilon)-1} h(\Theta(\tau^{\varepsilon}_{2\ell+1})) \,+ O(\varepsilon).$$

First, we have $\varepsilon\, N(T, \varepsilon) \xrightarrow[\varepsilon \downarrow 0]{} L^{\|X\|}(T)$ in probability, by Theorem VI.1.10 in REVUZ & YOR (1999). Next, by the strong law of large numbers, we have $\frac{1}{N} \sum_{\ell=0}^{N-1} h(\Theta(\tau^{\varepsilon}_{2\ell+1})) \xrightarrow[N \to \infty]{} \int_0^{2\pi} h(\theta) \boldsymbol{\nu}(\mathrm{d}\theta)$, a.e., for any fixed ε. By the definition of limit, we have $\sup_{n \geq N} \big| \big(\frac{1}{n} \sum_{\ell=0}^{n-1} h(\Theta(\tau^{\varepsilon}_{2\ell+1})) \big) - \int_0^{2\pi} h(\theta) \boldsymbol{\nu}(\mathrm{d}\theta) \big| \xrightarrow[N \to \infty]{} 0$, a.e., so this convergence is also valid in probability. Moreover, this convergence in probability is uniform in ε, because the distribution of the random variable $\sup_{n \geq N} \big| \big(\frac{1}{n} \sum_{\ell=0}^{n-1} h(\Theta(\tau^{\varepsilon}_{2\ell+1})) \big) - \int_0^{2\pi} h(\theta) \boldsymbol{\nu}(\mathrm{d}\theta) \big|$ does not depend on ε. Now it is not hard to see that we have the convergence in probability

$$\frac{1}{N(T, \varepsilon)} \sum_{\ell=0}^{N(T,\varepsilon)-1} h(\Theta(\tau^{\varepsilon}_{2\ell+1})) \;\xrightarrow[\varepsilon \downarrow 0]{}\; \int_0^{2\pi} h(\theta) \boldsymbol{\nu}(\mathrm{d}\theta)\,, \qquad \text{on the event } \Big\{ N(T, \varepsilon) \xrightarrow[\varepsilon \downarrow 0]{} \infty \Big\}.$$

Thus our claim holds on this event. On the complement of this event the terms $\frac{1}{N(T,\varepsilon)} \sum_{\ell=0}^{N(T,\varepsilon)-1} h(\Theta(\tau^{\varepsilon}_{2\ell+1}))$ stay bounded, and we have $\varepsilon\, N(T, \varepsilon) \xrightarrow[\varepsilon \downarrow 0]{} L^{\|X\|}(T) = 0$, thus

$$\sum_{\{\ell \,:\, \tau^{\varepsilon}_{2\ell+1} < T\}} \varepsilon\, g'_{\Theta(\tau^{\varepsilon}_{2\ell+1})}(0+) \;\xrightarrow[\varepsilon \downarrow 0]{}\; \Big(\int_0^{2\pi} g'_{\theta}(0+)\, \boldsymbol{\nu}(\mathrm{d}\theta) \Big) L^{\|X\|}(T) \;=\; 0\,, \qquad \text{in probability.}$$

This establishes our claim, and obtains the FREIDLIN-SHEU Formula (12.2) for the state process $X(\cdot)$ of the posited weak solution. With (12.2) just established, and (2.17) valid by assumption, we see that $X(\cdot)$ generates a probability measure on $(C[0, \infty)^2, \mathcal{B}(C[0, \infty)^2))$ which solves the local martingale problem associated with the triple $(\boldsymbol{\sigma}, \boldsymbol{b}, \boldsymbol{\mu})$, where $\boldsymbol{\mu}$ is defined as in Proposition 12.1. This proves Part (ii) of Theorem 9.1, assuming that the stopping times $\{\tau^{\varepsilon}_m\}_{m \in \mathbb{N}_0,\, \varepsilon > 0}$ are all finite with probability one.

12.2 Proof of Theorem 9.1(ii), Part B

• When the stopping times $\{\tau_m^\varepsilon\}_{m\in\mathbb{N}_0,\,\varepsilon>0}$ can be infinite, we proceed as follows.

Step 1: If $\mathbb{P}(\tau_0^\varepsilon<\infty)=0$, then $L^{\|X\|}(\cdot)\equiv 0$ and (12.2) holds for any $\boldsymbol{\nu}$. Thus the conclusion of Part (ii) of Theorem 9.1 is true for any probability measure $\boldsymbol{\mu}$ on $(\mathfrak{S},\,\mathcal{B}(\mathfrak{S}))$. If $\mathbb{P}(\tau_0^\varepsilon<\infty)>0$, we know from (2.17) that $X(\cdot)$ can reach the origin and leave it with positive probability. So we can pick up a ε_0 such that $\mathbb{P}(\tau_1^{\varepsilon_0}<\infty)>0$. Then for every $\varepsilon\in(0,\varepsilon_0]$, $\ell\in\mathbb{N}_0$, define the probability measure $\boldsymbol{\mu}_\ell^\varepsilon$ by

$$\boldsymbol{\mu}_\ell^\varepsilon(B)\,:=\,\frac{\mathbb{P}\big(\mathfrak{f}(X(\tau_{2\ell+1}^\varepsilon))\mathbf{1}_{\{\tau_{2\ell+1}^\varepsilon<\infty\}}\in B\big)}{\mathbb{P}\big(\tau_{2\ell+1}^\varepsilon<\infty\big)}\,,\qquad\forall\;B\in\mathcal{B}(\mathfrak{S})\,.$$

This is well-defined for $\ell=0$, by our choice of ε_0. If $\mathbb{P}\big(\tau_{2\ell+1}^\varepsilon<\infty\big)=0$ for some $\ell\geq 1$, we redefine $\boldsymbol{\mu}_\ell^\varepsilon$ by $\boldsymbol{\mu}_\ell^\varepsilon:=\boldsymbol{\mu}_0^\varepsilon$.

Step 2: It is straightforward but heavier in notation, to follow the steps of Proposition 12.1 and Lemma 12.1 and check that $\boldsymbol{\mu}_\ell^\varepsilon$ does not depend on either ε or ℓ. So we can define $\boldsymbol{\mu}:=\boldsymbol{\mu}_\ell^\varepsilon,\ \forall\,\varepsilon>0,\,\ell\in\mathbb{N}_0$. Now we enlarge the original probability space by means of a countable collection of $\mathfrak{S}$-valued I.I.D. random variables $\{\boldsymbol{\xi}_\ell^\varepsilon\}_{\varepsilon\in\mathbb{Q}^+,\ell\in\mathbb{N}_0}$ with common distribution $\boldsymbol{\mu}$, and independent of the $\sigma-$algebra $\mathcal{F}$. For every $\varepsilon\in\mathbb{Q}^+,\ell\in\mathbb{N}_0$, we define the $\mathfrak{S}$-valued random variable

$$\widetilde{\mathfrak{f}}\big(X(\tau_{2\ell+1}^\varepsilon)\big)\,:=\,\mathfrak{f}\big(X(\tau_{2\ell+1}^\varepsilon)\big)\,\mathbf{1}_{\{\tau_{2\ell+1}^\varepsilon<\infty\}}+\,\boldsymbol{\xi}_\ell^\varepsilon\,\mathbf{1}_{\{\tau_{2\ell+1}^\varepsilon=\infty\}}\,.$$

It is again straightforward but tedious, to check that for any $\varepsilon\in\mathbb{Q}^+$, the random variables $\big\{\,\widetilde{\mathfrak{f}}(X(\tau_{2\ell+1}^\varepsilon))\,\big\}_{\ell\in\mathbb{N}_0}$ are independent with common distribution $\boldsymbol{\mu}$. Then in the same way as in the last subsection, we can argue

$$\sum_{\{\ell\,:\,\tau_{2\ell+1}^\varepsilon<T\}}\varepsilon\,g'_{\Theta(\tau_{2\ell+1}^\varepsilon)}(0+)\;\underset{\varepsilon\downarrow 0}{\longrightarrow}\;\Big(\int_0^{2\pi}g'_\theta(0+)\,\boldsymbol{\nu}(\mathrm{d}\theta)\Big)L^{\|X\|}(T)\,,\qquad\text{in probability}$$

as $\varepsilon\downarrow 0$ along rationals. The proof of Part (ii) of Theorem 9.1 is now complete.

12.3 Proof of Theorem 9.1(iii)

Finally, let us argue Part (iii). Under the assumptions for Parts (ii) and (iii), let $\boldsymbol{\mu}$ be some probability measure which makes the conclusion in (ii) true. Then by Part (b) of Proposition 6.1, we know that $X(\cdot)$ also solves (6.4) with γ_i replaced by $\int_{\mathfrak{S}}\mathfrak{f}_i(z)\,\boldsymbol{\mu}(\mathrm{d}z)$. Thus we must have $\gamma_i=\int_{\mathfrak{S}}\mathfrak{f}_i(z)\,\boldsymbol{\mu}(\mathrm{d}z)$, which is (9.1), on the strength of $\mathbb{P}\big(L^{\|X\|}(\infty)>0\big)>0$. Moreover (2.18) also holds, namely

$$L^{R^A}(\cdot)\,\equiv\,\boldsymbol{\nu}(A)\,L^{\|X\|}(\cdot)\,,\qquad\forall\,A\in\mathcal{B}([0,2\pi))\,,$$

with $R^A(\cdot)=\|X(\cdot)\|\cdot\mathbf{1}_A\big(\arg\big(X(\cdot)\big)\big)$. Thanks to $\mathbb{P}\big(L^{\|X\|}(\infty)>0\big)>0$ again, we see from the above relationship that $X(\cdot)$ uniquely determines $\boldsymbol{\nu}$, thus also $\boldsymbol{\mu}$. The proof of Theorem 9.1 is now complete. □

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