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Stochastic integral representations and classification of sum- and max-infinitely divisible processes

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Introduced is the notion of minimality for spectral representations of sum- and max-infinitely divisible processes and it is shown that the minimal spectral representation on a Borel space exists and is unique. This fact is used to show that a stationary, stochastically continuous, sum- or max-i.d. random process on $\mathbb{R}^d$ can be generated by a measure-preserving flow on a σ-finite Borel measure space and that this flow is unique. This development makes it possible to extend the classification program of Rosiński (*Ann. Probab.* **23** (1995) 1163–1187) with a unified treatment of both sum- and max-infinitely divisible processes. As a particular case, a characterization of stationary, stochastically continuous, union-infinitely divisible random measurable subsets of $\mathbb{R}^d$ is obtained. Introduced and classified are several new max-i.d. random field models including fields of Penrose type and fields associated to Poisson line processes.

Keywords: infinitely divisible process; max-infinitely divisible process; measure-preserving flow; minimality; Poisson process; spectral representation; stochastic integral

1. Introduction

A stochastic process $X = \{X(t), t \in T\}$ is called *infinitely divisible* (i.d.) if for every $n \in \mathbb{N}$ it can be represented (in distribution) as a sum of n independent identically distributed (i.i.d.) processes. On the other hand, X is said to be *max-infinitely divisible* (max-i.d.) if for every $n \in \mathbb{N}$ it equals in distribution to the pointwise maximum of n i.i.d. processes. Infinitely divisible distributions and random vectors have been extensively studied in the literature, while the max-i.d. case is relatively less known. The important and widely studied classes of stable and max-stable processes arise as special cases of i.d. and max-i.d. processes, respectively. Recall that X is stable if for every $n \in \mathbb{N}$, the sum of n independent copies of X equals (in distribution), a rescaled and shifted version of the original process. The definition of max-stable processes is similar, with the addition replaced by componentwise maximum.

For the class of stable processes, a particularly rich representation and classification theory based on the notion of stochastic integral over a stable random measure was developed in the pioneering works of Hardin [10] and Rosiński [31] (see also [16,27,28,32,38] and the book [39]). In parallel with the stable case, the works [5,46] developed analogous classification theory for max-stable processes based on stochastic max-integrals [4,43]. In fact, the close connection between the sum- and max-stable cases can be formalized through the notion of association [11,45].

In this paper, we develop a general structure and classification theory that applies to both i.d. and max-i.d. processes in both discrete and continuous time. It is based on stochastic integrals over Poisson random measures. Our main motivation is to extend the classification program for sum-stable processes pioneered by Rosiński [31] to the infinitely divisible setting. We develop tools for the representation and study a variety of i.d. and max-i.d. models from a unified perspective.

Stochastic integral representations of i.d. processes were developed in the seminal works of Maruyama [21] and Rajput and Rosiński [29], while the max-i.d. case was addressed by Balkema *et al.* [2]. To the best of our knowledge, the structure theory based on such stochastic integral representations has not been much explored. A key problem in this context is to determine how two spectral representations of the same i.d. or max-i.d. process are related. In the special stable case, this problem is related to the structure of the isometries of L^α-spaces [9,10] and it was elegantly resolved in terms of the notion of *minimality*. In the setting of i.d. processes, these methods are not available. Instead, we prove a general result on the existence of conjugacy between equimeasurable families of functions (Lemma 5.1, below) and based on this result we define a corresponding notion of *minimality*. It turns out that two minimal spectral representations of the same i.d. process defined on σ-finite Borel spaces are related through a unique measure space isomorphism between the two spaces. This result is then used to show that a *stationary* i.d. process can be generated as a stochastic integral over a measure-preserving flow. This extends some classification results of Rosiński [31,32] on the spectral representations of stationary stable processes. The i.d. theory we develop here is in fact simpler (although more general) than the stable theory of [31,32] since we deal with *measure-preserving* rather than *non-singular* flows. Our results can be specialized to the stable case by using the Maharam construction from ergodic theory. This sheds more light on the subtle concept of minimality in the stable case.

Recall that the law of a finite-dimensional i.d. random vector is characterized by its Lévy triplet [40]. Here, we focus on the case where the i.d. random vectors have trivial Gaussian component and their laws are determined by their Lévy measures along with constant location vectors. In a pioneering work, Maruyama [21] considered the case of i.d. processes $X = \{X(t), t \in T\}$ and extended the concept of Lévy measure to the infinitely dimensional setting as a measure on $\mathbb{R}^T$. This extension is especially non-trivial when the set T is not countable because in this case several measurability issues arise. The Lévy measure introduced by Maruyama [21] could be used to establish some of the results of the present paper. Here, we chose to develop classification theory based on spectral representations in order to draw parallels with the abundant theory for stable processes.

Unlike the i.d. case, every one-dimensional distribution is max-i.d., but the situation changes in higher dimensions. Max-infinitely divisible random vectors are characterized by the so called *exponent measure*, which plays a role similar to that of the Lévy measure in the i.d. case (see, e.g., Chapter 5 in [30]). Two-dimensional max-i.d. distributions were introduced by Balkema and Resnick [3], the general d-dimensional case was considered by Gerritse [6] and Vatan [44] (the latter work studies also max-i.d. vectors with values in $\mathbb{R}^{\mathbb{N}}$). Representations in terms of suprema over a Poisson point process were obtained by Giné *et al.* [7] for max-i.d. processes with continuous sample paths and by Balkema *et al.* [2] for stochastically continuous processes. It seems that in the case of uncountable T, the general concept of exponent measure (parallel to [21]) has not been studied. In this paper, we develop the representation theory of max-i.d. processes further, for example, by proving existence and uniqueness of the minimal spectral representation and by constructing a representation over a measure-preserving flow for stationary processes. We illustrate our theory by introducing and classifying several new examples of max-i.d. processes. As a special case, we arrive at a representation result for stationary union infinitely divisible random sets, which may be of independent interest. All our examples have direct analogs in the sum-i.d. context.

The paper is organized as follows. In Section 2.1, we introduce minimal spectral representations for max-i.d. processes and show their existence and uniqueness under the general Condition S of separability in probability. Section 2.2 contains parallel results for i.d. processes. In Section 2.3, we discuss measurability of i.d. and max-i.d. processes. In Section 3, the developed theory is used to associate stationary i.d. and max-i.d. processes to measure preserving flows leading to extensions of known classification results on stable processes. In Section 4, we present several examples and applications. The connection between the new notion of minimal spectral representations and the existing ones for stable processes is demonstrated in Section 4.1. In Sections 4.2–4.5, we present new examples of stationary max-i.d. processes associated with dissipative, conservative, or null flows. In Section 4.6, we characterize the stationary union infinitely divisible random sets by relating them to max-i.d. processes. The proofs are given in Section 5.

2. Spectral representations of i.d. and max-i.d. processes

2.1. Spectral representations of max-i.d. processes

A stochastic process $X = \{X(t), t \in T\}$ defined on an index set T and taking values in $\mathbb{R}$ is called max-i.d., if for all $n \in \mathbb{N}$, there exist independent identically distributed (i.i.d.) processes $\{X_{i,n}(t), t \in T\}$, $i = 1, \dots, n$, such that

$$\bigl\{X(t), t \in T\bigr\} \stackrel{d}{=} \Bigl\{\max_{1\le i\le n} X_{i,n}(t), t \in T\Bigr\}. \tag{2.1}$$

Here, $\stackrel{d}{=}$ denotes the equality of finite-dimensional distributions. If $\{X(t), t \in T\}$ is a max-i.d. process, then for every collection of non-decreasing functions $\varphi_t : \mathbb{R} \to \mathbb{R}$, $t \in T$, the process $\{\varphi_t(X(t)), t \in T\}$ is also max-i.d. By choosing

$$\varphi_t(x) = \begin{cases} e^x, & \text{if } \operatorname{essinf} X(t) = -\infty, \\ x - \operatorname{essinf} X(t), & \text{if } \operatorname{essinf} X(t) > -\infty \end{cases}$$

we can always achieve that $\operatorname{essinf} \varphi_t(X(t)) = 0$. In the sequel, we therefore assume without loss of generality that $\operatorname{essinf} X(t) = 0$ for every $t \in T$.

Balkema *et al.* [2] gave a representation of max-i.d. processes in terms of stochastic max-integrals over a Poisson point process. This representation is, in general, non-unique; see Example 2.12 below. We will introduce the notion of *minimality* for representations of max-i.d. processes and prove that the minimal representation exists and is unique.

We recall the construction of Balkema *et al.* [2] in a form which is suitable for our purposes. Let $(\Omega, \mathcal{B}, \mu)$ be a σ-finite measure space. We denote by $\mathcal{L}^\vee = \mathcal{L}^\vee(\Omega, \mathcal{B}, \mu)$ the space of all measurable functions $f : \Omega \to \mathbb{R}$ such that $f \geq 0$ μ-a.e. and $\mu\{\omega : f(\omega) > a\}$ is finite for all $a > 0$. As usual, two functions are identified if they differ on a set of measure zero. Note that for every $f_1, f_2 \in \mathcal{L}^\vee$ and $c_1, c_2 \geq 0$ we have $\max(c_1 f_1, c_2 f_2) \in \mathcal{L}^\vee$. Next let us recall the definition of the max-integral from [2]. Let $\Pi_\mu = \{U_i, i \in J\}$ be a Poisson point process on the space $(\Omega, \mathcal{B})$ with intensity μ. Here, J is at most countable index set. For $f \in \mathcal{L}^\vee$ define the stochastic max-integral

$$I(f) \equiv \int_\Omega^\vee f \, \mathrm{d}\Pi_\mu := \sup_{i \in J} f(U_i). \tag{2.2}$$

Here, the supremum is taken over all atoms U_i of the Poisson process Π_μ. If Π_μ is empty, which can happen if $\mu(\Omega) < \infty$, then the supremum in the right-hand side is defined to be 0. From (2.2), one readily derives a formula for the joint distribution of the stochastic max-integrals: for all $f_1, \ldots, f_n \in \mathcal{L}^\vee$ and $x_1, \ldots, x_n \geq 0$ (not all of which are 0), we have

$$\begin{aligned} \mathbb{P}\{I(f_j) < x_j, 1 \leq j \leq n\} &= \mathbb{P}\left\{\Pi_\mu\left(\bigcup_{j=1}^n \{f_j \geq x_j\}\right) = 0\right\} \\ &= \exp\left\{-\mu\left(\bigcup_{j=1}^n \{f_j \geq x_j\}\right)\right\}. \end{aligned} \tag{2.3}$$

Observe that for any collection of deterministic functions $f_t \in \mathcal{L}^\vee, t \in T$, the process $\{I(f_t), t \in T\}$, is max-i.d. since the $X_{i,n}$'s in (2.1) can be defined by using independent copies of the same stochastic max-integrals but with respect to a Poisson point process with intensity $\frac{1}{n}\mu$.

Definition 2.1. *Let $X = \{X(t), t \in T\}$ be a max-i.d. process with* $\operatorname{essinf} X(t) = 0$ *for all $t \in T$. A collection of functions $\{f_t, t \in T\} \subset \mathcal{L}^\vee(\Omega, \mathcal{B}, \mu)$ is a spectral representation of the process X if we have the following equality of laws*:

$$\{X(t), t \in T\} \stackrel{d}{=} \left\{\int_\Omega^\vee f_t \, \mathrm{d}\Pi_\mu, t \in T\right\}, \tag{2.4}$$

where Π_μ is a Poisson point process on $(\Omega, \mathcal{B})$ with intensity μ.

Here, we focus on the general class of processes that are separable in probability in the sense of the following definition.

Definition 2.2. *A stochastic process $\{X(t), t \in T\}$ satisfies Condition* S *if there is an at most countable set $T_0 \subset T$ such that for all $t \in T$, there exists a sequence $\{t_n\}_{n \in \mathbb{N}} \subset T_0$, with $X(t_n) \to X(t)$ in probability.*

As shown in Balkema *et al.* [2], the convergence in probability for max-i.d. random variables is equivalent to convergence in measure of their spectral functions.

Proposition 2.3 (Balkema *et al.* [2]). *Let $f_n \in \mathcal{L}^\vee(\Omega, \mathcal{B}, \mu), n \in \mathbb{N}$. Then, there is a random variable ξ such that $I(f_n) \to \xi$ in probability as $n \to \infty$, if and only if, there exists $f \in \mathcal{L}^\vee(\Omega, \mathcal{B}, \mu)$ such that $f_n \to f$ in measure, as $n \to \infty$. In this case, $\xi = I(f)$ a.s.*

The proof follows from Theorems 4.4 and 4.5 in [2].

Remark 2.4. Observe that if $f_n \in \mathcal{L}^\vee(\Omega, \mathcal{B}, \mu)$ and $f_n \to f, n \to \infty$, in measure, then necessarily $f \in \mathcal{L}^\vee(\Omega, \mathcal{B}, \mu)$. Indeed, since for all $\epsilon > 0$, we have $\{f > \epsilon\} \subset \{f_n > \epsilon/2\} \cup \{|f - f_n| > \epsilon/2\}$, it follows that $\mu\{f > \epsilon\} < \infty$, for all $\epsilon > 0$. Thus, $\mathcal{L}^\vee(\Omega, \mathcal{B}, \mu)$ is closed with respect to convergence in measure, which in fact can be metrized by a version of the Ky Fan metric:

$$d_\mu(f, g) := \inf\bigl\{\epsilon > 0 : \mu\bigl(|f - g| \ge \epsilon\bigr) \le \epsilon\bigr\}. \tag{2.5}$$

Note that $\mu(|f - g| \ge \epsilon) < \infty$, for all $\epsilon > 0$ and $f, g \in \mathcal{L}^\vee(\Omega, \mathcal{B}, \mu)$. One can show that $\mathcal{L}^\vee(\Omega, \mathcal{B}, \mu)$ equipped with d_μ becomes a complete metric space. Thus, Proposition 2.3 entails that the stochastic max-integral operator is a homeomorphism of metric spaces. More precisely, $I : (\mathcal{L}^\vee(\Omega, \mathcal{B}, \mu), d_\mu) \to (\mathcal{L}^0(\mathbb{P}), d_{\mathrm{KF}})$ is a continuous bijection onto its image with a continuous inverse. Here, $\mathcal{L}^0(\mathbb{P})$ is the space of random variables on the probability space $(E, \mathcal{F}, \mathbb{P})$ on which the Poisson process Π_μ is defined. The space $\mathcal{L}^0(\mathbb{P})$ is endowed with the Ky Fan metric d_{KF} that metrizes the convergence in probability (see, e.g., (5.9) below).

Theorem 2.5 (Balkema *et al.* [2]). *Let $\{X(t), t \in \mathbb{R}^d\}$ be a max-i.d. process satisfying Condition* S. *There exists a spectral representation of X defined on $\mathbb{R}$ endowed with the Lebesgue measure.*

We will prove existence and uniqueness of the spectral representation under the following condition of minimality.

Definition 2.6. *A spectral representation $\{f_t, t \in T\} \subset \mathcal{L}^\vee(\Omega, \mathcal{B}, \mu)$ is called minimal if the following two conditions hold*:

(i) *The σ-algebra generated by $\{f_t, t \in T\}$ coincides with $\mathcal{B}$ up to μ-zero sets. That is, for every $B \in \mathcal{B}$, exists an $A \in \sigma\{f_t, t \in T\}$, such that $\mu(A \Delta B) = 0$.*
(ii) *There is no set $B \in \mathcal{B}$ such that $\mu(B) > 0$ and for every $t \in T$, $f_t = 0$ a.e. on B.*

If just the second condition holds, we will say that the representation has full support.

Remark 2.7. The first condition does not imply the second one: consider $\Omega = \{0, 1\}$ with counting measure, $T = \{1\}$, and $f_1(\omega) = \omega$.

Theorem 2.8. *Let $X = \{X(t), t \in T\}$ be a max-i.d. process satisfying Condition* S. *There exists a minimal spectral representation of X defined on* [0, 1] *endowed with a σ-finite Borel measure.*

We recall next several notions of isomorphisms from measure theory. For more details, see, for example, Chapter 22 in [41], page 167 in [8], or Chapter 15.4 in [37].

Definition 2.9. (i) *An* isomorphism *between two* measurable spaces $(\Omega_i, \mathcal{B}_i)$, $i = 1, 2$, *is a bijection* $\Phi : \Omega_1 \to \Omega_2$ *such that both* Φ *and* Φ^{-1} *are measurable.*

(ii) *A measurable space* $(\Omega, \mathcal{B})$ *is said to be a* Borel space *if it is isomorphic* (*in the sense of part* (i)) *to a complete separable metric space endowed with its Borel* σ*-algebra.*

(iii) *A Borel space endowed with a* σ*-finite measure will be called a* σ*-finite Borel space.*

(iv) *An* isomorphism (modulo null sets) *between two* measure spaces $(\Omega_i, \mathcal{B}_i, \mu_i)$, $i = 1, 2$, *is a bijection* $\Phi : \Omega_1 \setminus A_1 \to \Omega_2 \setminus A_2$, *where* $A_1 \in \mathcal{B}_1$ *and* $A_1 \in \mathcal{B}_2$ *are null sets, such that both* Φ *and* Φ^{-1} *are measurable and* $\mu_1(A) = \mu_2(\Phi(A))$, *for all measurable* $A \subset \Omega_1 \setminus A_1$. *Two isomorphisms* Φ, Ψ *are considered as equal modulo null sets if* $\Phi(\omega) = \Psi(\omega)$ *for* μ_1*-a.a.* $\omega \in \Omega_1$.

Remark 2.10. Any Borel space is isomorphic to the interval $[0, 1]$ endowed with the Borel σ-algebra or to an at most countable set endowed with the σ-algebra of all subsets. This result is known as Kuratowski's theorem (see, e.g., page 406 in [37]).

The next statement is the main result in this section. It establishes the uniqueness of the minimal spectral representation.

Theorem 2.11. *Let* $X = \{X(t), t \in T\}$ *be a max-i.d. process. Let also* $\{f_t^{(i)}, t \in T\}$ *be two minimal spectral representations of* X *defined on the spaces* $(\Omega_i, \mathcal{B}_i, \mu_i)$, $i = 1, 2$.

(i) *If* $(\Omega_1, \mathcal{B}_1, \mu_1)$ *is a* σ*-finite Borel space, then there is a measurable map* $\Phi : \Omega_2 \to \Omega_1$ *such that* $\mu_1 = \mu_2 \circ \Phi^{-1}$ *and for all* $t \in T$,

$$f_t^{(2)}(\omega) = f_t^{(1)} \circ \Phi(\omega) \qquad \text{for } \mu_2\text{-a.a. } \omega \in \Omega_2. \tag{2.6}$$

(ii) *If both* $(\Omega_i, \mathcal{B}_i, \mu_i)$, $i = 1, 2$ *are* σ*-finite Borel spaces, then the mapping* Φ *in part* (i) *is a measure space isomorphism and it is unique modulo null sets.*

Example 2.12. Our definition of the space $\mathcal{L}^\vee$ of integrands is more restrictive than that of Balkema *et al.* [2], who allow measurable functions $f : \Omega \to \mathbb{R}$ with $\mu\{f > a\} < \infty$, for *some* $a \in \mathbb{R}$ (and do not assume that $\operatorname{essinf} X_t = 0$). With the definition used in [2] the uniqueness may fail, even for minimal representations. Indeed, let $\Omega_1 = \Omega_2 = \mathbb{Z}$ be endowed with the counting measure. Set $T = \mathbb{Z} \cup \{*\}$ and define

$$f_t^{(1)}(\omega) = f_t^{(2)}(\omega) = \mathbb{1}_{\{t\}}(\omega) \qquad \text{if } t \neq *,$$
$$f_*^{(1)}(\omega) = 1, \qquad f_*^{(2)}(\omega) = \mathbb{1}_{\{\omega > 0\}} + \tfrac{1}{2}\mathbb{1}_{\{\omega \le 0\}}.$$

One verifies readily that $\{f_t^{(i)}, t \in T\}$, $i = 1, 2$, are minimal representations of the same max-i.d. process. However, there is no bijection $\Phi : \Omega_1 \to \Omega_2$ such that $f_*^{(1)} \circ \Phi = f_*^{(2)}$. Note that $f_*^{(2)} \notin \mathcal{L}^\vee$ and hence Theorem 2.11 does not apply. The constant $1/2$ in the definition of $f_*^{(2)}$ could in fact be replaced by any $0 < c < 1$. This example shows why it is important to require that the max-integrands f in $\mathcal{L}^\vee$ satisfy the condition $\mu\{f > a\} < \infty$ for *all* $a > 0$.

2.2. Spectral representations of i.d. processes

A process $\{X(t), t \in T\}$ is said to be infinitely divisible (i.d. or sum-i.d.) if for all $n \in \mathbb{N}$ it can be represented (in distribution) as a sum of n independent and identically distributed processes.

There is already a lot of literature on the spectral representations of i.d. processes (see, e.g., [21, 29,33]). Our aim here is to study the minimality and the uniqueness of the spectral representation. This is a key step which allows us to extend the classification program pioneered by Rosiński [31] in the stable case to the general i.d. context.

Let $(\Omega, \mathcal{B}, \mu)$ be a σ-finite measure space. The space of integrands $\mathcal{L}^+$ consists of all measurable $f : \Omega \to \mathbb{R}$ such that

$$\int_\Omega \min\{\varepsilon, |f(\omega)|^2\}\mu(\mathrm{d}\omega) < +\infty \tag{2.7}$$

for some (or, equivalently, any) $\varepsilon > 0$. Functions differing on a set of measure 0 are identified. Observe that $\mathcal{L}^+$ is a linear space since $1 \wedge (f+g)^2 \le 2(1 \wedge f^2 + 1 \wedge g^2)$. Following Maruyama [21], for $f \in \mathcal{L}^+$ define the stochastic integral

$$I(f) \equiv \int_\Omega^+ f\, \mathrm{d}\Pi_\mu := \lim_{\varepsilon \to 0+} \left\{ \sum_{i \in J} f(U_i) \mathbb{1}_{\{|f(U_i)| > \varepsilon\}} - \int_{\{|f| > \varepsilon\}} a(f)\, \mathrm{d}\mu \right\}, \tag{2.8}$$

where $\Pi_\mu = \{U_i, i \in J\}$ is a Poisson point process on $(\Omega, \mathcal{B})$ with intensity μ and

$$a(u) = \begin{cases} u, & |u| \le 1, \\ 1, & u > 1, \\ -1, & u < -1. \end{cases} \tag{2.9}$$

Note that the limit in (2.8) exists in the a.s. sense by the convergence theorem for L^2-bounded martingales. For $f_1, \ldots, f_n \in \mathcal{L}^+$ the joint distribution of the $I(f_j)$'s is characterized as follows. For all $\theta_1, \ldots, \theta_n \in \mathbb{R}$ we have

$$\mathbb{E}\mathrm{e}^{\mathrm{i}\sum_{j=1}^n \theta_j I(f_j)} = \exp\left\{ \int_\Omega \left(\mathrm{e}^{\mathrm{i}\sum_{j=1}^n \theta_j f_j(\omega)} - 1 - \mathrm{i}\sum_{j=1}^n \theta_j a\big(f_j(\omega)\big) \right) \mu(\mathrm{d}\omega) \right\}, \tag{2.10}$$

where i stands for the imaginary unit. In particular, it is easy to verify that for all $f, g \in \mathcal{L}^+$ and $c \in \mathbb{R}$ we have $I(f+g) = I(f) + I(g) + \text{const}$ and $I(cf) = cI(f) + \text{const}$, that is, the functional I is essentially linear up to additive constants (see also (5.8) below).

Definition 2.13. *Let $X = \{X(t), t \in T\}$ be an i.d. process with trivial Gaussian component defined on some index set T.*

(i) *A collection of functions $\{f_t, t \in T\} \subset \mathcal{L}^+(\Omega, \mathcal{B}, \mu)$ is a spectral representation of the process X if we have the following equality of laws*:

$$\{X(t), t \in T\} \stackrel{d}{=} \left\{ \int_\Omega^+ f_t\, \mathrm{d}\Pi_\mu + c(t), t \in T \right\}, \tag{2.11}$$

where Π_μ is a Poisson point process on $(\Omega, \mathcal{B}, \mu)$ and $c : T \to \mathbb{R}$ is some function.

(ii) *The spectral representation is called* minimal *if $\{f_t, t \in T\}$ satisfy both conditions of Definition 2.6.*

Theorem 2.14. *Let $\{X(t), t \in T\}$ be an i.d. process which has a trivial Gaussian component and satisfies Condition* S. *There exists a minimal spectral representation of X defined on* $[0, 1]$ *endowed with a σ-finite Borel measure.*

The proof of the above result (given in Section 5 below) utilizes the following truncated L^2-metric on the space $\mathcal{L}^+$:

$$d(f, g) \equiv d(f - g) := \left(\int_\Omega 1 \wedge (f - g)^2 \, \mathrm{d}\mu\right)^{1/2}, \qquad f, g \in \mathcal{L}^+. \tag{2.12}$$

Note that the triangle inequality follows from $1 \wedge |f + g| \le 1 \wedge |f| + 1 \wedge |g|$ and the triangle inequality in $L^2(\Omega)$.

Proposition 2.15. *For any σ-finite Borel space $(\Omega, \mathcal{B}, \mu)$, the space $(\mathcal{L}^+, d)$ is separable and complete.*

The next proposition shows that the metric d on the space $\mathcal{L}^+$ corresponds to convergence in probability on the space of stochastic integrals $\{I(f), f \in \mathcal{L}^+\}$.

Proposition 2.16. *For $f_n \in \mathcal{L}^+$ and $c_n \in \mathbb{R}$, we have that $I(f_n) + c_n$ converges to a random variable ξ in probability, as $n \to \infty$, if and only if, there exists some $f \in \mathcal{L}^+$ and $c \in \mathbb{R}$, such that $d(f_n - f) + |c_n - c| \to 0$, as $n \to \infty$. In this case, $\xi = I(f) + c$ a.s.*

The next theorem, which is analogous to Theorem 2.11, shows the uniqueness of the minimal spectral representation for i.d. processes.

Theorem 2.17. *Let $X = \{X(t), t \in T\}$ be an i.d. process. Let also $\{f_t^{(i)}, t \in T\}$ be two minimal spectral representations of X defined on the spaces $(\Omega_i, \mathcal{B}_i, \mu_i)$, $i = 1, 2$.*

(i) *If $(\Omega_1, \mathcal{B}_1, \mu_1)$ is a σ-finite Borel space, then there is a measurable map $\Phi : \Omega_2 \to \Omega_1$ such that $\mu_1 = \mu_2 \circ \Phi^{-1}$ and for all $t \in T$,*

$$f_t^{(2)}(\omega) = f_t^{(1)} \circ \Phi(\omega) \qquad \text{for } \mu_2\text{-a.a. } \omega \in \Omega_2. \tag{2.13}$$

(ii) *If both $(\Omega_i, \mathcal{B}_i, \mu_i)$, $i = 1, 2$ are σ-finite Borel spaces, then the mapping Φ in part* (i) *is a measure space isomorphism and it is unique modulo null sets.*

Remark 2.18. Theorems 2.11 and 2.17 require that both representations be minimal. Minimality can be enforced by replacing Ω_i by $\mathrm{supp}\{f_t^{(i)}, t \in T\}$, $i = 1, 2$, and letting $\mathcal{B}_i = \sigma\{f_t^{(i)}, t \in T\}$, $i = 1, 2$. To be able to apply the above results, however, at least one of the measure spaces should be Borel. This is neither automatic nor obvious for the new spaces $(\Omega_i, \mathcal{B}_i)$. If both spaces are Borel, then by part (ii) of Theorems 2.11 and 2.17 minimality ensures that the two representations are related through a unique measure space isomorphism as in (2.13). In fact, by part (i) of Theorems 2.11 and 2.17, this relation still holds for some not necessarily invertible map, provided that just the first representation is minimal. This last fact is an important technical tool, analogous to Remark 2.5 in Rosiński [31], in the stable case.

2.3. Measurability and stochastic continuity

When studying path properties or ergodicity, it is important or in fact necessary to work with measurable processes. Here, we establish necessary and sufficient conditions for the existence of measurable versions of max-i.d. and i.d. processes in terms of their spectral representations.

Let (T, ρ_T) be a separable metric space, equipped with its Borel σ-algebra $\mathcal{A}$. Consider a family of measurable functions $\{f_t, t \in T\}$ on $(\Omega, \mathcal{B}, \mu)$. This family is said to be *jointly measurable* if the map $(t, \omega) \mapsto f_t(\omega)$ is measurable with respect to the product σ-algebra $\mathcal{A} \otimes \mathcal{B} := \sigma(\mathcal{A} \times \mathcal{B})$. For the classical notions of measurability and strong separability of a stochastic process $X = \{X_t\}_{t\in T}$, we refer to Chapter 9 in [39]. The following result extends Proposition 4.1 in [46] (see also Theorem 11.1.1 in [39]). Its proof is given in [14].

Proposition 2.19. *Let $X = \{X(t), t \in T\}$ be a max-i.d.* (*i.d.*, *resp.*) *process with spectral representation $\{f_t, t \in T\} \subset \mathcal{L}^{\vee/+}(\Omega, \mathcal{B}, \mu)$ over a σ-finite measure space $(\Omega, \mathcal{B}, \mu)$ as in* (2.4) (*as in* (2.11), *resp.*). *The process X has a measurable modification if and only if the following two conditions hold*:

(i) *The family $\{f_t, t \in T\}$ has a jointly measurable modification*, *that is*, *there exists a $\mathcal{A} \otimes \mathcal{B}$-measurable mapping $(t, \omega) \mapsto g_t(\omega)$, such that $f_t = g_t \ (\mathrm{mod}\, \mu)$, for all $t \in T$.*

(ii) *The function $t \mapsto c(t)$ is measurable if X is i.d.*

If the process X has a measurable modification, *then it satisfies Condition* S, *and consequently*, *X has a spectral representation over a σ-finite Borel space. In this case*, *the measurable version of X and the jointly measurable version of $\{f_t, t \in T\}$ may be taken to be strongly separable.*

Remark 2.20. The last result shows that, if X has a measurable version, then this version as well as its corresponding jointly measurable representation $\{f_t\}_{t\in T}$ can be taken to be strongly separable (cf. Chapter 9 in [39]).

Remark 2.21. Proposition 3.1 in [35] states that any *measurable* and *stationary* random field $\{X(t), t \in \mathbb{R}^d\}$ is automatically continuous in probability.

This follows from a celebrated result due to Banach on Polish groups. Conversely, it is well known that stochastically continuous processes (indexed by separable metric spaces) have measurable modifications. Therefore, in the case of stationary random fields on $\mathbb{R}^d$ the assumptions of stochastic continuity and measurability are essentially equivalent.

3. Flow representations and ergodic decompositions for stationary i.d. and max-i.d. processes

In the stable and max-stable cases, the connections between spectral representations and ergodic theoretic decompositions of the underlying flows have lead to a wealth of decomposition and classification results. We will show that this theory naturally extends to the i.d. and max-i.d. setting. An alternative powerful approach from the perspective of *Poisson suspensions* and *factor maps* has been recently pioneered by Emmanuel Roy [33,34]. We expect that these tools can be used to develop an all-encompassing theory, but this is beyond the scope and goals of the present work. Here, we adopt an alternative approach, which is useful when the i.d. and max-i.d. processes are given through their stochastic integral representations.

3.1. Existence and uniqueness of flow representations

Let $\mathbb{T}$ denote either $\mathbb{Z}$ or $\mathbb{R}$. Consider the measure space $(\mathbb{T}^d, \mathcal{A}, \lambda)$, where λ is either the counting measure if $\mathbb{T} = \mathbb{Z}$ or the Lebesgue measure with $\mathcal{A}$ the Borel σ-algebra if $\mathbb{T} = \mathbb{R}$. A measure-preserving $\mathbb{T}^d$-action (or flow) on a measure space $(\Omega, \mathcal{B}, \mu)$ is a family $\{T_t\}_{t\in\mathbb{T}^d}$ of measure space isomorphisms $T_t : \Omega \to \Omega$ such that $T_0 = \mathrm{id}$ μ-a.e. and for every $t, s \in \mathbb{R}^d$, $T_t \circ T_s = T_{t+s}$ μ-a.e. The action is called *measurable* if $(t, \omega) \mapsto T_t(\omega)$ is a measurable map from $\mathbb{T}^d \times \Omega$ to Ω, where the former space is endowed with the product σ-algebra $\mathcal{A} \otimes \mathcal{B}$.

The next statement combines both the i.d. and max-i.d. cases and shows that one can associate stationary processes with measure-preserving actions. The common theme is the uniqueness.

Theorem 3.1. *Let $X = \{X(t), t \in \mathbb{T}^d\}$ be a stationary and stochastically continuous max-i.d. (resp., i.d., without Gaussian component) process with a representation $\{f_t, t \in T\} \subset \mathcal{L}^{\vee/+}(\Omega, \mathcal{B}, \mu)$ as in (2.4) (or (2.11), resp.). If the representation is minimal and the measure space $(\Omega, \mathcal{B}, \mu)$ is σ-finite Borel, then there exists a measurable and measure-preserving flow $\{T_t\}_{t\in\mathbb{T}^d}$ on $(\Omega, \mathcal{B}, \mu)$ such that for all $t \in \mathbb{T}^d$, we have*

$$f_t = f_0 \circ T_t, \qquad \mu\text{-a.e.} \tag{3.1}$$

In the i.d. case the function $c(t)$ in (2.11) is constant.

Proof. By stationarity, for every fixed $s \in \mathbb{T}^d$, both $\{f_t, t \in \mathbb{T}^d\}$ and $\{f_{t+s}, t \in \mathbb{T}^d\}$ are minimal spectral representations of X defined over the same σ-finite Borel space. By Theorems 2.11(ii) and 2.17(ii), there is a modulo μ unique automorphism T_s of the measure space $(\Omega, \mathcal{B}, \mu)$ such that for every $t \in \mathbb{T}^d$, $f_{s+t} = f_t \circ T_s$, μ-a.e. Let us show that for every $s_1, s_2 \in \mathbb{T}^d$, $T_{s_1+s_2} = T_{s_1} \circ T_{s_2}$, μ-a.e. Indeed, we have for every $t \in \mathbb{T}^d$,

$$f_t \circ T_{s_1+s_2} = f_{s_1+s_2+t} = f_t \circ (T_{s_1} \circ T_{s_2}), \qquad \mu\text{-a.e.}$$

By the uniqueness of the automorphisms, we have $T_{s_1+s_2} = T_{s_1} \circ T_{s_2}$, μ-a.e., which yields (3.1). In the sum-i.d. case, note also that the term $c(t)$ appearing in (2.11) does not depend on $t \in \mathbb{T}^d$ by stationarity.

This completes the proof in the case $\mathbb{T} = \mathbb{Z}$. In the case $\mathbb{T} = \mathbb{R}$ the flow $\{T_t\}_{t\in\mathbb{R}^d}$ constructed in this way need not in general be measurable (see, e.g., Example 3.6, below). However, one can argue as in [31], by using the works of Mackey [19] and Sikorski [41], that each T_t can be modified on a set of μ-measure zero so that the flow property is valid with probability one and the flow becomes measurable. Indeed, observe first that by Proposition 2.19, we may assume that the representation $\{f_t, t \in \mathbb{R}^d\}$ is jointly measurable. Now, consider the Boolean σ-algebra $\mathcal{B}_\mu$ whose elements are equivalence classes $[B]$ of sets $B \in \mathcal{B}$ with respect to the equality modulo μ-null sets. Following the argument on page 1168 of Rosiński [31], in order to apply Theorem 1 of [19], it is enough to show that for every finite measure $\widetilde{\nu}$ on the Boolean σ-algebra $\mathcal{B}_\mu$ and for every set $B \in \mathcal{B}$ the function

$$t \mapsto \widetilde{\nu}\big(\big[T_t(B)\big]\big)$$

is Borel measurable. For the finite measure ν on $(\Omega, \mathcal{B})$, induced by $\widetilde{\nu}$ as $\nu(B) := \widetilde{\nu}([B])$, we have $\nu \ll \mu$ and hence

$$\widetilde{\nu}\big(\big[T_t(B)\big]\big) = \int_\Omega (\mathbb{1}_B \circ T_{-t})(\omega)k(\omega)\mu(\mathrm{d}\omega), \tag{3.2}$$

where $k = \mathrm{d}\nu/\mathrm{d}\mu \in L^1(\Omega, \mathcal{B}, \mu)$ is the Radon–Nikodym density.

By minimality of the representation $\{f_t, t \in \mathbb{R}^d\}$, we have that there is a set $A \in \mathcal{B}$ such that $\mu(A \Delta B) = 0$ and $A \in \sigma\{f_{t_i}, i \in \mathbb{N}\}$, for some countable collection $\{t_i, i \in \mathbb{N}\} \subset \mathbb{R}^d$ and therefore, there exists a Borel function $g : \mathbb{R}^\mathbb{N} \to \mathbb{R}$, such that $\mathbb{1}_A = g(f_{t_1}, f_{t_2}, \ldots)$. We will show that the integral in (3.2) is a Borel measurable function of t. Since $f_{t_i - t} = f_{t_i} \circ T_{-t} \bmod \mu$, we have that $\mathbb{1}_A \circ T_{-t} = g(f_{t_1-t}, f_{t_2-t}, \ldots) \bmod \mu$, for every $t \in \mathbb{R}^d$. Hence, we have

$$\widetilde{\nu}\big(\big[T_t(B)\big]\big) = \int_\Omega g\big(f_{t_1-t}(\omega), f_{t_2-t}(\omega), \ldots\big)k(\omega)\mu(\mathrm{d}\omega) \qquad \text{for all } t \in \mathbb{R}^d. \tag{3.3}$$

Now, the joint measurability of $\{f_t, t \in \mathbb{R}^d\}$ implies the joint measurability of the integrand on the right-hand side of (3.3) as a function of t and ω. Hence, by Fubini's theorem, applied to (3.3), we obtain that $t \mapsto \widetilde{\nu}([T_t(B)])$ is Borel measurable. Now, proceeding as on page 1169 of [31], by using Theorem 1 of [19] and Theorem 32.5 of [41], we obtain that the flow $\{T_t, t \in \mathbb{R}^d\}$ has a jointly measurable modification. □

Theorem 3.1, as in the stable case (cf. [31]) motivates the following.

Definition 3.2. *A stationary max-i.d. or i.d. process* (*with trivial Gaussian part*) *X with spectral representation* $\{f_t, t \in \mathbb{T}^d\} \subset \mathcal{L}^{\vee/+}(\Omega, \mathcal{B}, \mu)$ *as in* (2.4) *or* (2.11), *is said to be generated by a measure-preserving flow* $\{T_t\}_{t\in\mathbb{T}^d}$ *on* $(\Omega, \mathcal{B}, \mu)$ *if*:

(i) *For every* $t \in \mathbb{T}^d$, $f_t = f_0 \circ T_t$ μ*-a.e.*
(ii) $\{f_t, t \in \mathbb{T}^d\}$ *has full support.*

In this case, we call the pair $(f_0, \{T_t\}_{t\in\mathbb{R}^d})$ *a flow representation of X on* $(\Omega, \mathcal{B}, \mu)$. *Furthermore, if* $\{f_t, t \in \mathbb{T}^d\}$ *is minimal, then we say that the flow representation is minimal.*

Remark 3.3. According to the above definition, a flow representation need not be minimal. The reason why we consider more general non-minimal flow representations is because minimality may not be easy to check or ensure in applications.

The next corollary follows immediately from Theorems 2.8, 2.14, and 3.1.

Corollary 3.4. *Let $\{X(t), t \in \mathbb{T}^d\}$ be a stationary max-i.d. or i.d.* (*without Gaussian component*) *process which is stochastically continuous* (*equivalently*: *has a measurable modification*). *Then, X has a minimal representation by a measurable flow on a σ-finite Borel space.*

The next result shows that the minimal flow representation associated with a stationary stochastically continuous max-i.d. or i.d. process is essentially unique up to a *flow isomorphism*. This fact will allow us to obtain structural results about the above two types of processes from ergodic theoretic properties of the associated flows.

Theorem 3.5. *Let $X = \{X(t), t \in \mathbb{T}^d\}$ be a stationary max-i.d. or i.d.* (*without Gaussian component*) *random field. If $\mathbb{T} = \mathbb{R}$, suppose in addition that X is stochastically continuous. If $(f_0^{(i)}, \{T_t^{(i)}\}_{t\in\mathbb{T}^d})$ are two minimal flow representations of X on σ-finite Borel spaces $(\Omega_i, \mathcal{B}_i, \mu_i)$, $i = 1, 2$, then there is a measure space isomorphism $\Phi : \Omega_1 \to \Omega_2$* (*defined modulo null sets*) *such that $f_0^{(1)} = f_0^{(2)} \circ \Phi$, μ_1-a.e., and for all $t \in \mathbb{T}^d$,*

$$\Phi \circ T_t^{(1)} = T_t^{(2)} \circ \Phi, \qquad \mu_1\text{-}a.e. \tag{3.4}$$

The isomorphism Φ is unique modulo null sets.

Proof. By assumption, $\{f_0^{(i)} \circ T_t^{(i)}, t \in \mathbb{T}^d\}$, $i = 1, 2$, are two minimal spectral representations of X. By the uniqueness of the minimal spectral representations over Borel spaces, there is a (modulo null sets) unique measure space isomorphism $\Phi : \Omega_1 \to \Omega_2$ such that for every $t \in \mathbb{T}^d$,

$$\big(f_0^{(1)} \circ T_t^{(1)}\big) = \big(f_0^{(2)} \circ T_t^{(2)}\big) \circ \Phi, \qquad \mu_1\text{-a.e.} \tag{3.5}$$

Replacing t by $t + s$ and taking the composition of both sides with $T_{-s}^{(1)}$ from the right, we obtain

$$\big(f_0^{(1)} \circ T_t^{(1)}\big) = \big(f_0^{(2)} \circ T_t^{(2)}\big) \circ \big(T_s^{(2)} \circ \Phi \circ T_{-s}^{(1)}\big), \qquad \mu_1\text{-a.e.} \tag{3.6}$$

It follows from (3.5) and (3.6) that Φ and $T_s^{(2)} \circ \Phi \circ T_{-s}^{(1)}$ are two measure space isomorphisms each linking the representations $\{f_0^{(i)} \circ T_t^{(i)}, t \in \mathbb{T}^d\}$, $i = 1, 2$. By the last statement of Theorem 2.11 or Theorem 2.17, these isomorphisms should be equal up to μ_1-zero sets. This yields (3.4). □

The following example shows that the stochastic continuity of the process X is an essential assumption for the measurability of the flow in Theorem 3.1.

Example 3.6. Take $\Omega = \mathbb{R}$, let $\mathcal{B}$ be the Borel σ-algebra and $\mu = \lambda$ the Lebesgue measure. Take any function $f_0 \in \mathcal{L}^{\vee}(\mathbb{R}, \mathcal{B}, \lambda)$, for concreteness let $f_0(\omega) = \mathrm{e}^{-\omega}\mathbb{1}_{\omega>0}$. We now construct a measure-preserving flow on $(\mathbb{R}, \mathcal{B}, \lambda)$ with "bad" properties. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a Hamel function, that is a *non-measurable* function which satisfies the Cauchy functional equation $\varphi(t + s) = \varphi(t) + \varphi(s)$ for all $t, s \in \mathbb{R}$. Define a map $T_t : \mathbb{R} \to \mathbb{R}$ by $T_t(\omega) = \omega - \varphi(t)$, for $s, t \in \mathbb{R}$. It is easy to check that $\{T_t\}_{t\in\mathbb{R}}$ is a measure-preserving (but not measurable) flow on $(\mathbb{R}, \mathcal{B}, \lambda)$. Consider now a max-i.d. process $\{X(t), t \in \mathbb{R}\}$ defined by $X(t) = I(f_t)$, where

$$f_t(\omega) = (f_0 \circ T_t)(\omega) = \mathrm{e}^{-(\omega-\varphi(t))}\mathbb{1}_{\omega>\varphi(t)}.$$

The process X defined above is a stationary max-i.d. process which satisfies Condition S. To see that Condition S is satisfied note that $\varphi(t) = ct$ for all $t \in \mathbb{Q}$ and some constant $c = \varphi(1)$. Hence, the collection $\{X(t), t \in \mathbb{Q}\}$ is dense in probability in $\{X(t), t \in \mathbb{R}\}$. The spectral representation $\{f_t, t \in \mathbb{R}\}$ constructed above is minimal and it is defined on a σ-finite Borel space. Note that the minimality follows from the fact that the σ-algebra generated by the functions $\mathrm{e}^{-(x-ct)}\mathbb{1}_{x>ct}, t \in \mathbb{Q}$, coincides with $\mathcal{B}$. This representation is generated by a measure-preserving flow $\{T_t\}_{t\in\mathbb{R}}$. On the other hand, the process X is not stochastically continuous (otherwise, the function φ would be continuous and hence, linear). By Remark 2.21, the process X has no jointly measurable modification. Consequently, by Proposition 2.19, the representation $\{f_t, t \in \mathbb{R}\}$ has no jointly measurable modification and the process X has no representation generated by a measurable measure-preserving flow.

3.2. Conservative–dissipative decompositions

Let $\{T_t\}_{t\in\mathbb{T}^d}$, with $\mathbb{T} = \mathbb{Z}$ or $\mathbb{R}$, be a measure-preserving and measurable $\mathbb{T}^d$-action on a σ-finite Borel space $(\Omega, \mathcal{B}, \mu)$. Suppose first that $\mathbb{T} = \mathbb{Z}$ and $d = 1$. Recall that a set $W \in \mathcal{B}$ is said to be wandering if $T_n(W)$, $n \in \mathbb{Z}$, are disjoint modulo μ. If $\Omega = \bigcup_{t\in\mathbb{Z}} T_n(W) \bmod \mu$, for some (maximal) wandering set W, then the flow is said to be dissipative. Conversely, a flow $\{T_n, n \in \mathbb{Z}\}$ is said to be conservative if it has no wandering sets of positive measure. In general, a flow may be neither purely conservative nor dissipative. The Hopf decomposition entails that $\Omega = C \cup D$, where $C \cap D = \varnothing$ and C and D are two T_1-invariant sets such that the restriction of T_1 is conservative on C and dissipative on D. In the continuous-parameter case $\mathbb{T} = \mathbb{R}$ and $d = 1$, one can show that the Hopf decompositions $\Omega = C_t \cup D_t$ corresponding to the measure preserving map T_t do not depend on $t \in \mathbb{R} \setminus \{0\}$, modulo μ (cf. [17,18,31]). Furthermore, a celebrated result due to Krengel implies that $\{T_t, t \in \mathbb{R}\}$ is dissipative if and only if it is isomorphic to a mixture of Lebesgue shifts, that is, $T_t \circ \Phi(s, v) = \Phi(s + t, v)$ for a measure space isomorphism $\Phi : (\mathbb{T} \times V, \mathcal{A} \otimes \mathcal{V}, \lambda\, \mathrm{d}\nu) \to (\Omega, \mathcal{B}, \mu)$.

The multi-parameter case $d \geq 2$ is more delicate since it is not obvious how to even define the conservative/dissipative component of the flow. In a series of works Rosiński, Samorodnitsky and Parthanil Roy [32,35,36] have shown that the Hopf decomposition and Krengel's characterization of dissipativity extend to the multi-parameter setting with $\mathbb{T}$ discrete and/or continuous. Here, we shall adopt the approach of Parthanil Roy [35] and say that the conservative (dissipative) component of the flow $\{T_t, t \in \mathbb{R}^d\}$ is that of the discrete skeleton $\{T_\gamma, \gamma \in \mathbb{Z}^d\}$ (see Proposition 2.1 therein). The flow is said to be conservative (dissipative, resp.) if its dissipative (conservative, resp.) component is trivial. The following characterization result may be taken as a definition of the Hopf decomposition of a $\mathbb{T}^d$-action.

Theorem 3.7 (Corollary 2.2 in [35]). *Let $\{T_t, t \in \mathbb{T}^d\}$ be a measure-preserving and measurable flow. Let also $h \in L^1(\Omega, \mathcal{B}, \mu)$ be positive μ-a.e. Then the conservative part of $\{T_t\}_{t\in\mathbb{T}^d}$ is modulo μ equal to*:

$$C := \left\{\omega \in \Omega : \int_{\mathbb{T}^d} h\big(T_t(\omega)\big)\lambda(\mathrm{d}t) = \infty\right\}.$$

And for the dissipative component, we have $D = \Omega \setminus C$.

Definition 3.8. *Let $X = \{X_t, t \in \mathbb{T}^d\}$ be a measurable stationary max-i.d. or i.d. random field (without Gaussian part) generated by a measurable flow $T = \{T_t, t \in \mathbb{T}^d\}$ on $(\Omega, \mathcal{B}, \mu)$ in the sense of Definition 3.2. We shall say that X is generated by a conservative (dissipative) flow if $\{T_t, t \in \mathbb{T}^d\}$ is conservative (dissipative).*

The following result shows that this definition does not depend on the choice of the flow representation. It provides, moreover, a useful integral test for identifying the conservative and dissipative parts of a flow. The situation is conceptually similar to the stable and max-stable cases (Corollary 4.2 in [31], Proposition 3.2 in [35], or Theorem 5.2 in [46]).

Theorem 3.9. *Consider a stationary max-i.d. (or i.d.) process X with a jointly measurable spectral representation $\{f_t, t \in \mathbb{T}^d\}$ of full support. The process X is generated by a conservative flow, if and only if, for every (equivalently any) nonnegative Borel function $\psi : [0,\infty) \to [0,\infty)$ such that $\psi(x) > 0$ for all $x > 0$, and $\int_\Omega \psi(|f_0|)\,\mathrm{d}\mu < \infty$, we have*

$$\int_{\mathbb{T}^d} \psi\big(\big|f_t(\omega)\big|\big)\lambda(\mathrm{d}t) = \infty \qquad \text{for } \mu\text{-a.e. } \omega. \tag{3.7}$$

Conversely, the process X is generated by a dissipative flow, if the latter integral is finite μ-a.e. for some (equivalently, every) ψ such that $\psi(x) > 0$ for all $x > 0$ and $\int_\Omega \psi(|f_0|)\,\mathrm{d}\mu < \infty$.

Consider a max-i.d. or an i.d. process $X = \{X(t), t \in \mathbb{T}^d\}$ with a measurable spectral representation of full support. Motivated by Theorem 3.9, let

$$C = \left\{\omega \in \Omega : \int_{\mathbb{T}^d} \psi\big(\big|f_t(\omega)\big|\big)\lambda(\mathrm{d}t) = \infty\right\} \quad \text{and} \quad D := \Omega \setminus C. \tag{3.8}$$

By restricting the spectral representation to the sets C and D, we obtain the following decomposition of X into a max/sum of two independent processes:

$$X \stackrel{d}{=} X_C \,\Box\, X_D, \qquad \Box \in \{\vee, +\}, \tag{3.9}$$

where $X_C(t) := I^{\vee/+}(\mathbb{1}_C f_t) + c$ and $X_D(t) := I^{\vee/+}(\mathbb{1}_D f_t) + c$, $t \in \mathbb{T}^d$, where $c = 0$ in the max-i.d. case.

By relation (5.17), as in the proof of Theorem 3.9, it follows that X_C and X_D are *stationary* and generated by conservative and dissipative flows, respectively. The next result shows that this decomposition is unique.

Corollary 3.10. *The conservative/dissipative decomposition* (3.9) *is unique in law.*

Proof. Let $\{f_t^{(i)}, t \in \mathbb{T}^d\}$ be two measurable representations of X of full support defined on $(\Omega_i, \mathcal{B}_i, \mu_i), i = 1, 2$. As in the proof of Theorem 3.9, there exist measure-preserving $\Phi_i : \Omega_i \to \widetilde{\Omega}, i = 1, 2$, such that for all $t \in \mathbb{T}^d$, we have $f_t^{(i)}(\omega) = g_t(\Phi_i(\omega))$, modulo μ_i $(= \widetilde{\mu} \circ \Phi_i^{-1}), i = 1, 2$, where $g_t = g_0 \circ T_t, t \in T$ is a measurable minimal spectral representation over the Borel σ-finite space $(\widetilde{\Omega}, \widetilde{\mathcal{B}}, \widetilde{\mu})$. Let C_i and D_i be defined as in (3.8) with f replaced by $f^{(i)}, i = 1, 2$. Then, as argued in the proof of Theorem 3.9 (cf. (5.17)), we have that $C_i = \Phi_i^{-1}(\widetilde{C})$, modulo μ_i, where $\widetilde{C}$ is the conservative part of the flow $\{T_t, t \in \mathbb{T}^d\}$. This fact since the Φ_i's are measure preserving implies that $\{\mathbb{1}_{\widetilde{C}} g_t, t \in \mathbb{T}^d\}$ is a spectral representation for both X_{C_1} and X_{C_2}, where $X_{C_i}(t) = I^{\vee/+}(\mathbb{1}_{C_i} f_t^{(i)}) + c, i = 1, 2$. Hence, $X_{C_1} \stackrel{d}{=} X_{C_2}$. One can similarly show that $X_{D_1} \stackrel{d}{=} X_{D_2}$. □

In view of (the multi-parameter version) of Krengel's characterization of dissipativity (see, e.g., [32] or Corollary 2.4 in [35]), we arrive at the following important result.

Corollary 3.11. *A measurable stationary max-i.d. or i.d. process X is generated by a dissipative flow if and only if it has a mixed moving maximum/average representation. That is, for some σ-finite Borel space $(V, \mathcal{V}, \nu)$ and $(\Omega, \mathcal{B}, \mu) = (\mathbb{T}^d \times V, \mathcal{A} \otimes \mathcal{V}, \lambda \otimes \nu)$, we have*

$$X \stackrel{d}{=} \left\{I^{\vee/+}(f_t) + c, t \in \mathbb{T}^d\right\}, \qquad \text{where } f_t(s, v) = f_0(t + s, v), (s, v) \in \Omega$$

for some $f_0 \in \mathcal{L}^{\vee/+}(\Omega, \mathcal{B}, \mu)$.

4. Examples of max-i.d. processes

4.1. Max-stable processes

Max-stable processes form a subclass of the max-i.d. processes. Fix $\alpha > 0$. A process $X = \{X(t), t \in T\}$ is called (α-Fréchet) max-stable if for every $n \in \mathbb{N}$ the process $X_1 \vee \cdots \vee X_n$ has the same law as $n^{1/\alpha} X$, where $X_1, \ldots, X_n$ are i.i.d. copies of X. The marginal distributions of X are α-Fréchet distributions of the form $\mathbb{P}[X(t) \le x] = \exp\{-\sigma^\alpha(t) x^{-\alpha}\}$, $x > 0$. Here, $\sigma(t) > 0$ is called the scale parameter of $X(t)$.

For max-stable processes, a theory of spectral representations has been developed; see [4, 5,11,42,46]. We will explain the connection to the max-i.d. spectral representations developed here. Let $L_+^\alpha(\Omega', \mathcal{B}', \mu')$ be the set of measurable functions $g : \Omega' \to [0, \infty)$ such that $\int_{\Omega'} g^\alpha \, d\mu' < \infty$. Let $\Omega = (0, \infty) \times \Omega'$ be equipped with the product σ-algebra $\mathcal{B}$ and with a measure $\mu = \alpha u^{-(\alpha+1)}\, du \mu'$. A collection of functions $\{g_t, t \in T\} \subset L^\alpha_+(\Omega', \mathcal{B}', \mu')$ is called a spectral representation of a max-stable process $\{X(t), t \in T\}$ if

$$\{X(t), t \in T\} \stackrel{d}{=} \left\{ \bigvee_{i \in \mathbb{N}} u_i g_t(\omega_i'), t \in T \right\}, \tag{4.1}$$

where $\{(u_i, \omega_i'), i \in \mathbb{N}\}$, are points of the Poisson process Π_μ with intensity μ on Ω. The process X, being also max-i.d., must admit a spectral representation in the sense of Section 2.1. This representation can be constructed as follows. Define $f_t : \Omega \to [0, \infty)$ by $f_t(u, \omega') = u g_t(\omega')$, $t \in T$. Then, (4.1) implies that $\{f_t, t \in T\}$ is a spectral representation of X viewed as a max-i.d. process.

A spectral representation $\{g_t, t \in T\} \subset L^\alpha_+(\Omega', \mathcal{B}', \mu')$ of a max-stable process X is called minimal (see [11,45,46]) if (i) $\operatorname{supp}\{g_t, t \in T\} = \Omega' \bmod \mu'$ and (ii) $\sigma\{g_t/g_s, t, s \in T\} = \mathcal{B}' \bmod \mu'$.

Lemma 4.1. *In the above context, if $\{g_t, t \in T\}$ is a minimal spectral representation of a max-stable process X, then $\{f_t, t \in T\}$ is a minimal spectral representation of X as a max-i.d. process.*

Proof. Notice that $(f_t/f_s)(x, y) = (g_t/g_s)(y)$, (with $0/0$ is interpreted as 0) does not depend on x. Therefore, $\rho(F) := \sigma\{f_t/f_s, t, s \in T\} = \mathbb{R}_+ \times \sigma\{g_t/g_s, t, s \in T\}$, which is $(\bmod \mu)$ equivalent to $\mathbb{R}_+ \times \mathcal{B}_{\Omega'} := \{\mathbb{R}_+ \times B : B \in \mathcal{B}_{\Omega'}\}$ by condition (ii). We also have that g_t is $(\bmod \mu)$ measurable with respect to (w.r.t.) $\mathbb{R}_+ \times \mathcal{B}_{\Omega'}$ $(\bmod \mu)$ and since $\rho(F) = \mathbb{R}_+ \times \mathcal{B}_{\Omega'}$ $(\bmod \mu)$, it follows that g_t is $(\bmod \mu)$ measurable w.r.t. $\sigma(F) := \sigma\{f_t, t \in T\}$ $(\supset \rho(F))$. Therefore, $(x, y) \mapsto x \mathbb{1}_{\{\operatorname{supp}(g_t)\}}(y) = f_t(x, y)/g_t(y)$ is $(\bmod \mu)$ measurable w.r.t. $\sigma(F)$. Now, the full support condition (i) implies also that $(x, y) \mapsto x$ is $(\bmod \mu)$ measurable w.r.t. $\sigma(F)$. This implies that $\mathcal{B}_{\mathbb{R}_+} \times \Omega'$ is included in $\sigma(F)$ $(\bmod \mu)$. Since also $\mathbb{R}_+ \times \mathcal{B}_{\Omega'}$ is $(\bmod \mu)$ included in $\sigma(F)$, it follows that $\mathcal{B}_{\mathbb{R}_+} \times \mathcal{B}_{\Omega'}$ is $(\bmod \mu)$ contained in $\sigma(F)$. This shows that $\mathcal{B}_{\mathbb{R}_+} \otimes \mathcal{B}_{\Omega'} \equiv \sigma(\mathcal{B}_{\mathbb{R}_+} \times \mathcal{B}_{\Omega'}) = \sigma(F)$ $(\bmod \mu)$. This shows that $\{f_t, t \in T\}$ is a minimal representation of X since condition (i) for $\{g_t, t \in T\}$ implies also the full support condition for the f_t's. □

Remark 4.2. Lemma 4.1 shows that all previous results on max-stable processes that rely on the notion of minimality can be obtained via the new notion of minimality. The following construction due to Maharam gives the precise connection between the "old" and "new" spectral representations in the case of stationary processes.

Let $X = \{X(t), t \in \mathbb{R}^d\}$ be a stationary stochastically continuous max-stable process with α-Fréchet margins. Then by [5], there is a *non-singular* flow T_t' on a σ-finite Borel space $(\Omega', \mathcal{B}', \mu')$ and a function $g_0 \in L^\alpha_+(\Omega', \mathcal{B}', \mu')$ such that $\{g_t, t \in \mathbb{R}^d\}$ is a minimal spectral representation of X, where

$$g_t = \left(\frac{d\mu' \circ T_t'}{d\mu'} \right)^{1/\alpha} g_0 \circ T_t', \qquad t \in \mathbb{R}.$$

(Recall that a measurable flow $\{T_t'\}_{t \in \mathbb{R}^d}$ is said to be non-singular if the measures $\mu' \circ T_t'$ and μ are equivalent.)

The process X, being max-stable, is also max-i.d. Let us construct the flow representation of X in the sense of Section 3. We shall employ the Maharam construction [1,20]. Let $\Omega = (0,\infty) \times \Omega'$ and consider the mappings $T_t : \Omega \to \Omega$ defined by

$$T_t\big(u,\omega'\big) := \left(\left(\frac{\mathrm{d}\mu' \circ T_t'}{\mathrm{d}\mu'}\big(\omega'\big)\right)^{1/\alpha} u, T_t'\big(\omega'\big)\right), \tag{4.2}$$

where $(t,\omega') \mapsto \mathrm{d}(\mu' \circ T_t')/\mathrm{d}\mu'(\omega')$ is a measurable version of the Radon–Nikodym derivatives (see, e.g., Theorem A.1 in [16]). It is easy to see that $\{T_t\}_{t\in\mathbb{R}}$ is a measurable flow, which is measure-preserving (see, e.g., [1,20]). Now, (4.2) implies that $(f_0, \{T_t\}_{t\in\mathbb{R}^d})$ is a flow representation of X in the sense of Section 3.

4.2. Independent random variables

Any collection $\{X(t), t \in T\}$ of independent random variables forms a max-i.d. process. To see this, take any $n \in \mathbb{N}$ and let $X_{i,n}(t)$, $1 \le i \le n$, $t \in T$, be independent random variables such that $\mathbb{P}[X_{i,n}(t) \le x] = (\mathbb{P}[X(t) \le x])^{1/n}$. Then, $\{X(t), t \in T\}$ has the same law as $\{\bigvee_{i=1}^n X_{i,n}(t), t \in T\}$, thus showing the max-i.d. property. Assume that T is countable. Then, Condition S is satisfied. The minimal spectral representation of $\{X(t), t \in T\}$ can be constructed as follows. As always, we assume that $\operatorname{essinf} X(t) = 0$ and, additionally, $\mathbb{P}[X(t) = 0] < 1$ for all $t \in T$. Let $\Omega = T \times (0,\infty)$ be endowed with the product of the power set 2^T and the Borel σ-algebra on $(0,\infty)$. Define a measure μ on Ω by $\mu(\{t\} \times [x,\infty)) = -\log \mathbb{P}[X(t) < x]$, $t \in T$, $x > 0$. In this way, Ω turns into a σ-finite Borel space. Define the functions $f_t : \Omega \to \mathbb{R}$, $t \in T$, by

$$f_t(s,x) = \begin{cases} x, & t = s, \\ 0, & t \ne s, \end{cases} \qquad s \in T, x > 0.$$

Then, $\{f_t, t \in T\}$ is a minimal spectral representation of $\{X_t, t \in T\}$. If $T = \mathbb{Z}$ and the random variables X_t are i.i.d., then X is stationary and we can define a (discrete time) flow representation by setting $T_t(s,x) = (s-t,x)$, $t \in \mathbb{Z}$, and noting that $f_t = f_0 \circ T_t$.

4.3. Mixed moving maximum processes

Here, we present a general probabilistic construction of mixed moving maxima max-i.d. processes. Similar construction applies in the i.d. context. Let $\{U_i, i \in \mathbb{N}\}$ (interpreted as storm centers) be the points of a Poisson process on $\mathbb{R}^d$ with constant intensity λ. Let $\{F(t), t \in \mathbb{R}^d\}$ be a measurable random process with values in $[0,\infty)$ such that for every $a > 0$, we have

$$\int_{\mathbb{R}^d} \mathbb{P}\big[F(t) > a\big]\,\mathrm{d}t < \infty. \tag{4.3}$$

Let F_n, $n \in \mathbb{N}$, be i.i.d. copies of F (storms), which are independent from the Poisson process $\{U_i, i \in \mathbb{N}\}$ of storm centers. Define a process $\{X(t), t \in \mathbb{R}^d\}$ by

$$X(t) = \sup_{i\in\mathbb{N}} F_i(t - U_i). \tag{4.4}$$

Condition (4.3) implies that X is a well-defined max-i.d. process, which is stationary by the translation invariance of the point process $\{U_i, i \in \mathbb{N}\}$. Indeed, without loss of generality, we can let $F_i(t) = f(t, V_i)$, where V_i are i.i.d. Uniform(0, 1) random variables and $f : \mathbb{R}^d \times [0,1] \to [0,\infty)$ is a Borel function. Thus, $\Pi_\mu = \{(U_i, V_i), i \in \mathbb{N}\}$ is a Poisson point process on $\mathbb{R}^d \times [0,1]$ with intensity $\mu(\mathrm{d}u\,\mathrm{d}v) = \lambda\,\mathrm{d}u\,\mathrm{d}v$. Relation (4.3) and Fubini's theorem guarantee that $f_t(u,v) := f(t-u,v) \in \mathcal{L}^+(\mathbb{R}^d \times [0,1], \mu)$, for all $t \in \mathbb{R}^d$, and hence

$$\left\{X(t), t \in \mathbb{R}^d\right\} \stackrel{d}{=} \left\{\int^{\vee}_{\mathbb{R}^d \times [0,1]} f(t-u,v)\Pi_\mu(\mathrm{d}u, \mathrm{d}v)\right\}$$

is well-defined. Clearly $f_t(u,v) = f_0(T_t(u,v))$, where $f_0(u,v) := f(-u,v)$ and $T_t(u,v) := (u-t, v)$, $t \in \mathbb{R}^d$, is the simple Lebesgue shift flow in the first coordinate, which is measurable and measure-preserving. This shows that the process X is stationary and in fact has a *mixed moving maximum* representation. The above discussion and Corollary 3.11 imply that the process X in (4.4) is generated by a dissipative flow.

4.4. Max-i.d. processes associated to Poisson line processes

Instead of taking points of a Poisson process as storm centers in (4.4), we can also take lines of a Poisson line process as storm centers. Let $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$ be identified with the unit circle. Each point (r,φ) in $\mathbb{L} := \mathbb{R} \times \mathbb{T}$ corresponds to an oriented line in $\mathbb{R}^2$ which passes through the point $(r\cos\varphi, r\sin\varphi)$ in the direction of the vector $(-\sin\varphi, \cos\varphi)$. In this way, $\mathbb{L}$ can be identified with the set of all oriented lines in $\mathbb{R}^2$. Take a Poisson point process $\{(r_i,\varphi_i), i \in \mathbb{N}\}$ on $\mathbb{L}$ whose intensity is $\lambda\,\mathrm{d}r \times \mathrm{d}\varphi$, where $\lambda > 0$ is constant. The corresponding random set of lines in $\mathbb{R}^2$ is called the Poisson line process and is interpreted as the set of storm centers. Its law is invariant with respect to translations of $\mathbb{R}^2$; see, for example, [15].

Let now $\{F(t), t \in \mathbb{R}\}$ be a measurable process with values in $[0,\infty)$ such that for every $a > 0$, we have $\int_{\mathbb{R}} \mathbb{P}[F(t) > a]\,\mathrm{d}t < \infty$. Let F_i, $i \in \mathbb{N}$, be i.i.d. copies of F (storms). Define a process $\{\eta(x,y), (x,y) \in \mathbb{R}^2\}$ by

$$\eta(x,y) = \max_{i \in \mathbb{N}} F_i(x\cos\varphi_i + y\sin\varphi_i - r_i). \tag{4.5}$$

Note that $|x\cos\varphi + y\sin\varphi - r|$ is the distance from the point (x,y) to the line corresponding to $(r,\varphi) \in \mathbb{L}$. As in the previous Section 4.3, without loss of generality, we have $F_i(t) = f(t, V_i)$, where V_i are i.i.d. Uniform(0, 1) random variables and hence

$$\begin{aligned}&\left\{\eta(x,y), (x,y) \in \mathbb{R}^2\right\}\\ &\stackrel{d}{=} \left\{\int^{\vee}_{\mathbb{R}\times[0,2\pi)\times[0,1]} f(x\cos\varphi + y\sin\varphi - r, v)\Pi_\mu(\mathrm{d}r, \mathrm{d}\varphi, \mathrm{d}v), (x,y) \in \mathbb{R}^2\right\},\end{aligned} \tag{4.6}$$

where $\Pi_\mu = \{(r_i, \varphi_i, v_i), i \in \mathbb{N}\}$ is a Poisson process on $\Omega := \mathbb{R} \times [0,2\pi) \times [0,1]$ with intensity $\mu(\mathrm{d}r, \mathrm{d}\varphi, \mathrm{d}v) = \lambda\,\mathrm{d}r\,\mathrm{d}\varphi\,\mathrm{d}v$. This representation together with the translation invariance of the Poisson line process readily implies the following result.

Proposition 4.3. *η is a stationary max-i.d. process on $\mathbb{R}^2$.*

One can construct also max-stable processes of this type. Fix $\alpha > 0$. Start with a Poisson process $\Phi = \{(r_i, \varphi_i, z_i), i \in \mathbb{N}\}$ on $\mathbb{L} \times (0, \infty)$ with intensity $\lambda\, \mathrm{d}r \times \mathrm{d}\varphi \times \alpha z^{-(\alpha+1)}\, \mathrm{d}z$. Let $\{F(t), t \in \mathbb{R}\}$ be a process with values in $[0, \infty)$ such that $\mathbb{E}\int_{\mathbb{R}} F^{\alpha}(r)\, \mathrm{d}r < \infty$. Let $\{F_i, i \in \mathbb{N}\}$ be independent copies of F. Define

$$\zeta(x, y) = \max_{i \in \mathbb{N}} z_i F_i(x \cos\varphi_i + y \sin\varphi_i - r_i). \tag{4.7}$$

Proposition 4.4. *$\{\zeta(x, y), (x, y) \in \mathbb{R}^2\}$ is a stationary max-stable process with α-Fréchet margins.*

Max-stability follows directly from the properties of the Poisson processes and stationarity is the consequence of the stationarity of the Poisson line process. For simplicity, we considered here processes based on Poisson lines in $\mathbb{R}^2$, but a similar construction is possible in $\mathbb{R}^d$, where the lines are replaced by k-dimensional affine subspaces in $\mathbb{R}^d$, $k < d$. For $k = 0$ we recover the mixed moving maxima processes, for $k \geq 1$, however, these processes are generated by a conservative flow. We show this next for the case of the Poisson line process ($k = 1$).

Proposition 4.5. *The Poisson line max-i.d. process in (4.5) is generated by a conservative flow.*

Proof. Without loss of generality, we may assume that $\mathrm{Leb}[t \in \mathbb{R} : F(t) > 0] > 0$, almost surely.

Indeed, let as above $F(t) = f(t, V)$ and $A := \{v \in [0, 1] : \mathrm{Leb}[t \in \mathbb{R} : f(t, v) > 0] = 0\}$. Suppose first that $\mathrm{Leb}(A) = 1$, that is the paths $t \mapsto F(t)$ are zero for almost all $t \in \mathbb{R}$, with probability one. For example, $F(t) = \mathbb{1}_B(t)$ for a set B of Lebesgue measure zero. In this case, we have that for all fixed $(x, y) \in \mathbb{R}^2$, the random variable $\eta(x, y)$ in (4.5) is almost surely zero.

On the other hand, if $0 < \mathrm{Leb}(A) < 1$, consider the process $G(t) = f(t, W)$, where $W \stackrel{d}{=} V | A^c$ have the conditional distribution of V restricted to the set $A^c := [0, 1] \setminus A$. By a thinning argument and replacing in (4.5), λ and F by $\lambda / \mathrm{Leb}(A^c)$ and G, respectively, we see that for all $(x, y) \in \mathbb{R}^2$, we have

$$\left\{\eta(x, y), (x, y) \in \mathbb{R}^2\right\} \stackrel{d}{=} \left\{\max_{i \in \mathbb{N}} G_i(x \cos\varphi_i + y \sin\varphi_i - r_i), (x, y) \in \mathbb{R}^2\right\},$$

where G_i's are independent copies of G. With probability one, however, the paths of the process $t \mapsto G(t)$ are positive over a set of positive Lebesgue measure. This shows that, without loss of generality, we can suppose that (4.5) holds with $\mathrm{Leb}[t \in \mathbb{R} : F(t) > 0] > 0$, almost surely.

Let now $\psi(x) > 0$, $x > 0$, be as in Theorem 3.9. In view of (4.6), we have that

$$J_\psi(\varphi, r, v) := \int_{\mathbb{R}^2} \psi\big(f(x \cos\varphi + y \sin\varphi - r, v)\big)\, \mathrm{d}x\, \mathrm{d}y = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \psi\big(f(\widetilde{x}, v)\big)\, \mathrm{d}\widetilde{x}\right) \mathrm{d}\widetilde{y},$$

where $\widetilde{x} := x \cos\varphi + y \sin\varphi - r$, and $\widetilde{y} := -x \sin\varphi + y \cos\varphi$.

Since $\mathrm{Leb}[t \in \mathbb{R} : F(t) = f(t, V) > 0] > 0$ almost surely, we have $\int_{\mathbb{R}} \psi(f(\widetilde{x}, V))\, \mathrm{d}\widetilde{x} > 0$ a.s. Hence, $J_\psi(\varphi, r, v) = \infty$ for almost all $(\varphi, r, v) \in \mathbb{R} \times [0, 2\pi) \times [0, 1]$, which means that the process η is generated by a conservative flow. $\square$

Remark 4.6. In general, we conjecture that the Poisson line process X for $k \geq 1$ is generated by a null-recurrent flow (see, e.g., Samorodnitsky [38], and also [11,33,46]).

4.5. Penrose min-i.d. random fields

The next family of examples generalizes the processes considered by Penrose [24–26]. Let $\Pi = \{U_i, i \in \mathbb{Z}\}$ be the points of Poisson process on $\mathbb{R}^k$ with a constant intensity λ. Let $\{\xi_i(t), t \in \mathbb{R}^d\}$, $i \in \mathbb{Z}$, be independent copies of a random field $\{\xi(t), t \in \mathbb{R}^d\}$ with values in $\mathbb{R}^k$ which has stationary increments. Let $|\cdot|$ be the Euclidean norm. Define

$$X(t) = \min_{i \in \mathbb{Z}} \left| U_i + \xi_i(t) \right|. \tag{4.8}$$

Proposition 4.7. *The process X is stationary, min-i.d. process (i.e., $-X$ is max-i.d.).*

The min-i.d. property follows directly from the fact that for every $n \in \mathbb{N}$, we can represent Π as a union of n independent Poisson processes with constant intensity $\frac{\lambda}{n}$. The stationarity of X follows from the stationarity of increments of ξ; see Proposition 2.1 in [12]. To construct concrete families of examples one may take $k = 1$ and ξ to be the zero-mean Gaussian process defined on $\mathbb{R}^d$ with covariance function $\mathbb{E}[\xi(t)\xi(s)] = \frac{\sigma^2}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H})$, where $H \in (0, 1]$ is the Hurst exponent and $\sigma^2 > 0$ (see Figure 1). Min-i.d. processes of this type appeared in [13] as limits of pointwise minima (in the sense of absolute value) of independent Gaussian processes.

One can also take $d = 1$, $k \in \mathbb{N}$ arbitrary and let ξ be the $\mathbb{R}^k$-valued standard Brownian motion. The next result shows that the resulting processes, which were introduced and studied by Penrose [24–26], are of mixed moving maximum type for $k \geq 3$ and are conservative for $k \leq 2$.

Proposition 4.8. *Let $X = \{X(t), t \in \mathbb{R}\}$ be as in (4.8), where $\{\xi(t), t \in \mathbb{R}\}$ is the standard Brownian motion in $\mathbb{R}^k$. The max-i.d. process $-X$ is generated by a conservative flow for $k = 1, 2$ and dissipative for $k \geq 3$.*

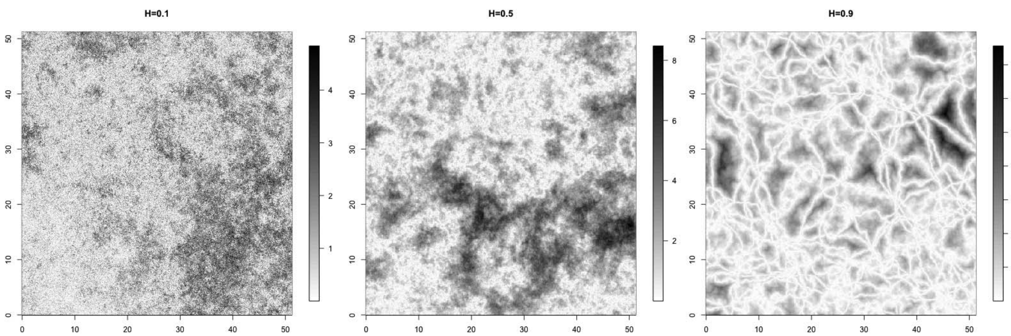

Figure 1. Realizations of Penrose-type min-i.d. random fields driven by isotropic Lévy fractional Brownian motions defined on $\mathbb{R}^2$ with Hurst exponents $H = 0.1$, 0.5, and 0.9, respectively, left to right.

Proof. We shall apply the integral test in Theorem 3.9 above. In the case $k = 1, 2$ the result follows from the neighborhood-recurrence property of the Brownian motion. In the case $k \geq 3$ we will use the fact that the Brownian motion in $\mathbb{R}^k$ is transient.

Consider the space $\Omega := \mathbb{R} \times C_0(\mathbb{R}, \mathbb{R}^k)$, equipped with the product of the Borel σ-algebras, where $C_0(\mathbb{R}, \mathbb{R}^k)$ is the space of $\mathbb{R}^k$-valued continuous functions on $\mathbb{R}$ which vanish at 0. Consider the Poisson point process $\Pi = \{(U_i, \xi_i), i \in \mathbb{Z}\}$ on Ω with intensity $\mu(\mathrm{d}u, \mathrm{d}v) = \lambda(\mathrm{d}u)\mathbb{P}_\xi(\mathrm{d}v)$, where λ is the Lebesgue measure on R and $\mathbb{P}_\xi$ is the law of ξ on $C_0(\mathbb{R}, \mathbb{R}^k)$. Therefore, for the max-i.d. process $-X$ we obtain the spectral representation

$$-X \stackrel{d}{=} \left\{ \int_\Omega^\vee f_t(u, v)\Pi_\mu(\mathrm{d}u, \mathrm{d}v) \right\}_{t \in \mathbb{R}}, \qquad \text{with } f_t(u, v) = -\left|u + v(t)\right|,$$

where $v = (v(t))_{t \in \mathbb{R}} \in C_0(\mathbb{R}, \mathbb{R}^k)$. Observe that since $\mathbb{P}_\xi[v(0) = 0] = 1$, we have $\int_\Omega f_0 \, \mathrm{d}\mu = \int_\mathbb{R} \mathrm{e}^{-|u|} \, \mathrm{d}u < \infty$ and one can take $\psi(x) := \mathrm{e}^x$ in (3.7).

Consider first the transient case $k \geq 3$. By the Dvoretzky–Erdös criterion (see, e.g., Theorem 3.22 in [23]) by taking $g(r) = r^{1/3}$, we obtain that $\int_1^\infty g(r)^{k-2} r^{-k/2} \, \mathrm{d}r < \infty$, and therefore $\liminf_{|t| \to \infty} |\xi(t)|/g(t) = \infty$, with probability one. Thus, for $\mathbb{P}_\xi$-almost all v, we have

$$\psi\big(f_t(u, v)\big) = \mathrm{e}^{-|u+v(t)|} \leq \exp\big\{-|t|^{1/3}\big\}$$

for all sufficiently large $|t|$. Since the latter bound is integrable, we obtain that $\int_\mathbb{R} \psi(f_t(u, v)) \, \mathrm{d}t < \infty$ for μ-almost all $(u, v) \in \Omega$. This, in view of Theorem 3.9 implies that $-X$ is generated by a dissipative flow.

Suppose now $k \leq 2$. By the neighborhood recurrence of the Brownian motion [23] in dimensions $k = 1, 2$, the time which the Brownian motion spends in any open set is infinite with probability 1. It follows immediately that $\int_\mathbb{R} \psi(f_t(u, v)) \, \mathrm{d}t = \infty$ for μ-almost every $(u, v) \in \Omega$. By Theorem 3.9 this implies that $-X$ is generated by a conservative flow. □

4.6. Stationary union-i.d. random sets

A measurable process $\{X(t), t \in \mathbb{R}^d\}$ taking only values 0, 1 can be identified with the random set $S := \{t \in \mathbb{R}^d : X(t) = 1\}$. Note that we do not require the sets to be, say, closed. If the process X is max-i.d., then the corresponding random set S is *union-i.d.* and vice versa. This means that for every $n \in \mathbb{N}$ we can find i.i.d. random sets $C_1, \ldots, C_n$ such that S has the same finite-dimensional distributions as $C_1 \cup \cdots \cup C_n$; see [22], Chapter 4. The process X is stochastically continuous iff the random set S is stochastically continuous in the following sense: for every $t \in \mathbb{R}^d$, $\lim_{s \to t} \mathbb{P}[t \in S, s \notin S] = \lim_{s \to t} \mathbb{P}[t \notin S, s \in S] = 0$. Using Theorem 3.1, we can describe all stationary stochastically continuous union-i.d. random sets.

Theorem 4.9. *Let S be a stationary, stochastically continuous, union-i.d. random set in $\mathbb{R}^d$. Then there is a σ-finite Borel space $(\Omega, \mathcal{B}, \mu)$, a measurable, measure-preserving $\mathbb{R}^d$-action $\{T_t\}_{t \in \mathbb{R}^d}$ on $(\Omega, \mathcal{B}, \mu)$, and a set $A \in \mathcal{B}$ with $\mu(A) < \infty$ such that*

$$S \stackrel{d}{=} \big\{t \in \mathbb{R}^d : \Pi_\mu\big(T_t^{-1}(A)\big) \neq 0\big\}, \tag{4.9}$$

where Π_μ is a Poisson random measure on $(\Omega, \mathcal{B})$ with intensity measure μ.

Proof. Let $\{X(t), t \in \mathbb{R}^d\}$ be the $\{0,1\}$-valued max-i.d. process corresponding to S, that is, $X(t) = \mathbb{1}_{t \in S}$. Then, X has a flow representation in the sense of Theorem 3.1. The function f_0 in this representation takes only values 0, 1, $(\mathrm{mod}\, \mu)$. That is, $f = \mathbb{1}_A$ for some set $A \in \mathcal{B}$. Note that A has finite measure since $f \in \mathcal{L}^\vee$. Since $f_0 \circ T_t = \mathbb{1}_{T_t^{-1}A}$, the statement of the theorem follows. □

Example 4.10. Let Ω be the space $\mathbb{R}^d$ endowed with the Lebesgue measure. Consider a flow $T_t(\omega) = \omega - t, \omega, t \in \mathbb{R}^d$. Let $A \subset \mathbb{R}^d$ be a Borel set of finite measure and let $\Pi = \{U_i, i \in \mathbb{N}\}$ be a unit intensity Poisson process on $\mathbb{R}^d$. Then, the corresponding union-i.d. stationary random set has the form $S = \bigcup_{i \in \mathbb{N}}(U_i - A)$ and it is known in the literature as the Boolean model with (non-random) grain A. More generally, one can let $\Omega := \mathbb{R}^d \times E$ be the product of $\mathbb{R}^d$ with some probability space E and define $T_t(x, y) := (x - t, y), x, t \in \mathbb{R}^d, y \in E$ as the shift of the first coordinate. By taking a random set $A = A(y)$, one obtains a *mixed* or random grain Boolean model, which is similar to and in fact corresponds to the level-set of a mixed moving maxima random field model.

5. Proofs

5.1. Lemma on conjugacy between collections of functions

The following lemma is used in the proofs of Theorems 2.11 and 2.17.

Lemma 5.1. *Let $(\Omega_i, \mathcal{B}_i, \mu_i), i = 1, 2$ be two measure spaces. Consider two families of measurable functions $f_t^{(i)} : \Omega_i \to \mathbb{R}, t \in T$, $i = 1, 2$, and define two measurable mappings $F_i : (\Omega_i, \mathcal{B}_i) \to (\mathbb{R}^T, \mathcal{B})$ by $F_i(\omega) = (f_t^{(i)}(\omega))_{t \in T}$, $\omega \in \Omega_i$, $i = 1, 2$. Here, $\mathcal{B}$ is the product σ-algebra on $\mathbb{R}^T$. Assume that*

1. $\sigma\{f_t^{(i)}, t \in T\} = \mathcal{B}_i \,\mathrm{mod}\, \mu_i$, $i = 1, 2$.
2. *The induced measures $\mu_1 \circ F_1^{-1}$ and $\mu_2 \circ F_2^{-1}$ are equal on $(\mathbb{R}^T, \mathcal{B})$.*

Then, the following two claims are true:

- (i) *If $(\Omega_1, \mathcal{B}_1)$ is a Borel space, then there exists a measurable map $\Phi : \Omega_2 \to \Omega_1$ such that $\mu_1 = \mu_2 \circ \Phi^{-1}$ and for all $t \in T$, we have $f_t^{(2)} = f_t^{(1)} \circ \Phi$, μ_1-a.e.*
- (ii) *If both $(\Omega_i, \mathcal{B}_i), i = 1, 2$ are Borel spaces, then the mapping in part* (i) *is a measure space isomorphism and it is unique (modulo null sets).*

Proof. It will be convenient to identify the sets in $\mathcal{B}_i$ that are equal modulo μ_i, $i = 1, 2$. Formally, let $\mathcal{I}_i \subset \mathcal{B}_i$ be the σ-ideals of μ_i-null sets in the spaces $(\Omega_i, \mathcal{B}_i, \mu_i)$ (see, e.g., Chapter II.21 in [41]) and let $[\mathcal{B}_i] := \mathcal{B}_i / \mathcal{I}_i$ be the corresponding factor σ-fields, $i = 1, 2$. The elements of $[\mathcal{B}_i]$ are the equivalence classes $[B] = \{A \in \mathcal{B}_i : \mu_i(A \Delta B) = 0\}$, where $B \in \mathcal{B}_i$, $i = 1, 2$.

We shall define next a σ-isomorphism $U:[\mathcal{B}_1]\to[\mathcal{B}_2]$, that is, a bijective mapping that preserves countable unions and complements. For all $B\in\mathcal{B}_1$, we set

$$U\big([B]\big):=\big[F_2^{-1}(A)\big],\qquad \text{where } \big[F_1^{-1}(A)\big]=[B]. \tag{5.1}$$

Note that such an $A\in\mathcal{B}$ exists since by assumption $F_1^{-1}(\mathcal{B})=\sigma\{f_t^{(1)},t\in T\}=\mathcal{B}_1 \bmod \mu_1$. One can readily see that the mapping U is a well-defined σ-isomorphism. Indeed, since $\mu_1\circ F_1^{-1}=\mu_2\circ F_2^{-1}$, for every $A',A''\in\mathcal{B}$,

$$\begin{aligned}\mu_1\big(F_1^{-1}(A')\Delta F_1^{-1}(A'')\big)&=\mu_1\big(F_1^{-1}(A'\Delta A'')\big)\\&=\mu_2\big(F_2^{-1}(A'\Delta A'')\big)=\mu_2\big(F_2^{-1}(A')\Delta F_2^{-1}(A'')\big).\end{aligned}$$

Thus, $F_1^{-1}(A')=F_1^{-1}(A'') \bmod \mu_1$, if and only if $F_2^{-1}(A')=F_2^{-1}(A'') \bmod \mu_2$, and the definition of U does not depend on the choice of the representative B of the equivalence class $[B]$ and on the choice of A in (5.1). This shows, moreover, that $[B']=[B'']$ if and only if $U([B'])=U([B''])$, that is, U is *injective*. On the other hand, since $F_2^{-1}(\mathcal{B})=\sigma\{f_t^{(2)},t\in T\}=\mathcal{B}_2 \bmod \mu_2$, for all $B\in\mathcal{B}_2$, we have $[F_2^{-1}(A)]=[B]$, for some $A\in\mathcal{B}$ and hence $U([F_1^{-1}(A)])=[B]$. This shows that U is *onto* and hence a bijection. Also, since $\mu_1(F_1^{-1}(A))=\mu_2(F_2^{-1}(A))$, we have by (5.1) that U is measure-preserving. Since U clearly preserves the countable unions and complements, it is a σ-isomorphism.

Under the assumption of part (i)*, we have that* $(\Omega_1,\mathcal{B}_1)$ *is a Borel space.* Then, Theorem 32.5 of [41] implies that the σ-isomorphism U is induced by a measurable point mapping $\Phi:\Omega_2\to\Omega_1$ in the following sense:

$$U\big([B]\big)=\big[\Phi^{-1}(B)\big],\qquad B\in\mathcal{B}_1. \tag{5.2}$$

Clearly, since U is a σ-isomorphism, we also have that $\mu_1=\mu_2\circ\Phi^{-1}$.

Let us fix some $t\in T$ and show that $f_t^{(2)}=f_t^{(1)}\circ\Phi$ holds μ_2-a.e. Let I be a Borel subset of $\mathbb{R}$ and consider the cylinder set $A=\{\varphi:T\to\mathbb{R}:\varphi(t)\in I\}\subset\mathbb{R}^T$. We have

$$\begin{aligned}\big[\big(f_t^{(1)}\circ\Phi\big)^{-1}(I)\big]&=\big[\Phi^{-1}\big(\big(f_t^{(1)}\big)^{-1}(I)\big)\big]=U\big(\big[\big(f_t^{(1)}\big)^{-1}(I)\big]\big)\\&=U\big(\big[F_1^{-1}(A)\big]\big)=\big[F_2^{-1}(A)\big]=\big[\big(f_t^{(2)}\big)^{-1}(I)\big].\end{aligned} \tag{5.3}$$

Assume that $f_t^{(2)}\neq f_t^{(1)}\circ\Phi$ on $D\in\mathcal{B}_2$ with $\mu_2(D)>0$. Then we can find an $\varepsilon>0$ and a measurable set $D'\subset D$ with $\mu_2(D')>0$ such that $|f_t^{(2)}-f_t^{(1)}\circ\Phi|>\varepsilon$ everywhere on D'. Further, we can find a $k\in\mathbb{Z}$ and a measurable set $D''\subset D'$ with $\mu_1(D'')>0$ such that with $I=[k\varepsilon,(k+1)\varepsilon)$, we have $f_t^{(2)}\in I$ everywhere on D''. It then follows that $f_t^{(1)}\circ\Phi\notin I$ on D''. But this contradicts (5.3), which implies $f_t^{(2)}=f_t^{(1)}\circ\Phi$, μ_1-a.e.

Now, we turn to proving part (ii). That is, that Φ a measure space isomorphism and unique (modulo null sets) under the additional assumption that $(\Omega_2,\mathcal{B}_2)$ is a Borel space. By applying the above argument to the σ-isomorphism $U^{-1}:[\mathcal{B}_2]\to[\mathcal{B}_1]$, we obtain that there exists a measurable, measure-preserving $\widetilde{\Phi}: \Omega_1 \to \Omega_2$, such that

$$U^{-1}\big([B]\big) = \big[\widetilde{\Phi}^{-1}(B)\big] \qquad \text{for all } B \in \mathcal{B}_2.$$

Therefore, $\Psi := \Phi \circ \widetilde{\Phi}: \Omega_1 \to \Omega_1$ is measurable and since $U \circ U^{-1} \equiv \mathrm{id}$, we have that $[\Psi(A)] = [A]$ for all $A \in \mathcal{B}_1$. We will use the fact that $(\Omega_1, \mathcal{B}_1)$ is a Borel space to show that $\Psi = \mathrm{id} \bmod \mu_1$, which will imply that Φ is a measure space isomorphism (Definition 2.9).

By Kuratowski's theorem, $(\Omega_1, \mathcal{B}_1)$ is isomorphic to either $(E, 2^E)$, where E is an at most countable set, or $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ – the real line equipped with the Borel σ-algebra. The discrete case is trivial. Suppose now the latter is true and without loss of generality let $(\Omega_1, \mathcal{B}_1) \equiv (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$. Let $\varepsilon > 0$ be arbitrary and suppose that $\mu_1(\{|\Psi - \mathrm{id}| > \varepsilon\}) > 0$, then for some $k \in \mathbb{Z}$, we have for $D := \{|\Psi - \mathrm{id}| > \varepsilon\} \cap [k\varepsilon, (k+1)\varepsilon)$ that $\mu_1(D) > 0$. But then $\Psi(x) \notin [k\varepsilon, (k+1)\varepsilon)$, for all $x \in D$, and hence $\Psi(D) \cap D = \varnothing$. This contradicts the fact that $[\Psi(D)] = [D]$ because $\mu_1(D) > 0$. Since $\varepsilon > 0$ was arbitrary, it follows that $\Psi = \mathrm{id} \bmod \mu_1$ and hence $\Phi^{-1} = \widetilde{\Phi} \bmod \mu_1$.

To complete the proof, we need to show the uniqueness of Φ. Assume that $\Phi_*: \Omega_2 \to \Omega_1$ is another measure space isomorphism such that for all $t \in T$, $f_t^{(2)} = f_t^{(1)} \circ \Phi_*$, μ_2-a.e. Then, relation (5.3) holds with Φ replaced by Φ_*, which implies that Φ_* induces the same σ-isomorphism U as Φ. Since Φ_* is a measure space isomorphism, the measurable map $\widetilde{\Phi} := (\Phi_*)^{-1}$ induces the σ-isomorphism U^{-1} and hence $\Psi := \Phi \circ \widetilde{\Phi}$ induces the identity σ-isomorphism on the Borel space $(\Omega_1, \mathcal{B}_1)$. As argued above, this implies that $\Phi \circ \widetilde{\Phi} = \mathrm{id}$, $(\bmod \mu_1)$. □

5.2. Proofs in the max-i.d. case

Proof of Theorem 2.8. Write $\mathbb{R}_+ = [0, \infty)$. Let T_0 be the at most countable set appearing in Condition S. Let $\mathbb{R}_+^{T_0}$ be the space of functions $\varphi: T_0 \to \mathbb{R}_+$ endowed with the product σ-algebra $\mathcal{B}$. Denote by ν the exponent measure of the process $\{X(t), t \in T_0\}$; see Vatan [44]. It is a σ-finite measure on $\mathbb{R}_+^{T_0}$ such that for every $t_1, \ldots, t_n \in T_0$ and $x_1, \ldots, x_n > 0$ we have

$$\mathbb{P}\big\{X(t_j) < x_j, 1 \le j \le n\big\} = \exp\left\{-\nu\left(\bigcup_{j=1}^{n}\big\{\varphi \in \mathbb{R}_+^{T_0}: \varphi(t_j) \ge x_j\big\}\right)\right\}. \tag{5.4}$$

We agree that $\nu(\{0\}) = 0$ (which is different from [44]). Taking the coordinate mappings $f_t: \mathbb{R}_+^{T_0} \to \mathbb{R}$, $f_t(\varphi) = \varphi(t)$, $t \in T_0$, we therefore obtain a spectral representation of $\{X(t), t \in T_0\}$ on $(\mathbb{R}_+^{T_0}, \mathcal{B})$. To see this, compare (2.3) and (5.4). Let $t \in T$ be arbitrary. Condition S states that there exists a sequence $\{t_n\}_{n \in \mathbb{N}} \subset T_0$ such that $X(t_n) \to X(t)$ in probability. Thus, the sequence $X(t_n)$ is Cauchy in probability. By the equality of the finite-dimensional distributions, the sequence $I(f_{t_n})$ is Cauchy in probability, and therefore, it converges in probability. By Theorem 4.5 in [2], there is a function $f_t \in \mathcal{L}^{\vee}(\mathbb{R}_+^{T_0}, \mathcal{B}, \nu)$ such that $I(f_{t_n})$ converges in probability to $I(f_t)$. Theorem 4.4 of [2] implies that the finite-dimensional distributions of $\{I(f_t), t \in T\}$ and $\{X(t), t \in T\}$ are equal, that is, the collection $\{f_t, t \in T\}$ is a spectral representation of $\{X(t), t \in T\}$ on $(\mathbb{R}_+^{T_0}, \mathcal{B})$. Since the coordinate functions f_t, $t \in T_0$, generate the product σ-algebra $\mathcal{B}$, and $\nu(\bigcap_{t\in T_0}\{f_t = 0\}) = \nu(\{0\}) = 0$, this representation is minimal. To complete the proof note that by Kuratowski's theorem, for at most countable T_0, the measurable space $(\mathbb{R}_+^{T_0}, \mathcal{B})$ is isomorphic to $[0, 1]$ endowed with the Borel σ-algebra. □

Proof of Theorem 2.11. As in Lemma 5.1, we define two measurable mappings $F_i : (\Omega_i, \mathcal{B}_i) \to (\mathbb{R}^T, \mathcal{B})$ by

$$F_i(\omega) = \big(f_t^{(i)}(\omega)\big)_{t\in T}, \qquad \omega \in \Omega_i, i = 1, 2.$$

The first condition of Lemma 5.1 is satisfied by the assumption of minimality. We will show that the induced measures $\mu_1 \circ F_1^{-1}$ and $\mu_2 \circ F_2^{-1}$ are equal on $(\mathbb{R}^T, \mathcal{B})$. We will prove that for all $t_1, \dots, t_n \in T$ and all intervals $[x_1, y_1), \dots, [x_n, y_n) \subset \mathbb{R}$ we have

$$\mu_1\left(\bigcap_{j=1}^{n}\{x_j \le f_{t_j}^{(1)} < y_j\}\right) = \mu_2\left(\bigcap_{j=1}^{n}\{x_j \le f_{t_j}^{(2)} < y_j\}\right). \tag{5.5}$$

Recall that $\{f_t^{(i)}, t \in T\}$, $i = 1, 2$, are spectral representations of the same process X. By (2.3), we have that for all $x_1, \dots, x_n > 0$,

$$\mu_1\left(\bigcup_{j=1}^{n}\{f_{t_j}^{(1)} \ge x_j\}\right) = \mu_2\left(\bigcup_{j=1}^{n}\{f_{t_j}^{(2)} \ge x_j\}\right). \tag{5.6}$$

Note that $\mu_i(\{f_t^{(i)} > x\}) < \infty$ for all $x > 0$, $t \in T$, since $f_t^{(i)} \in \mathcal{L}^\vee$. Using this fact and the inclusion–exclusion formula, we obtain that relation (5.6) is also valid with the unions therein replaced by intersections. This proves that (5.5) holds provided that $0 < x_j < y_j$ for all $j = 1, \dots, n$. Note that this argument breaks down if $x_j = 0$ for some j since we cannot apply the inclusion–exclusion formula to sets of infinite measure. To show that the measures $\mu_1 \circ F_1^{-1}$ and $\mu_2 \circ F_2^{-1}$ agree on the "boundary" of $\mathbb{R}_+^T$ we need a separate argument.

We now show that (5.5) continues to hold even if some of the x_j's are allowed to be zero. We do not need to consider the case of negative x_j's since $f_t^{(i)} \ge 0$, μ_i-a.e., by definition of $\mathcal{L}^\vee$. By letting some of the x_j's go to 0 and using continuity of measure we obtain that (5.5) continues to hold if some of the sets of the form $\{x_j \le f_{t_j}^{(i)} < y_j\}$ therein are replaced by $\{0 < f_{t_j}^{(i)} < y_j\}$. By additivity of measure, the proof of (5.5) in full generality will be completed if we show that (5.5) continues to hold if some of the sets of the form $\{x_j \le f_{t_j}^{(i)} < y_j\}$ therein are replaced by $\{f_{t_j}^{(i)} = 0\}$. Let us make this statement precise. Take $l, m \in \mathbb{N}_0$, $s_1, \dots, s_l \in T$, $r_1, \dots, r_m \in T$ and $0 < u_1 < v_1, \dots, 0 < u_m < v_m$. Define two measurable sets $C_i \subset \Omega_i$, $i = 1, 2$, by

$$C_i = A_i \cap B_i, \qquad A_i = \bigcap_{k=1}^{l}\{f_{s_k}^{(i)} = 0\}, \qquad B_i = \bigcap_{j=1}^{m}\{u_j \le f_{r_j}^{(i)} < v_j\}, \qquad i = 1, 2.$$

We will show that $\mu_1(C_1) = \mu_2(C_2)$. Suppose first that $m \neq 0$. Then, $\mu_i(C_i) = \mu_i(B_i) - \mu_i(B_i \cap D_i)$, where

$$D_i = \bigcup_{k=1}^{l} \{f_{s_k}^{(i)} > 0\} = \bigcup_{n \in \mathbb{N}} D_{i,n}, \qquad D_{i,n} = \bigcup_{k=1}^{l} \left\{ \frac{1}{n} \le f_{s_k}^{(i)} < n \right\}, \qquad i = 1, 2.$$

We have already shown that (5.5) holds if $x_j > 0$ for all $j = 1, \ldots, n$. This implies that $\mu_1(B_1) = \mu_2(B_2)$ (where both terms are finite since $m \neq 0$). Also, by the inclusion–exclusion formula, $\mu_1(B_1 \cap D_{1,n}) = \mu_2(B_2 \cap D_{2,n})$ for every $n \in \mathbb{N}$. Note that $D_{i,1} \subset D_{i,2} \subset \cdots$. Letting $n \to \infty$ and using the continuity of measure, we obtain $\mu(B_1 \cap D_1) = \mu_2(B_2 \cap D_2)$. This proves that $\mu_1(C_1) = \mu_2(C_2)$ in the case $m \neq 0$.

Consider now the case $m = 0$. In this case it is possible that $\mu_i(B_i) = \infty$ and the above argument breaks down. We show that $\mu_1(C_1) = \mu_2(C_2)$, or, equivalently, $\mu_1(A_1) = \mu_2(A_2)$. We will use the minimality and an exhaustion argument (cf. Lemma 1.0.7 in [1]) to show that there is a sequence $q_1, q_2, \ldots \in T$ such that

$$\mu_i\left(\bigcap_{n \in \mathbb{N}} \{f_{q_n}^{(i)} = 0\} \right) = 0, \qquad i = 1, 2. \tag{5.7}$$

Fix $i \in \{1, 2\}$. Since the measure μ_i is σ-finite, we can represent Ω_i as a disjoint union of sets $E_1, E_2, \ldots \in \mathcal{B}_i$ such that $\mu_i(E_k) < \infty$, $k \in \mathbb{N}$. Let $e_k = \inf_Q \mu_i(\bigcap_{q \in Q} \{f_q^{(i)} = 0\} \cap E_k)$, where the infimum is taken over all at most countable sets $Q \subset T$. Clearly, $e_k < \infty$. For every $n \in \mathbb{N}$ we can find at most countable $Q_{kn} \subset T$ such that $\mu_i(\bigcap_{q \in Q_{kn}} \{f_q^{(i)} = 0\} \cap E_k) < e_k + \frac{1}{n}$. Since $Q_k := \bigcup_{n \in \mathbb{N}} Q_{kn}$ is at most countable, we have $e_k = \mu_i(F_k)$, where $F_k = \bigcap_{q \in Q_k} \{f_q^{(i)} = 0\} \cap E_k$. It follows that for every $t \in T$, $f_t^{(i)} = 0$ a.e. on F_k. Otherwise, we could consider $Q_k \cup \{t\}$ and arrive at a contradiction. By the assumption of minimality this implies that, we must have $e_k = 0$. This holds for every $k \in \mathbb{N}$. The proof of (5.7) is completed by taking the union of the collections Q_k, $k \in \mathbb{N}$.

Consider measurable sets

$$G_{i,p} = A_i \cap \left(\bigcap_{k=1}^{p-1} \{f_{q_k}^{(i)} = 0\} \right) \cap \{f_{q_p}^{(i)} > 0\}, \qquad p \in \mathbb{N}, i = 1, 2.$$

We have $\mu_1(G_{1,p}) = \mu_2(G_{2,p})$ for every $p \in \mathbb{N}$. Indeed, by continuity of measure,

$$\mu_i(G_{i,p}) = \lim_{n \to \infty} \mu_i\left(A_i \cap \left(\bigcap_{k=1}^{p-1} \{f_{q_k}^{(i)} = 0\} \right) \cap \left\{ \frac{1}{n} \le f_{q_p}^{(i)} < n \right\} \right).$$

The right-hand side does not depend on $i = 1, 2$ as a particular case of $\mu_1(C_1) = \mu_2(C_2)$ in the case $m > 0$. It follows from (5.7) that

$$\mu_1(A_1) = \sum_{p=1}^{\infty} \mu_1(G_{1,p}) = \sum_{p=1}^{\infty} \mu_2(G_{2,p}) = \mu_2(A_2).$$

This completes the proof of (5.5).

It follows now from (5.5) that the measures $\mu_1 \circ F_1^{-1}$ and $\mu_2 \circ F_2^{-1}$ coincide on the semiring $\mathcal{C}$ consisting of sets of the form $\bigcap_{j=1}^n \{\varphi : T \to \mathbb{R} : x_j \le \varphi(t_j) < y_j\}$, where $t_1, \dots, t_n \in T$, $[x_1, y_1), \dots, [x_n, y_n) \subset \mathbb{R}$. Note that $\mathcal{C}$ generates the product σ-algebra $\mathcal{B}$. Also, by (5.7), we can represent $\mathbb{R}^T$ as

$$\mathbb{R}^T = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} \big\{\varphi : T \to \mathbb{R} : k^{-1} \le \varphi(q_n) < k\big\} \bmod \mu_1 \circ F_1^{-1} \text{ and } \mu_2 \circ F_2^{-1}.$$

Note that the sets in the union in the right-hand side have finite $\mu_1 \circ F_1^{-1}$ (and $\mu_2 \circ F_2^{-1}$) measure and belong to the semiring $\mathcal{C}$. The uniqueness of the extension of measure theorem yields that $\mu_1 \circ F_1^{-1} = \mu_2 \circ F_2^{-1}$. The assumptions of Lemma 5.1 are verified. Lemma 5.1 yields (2.6) and completes the proof of the theorem. □

5.3. Proofs in the i.d. case

We start with discussing some properties of the spectral representation. Note first that the functional I is not additive. Nevertheless, by (2.10) it follows that for all $f, g \in \mathcal{L}^+$

$$\begin{aligned} I(f) + I(g) &= I(f+g) + \gamma(f,g), \qquad \text{where} \\ \gamma(f,g) &:= \int_\Omega \big(a(f+g) - a(f) - a(g)\big)\,\mathrm{d}\mu. \end{aligned} \tag{5.8}$$

The next result shows that γ in (5.8) is well defined and that this constant correction term can be controlled in terms of the metric d.

Lemma 5.2. *For all $f, g \in \mathcal{L}^+$, we have that $\int_\Omega |a(f) + a(g) - a(f+g)|\,\mathrm{d}\mu < \infty$. Moreover, for γ and d as in (5.8) and (2.12), we have*

$$\big|\gamma(f,g)\big| \le 3\big(d(f+g)\big)^2 + 2\big(d(f) + d(g)\big)d(f+g).$$

Proof. Consider the integral defining $\gamma(f,g)$ over the sets $A := \{|f+g| > 1\}$, $B := A^c \cap \{|f| \le 1\} \cap \{|g| \le 1\}$ and $C := A^c \cap (\{|f| > 1\} \cup \{|g| > 1\})$, which form a disjoint partition of Ω.

Observe that over B the integrand is zero, since $a(f) = f, a(g) = g$ and $a(f+g) = f+g$ whenever $|f| \le 1, |g| \le 1$, and $|f+g| \le 1$. Note, on the other hand that the set $B^c = A \cup C \subset \{|f+g| > 1\} \cup \{|f| > 1\} \cup \{|g| > 1\}$ has a finite μ measure because f, g and $f+g$ belong to $\mathcal{L}^+$. Since $|a(f+g) - a(f) - a(g)| \le 3$ it therefore follows that $\int_\Omega |a(f+g) - a(f) - a(g)|\,\mathrm{d}\mu < \infty$ and $\gamma(f,g)$ is well defined.

Over A, we have that

$$\int_{\{|f+g|>1\}} \big|a(f+g) - a(f) - a(g)\big|\,\mathrm{d}\mu \le 3\mu\big\{|f+g| > 1\big\} \le 3\big(d(f+g)\big)^2.$$

Now, focus on the set C. The function a in (2.9) is Lipschitz and in fact $|a(x)+a(y)| \le |x+y|$ for all $x, y \in \mathbb{R}$. Therefore, $|a(f+g)-a(f)-a(g)| \le 2|f+g|$, and hence

$$\int_C \big|a(f+g)-a(f)-a(g)\big|\,\mathrm{d}\mu \le 2\int_{\{|f+g|\le 1\}} |f+g|(\mathbb{1}_{\{|f|>1\}}+\mathbb{1}_{\{|g|>1\}})\,\mathrm{d}\mu$$
$$\le 2d(f+g)\big(d(f)+d(g)\big),$$

where the last relation follows from the Cauchy–Schwartz inequality and the fact that $\mu\{|f|>1\} \le d(f)^2$. Combining the above two bounds, we obtain the desired inequality. □

Relation (5.8) readily implies the following result on the sum of two spectral representations over the same space.

Proposition 5.3. *Consider two i.d. processes* $X_t^{(i)} := I(f_t^{(i)}) + c_t^{(i)}, t \in T$, *where* $\{f_t^{(i)}\}_{t\in T} \subset \mathcal{L}^+(\Omega, \mathcal{B}, \mu)$, *and* $c_t^{(i)} \in \mathbb{R}, i = 1, 2$. *Then, their sum has the following spectral representation*:

$$\big\{X_t^{(1)} + X_t^{(2)}\big\}_{t\in T} \stackrel{d}{=} \big\{I\big(f_t^{(1)} + f_t^{(2)}\big) + c_t^{(1)} + c_t^{(2)} + \gamma\big(f_t^{(1)}, f_t^{(2)}\big), t \in T\big\}.$$

Recall now that convergence in probability is metrized by the Ky Fan distance which is given by

$$d_{\mathrm{KF}}(\xi, \eta) \equiv d_{\mathrm{KF}}(\xi-\eta) := \inf\big\{\delta > 0 : \mathbb{P}\big\{|\xi-\eta| \ge \delta\big\} \le \delta\big\}. \tag{5.9}$$

The next proposition shows that the metric d on the space of integrands is comparable to the metric d_{KF} on the space of integrals.

Proposition 5.4. *For all* $f \in \mathcal{L}^+$, *we have*

$$d_{\mathrm{KF}}\big(I(f)\big) \le 2d(f)^{2/3} \quad \text{and} \quad 1-\mathrm{e}^{-cd(f)^2} \le 2d_{\mathrm{KF}}\big(I(f)-I(f)'\big) \le 4d_{\mathrm{KF}}\big(I(f)\big), \tag{5.10}$$

with $c = 1-\sin(1)$, *where* $I(f)'$ *is an independent copy of* $I(f)$.

The following elementary inequality is used in the proof of Proposition 5.4.

Lemma 5.5. *Let* X *be a symmetric random variable. Then*

$$\sup_{|\theta|\le 1}\big(1-\mathbb{E}\mathrm{e}^{\mathrm{i}\theta X}\big) \le 2d_{\mathrm{KF}}(X). \tag{5.11}$$

Proof of Lemma 5.5. Since X is symmetric, we have that its characteristic function $\phi_X(\theta) = \mathbb{E}\mathrm{e}^{\mathrm{i}\theta X}, \theta \in \mathbb{R}$ is real and

$$\big(1-\phi_X(\theta)\big) = \int_{-\infty}^{\infty}\big(1-\cos(\theta x)\big)F_X(\mathrm{d}x).$$

Note that $0 \le 1-\cos(u) \le u^2/2$, for all $u \in \mathbb{R}$. Thus, with $\varepsilon \in (0,1]$, we have

$$\big(1-\phi_X(\theta)\big) \le \frac{|\theta\varepsilon|^2}{2}\int_{-\varepsilon}^{\varepsilon} F_X(\mathrm{d}x) + \int_{|x|\ge\varepsilon} F_X(\mathrm{d}x) \le \varepsilon + \mathbb{P}\big\{|X|\ge\varepsilon\big\}$$

for all $|\theta| \le 1 \le \sqrt{2/\varepsilon}$. The inequality (5.11) follows from the definition (5.9) of the Ky Fan distance functional. □

Proof of Proposition 5.4. We first prove the second inequality in (5.10). Let $X := I(f) - I(f)'$, where $I(f)'$ is an independent copy of $I(f)$. Thus, in view of (2.10), X is symmetric with characteristic function

$$\phi_X(\theta) = \big|\mathbb{E}\mathrm{e}^{\mathrm{i}\theta I(f)}\big|^2 = \exp\bigg\{-2\int_\Omega \big(1-\cos(\theta f)\big)\,\mathrm{d}\mu\bigg\}, \qquad \theta \in \mathbb{R}. \tag{5.12}$$

Now, by Lemma 5.5, we obtain

$$0 \le \sup_{|\theta|\le 1}\big(1-\phi_X(\theta)\big) \le 2d_{\mathrm{KF}}(X).$$

Thus, in view of (5.12), using the fact that the function $u \mapsto 1-\mathrm{e}^{-2u}$, $u \ge 0$ is strictly increasing, the above supremum can be taken inside the exponential, and hence

$$1-\mathrm{e}^{-2A} := 1-\exp\bigg\{-2\sup_{|\theta|\le 1}\int_\Omega \big(1-\cos(\theta f)\big)\,\mathrm{d}\mu\bigg\} \le 2d_{\mathrm{KF}}(X). \tag{5.13}$$

We will focus on the term A above and obtain a lower bound for it. Notice that

$$\sup_{|\theta|\le 1}\int_{\{|f|\le 1\}}\big(1-\cos(\theta f)\big)\,\mathrm{d}\mu + \sup_{|\theta|\le 1}\int_{\{|f|>1\}}\big(1-\cos(\theta f)\big)\,\mathrm{d}\mu \le A + A \equiv 2A.$$

Since $x^2/3 \le 1-\cos(x)$, $|x| \le 1$, for the first term above, we have

$$\frac{1}{3}\int_{\{|f|\le 1\}} |f|^2\,\mathrm{d}\mu = \sup_{|\theta|\le 1}\frac{\theta^2}{3}\int_{\{|f|\le 1\}} |f|^2\,\mathrm{d}\mu \le \sup_{|\theta|\le 1}\int_{\{|f|\le 1\}}\big(1-\cos(\theta f)\big)\,\mathrm{d}\mu.$$

On the other hand, over the set $\{|f|>1\}$, we apply the inequality $\sup_{|\theta|\le 1}(1-\cos(\theta f)) \ge \int_0^1 (1-\cos(\theta f))\,\mathrm{d}\theta = 1-\sin(f)/f$. By combining these two lower bounds, we obtain

$$\frac{1}{3}\int_{\{|f|\le 1\}} |f|^2\,\mathrm{d}\mu + \int_{\{|f|>1\}}\bigg(1-\frac{\sin(f)}{f}\bigg)\,\mathrm{d}\mu \le 2A. \tag{5.14}$$

Also, since $1-\sin(x)/x \ge 1-\sin(1) =: c \approx 0.1585 > 0$, for all $|x| \ge 1$, we obtain further that

$$cd(f)^2 \le \frac{1}{3}\int_{\{|f|\le 1\}} |f|^2\,\mathrm{d}\mu + \int_{\{|f|>1\}}\bigg(1-\frac{\sin(f)}{f}\bigg)\,\mathrm{d}\mu.$$

In view of (5.13), (5.14), and the monotonicity of $u \mapsto 1 - \mathrm{e}^{-u}$, we obtain $1 - \mathrm{e}^{-cd(f)^2} \le 2d_{\mathrm{KF}}(X)$, which, since $d_{\mathrm{KF}}(X) \equiv d_{\mathrm{KF}}(I(f) - I(f)') \le 2d_{\mathrm{KF}}(I(f))$, yields the second inequality in (5.10).

We now establish the first inequality in (5.10). Let $d := d(f) \equiv (\int_\Omega 1 \wedge |f|^2 \,\mathrm{d}\mu)^{1/2}$, $f \in \mathcal{L}^+$ and consider the sets $A = \{|f| \ge 1\}$ and $B = \{|f| < 1\}$. Note that $\mu(A) < \infty$ and recall by (2.8) that $I(f\mathbb{1}_A) = \int_A f \,\mathrm{d}\Pi_\mu - \int_A a(f)\,\mathrm{d}\mu$. From the definition of a and d, see (2.9) and (2.12), it follows that $|\int_A a(f)\,\mathrm{d}\mu| \le \mu(A) \le d^2$ and therefore

$$\mathbb{P}\{|I(f\mathbb{1}_A)| > d^2\} \le \mathbb{P}\left\{\left|\int_A f \,\mathrm{d}\Pi_\mu\right| \ne 0\right\} \le 1 - \mathrm{e}^{-\mu(A)} \le 1 - \mathrm{e}^{-d^2}. \tag{5.15}$$

The second inequality follows from the fact that $\int_A f \,\mathrm{d}\Pi_\mu$ is non-zero only when the Poisson point process Π_μ has at least one point in the set A. Also, $I(f\mathbb{1}_B)$ has (by definition) expectation 0 and variance $\int_B f^2 \,\mathrm{d}\mu \le d^2$. Thus, by the Chebyshev's inequality,

$$\mathbb{P}\{|I(f\mathbb{1}_B)| > d^{2/3}\} \le d^{2/3}. \tag{5.16}$$

Since $I(f) = I(f\mathbb{1}_A) + I(f\mathbb{1}_B)$, by (5.15) and (5.16), in the case $d \le 1$, we get

$$\begin{aligned}\mathbb{P}\{|I(f)| > 2d^{2/3}\} &\le \mathbb{P}\{|I(f)| > d^2 + d^{2/3}\} \\ &\le \mathbb{P}\{|I(f\mathbb{1}_A)| > d^2\} + \mathbb{P}\{|I(f\mathbb{1}_B)| > d^{2/3}\} \\ &\le 1 - \mathrm{e}^{-d^2} + d^{2/3} \\ &\le 2d^{2/3}.\end{aligned}$$

Hence $d_{\mathrm{KF}}(I(f)) \le 2d^{2/3}$, provided that $d \le 1$. This, since $d_{\mathrm{KF}}(I(f)) \le 1$ implies the first inequality in (5.10). □

Proof of Proposition 2.15. The proof is standard. Let $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{L}^+$ be a Cauchy sequence in d. Then, for all $\varepsilon \in (0, 1)$, we have

$$\mu\{|f_m - f_n| > \varepsilon\} \le \frac{1}{\varepsilon^2}\int_\Omega 1 \wedge |f_m - f_n|^2 \,\mathrm{d}\mu = \frac{d(f_m, f_n)^2}{\varepsilon^2} \to 0,$$

as $m, n \to \infty$, which shows that $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy in measure. Hence, there exists a subsequence $\{n_k\}_{k\in\mathbb{N}}$ and a measurable function f, such that $f_{n_k} \to f$, as $n_k \to \infty$, μ-a.e. Now, by the Fatou's lemma, we obtain

$$d(f_{n_k}, f)^2 = \int_\Omega 1 \wedge |f_{n_k} - f|^2 \,\mathrm{d}\mu \le \liminf_{\ell\to\infty} \int_\Omega 1 \wedge |f_{n_k} - f_{n_\ell}|^2 \,\mathrm{d}\mu = \liminf_{\ell\to\infty} d(f_{n_k}, f_{n_\ell})^2.$$

This inequality implies that $d(f_{n_k}, f) < \infty$, and hence $f \in \mathcal{L}^+$, because $d(f, 0) \le d(f, f_{n_k}) + d(f_{n_k}, 0) < \infty$. Since $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy in the metric d, we also have that $d(f_{n_k}, f) \to 0$, as $n_k \to \infty$, and hence $d(f_n, f) \to 0$, as $n \to \infty$. Thereby proving that the metric d is complete.

Let now $(\Omega, \mathcal{B})$ be Borel. Recall that the measure μ is σ-finite. Then the space $L^2 = L^2(\Omega, \mathcal{B}, \mu)$ $(\subset \mathcal{L}^+)$ equipped with the usual L^2-norm is separable and let $\{f_n\}_{n\in\mathbb{N}}$ be a dense subset of L^2. By (2.12), for all $f \in \mathcal{L}^+$, $f_n \in L^2$, and $K > 0$, we have

$$\begin{aligned} d(f, f_n)^2 &= \int_\Omega 1 \wedge |f - f_n|^2 \, \mathrm{d}\mu \\ &\le \int_{\{|f|\le K\}} |f - f_n|^2 \, \mathrm{d}\mu + \int_{\{|f|>K\}} \mathrm{d}\mu \\ &\le \int_\Omega (f \mathbb{1}_{\{|f|\le K\}} - f_n)^2 \, \mathrm{d}\mu + \mu\big\{|f| > K\big\}. \end{aligned}$$

Since $f \in \mathcal{L}^+$, we have that $f \mathbb{1}_{\{|f|\le K\}} \in L^2$ and $\mu\{|f| > K\} \to 0$, as $K \to \infty$. Thus, by picking large enough K and a suitable f_n, one can make $d(f, f_n)$ arbitrarily small, showing that $\{f_n\}_{n\in\mathbb{N}}$ is also dense in the metric space $(\mathcal{L}^+, d)$, thereby proving separability. □

Proof of Proposition 2.16. Suppose first that $d(f_n - f) + |c_n - c| \to 0$, as $n \to \infty$. Then, by Slutsky's theorem, it is enough to show that $I(f_n)$ converges in probability to $I(f)$, as $n \to \infty$. By (5.8), we have that $I(f_n) - I(f) = I(f_n - f) + \gamma(f_n, -f)$. Proposition 5.4 and the assumption $d(f_n - f) \to 0$ imply that $I(f_n - f) \overset{\mathbb{P}}{\to} 0, n \to \infty$. It remains to show that $\gamma(f, -f_n) \to 0$, as $n \to \infty$. By the triangle inequality for d, we have $|d(f_n) - d(f)| \le d(f_n - f) \to 0$, as $n \to \infty$, and in particular $d(f_n), n \in \mathbb{N}$ is bounded. Thus, by Lemma 5.2 applied to f and $g := -f_n$, we obtain $\gamma(f, -f_n) \to 0$, as $n \to \infty$. This completes proof of the 'if' part.

To prove the 'only if' part, suppose that $I(f_n) + c_n \overset{\mathbb{P}}{\to} \xi, n \to \infty$, set $\xi_{m,n} := I(f_m) - I(f_n) + c_m - c_n$, and let $\xi'_{m,n}$ be and independent copy of $\xi_{m,n}$. Then, by using (2.10) we obtain that

$$\xi_{m,n} - \xi'_{m,n} \overset{d}{=} I(f_m - f_n) - I(f_m - f_n)',$$

where $I(f_m - f_n)'$ is an independent copy of $I(f_m - f_n)$. Now, by the second bound in (5.10) of Proposition 5.4 applied to $f := f_m - f_n$, we obtain

$$1 - \mathrm{e}^{-cd(f_m - f_n)^2} \le 2d_{\mathrm{KF}}\big(I(f_m - f_n) - I(f_m - f_n)'\big) \equiv 2d_{\mathrm{KF}}\big(\xi_{m,n} - \xi'_{m,n}\big) \le 4d_{\mathrm{KF}}(\xi_{m,n}).$$

The right-hand side of the last inequality vanishes, as $m, n \to \infty$, since the sequence $\{I(f_n) + c_n, n \in \mathbb{N}\}$ converges in probability and therefore it is Cauchy in the Ky Fan metric. This implies that $d(f_m - f_n) \to 0, m, n \to \infty$, and since $(\mathcal{L}^+, d)$ is complete (Proposition 2.15), there is an $f \in \mathcal{L}^+$, such that $d(f_n - f) \to 0, n \to \infty$. Therefore, by the already established 'if' part, it follows that $I(f_n) \overset{\mathbb{P}}{\to} I(f), n \to \infty$. This, and the fact that $I(f_n) + c_n \overset{\mathbb{P}}{\to} \xi, n \to \infty$ imply (by Slutsky) that the sequence c_n converges to a constant c and $\xi = I(f) + c$. This completes the proof. □

Proof of Theorem 2.14. Let T_0 be the at most countable subset of T appearing in Condition S. Consider the space $\mathbb{R}^{T_0}$, equipped with the product σ-algebra $\mathcal{B}$. Following [21] (see also [33]), let μ be the Lévy measure of $\{X(t), t \in T_0\}$ on $\mathbb{R}^{T_0}$. For $t \in T_0$, we define the coordinate mappings $f_t : \mathbb{R}^{T_0} \to \mathbb{R}$ by $f_t(\varphi) = \varphi(t)$, where $\varphi : T_0 \to \mathbb{R}$, $\varphi \in \mathbb{R}^{T_0}$. Then, $\{f_t, t \in T_0\}$ is a spectral representation of $\{X(t), t \in T_0\}$ by the properties of the Lévy measure.

For $t \notin T$, observe that by Condition S, there exists a sequence $\{t_n\} \subset T_0$, such that $X(t_n)$ converges in probability to $X(t)$, as $n \to \infty$. In other words, $I(f_n) + c_n$ converges in probability to $I(f) + c$, for some c_n and c. Thus, by Proposition 2.16, the sequence of functions f_{t_n} has a limit in $(\mathcal{L}^+, d)$, as $n \to \infty$. We take this limit to be the spectral function f_t.

Notice that the so-defined spectral representation is minimal. Indeed, the σ-algebra $\sigma\{f_t, t \in T\}$ coincides with the product σ-algebra $\mathcal{B}$ on $\mathbb{R}^{T_0}$. We also have that $\operatorname{supp}\{f_t, t \in T_0\} = \mathbb{R}^{T_0}$ (mod μ) because $\bigcap_{t \in T_0}\{f_t = 0\} = \{0\}$, a set whose Lévy measure is 0 by convention. To complete the proof, observe that the measurable space $(\mathbb{R}^{T_0}, \mathcal{B})$ is Borel by Kuratowski's theorem. □

Proof of Theorem 2.17. We are going to apply Lemma 5.1. Define the measurable mappings $F_i : (\Omega_i, \mathcal{B}_i) \to (\mathbb{R}^T, \mathcal{B})$ by

$$F_i(\omega) = \big(f_t^{(i)}(\omega)\big)_{t \in T}, \qquad \omega \in \Omega_i, i = 1, 2.$$

Minimality implies that the first condition of Lemma 5.1 is satisfied. We prove that $\mu_1 \circ F_1^{-1} = \mu_2 \circ F_2^{-1}$. Let $t_1, \dots, t_n \in T$ and observe that in view of (2.10) we have

$$\begin{aligned} \mathbb{E} e^{\mathrm{i} \sum_{j=1}^n \theta_j X(t_j)} &= \mathbb{E} \exp\left\{ \mathrm{i} \sum_{j=1}^n \theta_j \big(I\big(f_{t_j}^{(i)}\big) + c_j^{(i)}\big) \right\} \\ &= \exp\left\{ \mathrm{i} \sum_{j=1}^n c_j^{(i)} \theta_j + \int_{\mathbb{R}^n} \left(e^{\mathrm{i} \sum_{j=1}^n \theta_j x_j} - \mathrm{i} \sum_{j=1}^n \theta_j a(x_j) - 1 \right) \big(\mu_i \circ G_i^{-1}\big)(\mathrm{d}x) \right\}, \end{aligned}$$

where $G_i = (f_{t_j}^{(i)})_{j=1}^n : \Omega_i \to \mathbb{R}^n$ and $c_1^{(i)}, \dots, c_n^{(i)} \in \mathbb{R}$ are constants, $i = 1, 2$. The last relation and the uniqueness of the Lévy measure of the i.d. random vector $(X(t_j))_{j=1}^n$ shows that $(\mu_1 \circ G_1)^{-1}(A) = (\mu_2 \circ G_2)^{-1}(A)$ for all Borel sets $A \subset \mathbb{R}^n \setminus \{0\}$. We need to show that $(\mu_1 \circ G_1)^{-1}(\{0\}) = (\mu_2 \circ G_2)^{-1}(\{0\})$. As in the proof of Theorem 2.11 we can find a sequence $q_1, q_2, \ldots \in T$ such that $\mu_i(\bigcap_{j \in \mathbb{N}}\{f_{t_j}^{(i)}\}) = 0$, $i = 1, 2$. Consider measurable sets

$$E_{i,p} = G_i^{-1}\big(\{0\}\big) \cap \left(\bigcap_{j=1}^{p-1} \big\{f_{q_j}^{(i)} = 0\big\} \right) \cap \big\{f_{q_p}^{(i)} \neq 0\big\}.$$

For every p, we have shown that $\mu_1(E_{1,p}) = \mu_2(E_{2,p})$. It follows that

$$\mu_1\big(G_1^{-1}\big(\{0\}\big)\big) = \sum_{p=1}^{\infty} \mu_1(E_{1,p}) = \sum_{p=1}^{\infty} \mu_2(E_{2,p}) = \mu_2\big(G_2^{-1}\big(\{0\}\big)\big).$$

This proves that $(\mu_1 \circ G_1)^{-1}(A) = (\mu_2 \circ G_2)^{-1}(A)$ for all Borel sets $A \subset \mathbb{R}^n$. In other words, the measures $\mu_1 \circ F_1^{-1}$ and $\mu_2 \circ F_2^{-1}$ are equal on the semiring $\mathcal{C}$ consisting of subsets $\{\varphi : T \to \mathbb{R} : (\varphi(t_j))_{j=1}^n \in A\}$, where $A \subset \mathbb{R}^n$ is Borel. This semiring generates the product σ-algebra $\mathcal{B}$. Also, we have a decomposition

$$\mathbb{R}^T = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} \{\varphi : T \to \mathbb{R} : k^{-1} \le |\varphi(q_n)| \le k\} \bmod \mu_1 \circ F_1^{-1} \text{ and } \mu_2 \circ F_2^{-1}.$$

Note that the sets on the right-hand side have finite $\mu_1 \circ F_1^{-1}$ (and $\mu_2 \circ F_2^{-1}$) measure and belong to the semiring $\mathcal{C}$. By the uniqueness of measure extension theorem, the measures $\mu_1 \circ F_1^{-1}$ and $\mu_2 \circ F_2^{-1}$ are equal. Lemma 5.1 completes the proof. □

5.4. Proof of Theorem 3.9

By Theorems 2.8, 2.14 and 3.1, the process X has a minimal spectral representation $g_t := g_0 \circ T_t, t \in \mathbb{T}^d$ over a σ-finite Borel space $(\widetilde{\Omega}, \widetilde{\mathcal{B}}, \widetilde{\mu})$, where $\{T_t, t \in \mathbb{T}^d\}$ is a measure preserving and measurable flow (see also Proposition 2.19).

Since the spectral representation $\{f_t, t \in T\} \subset \mathcal{L}^{\vee/+}(\Omega, \mathcal{B}, \mu)$ is of full support, it is minimal if we set $\mathcal{B} = \sigma\{f_t, t \in T\}$. Even though $\mathcal{B}$ may not be Borel, Theorems 2.11(i) and 2.17(i) imply that there exists a measurable measure-preserving mapping $\Phi : (\Omega, \mathcal{B}) \to (\widetilde{\Omega}, \widetilde{\mathcal{B}})$, such that for all $t \in \mathbb{T}^d$, $g_t \circ \Phi = f_t$ μ-a.e. Therefore, by using the joint measurability of the two representations and appealing to Fubini, we see that

$$\begin{aligned}\int_{\mathbb{T}^d} \psi(|f_t(\omega)|)\lambda(\mathrm{d}t) &= \int_{\mathbb{T}^d} \psi(|g_t(\Phi(\omega))|)\lambda(\mathrm{d}t) \\ &\equiv \int_{\mathbb{T}^d} \psi(|g_0 \circ T_t(\Phi(\omega))|)\lambda(\mathrm{d}t) = \infty, \qquad \mu\text{-a.e.}\end{aligned} \tag{5.17}$$

which, since $\widetilde{\mu} = \mu \circ \Phi^{-1}$, shows that relation (3.7) is equivalent to $\int_{\mathbb{T}^d} \psi(|g_0 \circ T_t(\widetilde{\omega})|)\lambda(\mathrm{d}t) = \infty$, $\widetilde{\mu}$-a.e. Thus, using the criterion in Theorem 3.7 one can relate (3.7) to the conservativity of the flow. More precisely, proceeding as in the proof of Proposition 3.2 in [35], let

$$h(\widetilde{\omega}) := \sum_{\gamma \in \mathbb{Z}^d} a_\gamma \int_{\gamma + [0,1)^d} \psi(|g_0 \circ T_t(\widetilde{\omega})|)\lambda(\mathrm{d}t),$$

where $a_\gamma > 0$ and $\sum_{\gamma \in \mathbb{Z}^d} a_\gamma = 1$. By Fubini's theorem, the full support condition on $\{g_t, t \in T\}$ implies that $h \in \mathcal{L}^1(\widetilde{\Omega}, \widetilde{\mathcal{B}}, \widetilde{\mu})$ and $h > 0$, $\widetilde{\mu}$-a.e. Observe also by applying Fubini again and using the facts that λ is shift-invariant and the flow $\{T_t\}_{t \in \mathbb{T}^d}$ is measure-preserving

$$\sum_{\beta \in \mathbb{Z}^d} h \circ T_\beta(\widetilde{\omega}) = \int_{\mathbb{T}^d} \psi(|g_0 \circ T_t(\widetilde{\omega})|)\lambda(\mathrm{d}t).$$

Theorem 3.7, applied to the discrete flow $\{T_\beta\}_{\beta\in\mathbb{Z}^d}$ shows that is conservative if and only if (3.7) holds, which completes the proof.

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