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Stochastic Integration of Operator-Valued Functions with Respect to Banach Space-Valued Brownian Motion

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Abstract Let E be a real Banach space with property (α) and let W_Γ be an E-valued Brownian motion with distribution Γ. We show that a function $\Psi : [0, T] \to \mathscr{L}(E)$ is stochastically integrable with respect to W_Γ if and only if Γ-almost all orbits Ψx are stochastically integrable with respect to a real Brownian motion. This result is derived from an abstract result on existence of Γ-measurable linear extensions of γ-radonifying operators with values in spaces of γ-radonifying operators. As an application we obtain a necessary and sufficient condition for solvability of stochastic evolution equations driven by an E-valued Brownian motion.

Keywords Stochastic integration in Banach spaces · γ-Radonifying operators · Property (α) · Measurable linear extensions · Stochastic evolution equations

Mathematics Subject Classifications (2000) Primary 60H05 · Secondary 35R15 · 47B10 · 60H15

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1 Introduction

The problem of stochastic integration in general Banach spaces has been considered by many authors, cf. [3, 17, 24, 25]. In [18] the authors constructed a theory of stochastic integration with respect to a H-cylindrical Brownian motion for functions with values in the space of bounded operators $\mathscr{L}(H, E)$, where H is a separable real Hilbert space and E a real Banach space. As was explained there, from this theory one obtains conditions for stochastic integrability of $\mathscr{L}(E)$-valued functions Ψ with respect to E-valued Brownian motions. The purpose of this paper is to address the following natural question which was left open in [18]: if Ψ is stochastically integrable with respect to an E-valued Brownian motions, is it true that for 'most' $x \in E$ the orbits Ψx are stochastically integrable with respect to a real Brownian motion? In the formulation of our main result, W denotes a scalar Brownian motion and W_Γ an E-valued Brownian motion with distribution Γ, i.e., Γ is the unique Gaussian Radon measure on E such that

$$\mathbb{P}\{W_\Gamma(t) \in A\} = \Gamma(A/\sqrt{t}) \qquad \forall t \geqslant 0, \quad A \subseteq E \text{ Borel.}$$

Theorem 1.1 *Let E have property (α). For an operator-valued function $\Psi : [0, T] \to \mathscr{L}(E)$ the following assertions are equivalent:*

(1) *Ψ is stochastically integrable with respect to W_Γ;*
(2) *Ψx is stochastically integrable with respect to W for Γ-almost all $x \in E$.*

In this situation we have

$$\mathbb{E}\left\|\int_0^T \Psi \, dW_\Gamma\right\|^2 \eqsim \int_E \mathbb{E}\left\|\int_0^T \Psi x \, dW\right\|^2 d\Gamma(x) \tag{1.1}$$

with proportionality constants depending on E only.

Here and in the rest of the paper we write $X \eqsim Y$ if there exist constants $0 < c \leqslant C < \infty$, depending on E only, such that $cX \leqslant Y \leqslant CX$. The notations $\gtrsim$ and $\lesssim$ are defined in a similar way.

By considering step functions it is easy to see that property (α) is also necessary for the two-sided estimate 1.1.

Property (α) has been introduced by Pisier [20] in connection with the geometry of Banach spaces and will be discussed in Section 2. This property has proved its importance in connection with operator-valued Fourier multipliers [4, 11, 15, 28] and operator algebras [19].

The proof of Theorem 1.1 has two main ingredients. The first is to show that there is a canonical way to associate with Ψ an 'orbit operator' which acts as a γ-radonifying operator from the reproducing kernel Hilbert space H_Γ into $\gamma(L^2(0, T); E)$, the space of all γ-radonifying operators from $L^2(0, T)$ into E. In the opposite direction we have a 'tensor operator' piecing together the orbits through the points in H_Γ. Both operators are constructed, in an abstract setting, in Section 3. The second ingredient for the proof of Theorem 1.1 is a result on the existence of Γ-measurable linear extensions of certain γ-radonifying operators acting from H_Γ into another Banach space, which is proved in Section 4. After introducing the concept of a representable operator, in Section 5 we prove abstract one-sided versions of Theorem 1.1. In Section 6, these are worked out in the setting of stochastic integration and Theorem 1.1 is proved.

The results of this paper can be applied to, and are in fact motivated by, the study of linear stochastic evolution equations in Banach spaces. To illustrate this point, in the final Section 7 we obtain a necessary and sufficient condition for existence of solutions for stochastic linear evolution equations in terms of stochastic integrability properties of the orbits of the semigroup governing the deterministic part of the equation.

2 Property (α)

Let $(r'_m)_{m\geqslant 1}$ and $(r''_n)_{n\geqslant 1}$ be mutually independent Rademacher sequences on probability spaces $(\Omega', \mathbb{P}')$ and $(\Omega'', \mathbb{P}'')$. A Banach space E is said to have the *Rademacher property* (α) if

$$\mathbb{E}'\mathbb{E}''\left\|\sum_{m,n=1}^{N}\varepsilon_{mn}r'_m r''_n x_{mn}\right\|^2 \lesssim \mathbb{E}'\mathbb{E}''\left\|\sum_{m,n=1}^{N} r'_m r''_n x_{mn}\right\|^2$$

for all $N \geqslant 1$ and all choices $\varepsilon_{mn} \in \{-1, 1\}$ and $x_{mn} \in E$. This property was introduced by Pisier [20], who proved that every Banach space with local unconditional structure and finite cotype has property (α). In particular, every Banach lattice with finite cotype has property (α). Explicit examples of spaces with property (α) are Hilbert spaces and the L^p-spaces with $1 \leqslant p < \infty$.

By replacing the rôle of Rademacher sequences $(r'_m)_{m\geqslant 1}$ and $(r''_n)_{n\geqslant 1}$ by orthogaussian sequences $(g'_m)_{m\geqslant 1}$ and $(g''_n)_{n\geqslant 1}$, in a similar way we define Banach spaces with the *Gaussian property* (α). The following proposition relates both definitions. In its formulation, and in the rest of the paper, $(r_{mn})_{m,n\geqslant 1}$ and $(g_{mn})_{m,n\geqslant 1}$ denote doubly indexed Rademacher sequences and orthogaussian sequences, respectively.

Proposition 2.1 *For a Banach space E, the following assertions are equivalent:*

(1) *E has the Rademacher property (α);*
(2) *E has the Gaussian property (α);*
(3) *For all $N \geqslant 1$ and all sequences $(x_{mn})_{m,n=1}^N$ in E we have*

$$\mathbb{E}\left\|\sum_{m,n=1}^{N} r_{mn}x_{mn}\right\|^2 \eqsim \mathbb{E}'\mathbb{E}''\left\|\sum_{m,n=1}^{N} r'_m r''_n x_{mn}\right\|^2;$$

(4) *For all $N \geqslant 1$ and all sequences $(x_{mn})_{m,n=1}^N$ in E we have*

$$\mathbb{E}\left\|\sum_{m,n=1}^{N} g_{mn}x_{mn}\right\|^2 \eqsim \mathbb{E}'\mathbb{E}''\left\|\sum_{m,n=1}^{N} g'_m g''_n x_{mn}\right\|^2.$$

If E satisfies these equivalent conditions, then E has finite cotype.

Proposition 2.1 is part of mathematical folklore and can be proved by standard randomization techniques, observing that both Rademacher and Gaussian property (α) imply finite cotype and using the well-known fact that Rademacher and Gaussian sums are equivalent in spaces with finite cotype. Henceforth we shall say that E has *property* (α) if it satisfies the equivalent condition of the proposition.

Let $(r_n)_{n\geqslant 1}$ be a Rademacher sequence on a probability space $(\Omega, \mathbb{P})$. For a random variable $\xi \in L^2(\Omega; E)$ we define

$$\pi_N \xi := \sum_{n=1}^{N} r_n \mathbb{E}\,(r_n \xi).$$

Each π_N is a projection on $L^2(\Omega; E)$. The space E is *K-convex* if $\sup_{N\geqslant 1} \|\pi_N\| < \infty$. If E is K-convex, then the strong operator limit $\pi := \lim_{N\to\infty} \pi_N$ exists and defines a projection on $L^2(\Omega; E)$ of norm $\|\pi\| = \sup_{N\geqslant 1} \|\pi_N\|$. It is not hard to see that E is K-convex if and only if E^* is K-convex [6, Corollary 13.7]. For a detailed treatment of K-convexity we refer to the monographs [6, 22].

Recall that a Banach space E is said to be *B-convex* if there exist an integer $N \geqslant 2$ and a real number $\delta \in (0, 1)$ such that for all $x_1, \ldots, x_N \in E$ we can choose $\varepsilon_1, \ldots, \varepsilon_N \in \{-1, 1\}$ in such a way that

$$\frac{1}{N}\left\| \sum_{n=1}^{N} \varepsilon_n x_n \right\| \leqslant (1-\delta) \max_{1\leqslant n\leqslant N} \|x_n\|.$$

It is a deep result due to Pisier [21], see also [6, Theorems 13.10, 13.15], that a for a Banach space E the following properties are equivalent:

- E is K-convex;
- E is B-convex;
- E has non-trivial type.

Proposition 2.2 *Let E be K-convex. Then E has property (α) if and only if its dual E^* has property (α).*

Proof For a Banach space Y, let $\mathrm{Rad}_\Omega(\mathrm{Y})$ denote the closed linear span of all finite Rademacher sums $\sum_{n=1}^{N} r_n y_n$ in $L^2(\Omega; Y)$; this is precisely the range of the Rademacher projection π. Here, and in the rest of the proof, the Rademacher sequence is assumed to be fixed. By condition (3) in Proposition 2.1, Y has property (α) if and only if there is a canonical isomorphism of Banach spaces

$$\mathrm{Rad}_\Omega(\mathrm{Y}) \simeq \mathrm{Rad}_{\Omega'}(\mathrm{Rad}_{\Omega''}(\mathrm{Y})).$$

It follows from [22, Proposition 2.7] and a limiting argument that there is a canonical isomorphism

$$(\mathrm{Rad}_\Omega(\mathrm{Y}))^* \simeq \mathrm{Rad}_\Omega(\mathrm{Y}^*).$$

Now let E be K-convex. If E has property (α), then

$$\begin{aligned}\mathrm{Rad}_\Omega(\mathrm{E}^*) &\simeq (\mathrm{Rad}_\Omega(\mathrm{E}))^* \simeq (\mathrm{Rad}_{\Omega'}(\mathrm{Rad}_{\Omega''}(\mathrm{E})))^* \\ &\simeq \mathrm{Rad}_{\Omega'}((\mathrm{Rad}_{\Omega''}(\mathrm{E}))^*) \simeq \mathrm{Rad}_{\Omega'}(\mathrm{Rad}_{\Omega''}(\mathrm{E}^*)),\end{aligned}$$

where in the third step we used that $L^2(\Omega''; E)$ is K-convex, and therefore also its closed subspace $\mathrm{Rad}_{\Omega''}(\mathrm{E})$. It follows that E^* has property (α).

The K-convexity of E implies the K-convexity of E^*. Hence if E^* has property (α), then by what we just proved E^{**} has property (α), and therefore also its closed subspace E. □

We shall use property (α) through condition (4) of Proposition 2.1. Since most of our results require only a one-sided estimate we define:

Definition 2.3 Let E be a Banach space.

(1) E has the *Rademacher property* (α^-) if there is a constant C^- such that for all finite sequences $(x_{mn})_{m,n=1}^N$ in E we have

$$\mathbb{E}'\mathbb{E}''\left\|\sum_{m,n=1}^N r'_m r''_n x_{mn}\right\|^2 \leqslant (C^-)^2\mathbb{E}\left\|\sum_{m,n=1}^N r_{mn}x_{mn}\right\|^2.$$

(2) E has the *Rademacher property* (α^+) if there is a constant C^+ such that for all finite sequences $(x_{mn})_{m,n=1}^N$ in E we have

$$\mathbb{E}\left\|\sum_{m,n=1}^N r_{mn}x_{mn}\right\|^2 \leqslant (C^+)^2\mathbb{E}'\mathbb{E}''\left\|\sum_{m,n=1}^N r'_m r''_n x_{mn}\right\|^2.$$

The best possible constants in (1) and (2) are called the *Rademacher property* $(\alpha^\mp)$ *constants*. The corresponding notions of *Gaussian property* (α^-) and (α^+) are defined analogously. The best possible constants are called the *Gaussian property* $(\alpha^\mp)$ *constants*.

Example 2.4 The Schatten class S_p has property (α) if and only if $p=2$. Furthermore, S_p has the Rademacher property (α^+), but not the Rademacher property (α^-), for $p\in[1,2)$; it has the Rademacher property (α^-), but not the Rademacher property (α^+), for $p\in(2,\infty)$. This can be deduced from the estimates in [16], [23, Section 6]. Since S_p has finite cotype for $p\in(1,\infty)$ [7, 26], Rademacher sums and Gaussian sums in S_p are comparable and the observations just made also hold for the Gaussian properties (α^-) and (α^+).

The space c_0 fails both the Rademacher properties (α^-) and (α^+) and the Gaussian properties (α^-) and (α^+). To see why, observe that if c_0 had one of these properties, then every Banach space would have them, since every Banach space is finitely representable in c_0. But this would contradict the above Example. As a consequence we obtain the following result, which was kindly pointed out to us by Professor Stanisław Kwapień.

Proposition 2.5 *Let E be a Banach space.*

(1) *E has the Rademacher property (α^-) if and only if it has the Gaussian property (α^-);*
(2) *E has the Rademacher property (α^+) if and only if it has the Gaussian property (α^+).*

If E has either one of these properties, then E has finite cotype.

Proof If E has Rademacher (resp. Gaussian) property $(\alpha^{\pm})$, then by the above discussion c_0 cannot be finitely representable in E. It follows that E has finite cotype. But then Rademacher sums and Gaussian sums in E are comparable, and E has the Gaussian (resp. Rademacher) property $(\alpha^{\pm})$. □

In view of this proposition, henceforth we shall simply speak of *property* (α^-) and *property* (α^+).

Corollary 2.6 *For a Banach lattice E, the following assertions are equivalent:*

(1) *E has property* (α^-)*;*
(2) *E has property* (α^+)*;*
(3) *E has property* (α)*;*
(4) *E has finite cotype.*

Proof (4)⇒(3) follows from Pisier's result mentioned at the beginning of this section. The remaining implications follow from Proposition 2.5. □

The next one-sided version of Proposition 2.2 holds:

Proposition 2.7 *Let E be K-convex.*

(1) *E has property* (α^-) *if and only if its dual* E^* *has property* (α^+)*;*
(2) *E has property* (α^+) *if and only if its dual* E^* *has property* (α^-).

Proof This is proved in the same way as Proposition 2.2 by noting that a Banach space Y has property (α^-) if and only if $\mathrm{Rad}_\Omega(Y) \hookrightarrow \mathrm{Rad}_{\Omega'}(\mathrm{Rad}_{\Omega''}(Y))$, and Y has property (α^+) if and only if $\mathrm{Rad}_{\Omega'}(\mathrm{Rad}_{\Omega''}(Y)) \hookrightarrow \mathrm{Rad}_\Omega(Y)$. □

3 Spaces of γ-Radonifying Operators

At several occasions we shall use the fact, due to Itô and Nisio [10, 13, 14], that various types of convergence of sums of E-valued independent symmetric random variables are equivalent. For the reader's convenience we recall the precise formulation of this result.

Theorem 3.1 (Itô-Nisio theorem) *Let E be a Banach space and let* $(\xi_n)_{n\geqslant 1}$ *be a sequence of E-valued independent symmetric random variables. For the partial sums* $S_n := \sum_{j=1}^n \xi_j$ *the following assertions are equivalent:*

(1) *There exists an E-valued random variable S such that for all* $x^* \in E^*$ *we have* $\lim_{n\to\infty}\langle S_n, x^*\rangle = \langle S, x^*\rangle$ *almost surely;*
(2) *There exists an E-valued random variable S such that* $\lim_{n\to\infty} S_n = S$ *almost surely;*
(3) *There exists an E-valued random variable S such that* $\lim_{n\to\infty} S_n = S$ *in probability;*

(4) *There exists a Radon probability measure μ on E such that for all $x^* \in E^*$ the Fourier transforms satisfy* $\lim_{n\to\infty} \widehat{\mu_n}(x^*) = \widehat{\mu}(x^*)$.

If $\mathbb{E}\,\|S\|^p < \infty$ *for some* $1 \leqslant p < \infty$, *then*

$$\mathbb{E} \sup_{n\geqslant 1} \|S_n\|^p \leqslant 2 \sup_{n\geqslant 1} \mathbb{E}\,\|S_n\|^p \leqslant 2\mathbb{E}\,\|S\|^p$$

and $\lim_{n\to\infty} \mathbb{E}\,\|S_n - S\|^p = 0$.

It should perhaps be emphasized that E-valued random variables are always assumed to be strongly measurable.

In the rest of this paper, H is a separable real Hilbert space and E a real Banach space. A bounded operator $T \in \mathscr{L}(H, E)$ is said to be *γ-radonifying* if for some orthonormal basis $(h_n)_{n\geqslant 1}$ of H the Gaussian series $\sum_{n\geqslant 1} g_n T h_n$ converges in $L^2(\Omega; E)$. This definition is independent of the choice of the sequence $(g_n)_{n\geqslant 1}$ and the basis $(h_n)_{n\geqslant 1}$. The sum $X := \sum_{n\geqslant 1} g_n T h_n$ is Gaussian distributed with variance

$$\mathbb{E}\,\langle X, x^*\rangle^2 = \langle TT^* x^*, x^*\rangle \qquad \forall x^* \in E^*.$$

Thus, the distribution Γ of X is a Gaussian Radon measure on E with covariance operator TT^*. By a change of variables we see that

$$\mathbb{E}\,\|X\|^2 = \int_E \|x\|^2 \, d\Gamma(x) = \mathbb{E}\left\|\sum_{n\geqslant 1} g_n T h_n\right\|^2.$$

In particular, the right-hand side expression does not depend upon the choice of the basis $(h_n)_{n\geqslant 1}$. Thus we may define

$$\gamma(T) := \left(\mathbb{E}\left\|\sum_{n\geqslant 1} g_n T h_n\right\|^2\right)^{\frac{1}{2}}.$$

This defines a norm γ on the linear space $\gamma(H, E)$ of all γ-radonifying operators from H into E. Endowed with this norm, $\gamma(H, E)$ is a Banach space which has the following ideal property: if $R : \tilde{H} \to H$ is bounded, $T : H \to E$ is γ-radonifying, and $S : E \to \tilde{E}$ is bounded, then $S \circ T \circ R : \tilde{H} \to \tilde{E}$ is γ-radonifying and

$$\gamma(S \circ T \circ R) \leqslant \|S\|\,\gamma(T)\,\|R\|.$$

The following proposition gives two useful characterizations of radonifying operators. Recall that the field of cylindrical sets in H supports a unique finitely additive Gaussian measure, denoted by Γ_H, such that each of its finite dimensional orthogonal projections is a standard Gaussian measure.

Proposition 3.2 *For a bounded operator $T \in \mathscr{L}(H, E)$ the following assertions are equivalent:*

(1) *T is γ-radonifying;*
(2) *The series $\sum_{n\geqslant 1} g_n\, T h_n$ converges almost surely;*
(3) *The finitely additive image measure $T(\Gamma_H)$ admits an extension to a Gaussian Radon measure on E.*

In this situation the extension in (3) *is unique and its covariance operator equals* TT^*.

Proof This result is well-known; we sketch a proof for the reader's convenience. The implication (1)⇒(2) follows from the Itô-Nisio theorem, whereas the converse follows from the dominated convergence theorem and the fact that $X := \sum_{n\geqslant 1} g_n\, Th_n$ is Gaussian and therefore square integrable by Fernique's theorem. For (2)⇒(3), take the distribution of the Gaussian random variable $X := \sum_{n\geqslant 1} g_n\, Th_n$. The converse implication (3)⇒(2) follows again from the Itô-Nisio theorem. □

The Hilbert space tensor product of two separable real Hilbert spaces H and H' will be denoted by $H\overline{\otimes}H'$. Given elements $T \in \gamma(H\overline{\otimes}H', E)$ and $h_0 \in H$, we can define an element $T_{h_0} \in \gamma(H'; E)$ by

$$T_{h_0}h' := T(h_0 \otimes h').$$

If $(h_n)_{n\geqslant 1}$ is an orthonormal basis for H such that $\|h_0\|_H h_1 = h_0$ and $(h'_m)_{m\geqslant 1}$ an orthonormal basis for H', then by the Kahane contraction principle, $T_{h_0} \in \gamma(H', E)$ and

$$\begin{aligned}\gamma^2(T_{h_0}) = \mathbb{E}\left\|\sum_{m\geqslant 1} g_m T_{h_0}h'_m\right\|^2 &= \mathbb{E}\left\|\sum_{m\geqslant 1} g_m T(h_0 \otimes h'_m)\right\|^2 \\ &\leqslant \mathbb{E}\left\|\sum_{n,m\geqslant 1} g_{mn} T(h_n \otimes h'_m)\right\|^2 \|h_0\|_H^2 = \gamma^2(T)\|h_0\|_H^2.\end{aligned}$$

Using this construction, with an element $T \in \gamma(H\overline{\otimes}H', E)$ we can associate an operator $O_T : H \to \gamma(H'; E)$ by

$$O_T : h \mapsto T_h.$$

In the following theorem we study the properties of the operator $O : T \to O_T$.

Theorem 3.3

(1) *If E has property (α^-), each O_T belongs to $\gamma(H, \gamma(H', E))$ and the operator $O : T \to O_T$ is bounded from $\gamma(H\overline{\otimes}H', E)$ into $\gamma(H, \gamma(H', E))$ and we have*

$$\|O\| \leqslant C_\gamma^-,$$

where C_γ^- denotes the Gaussian property (α^-) constant of E.

(2) *If $\dim H = \dim H' = \infty$ and $O : T \to O_T$ defines a bounded operator from $\gamma(H\overline{\otimes}H', E)$ into $\gamma(H, \gamma(H', E))$, then E has property (α^-) and the Gaussian property (α^-) constant of E satisfies*

$$C_\gamma^- \leqslant \|O\|.$$

Proof

(1) Let E have property (α^-). First we show that $O_T \in \gamma(H, \gamma(H', E))$ for all $T \in \gamma(H\overline{\otimes}H', E)$ and that $O : T \mapsto O_T$ maps $\gamma(H\overline{\otimes}H', E)$ boundedly into $\gamma(H, \gamma(H', E))$.

Choose orthonormal bases $(h_m)_{m\geqslant 1}$ and $(h'_n)_{n\geqslant 1}$ for H and H', respectively. Then by property (α^-),

$$\begin{aligned}\mathbb{E}'\left\|\sum_{m=M}^{N} g'_m O_T h_m\right\|^2_{\gamma(H',E)} &= \mathbb{E}'\mathbb{E}''\left\|\sum_{n\geqslant 1}\sum_{m=M}^{N} g'_m g''_n T_{h_m} h'_n\right\|^2\\ &= \lim_{K\to\infty}\mathbb{E}'\mathbb{E}''\left\|\sum_{n=1}^{K}\sum_{m=M}^{N} g'_m g''_n T(h_m\otimes h'_n)\right\|^2\\ &\leqslant (C_\gamma^-)^2\lim_{K\to\infty}\mathbb{E}\left\|\sum_{n=1}^{K}\sum_{m=M}^{N} g_{mn} T(h_m\otimes h'_n)\right\|^2. \end{aligned} \tag{3.1}$$

Notice that the square expectations on the right-hand side increase with K by Kahane's contraction principle. Since by assumption we have $T\in\gamma(H\overline{\otimes}H',E)$, the sum $\sum_{m,n\geqslant 1} g_{mn}\,T(h_m\otimes h'_n)$ converges in $L^2(\Omega;E)$. It follows that the right-hand side of Eq. 3.1 tends to 0 as $M,N\to\infty$. This shows that the sum $\sum_{m\geqslant 1} g'_m O_T h_m$ converges in $L^2(\Omega',\gamma(H',E))$. Hence, $O_T\in\gamma(H,\gamma(H',E))$. Moreover,

$$\begin{aligned}\|O_T\|^2_{\gamma(H,\gamma(H',E))} &= \mathbb{E}'\mathbb{E}''\left\|\sum_{m,n\geqslant 1} g'_m g''_n T(h_m\otimes h'_n)\right\|^2\\ &\leqslant (C_\gamma^-)^2\,\mathbb{E}\left\|\sum_{m,n\geqslant 1} g_{mn} T(h_m\otimes h'_n)\right\|^2 = (C_\gamma^-)^2\,\gamma^2(T),\end{aligned}$$

from which it follows that $O: T\mapsto O_T$ is bounded. This proves (1).

(2) Assume that $\dim H = \dim H' = \infty$ and fix orthonormal bases $(h_m)_{m\geqslant 1}$ and $(h'_n)_{n\geqslant 1}$ for H and H', respectively. Choose $N\geqslant 1$ arbitrary and fix vectors $x_{mn}\in E$ for $1\leqslant m,n\leqslant N$. Define $T\in\gamma(H\overline{\otimes}H',E)$ by

$$T(h_m\otimes h'_n) := \begin{cases} x_{mn}, & 1\leqslant m,n\leqslant N,\\ 0, & \text{else.}\end{cases}$$

Then,

$$\gamma^2(T) = \mathbb{E}\left\|\sum_{m,n\geqslant 1} g_{mn} T(h_m\otimes h'_n)\right\|^2 = \mathbb{E}\left\|\sum_{m,n=1}^{N} g_{mn} x_{mn}\right\|^2. \tag{3.2}$$

On the other hand,

$$\begin{aligned}\|O_T\|^2_{\gamma(H,\gamma(H',E))} &= \mathbb{E}'\mathbb{E}''\left\|\sum_{m,n=1}^{N} g'_m g''_n T(h_m\otimes h'_n)\right\|^2\\ &= \mathbb{E}'\mathbb{E}''\left\|\sum_{m,n=1}^{N} g'_m g''_n x_{mn}\right\|^2.\end{aligned} \tag{3.3}$$

By comparing Eqs. 3.2 and 3.3 we see that E has property (α^-) if O is bounded. This proves (2). □

With an operator $T \in \gamma(H, \gamma(H', E))$ we may associate a linear map $\tau_T : H \otimes H' \to E$ by

$$\tau_T(h \otimes h') := (Th)h'.$$

By reversing the estimates in the proof of Theorem 3.3 we obtain:

Theorem 3.4

(1) *If E has property (α^+), each τ_T extends to an element of $\gamma(H\overline{\otimes}H', E)$ and the operator $\tau : T \to \tau_T$ is bounded from $\gamma(H, \gamma(H', E))$ into $\gamma(H\overline{\otimes}H', E)$ and we have*

$$\|\tau\| \leqslant C_\gamma^+,$$

where C_γ^+ denotes the Gaussian property (α^+) constant of E.

(2) *If $\dim H = \dim H' = \infty$ and $\tau : T \to \tau_T$ defines a bounded operator from $\gamma(H, \gamma(H', E))$ into $\gamma(H\overline{\otimes}H', E)$, then E has property (α^+) and the Gaussian property (α^+) constant of E satisfies*

$$C_\gamma^+ \leqslant \|\tau\|.$$

Noting that the maps O and τ are inverse to each other we recover the following result from [11]:

Corollary 3.5

(1) *If E has property (α), the operators $O : \gamma(H\overline{\otimes}H', E) \to \gamma(H, \gamma(H', E))$ and $\tau : \gamma(H, \gamma(H', E)) \to \gamma(H\overline{\otimes}H', E)$ are isomorphisms and $\tau = O^{-1}$.*

(2a) *If $\dim H = \dim H' = \infty$ and O defines an isomorphism from $\gamma(H\overline{\otimes}H', E)$ into $\gamma(H, \gamma(H', E))$, then E has property (α) and O is surjective.*

(2b) *If $\dim H = \dim H' = \infty$ and τ defines an isomorphism from $\gamma(H, \gamma(H', E))$ into $\gamma(H\overline{\otimes}H', E)$, then E has property (α) and τ is surjective.*

4 Γ-Measurable Extensions

Let Γ be a Gaussian Radon measure on E with reproducing kernel Hilbert space (RKHS) (i_Γ, H_Γ). Thus H_Γ is the completion of the range of Q_Γ, the covariance operator of Γ, with respect to the inner product $(Q_\Gamma x^*, Q_\Gamma y^*) \mapsto \langle Q_\Gamma x^*, y^*\rangle$ and $i_\Gamma : H_\Gamma \hookrightarrow E$ is the inclusion operator. We recall that $Q_\Gamma = i_\Gamma i_\Gamma^*$, where we identify H_Γ and its dual.

Let $\mathscr{E}(E)$, $\mathscr{B}(E)$, and $\mathscr{B}_\Gamma(E)$ denote respectively the σ-algebra in E generated by E^*, the Borel σ-algebra of E, and the completion of $\mathscr{B}(E)$ with respect to Γ. A Γ-*measurable set* is a set in $\mathscr{B}_\Gamma(E)$. We will need the following well-known fact, cf. [2, Theorems 2.4.7 and 3.6.1].

Proposition 4.1 *H_Γ coincides with the intersection of all Γ-measurable subspaces E_0 of E satisfying $\Gamma(E_0) = 1$. Furthermore, $\Gamma(\overline{H_\Gamma}) = 1$, where the closure is taken with respect to the norm of E.*

Following [2, Definitions 2.10.1, 3.7.1], a mapping $\phi : E \to F$, where F is another Banach space, is said to be Γ-*measurable* if it is equal Γ-almost everywhere to a $\mathscr{B}_\Gamma(E)/\mathscr{E}(F)$ measurable mapping $\tilde{\phi} : E \to F$. A Γ-*measurable linear functional on* E is a mapping $\phi : E \to \mathbb{R}$ that is equal Γ-almost everywhere to a linear $\mathscr{B}_\Gamma(E)$ measurable mapping $\tilde{\phi} : E \to \mathbb{R}$.

Every $h_0 \in H_\Gamma$ induces a bounded linear map from H_Γ to $\mathbb{R}$ by $h \mapsto [h, h_0]_{H_\Gamma}$. It is well-known [2, Section 2.10] that this map admits an extension to a Γ-measurable linear functional $\overline{h_0}$ on E, and this extension is Γ-essentially unique in the sense that any two Γ-measurable linear extensions of h_0 agree Γ-almost everywhere; notice that we implicitly identify H_Γ with its image in E under i_Γ. As a random variable on the probability space (E, Γ), $\overline{h_0}$ is centred Gaussian with variance $\mathbb{E}\,(\overline{h_0})^2 = \|h_0\|^2_{H_\Gamma}$. Furthermore, if $h_0, \ldots, h_N$ are orthonormal in H_Γ, then the random variables $\overline{h_0}, \ldots, \overline{h_0}$ are independent.

The following result extends this to γ-radonifying operators from H_Γ into a Banach space F. For Hilbert spaces F, the implication (1)⇒(2) is due to Feyel and de la Pradelle [8]; see also [2, Theorem 3.7.6].

Theorem 4.2 *Let F be a real Banach space. For a bounded linear operator $T : H_\Gamma \to F$ the following assertions are equivalent:*

(1) *$T \in \gamma(H_\Gamma, F)$;*
(2) *T admits an extension to a Γ-measurable linear mapping $\overline{T} : E \to F$.*

In this situation we have $\overline{T} \in L^2(E, \Gamma; F)$ and

$$\int_E \|\overline{T}x\|^2 \, d\Gamma(x) = \gamma^2(T).$$

Moreover, the image measure $\overline{T}\Gamma$ is a Gaussian Radon measure on F with covariance operator TT^*. Finally, if $\overline{\overline{T}}$ is another Γ-measurable linear extension of T, then $\overline{\overline{T}} = \overline{T}$ Γ-almost surely.

Proof Fix an orthonormal basis $(h_n)_{n\geqslant 1}$ for H_Γ.
(1)⇒(2) The series $\sum_{n\geqslant 1} \overline{h_n}x\, Th_n$ converges for Γ-almost all $x \in E$. Indeed, this follows from the Itô-Nisio theorem, the observation that the sequence $(\overline{h_n})_{n\geqslant 1}$ is orthogaussian on the probability space (E, Γ), and the fact that T is γ-radonifying. Since each term in the series is equal Γ-almost everywhere to a linear Γ-measurable function on E, it follows that there exists a Γ-measurable subspace E_0 of E of full Γ-measure on which the series converges pointwise. We define $\overline{T}$ on E_0 to be its sum and extend $\overline{T}$ in a linear way to all of E by choosing a linear subspace Y of E such that E is the algebraic direct sum of E_0 and Y and putting $\overline{T}(x + y) := \overline{T}x$ for $x \in E_0$ and $y \in Y$ (cf. the remark after [2, Definition 2.10.1]). The resulting map $\overline{T} : E \to F$ is linear, Γ-measurable, and extends T.

(2)⇒(1) First we claim that

$$\overline{T}x = \sum_{n\geqslant 1} \overline{h_n}x\, Th_n \quad \text{for } \Gamma\text{-almost all } x \in E; \tag{4.1}$$

this will also settle the uniqueness part.

Fix $y^* \in F^*$ arbitrary. From $\sum_{n\geqslant 1}\langle Th_n, y^*\rangle^2 = \|T^*y^*\|^2_{H_\Gamma} < \infty$ it follows that the Gaussian series $\sum_{n\geqslant 1}\overline{h_n}x\,\langle Th_n, y^*\rangle$ converges in $L^2(E,\Gamma)$ and, by the Itô-Nisio theorem, Γ-almost surely. Since each term in the series is equal Γ-almost everywhere to a Γ-measurable linear function on E, there exists a Γ-measurable subspace E_0 of full Γ-measure on which the series converges pointwise. Define $\overline{T_{y^*}} : E_0 \to \mathbb{R}$ by

$$\overline{T_{y^*}}x := \sum_{n\geqslant 1}\overline{h_n}x\,\langle Th_n, y^*\rangle$$

and extend this definition to all of E as in the previous step. Since $\Gamma(E_0) = 1$ we have $H_\Gamma \subseteq E_0$ by Proposition 4.1. Noting that for $h \in H_\Gamma$ we have

$$\overline{T_{y^*}}h = \sum_{n\geqslant 1}\overline{h_n}h\,\langle Th_n, y^*\rangle = \sum_{n\geqslant 1}[h_n, h]_{H_\Gamma}\,[h_n, T^*y^*]_{H_\Gamma} = \langle Th, y^*\rangle$$

it follows that both $x \mapsto \overline{T_{y^*}}x$ and $x \mapsto \langle\overline{T}x, y^*\rangle$ are Γ-measurable linear extensions of $h \mapsto \langle Th, y^*\rangle$. Hence by Γ-essential uniqueness it follows that $\langle\overline{T}x, y^*\rangle = \overline{T_{y^*}}x$ for Γ-almost all $x \in E$. We conclude that

$$\sum_{n\geqslant 1}\overline{h_n}x\,\langle Th_n, y^*\rangle = \langle\overline{T}x, y^*\rangle$$

Γ-almost all $x \in E$. Since this holds for all $y \in F^*$, the Itô-Nisio theorem now implies the claim.

Since $(\overline{h}_n)_{n\geqslant 1}$ is orthogaussian, Proposition 3.2 implies that $T \in \gamma(H_\Gamma, E)$.

It follows from the representation (4.1) that $\overline{T}$ is Gaussian as a random variable on (E,Γ). Hence $\overline{T} \in L^2(E,\Gamma; F)$ by Fernique's theorem, and the orthogaussianity of $(\overline{h_n})_{n\geqslant 1}$ implies that

$$\int_E \|\overline{T}x\|^2\, d\Gamma(x) = \int_E \left\|\sum_{n\geqslant 1}\overline{h_n}x\, Th_n\right\|^2 d\Gamma(x) = \gamma^2(T).$$

Since T is γ-radonifying, it follows from Proposition 3.2 that $T\Gamma_H = \overline{T}i_\Gamma\Gamma_H = \overline{T}\Gamma$ is a Gaussian Radon measure on F with covariance TT^*. □

5 Representability and Orbits

Let $(M, \mathscr{M}, \mu)$ be a fixed separable measure space. Recall that this means that there exists a countable family of sets of finite μ-measure generating the underlying σ-algebra $\mathscr{M}$ of M, or equivalently, that $L^2(M)$ is separable. For notational convenience we shall always write $L^2 := L^2(M)$ and $L^2_F := L^2(M; F)$ when F is a Hilbert space or a Banach space.

In this section we will apply the results of the previous section to the special case $H' = L^2$. We will use the simple fact that the Hilbert space tensor product $H\overline{\otimes}L^2$ can be identified in a natural way with the space L^2_H.

A function $\Phi : M \to \mathscr{L}(H, E)$ will be called *weakly* L^2_H if $\Phi^* x^* \in L^2_H$ for all $x^* \in E^*$. Here, $(\Phi^* x^*)(t) := \Phi^*(t)x^*$. Such a function *represents* an operator $T \in \gamma(L^2_H, E)$ if Φ is weakly L^2_H and for all $x^* \in E^*$ we have $T^* x^* = \Phi^* x^*$ in L^2_H. Note that T is uniquely determined by Φ. In the converse direction, if both Φ and $\tilde{\Phi}$ represent T, then $\Phi^* x^* = \tilde{\Phi}^* x^*$ in L^2_H for all $x^* \in E^*$.

If Φ represents T, then for all $f \in L^2_H$ and $x^* \in E^*$ we have

$$\langle Tf, x^* \rangle = \int_M \langle \Phi f, x^* \rangle \, d\mu.$$

As a result, the function Φf is Pettis integrable and $Tf = \int_M \Phi f \, d\mu$. In other words, T is a Pettis integral operator with 'kernel' Φ. The idea to study functions through their associated integral operators was introduced in [11].

For $H = \mathbb{R}$ we identify $\mathscr{L}(\mathbb{R}, E)$ with E in the canonical way. Under this identification, a function $\phi : M \to E$ represents an operator $T \in \gamma(L^2, E)$ if for all $x^* \in E^*$ we have $\langle \phi, x^* \rangle \in L^2$ and $T^* x^* = \langle \phi, x^* \rangle$ in L^2. The proof of the following observation is left to the reader:

Proposition 5.1 *If* $\Phi : M \to \mathscr{L}(H, E)$ *represents* $T \in \gamma(L^2_H, E) \simeq \gamma(H\overline{\otimes}L^2, E)$*, then for all* $h \in H$ *the function* $\Phi h : M \to E$ *represents* $T_h \in \gamma(L^2, E)$*.*

The orbits of a function $\Psi : M \to \mathscr{L}(E)$ define functions $\Psi x : M \to E$. In this section we shall combine the above ideas with the results of the previous two sections to study the following question: given a Gaussian Radon measure Γ on E such that $\Psi \circ i_\Gamma$ represents an operator in $\gamma(L^2_{H_\Gamma}, E)$, do the orbits Ψx represent operators in $\gamma(L^2, E)$ and vice versa?

Theorem 5.2 *Suppose* E *has property* (α^-) *and let* Γ *be a Gaussian Radon measure on* E *with RKHS* (i_Γ, H_Γ)*. If* $\Psi \circ i_\Gamma$ *represents an operator* T_Γ *in* $\gamma(L^2_{H_\Gamma}, E)$ *then* Γ*-almost every orbit* Ψx *represents an element* T_x *of* $\gamma(L^2, E)$*, and*

$$\int_E \gamma^2(T_x) \, d\Gamma(x) \leqslant (C_\gamma^-)^2 \gamma^2(T_\Gamma).$$

Proof Let T be the operator in $\gamma(L^2_{H_\Gamma}, E)$ represented by $\Psi \circ i_\Gamma$. By Theorem 3.3, $O_T : H_\Gamma \to \gamma(L^2, E)$ is γ-radonifying. Therefore it has a Γ-essentially unique Γ-measurable linear extension $\overline{O_T} : E \to \gamma(L^2, E)$. We will show that for Γ-almost all $x \in E$ the orbit Ψx represents the operator $\overline{O_T} x$.

Fix an orthonormal basis $(h_n)_{n \geqslant 1}$ for H_Γ. By Theorem 4.2, for Γ-almost all $x \in E$ we have

$$\overline{O_T} x = \sum_{n \geqslant 1} \overline{h_n} x \, O_T h_n. \tag{5.1}$$

Also, for Γ-almost all $x \in E$ we have

$$x = \sum_{n \geqslant 1} \overline{h_n} x \, i_\Gamma h_n. \tag{5.2}$$

This can be derived from Theorem 4.2 applied to the γ-radonifying operator i_Γ, or by more direct arguments based on the Karhunen–Loève expansion of E-valued Gaussian variables.

Fix any $x \in E$ for which both Eqs. 5.1 and 5.2 hold. By Proposition 5.1,

$$(\overline{O_T x})^* x^* = \sum_{n\geqslant 1} \overline{h_n}x\,(O_T h_n)^* x^* = \sum_{n\geqslant 1} \overline{h_n}x\, T_{h_n}^* x^* = \sum_{n\geqslant 1} \overline{h_n}x\,\langle(\Psi \circ i_\Gamma)h_n, x^*\rangle$$
$$= \sum_{n\geqslant 1} \overline{h_n}x\,\langle i_\Gamma h_n, \Psi^* x^*\rangle = \langle x, \Psi^* x^*\rangle = \langle \Psi x, x^*\rangle$$

with all identities in the sense of L^2. This proves the first part of the theorem, with $T_x = \overline{O_T}x$. The second part follows from this by using the identity of Theorem 4.2 and then Theorem 3.3:

$$\int_E \gamma^2(\overline{O_T}x)\,d\Gamma(x) = \int_E \|\overline{O_T}x\|^2_{\gamma(L^2,E)}\,d\Gamma(x)$$
$$= \|O_T\|^2_{\gamma(H_\Gamma,\gamma(L^2,E))} \leqslant (C_\gamma^-)^2 \|T\|^2_{\gamma(L^2_{H_\Gamma},E)} = (C_\gamma^-)^2\gamma^2(T). \quad \square$$

In the opposite direction we have the following result.

Theorem 5.3 *Suppose E has property (α^+) and let Γ be a Gaussian Radon measure on E with RKHS (i_Γ, H_Γ). If Γ-almost every orbit Ψx represents an operator T_x in $\gamma(L^2, E)$, then $\Psi \circ i_\Gamma$ represents an element T_Γ of $\gamma(L^2_{H_\Gamma}, E)$ and*

$$\gamma^2(T_\Gamma) \leqslant (C_\gamma^+)^2 \int_E \gamma^2(T_x)\,d\Gamma(x) < \infty.$$

Proof Let $T : H_\Gamma \to \gamma(L^2, E)$ be defined by $Th := T_h$ (more accurately, $Th := T_{i_\Gamma h}$, but as before we identify H_Γ with its image in E under i_Γ). The subspace E_0 of E consisting of all $x \in E$ for which Φx represents an element of $\gamma(L^2, E)$ is linear and by assumption we have $\Gamma(E_0) = 1$. Proposition 4.1 implies that $H_\Gamma \subseteq E_0$. Defining $\overline{T} : E_0 \to \gamma(L^2, E)$ by $\overline{T}x := T_x$, and by extending $\overline{T}$ to a linear mapping on E as in the proof of Theorem 4.2, we obtain a Γ-measurable linear extension of T to E which belongs to $L^2(E, \Gamma; \gamma(L^2, E))$, and therefore Theorem 4.2 shows that $T \in \gamma(H_\Gamma, \gamma(L^2, E))$. Since E has property (α^+), Theorem 3.4 allows us to identify T with an element S of $\gamma(L^2_{H_\Gamma}, E)$. It remains to verify that this element S is represented by $\Psi \circ i_\Gamma$. For this we need to check that for all $x^* \in E^*$ we have $S^* x^* = i_\Gamma^* \Psi^* x^*$ in $L^2_{H_\Gamma}$. Using the notation of Theorem 3.4, for all $g \in L^2$ and $x^*, y^* \in E^*$ we have

$$\langle S(g \otimes i_\Gamma^* y^*), x^*\rangle = \langle \tau_T(g \otimes i_\Gamma^* y^*), x^*\rangle = \langle T(i_\Gamma^* y^*)g, x^*\rangle = \langle T_{i_\Gamma i_\Gamma^* y^*} g, x^*\rangle$$
$$= \int_M g\,\langle \Psi i_\Gamma i_\Gamma^* y^*, x^*\rangle\,d\mu = \int_M [g \otimes i_\Gamma^* y^*, i_\Gamma^* \Psi^* x^*]_{H_\Gamma}\,d\mu.$$

This proves the result for all functions in $L^2_{H_\Gamma}$ of the form $g \otimes i_\Gamma^* y^*$. Since these span a dense subspace of $L^2_{H_\Gamma}$, it follows that $i_\Gamma^* \Psi^* x^* \in L^2_{H_\Gamma}$ and $\|i_\Gamma^* \Psi^* x^*\|_{L^2_{H_\Gamma}} \leqslant \|S\|_{\mathscr{L}(L^2_{H_\Gamma},E)}\|x^*\|$. The general case follows by approximation. This proves the first part of the theorem.

The left-hand side inequality in the second part is proved as in Theorem 5.2. The right-hand side inequality expresses the fact, already observed, that $\overline{T} \in L^2(E, \Gamma; \gamma(L^2, E))$. □

If Ψx represents an element T_x of $\gamma(L^2, E)$ for all $x \in E_0$, where E_0 is a subset of the second category in E, then a closed graph argument shows that the map $x \mapsto T_x$ is bounded and we obtain a simpler direct proof of Theorem 5.3.

Recall from Proposition 4.1 that for any Gaussian Radon measure Γ on E we have $\Gamma(\overline{H_\Gamma}) = 1$. Thus, in Theorem 5.3 it is enough to consider the orbits Ψx with $x \in \overline{H_\Gamma}$. In general, the conditions of Theorem 5.2 do not imply that Ψx represents an element of $\gamma(L^2, E)$ for all $x \in \overline{H_\Gamma}$, however; a counterexample is given at the end of the paper.

A family of operators $\mathscr{S} \subseteq \mathscr{L}(E)$ is called *γ-bounded* if there exists a constant $C \geqslant 0$ such that for all finite sequences $(S_n)_{n=1}^N \subseteq \mathscr{S}$ and $(x_n)_{n=1}^N \subseteq E$ we have

$$\mathbb{E}\left\|\sum_{n=1}^N g_n\, S_n x_n\right\|^2 \leqslant C^2 \mathbb{E}\left\|\sum_{n=1}^N g_n x_n\right\|^2.$$

The concept of *R-boundedness* is defined similarly by replacing the Gaussian variables by Rademacher variables. By a simple randomization argument, every R-bounded family is γ-bounded, and the converse is true in spaces with finite cotype. An overview of examples of R-bounded families (and thus of γ-bounded families) is presented in [5, 12].

Let $\Psi : M \to \mathscr{L}(E)$ be a function with the property that $\Psi x : M \to E$ represents an element of $\gamma(L^2, E)$ for all $x \in E$. By the remarks at the beginning of Section 5, for each $g \in L^2$ we may define an operator $\Psi(g) \in \mathscr{L}(E)$ by

$$\Psi(g)x := \int_M g\, \Psi x \, d\mu,$$

where the integral is defined as a Pettis integral. For spaces with property (α) and under somewhat stronger assumptions on Ψ, the following result was obtained independently by Haak [9, Korollar 3.7.9] with a different proof. It generalizes a result of Le Merdy [15] for L^p-spaces. It gives a necessary condition in order that Ψx represent an element of $\gamma(L^2, E)$ for all $x \in E$.

Theorem 5.4 *Let E have property (α^+) and let $\Psi : M \to \mathscr{L}(E)$ be a function with the property that Ψx represents an element T_x of $\gamma(L^2, E)$ for all $x \in E$. Then the family $\{\Psi(g) : \|g\|_{L^2} \leqslant 1\}$ is γ-bounded.*

Proof The mapping $x \mapsto T_x$ is closed. To prove this, suppose that $\lim_{n\to\infty} x_n = x$ in E and $\lim_{n\to\infty} T_{x_n} = T$ in $\gamma(L^2, E)$. We have to show that $T = T_x$, i.e., that Ψx represents T. For all $x^* \in E^*$ we have $\lim_{n\to\infty}\langle \Psi x_n, x^*\rangle = \lim_{n\to\infty} T^*_{x_n} x^* = T^* x^*$ with convergence in L^2. For each $x^* \in E^*$ we can pass to a μ-almost everywhere convergent subsequence and conclude that $\langle \Psi x, x^*\rangle = \lim_{k\to\infty}\langle \Psi x_{n_k}, x^*\rangle = T^* x^*$ μ-almost everywhere on M. But this means that Ψx represents T and the claim is proved. By the closed graph theorem, there exists a constant $K \geqslant 0$ such that $\|T_x\|_{\gamma(L^2,E)} \leqslant K\|x\|$ for all $x \in E$.

Next we observe that if α_{ij}, $i \geqslant 1$, $j = 1, \ldots, N$, are real numbers satisfying $\sum_{i\geqslant 1} \alpha_{ij}^2 \leqslant 1$ $(j = 1, \ldots, N)$, then for all $x_{ij} \in \mathbb{E}$, $i \geqslant 1$, $j = 1, \ldots, N$, we have

$$\mathbb{E}\left\|\sum_{i\geqslant 1}\sum_{j=1}^{N}\alpha_{ij}g_j x_{ij}\right\|^2 \leqslant \mathbb{E}\left\|\sum_{i\geqslant 1}\sum_{j=1}^{N} g_{ij}x_{ij}\right\|^2. \tag{5.3}$$

Indeed, this follows from Anderson's inequality [1], [2, Corollary 3.3.7], noting that for all $x^* \in E^*$ we have

$$\begin{aligned}\mathbb{E}\left(\sum_{i\geqslant 1}\sum_{j=1}^{N}\alpha_{ij}g_j\langle x_{ij}, x^*\rangle\right)^2 &= \sum_{j=1}^{N}\left(\sum_{i\geqslant 1}\alpha_{ij}\langle x_{ij}, x^*\rangle\right)^2 \\ &\leqslant \sum_{j=1}^{N}\left(\sum_{i\geqslant 1}\alpha_{ij}^2\right)\left(\sum_{i=1}^{\infty}\langle x_{ij}, x^*\rangle^2\right) \leqslant \sum_{j=1}^{N}\sum_{i\geqslant 1}\langle x_{ij}, x^*\rangle^2 = \mathbb{E}\left(\sum_{j=1}^{N}\sum_{i\geqslant 1} g_{ij}\langle x_{ij}, x^*\rangle\right)^2.\end{aligned}$$

Let $(f_i)_{i\geqslant 1}$ be an orthonormal basis for L^2 and pick $\varphi_j \in L^2$ of norm $\leqslant 1$. With $\alpha_{ij} := [f_i, \varphi_j]_{L^2}$, by Eq. 5.3 and property (α^+) we obtain

$$\begin{aligned}\mathbb{E}\left\|\sum_{j=1}^{N} g_j\Psi(\varphi_j)x_j\right\|^2 &= \mathbb{E}\left\|\sum_{i\geqslant 1}\sum_{j=1}^{N} g_j[f_i, \varphi_j]_{L^2}\Psi(f_i)x_j\right\|^2 \leqslant \mathbb{E}\left\|\sum_{i\geqslant 1}\sum_{j=1}^{N} g_{ij}\Psi(f_i)x_j\right\|^2 \\ &= \mathbb{E}\left\|\sum_{i\geqslant 1}\sum_{j=1}^{N} g_{ij}\int_M f_i\,\Psi x_j\,d\mu\right\|^2 \leqslant (C_\gamma^+)^2\mathbb{E}\,\mathbb{E}'\left\|\sum_{i\geqslant 1} g_i\int_M f_i\,\Psi\sum_{j=1}^{N} g_j x_j\,d\mu\right\|^2 \\ &= (C_\gamma^+)^2\mathbb{E}\,\mathbb{E}'\left\|\sum_{i\geqslant 1} g_i T_{\sum_{j=1}^{N} g_j x_j} f_i\right\|^2 = (C_\gamma^+)^2\mathbb{E}\left\|T_{\sum_{j=1}^{N} g_j x_j}\right\|_{\gamma(L^2, E)}^2 \\ &\leqslant (C_\gamma^+)^2 K^2\,\mathbb{E}\left\|\sum_{j=1}^{N} g_j x_j\right\|^2,\end{aligned}$$

where C_γ^+ is the Gaussian property (α^+) constant of E. □

6 Stochastic Integration

The space $\gamma(L_H^2, E)$ provides the natural setting for the theory of stochastic integration of functions with values in $\mathscr{L}(H, E)$. Before we make this statement precise we first recall some results from [18].

A *random Gaussian measure on* $(M, \mathscr{M}, \mu)$ is an $L^2(\Omega)$-valued measure W on $\mathscr{M}$ with the following properties:

(1) For every $A \in \mathscr{M}$ the random variable $W(A)$ is centred Gaussian;
(2) For every disjoint pair $A, B \in \mathscr{M}$, the random variables $W(A)$ and $W(B)$ are independent;
(3) For every pair $A, B \in \mathscr{M}$ we have $\mathbb{E}\,(W(A)W(B)) = \mu(A \cap B)$.

If M is a finite or infinite time interval in $\mathbb{R}_+$, then W is just a Brownian motion. If M is a finite or infinite rectangle in $\mathbb{R}^2_+$, then W is a Brownian sheet.

Let H be a separable real Hilbert space. A *H-cylindrical random Gaussian measure on* $(M, \mathscr{M}, \mu)$ is a family $W_H = \{W_H h\}_{h\in H}$ of random Gaussian measures on $\mathscr{M}$ such that

$$\mathbb{E}\,(W_H h(A)\, W_H g(A)) = [h, g]_H\, \mu(A \cap B) \qquad \forall\, h, g \in H, \quad A, B \in M.$$

If M is a finite or infinite interval in $\mathbb{R}_+$, then W_H is usually called a *H-cylindrical Brownian motion*.

We can define a stochastic integral of L^2_H-functions f with respect to W_H as follows. For step functions $f = \sum_{j=1}^n 1_{A_j} \otimes h_j$ we set

$$\int_M f\, dW := \sum_{j=1}^n W_H h_j(A_j).$$

Noting that

$$\mathbb{E}\left\| \int_M f\, dW_H \right\|^2 = \int_M \|f\|_H^2\, d\mu$$

we extend the definition to arbitrary $f \in L^2_H$ by an approximation argument.

A function $\Phi : M \to \mathscr{L}(H, E)$ is called *stochastically integrable* with respect to a H-cylindrical random Gaussian measure W_H if Φ is weakly L^2_H and for every set $A \in \mathscr{M}$ there exists an E-valued random variable Y_A such that for all $x^* \in E^*$ we have

$$\langle Y_A, x^* \rangle = \int_A \Phi^* x^*\, dW_H.$$

In this situation, Y_A is uniquely defined up to a null set and we define $Y_A = \int_A \Phi\, dW_H$. It is shown in [18] that Φ is stochastically integrable with respect to W_H if and only if Φ represents an element of $\gamma(L^2_H, E)$.

Theorem 6.1 *Let W_H be a H-cylindrical Brownian motion and let $J_{W_H} : L^2_H \to L^2(\Omega; \mathbb{P})$ denote the Itô isometry:*

$$J_{W_H} f := \int_M f\, dW_H.$$

There exists a unique bounded operator $J^E_{W_H}$ from $\gamma(L^2_H, E)$ into $L^2(\Omega; E)$ which makes the following diagram commute for every $x^* \in E^*$:

$$\begin{array}{ccc} \gamma(L^2_H, E) & \xrightarrow{J^E_{W_H}} & L^2(\Omega; E) \\ {\scriptstyle x^*}\downarrow & & \downarrow{\scriptstyle x^*} \\ L^2_H & \xrightarrow[J_{W_H}]{} & L^2(\Omega, \mathbb{P}) \end{array}$$

This operator $J^E_{W_H}$ is an isometry.

Here the left vertical arrow represents the mapping $T \mapsto T^* x^*$.

Proof Suppose first that $T \in \gamma(L^2_H, E)$ is represented by a function Φ. By [18] there exists a unique random variable $J^E_{W_H} T \in L^2(\Omega; E)$ with the property that

$$\langle J^E_{W_H} T, x^* \rangle = \int_M \Phi^* x^* \, dW_H \qquad \forall x^* \in E^*$$

and we have $\|J^E_{W_H} T\|_{L^2(\Omega; E)} = \gamma(T)$. By an easy approximation argument one sees that the representable operators are dense in $\gamma(L^2_H, E)$. Therefore $J^E_{W_H}$ extends uniquely to an isometry from $\gamma(L^2_H, E)$ into $L^2(\Omega; E)$. In view of the identity

$$\int_M \Phi^* x^* \, dW_H = \int_M T^* x^* \, dW_H = J_{W_H} T^* x^*,$$

the operator $J^E_{W_H}$ has the required properties. □

The following theorem is an abstract version of the identity in [18, Theorem 4.3] with a somewhat simplified proof.

Proposition 6.2 *Let $T \in \gamma(L^2_H, E)$ and let $(h_n)_{n \geqslant 1}$ be an orthonormal basis for H. Then,*

$$\sum_{n \geqslant 1} J^E_{W_H h_n} T_{h_n} = J^E_{W_H} T,$$

where the sum converges almost surely and unconditionally in $L^2(\Omega; E)$.

Proof For $n \geqslant m \geqslant 1$ let π_{mn} denote the orthogonal projection in H onto the span of $\{h_m, \ldots, h_n\}$. For a function $f \in L^2_H$ we let $(\pi_{mn} f)(t) := \pi_{mn}(f(t))$.

For $f \in L^2_H$ of the form $1_A \otimes h_k$ we have

$$\sum_{j=m}^{n} \int_M [f, h_j]_H \, dW_H h_j = W_H h_k(A) = \int_M f \, dW_H = \int_M \pi_{mn} f \, dW_H$$

if $m \leqslant k \leqslant n$, and

$$\sum_{j=m}^{n} \int_M [f, h_j]_H \, dW_H h_n = 0 = \int_M \pi_{mn} f \, dW_H$$

otherwise. The span of all such f being dense in L^2_H, it follows that

$$\sum_{j=m}^{n} \int_M [f, h_j]_H \, dW_H h_j = \int_M \pi_{mn} f \, dW_H \qquad \forall f \in L^2_H.$$

Since $\lim_{m,n\to\infty} \pi_{mn} f = 0$ in L^2_H for all $f \in L^2_H$, it follows that

$$\sum_{j\geqslant 1} \int_M [f, h_j]_H \, dW_H h_j = \int_M f \, dW_H \qquad \forall f \in L^2_H,$$

where the sum converges in $L^2(\Omega, \mathbb{P})$. Using this identity, for all $x^* \in E^*$ we obtain

$$\begin{aligned} \langle J^E_{W_H} T, x^* \rangle &= \int_M T^* x^* \, dW_H = \sum_{j\geqslant 1} \int_M [T^* x^*, h_j]_H \, dW_H h_j \\ &= \sum_{j\geqslant 1} \int_M T^*_{h_j} x^* \, dW_H h_j = \sum_{j\geqslant 1} \langle J^E_{W_H h_j} T_{h_j}, x^* \rangle. \end{aligned}$$

Hence, by the Itô-Nisio theorem,

$$\sum_{j\geqslant 1} J^E_{W_H h_j} T_{h_j} = J^E_{W_H} T$$

where the sum converges almost surely and in $L^2(\Omega; E)$. The $L^2(\Omega; E)$ convergence is unconditional by observing that every permutation of $(h_n)_{n\geqslant 1}$ is again an orthonormal basis for H. □

Alternatively, a proof avoiding the use of the Itô-Nisio theorem could be based on [18, Proposition 6.1] and the vector-valued martingale convergence theorem.

Finally we consider the space $\gamma(L^2_{H_\Gamma}, E)$, where Γ is a centred Gaussian Radon measure on E with RKHS (i_Γ, H_Γ). We will show that this space provides the natural setting for integration of $\mathscr{L}(E)$-valued function with respect to an *E-valued random Gaussian measure on* $(M, \mathscr{M}, \mu)$ *with distribution* Γ, by which we mean a measure W_Γ on $\mathscr{M}$ with values in $L^2(\Omega; E)$ such that the $L^2(\Omega)$-valued measures $\langle W_\Gamma, x^* \rangle$ are random Gaussian measures on M satisfying

$$\mathbb{E} \langle W_\Gamma(A), x^* \rangle \langle W_\Gamma(B), y^* \rangle = \langle Q_\Gamma x^*, y^* \rangle \, \mu(A \cap B) \qquad \forall x^*, y^* \in E^*, \quad A, B \in \mathscr{M},$$

where $Q_\Gamma \in \mathscr{L}(E^*, E)$ is the covariance of the Gaussian Radon measure Γ. For a step function $\Psi : M \to \mathscr{L}(E)$ of the form $\Psi = \sum_{j=1}^n 1_{A_j} \otimes U_j$, with the $A_j \in \mathscr{M}$ disjoint and $U_j \in \mathscr{L}(E)$, we define

$$\int_M \Psi \, dW_\Gamma := \sum_{j=1}^{n} U_j(W_\Gamma(A_j)). \tag{6.1}$$

Let (i_Γ, H_Γ) denote the RKHS associated with Q_Γ and let W_{H_Γ} denote the H_Γ-cylindrical random Gaussian measure canonically associated with W_Γ via

$$W_{H_\Gamma} i^*_\Gamma y^* := \langle W_\Gamma, y^* \rangle \qquad (y^* \in E^*).$$

In view of the identity $\mathbb{E}(W_{H_\Gamma} i_\Gamma^* y^*)^2 = \mathbb{E}\langle W_\Gamma, y^*\rangle^2 = \|i_\Gamma x^*\|_{H_\Gamma}^2$, this defines W_{H_Γ} uniquely as a bounded operator from H_Γ to $L^2(\Omega)$. For all $x^* \in E^*$ we have

$$\left\langle \int_M \Psi \, dW_\Gamma, x^* \right\rangle = \sum_{j=1}^n W_\Gamma i_\Gamma^* U_j^* x^*(A_j) = \left\langle \int_M \Psi \circ i_\Gamma \, dW_{H_\Gamma}, x^* \right\rangle.$$

We call an element $T \in \gamma(L^2_{H_\Gamma}, E)$ *representable on* E if there exists a function $\Psi : M \to \mathscr{L}(E)$ such that the function $\Psi \circ i_\Gamma : M \to \mathscr{L}(H_\Gamma, E)$ represents T. If Ψ is an $\mathscr{L}(E)$-valued function representing an element $T \in \gamma(L^2_{H_\Gamma}, E)$ on E, we may define the *stochastic integral* of Ψ with respect to W_Γ by

$$\int_M \Psi \, dW_\Gamma := \int_M \Psi \circ i_\Gamma \, dW_{H_\Gamma}.$$

For step functions, this definition is consistent with the one in Eq. 6.1. Note that under these assumptions we have

$$\mathbb{E} \left\| \int_M \Psi \, dW_\Gamma \right\|^2 = \gamma^2(T).$$

Furthermore, if for some $x \in E$ the function $\Psi x : M \to E$ is stochastically integrable with respect to a random Gaussian measure W, and if Ψx is represented by $T_x \in \gamma(L^2, E)$, then

$$\mathbb{E} \left\| \int_M \Psi x \, dW \right\|^2 = \gamma^2(T_x).$$

Hence, using Theorem 6.1 we may now reformulate Theorems 5.2 and 5.3 as follows.

Theorem 6.3 *Let W be an arbitrary random Gaussian measure on M, let E have property (α^-), and let W_Γ be an E-valued random Gaussian measure on M with distribution Γ. If $\Psi : M \to \mathscr{L}(E)$ is stochastically integrable with respect to W_Γ, then for Γ-almost all $x \in E$ the orbit $\Psi x : M \to E$ is stochastically integrable with respect to W and we have*

$$\int_E \mathbb{E} \left\| \int_M \Psi x \, dW \right\|^2 d\Gamma(x) \leqslant (C_\gamma^-)^2 \, \mathbb{E} \left\| \int_M \Psi \, dW_\Gamma \right\|^2.$$

Theorem 6.4 *Let W be an arbitrary random Gaussian measure on M, let E have property (α^+), and let W_Γ be an E-valued random Gaussian measure on M with distribution Γ. Let $\Psi : M \to \mathscr{L}(E)$ be weakly $L^2_{H_\Gamma}$. If for Γ-almost all $x \in E$ the orbit Ψx of the function $\Psi : M \to \mathscr{L}(E)$ is stochastically integrable with respect to W, then Ψ is stochastically integrable with respect to W_Γ and we have*

$$\mathbb{E} \left\| \int_M \Psi \, dW_\Gamma \right\|^2 \leqslant (C_\gamma^+)^2 \int_E \mathbb{E} \left\| \int_M \Psi x \, dW \right\|^2 d\Gamma(x) < \infty.$$

Together, these theorems prove Theorem 1.1. By considering simple functions it is seen that the properties $(\alpha^\pm)$ cannot be omitted.

7 An Application to Stochastic Linear Evolution Equations

In this final section we sketch a simple application of our results to stochastic linear evolution equations in Banach spaces.

Let A be the infinitesimal generator of a C_0-semigroup $S = \{S(t)\}_{t\geqslant 0}$ of bounded linear operators on a real Banach space E, let H be a separable real Hilbert space and let $B \in \mathscr{L}(H, E)$ be bounded and linear. We consider the following stochastic initial value problem:

$$\begin{aligned} dU(t) &= AU(t)\,dt + B\,dW_H(t), \qquad t \in [0, T], \\ U(0) &= u_0, \end{aligned} \tag{7.1}$$

where W_H is a H-cylindrical Brownian motion on $[0, T]$. A *weak solution* of Eq. 7.1 is a measurable adapted E-valued process $U = \{U(t, u_0)\}_{t\in[0,T]}$ such that the following two conditions are satisfied:

(1) almost surely, $t \mapsto U(t, u_0)$ is integrable on $[0, T]$;
(2) for all $t \in [0, T]$ and $x^* \in D(A^*)$ (the domain of the adjoint operator A^*) we have, almost surely,

$$\langle U(t, u_0), x^*\rangle = \langle u_0, x^*\rangle + \int_0^t \langle U(s, u_0), A^*x^*\rangle\, ds + W_H B^* x^*([0, t]).$$

By the results in [3, 18], the problem (7.1) has a (necessarily unique) weak solution $\{U(t, u_0)\}_{t\in[0,T]}$ if and only if the $\mathscr{L}(H, E)$-valued function $S \circ B$ is stochastically integrable with respect to W_H. If $B \in \gamma(H, E)$, then BB^* is the covariance of a Gaussian measure Γ on E and there exists an E-valued Brownian motion W_Γ such that for all $x^* \in E^*$ we have $W_H B^* x^* = \langle W_\Gamma, x^*\rangle$. The results of the previous section now give the following necessary and sufficient condition for existence of a weak solution:

Theorem 7.1 *Let S be a C_0-semigroup on a real Banach space E with property* (α)*. Assume that $B \in \gamma(H, E)$. With the above notations, the following assertions are equivalent:*

(1) *Problem* 7.1 *admits a unique weak solution on* $[0, T]$*;*
(2) *The semigroup S is stochastically integrable on* $[0, T]$ *with respect to W_Γ;*
(3) *The operator-valued function $S \circ B$ is stochastically integrable on* $[0, T]$ *with respect to W_H;*
(4) *For Γ-almost all $x \in E$, the orbit Sx is stochastically integrable on* $[0, T]$ *with respect to W.*

Here, as always, W a real-valued Brownian motion.

Proof The equivalence (1)$\Leftrightarrow$(2) has been noted above and does not depend on the assumption that $B \in \gamma(H, E)$.

Noting that $BB^* = i_\Gamma i_\Gamma^*$ we obtain a H_Γ-cylindrical Brownian motion W_{H_Γ} by putting $W_{H_\Gamma} i_\Gamma^* x^* := \langle W_\Gamma, x^*\rangle$. By definition, (2) is then equivalent to

(2′) The operator-valued function $S \circ i_\Gamma$ is stochastically integrable on $[0, T]$ with respect to W_{H_Γ}.

Now $S \circ B$ represents an element of $\gamma(L^2_H, E)$ if and only if $S \circ i_\Gamma$ represents an element of $\gamma(L^2_{H_\Gamma}, E)$; this follows from the fact [18] that both conditions are equivalent to the existence of a centred Gaussian Radon measure on E whose covariance operator $R \in \mathscr{L}(E^*, E)$ is given by

$$\langle Rx^*, x^* \rangle = \int_0^T \|(S(t)B)^* x^*\|_H^2 \, ds = \int_0^T \|(S(t)i_\Gamma)^* x^*\|_{H_\Gamma}^2 \, ds.$$

The equivalence (2)⇔(3) follows.

The equivalence (2′)⇔(4) is a direct consequence of Theorems 6.3 and 6.4. □

Notice that the implication (1)⇒(2) uses only property (α^-) and (2)⇒(1) uses only property (α^+).

We now return to Theorem 5.2, where property (α^-) was shown to imply that Γ-almost all orbits represent elements of $\gamma(L^2, E)$. The following semigroup example shows that in general it is not true that all orbits represent elements of $\gamma(L^2, E)$.

Example 7.2 For $1 < p < 2$ we consider the rotation group S on $E = L^p(\mathbb{T})$, where $\mathbb{T}$ denotes the unit circle. Its generator will be denoted by A. For a fixed function $x \in E$ let W_x denote the E-valued Brownian defined by

$$W_x(t) := w(t)x,$$

where w is a given standard Brownian motion. As is shown in [18], the problem

$$\begin{aligned} dU(t) &= AU(t)\,dt + dW_x(t), \qquad t \in [0, 2\pi], \\ U(0) &= u_0, \end{aligned}$$

admits a weak solution if and only if $x \in L^2(\mathbb{T})$. By the results of [18] (cf. the discussion preceding Theorem 6.1) this may be reformulated as saying that the orbit Sx represents an element S_x of $\gamma(L^2, E)$ if and only if $x \in L^2(\mathbb{T})$; moreover, for $x \in L^2(\mathbb{T})$ we have

$$\gamma(S_x) \eqsim \|x\|_{L^2(\mathbb{T})} \tag{7.2}$$

with proportionality constants depending on p only.

Choose an orthonormal basis $(x_n)_{n\geqslant 1}$ for $L^2(\mathbb{T})$ and define the operators $B_n : \mathbb{R} \to E$ by $B_n r := r x_n$, where we think of the x_n as elements of E. Let $Q_n := B_n \circ B_n^*$, let $(\lambda_n)_{n\geqslant 1}$ be a sequence of strictly positive real numbers satisfying $\sum_{n\geqslant 1} \sqrt{\lambda_n} < \infty$ and define $Q \in \mathscr{L}(E^*, E)$ by

$$Qx^* := \sum_{n\geqslant 1} \lambda_n Q_n x^* = \sum_{n\geqslant 1} \lambda_n \langle x_n, x^* \rangle x_n, \qquad x^* \in E^*.$$

By [27, Exercise III.2.5], Q is the covariance of a centred Gaussian measure Γ. Let (i_Γ, H_Γ) be its RKHS. One easily checks that

$$H_\Gamma = \left\{ h \in L^2(\mathbb{T}) : \sum_{n\geqslant 1} \frac{1}{\lambda_n} [h, x_n]^2_{L^2(\mathbb{T})} < \infty \right\}$$

and

$$\|h\|_{H_\Gamma}^2 = \sum_{n\geqslant 1} \frac{1}{\lambda_n}[h, x_n]_{L^2(\mathbb{T})}^2, \qquad h \in H_\Gamma.$$

In particular, H_Γ is dense in E. Thus, $\overline{H_\Gamma} = E$. We claim that $S \circ i_\Gamma$ represents an element of $\gamma(L^2_{H_\Gamma}, E)$. To see this, we notice that $h_n := \sqrt{\lambda_n} x_n$ defines an orthonormal basis $(h_n)_{n\geqslant 1}$ for H_Γ. Using the Kahane-Khintchine inequalities, Proposition 6.2, and Eq. 7.2, we estimate

$$\begin{aligned}\gamma(S\circ i_\Gamma) &= \left(\mathbb{E}\left\|\int_0^{2\pi} S\circ i_\Gamma\, dW_{H_\Gamma}\right\|^2\right)^{\frac12} \eqsim \mathbb{E}\left\|\int_0^{2\pi} S\circ i_\Gamma\, dW_{H_\Gamma}\right\| \\ &\leqslant \sum_{n\geqslant 1}\mathbb{E}\left\|\int_0^{2\pi} Sh_n\, dW_{H_\Gamma} h_n\right\| \eqsim \sum_{n\geqslant 1}\left(\mathbb{E}\left\|\int_0^{2\pi} Sh_n\, dW_{H_\Gamma} h_n\right\|^2\right)^{\frac12} \\ &= \sum_{n\geqslant 1}\gamma(S_{h_n}) \eqsim \sum_{n\geqslant 1}\sqrt{\lambda_n} < \infty,\end{aligned}$$

with all constants depending on p only. This argument is somewhat formal and can be made rigorous by using finite dimensional projections as in the proof of Proposition 6.1. On the other hand we just saw that Sx represents an element of $\gamma(L^2; E)$ only when $x \in L^2(\mathbb{T})$.

Since $L^2(\mathbb{T})$ is of the first category in $L^p(\mathbb{T})$, this example shows that in Theorem 5.2 the set of all $x \in \overline{H_\Gamma}$ for which Ψx represents an element of $\gamma(L^2; E)$ can be of the first category in $\overline{H_\Gamma}$.

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