Stochastic integration with respect to the fractional Brownian motion

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Abstract

We develop a stochastic calculus for the fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$ using the techniques of the Malliavin calclulus. We establish estimates in L^p, maximal inequalities and a continuity criterion for the stochastic integral. Finally, we derive an Itô's formula for integral processes.

Keywords: Fractional Brownian motion. Stochastic integral. Malliavin calculus. Maximal inequalities. Itô's formula.

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1 Introduction

The fractional Brownian motion (fBm) of Hurst parameter $H \in (0,1)$ is a centered Gaussian process $B = \{B_t, t \geq 0\}$ with the covariance function (see [15])

$$R_H(t,s) = \frac{1}{2}\left(s^{2H} + t^{2H} - |t-s|^{2H}\right). \tag{1}$$

Notice that if $H = \frac{1}{2}$, the process B is a standard Brownian motion. From (1) it follows that $E|B_t - B_s|^2 = |t-s|^{2H}$ and, as a consequence, B has $\alpha-$Hölder continuous paths for all $\alpha < H$.

If $H \neq \frac{1}{2}$, the process B is not a semimartingale and we cannot apply the stochastic calculus developed by Itô in order to define stochastic integrals with respect to B. Different approaches have been used in order to construct a stochastic calculus with respect to B and we can mention the following contributions to this problem:

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- Lin [13] and Dai and Heide [5] defined stochastic integrals with respect to the fractional Brownian motion with parameter $H > \frac{1}{2}$ using a pathwise Riemann-Stieltjes method. The integrator must have finite p-variation where $\frac{1}{p} + H > 1$.

- The stochastic calculus of variations (see [18]) with respect to the Gaussian process B is a powerful technique that can be used to define stochastic integrals. More precisely, as in the case of the Brownian motion, the divergence operator with respect to B can be interpreted as a stochastic integral. This idea has been developed by Decreusefond and Üstünel [6, 7], Carmona and Coutin [4], Alòs, Mazet and Nualart [2, 3], Duncan, Hu and Pasik-Duncan [8] and Hu and Øksendal [10]. The integral constructed by this method has zero mean, and can be obtained as the limit of Riemann sums defined using Wick products.

- Using the notions of fractional integral and derivative, Zähle has introduced in [25] a pathwise stochastic integral with respect to the fBm B with parameter $H \in (0,1)$. If the integrator has λ-Hölder continuous paths with $\lambda > 1 - H$, then this integral can be interpreted as a Riemann-Stieltjes integral and coincides with the forward and Stratonovich integrals studied in [2] and [1].

The purpose of our paper is to develop a stochastic calculus with respect to the fractional Brownian motion B with Hurst parameter $H > \frac{1}{2}$ using the techniques of the Malliavin calculus. Unlike some previous works (see, for instance, [3]) we will not use the integral representation of B as a stochastic integral with respect to a Wiener process. Instead of this we will rely on the intrinsic Malliavin calculus with respect to B.

The organization of the paper is as follows. Section 2 contains some preliminaries on the fBm, and in Section 3 we review the basic facts on Malliavin calculus that will be used in order to define stochastic integrals. In Section 4 we show the existence of the symmetric stochastic integral in the sense of Russo and Vallois [21] under smoothness conditions on the integrand, in the sense of the Malliavin calculus. This symmetric integral turns out to be equal to the divergence operator plus a trace term involving the derivative operator. In Section 5, applying Meyer's inequalities and the results of [16], we derive L^p and maximal inequalities for the divergence integral with respect to fBm. These estimates allow us to deduce continuity results for the integral process in Section 6. Finally, in Section 7 we establish an Itô's formula for the divergence process.

2 Preliminaries on the fBm

Fix $H \in (1/2, 1)$. Let $B = \{B_t, t \in [0,T]\}$ be a fractional Brownian motion with parameter H. That is, B is a zero mean Gaussian process with the covariance (1). We assume that B is defined in a complete probability space $(\Omega, \mathcal{F}, P)$. We denote by $\mathcal{E} \subset \mathcal{H}$ the set of step functions on $[0,T]$. Let $\mathcal{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the scalar product

$$\left\langle \mathbf{1}_{[0,t]}, \mathbf{1}_{[0,s]} \right\rangle_{\mathcal{H}} = R_H(t,s).$$

The mapping $\mathbf{1}_{[0,t]} \longrightarrow B_t$ can be extended to an isometry between $\mathcal{H}$ and the Gaussian space $H_1(B)$ associated with B. We will denote this isometry by $\varphi \longrightarrow B(\varphi)$.

It is easy to see that

$$R_H(t,s) = \alpha_H \int_0^t \int_0^s |r-u|^{2H-2} du dr, \tag{2}$$

where $\alpha_H = H(2H-1)$. Formula (2) implies that

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = \alpha_H \int_0^T \int_0^T |r-u|^{2H-2} \varphi_r \psi_u du dr \tag{3}$$

for any pair of step functions φ and ψ in $\mathcal{E}$.

For $r > u$ we can write

$$\begin{aligned} (r-u)^{2H-2} &= \beta(2-2H, H-\frac{1}{2})(ru)^{H-\frac{1}{2}} \\ &\quad \times \int_0^u v^{1-2H}(r-v)^{H-\frac{3}{2}}(u-v)^{H-\frac{3}{2}} dv, \end{aligned} \tag{4}$$

where β denotes the Beta function. Let us show the equality (4). By means of the change of variables $z = \frac{r-v}{u-v}$ and $x = \frac{r}{uz}$, we obtain

$$\begin{aligned} & \int_0^u v^{1-2H}(r-v)^{H-\frac{3}{2}}(u-v)^{H-\frac{3}{2}} dv \\ = \; & (r-u)^{2H-2} \int_{\frac{r}{u}}^{\infty} (zu-r)^{1-2H} z^{H-\frac{3}{2}} dz \\ = \; & (ru)^{\frac{1}{2}-H}(r-u)^{2H-2} \int_0^1 (1-x)^{1-2H} x^{H-\frac{3}{2}} dx \\ = \; & \beta(2-2H, H-\frac{1}{2})(ru)^{\frac{1}{2}-H}(r-u)^{2H-2}. \end{aligned}$$

Consider the square integrable kernel

$$K_H(t,s) = c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du,$$

where $c_H = \left[\frac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}\right]^{1/2}$ and $t > s$.

Notice that

$$\frac{\partial K_H}{\partial t}(t,s) = c_H \left(\frac{t}{s}\right)^{H-\frac{1}{2}} (t-s)^{H-\frac{3}{2}} \geq 0. \tag{5}$$

Taking into account formulas (4) and (2) we deduce that this kernel verifies

$$\begin{aligned}
& \int_0^{t\wedge s} K_H(t,u) K_H(s,u) du \\
= \; & c_H^2 \int_0^{t\wedge s} \left(\int_u^t (y-u)^{H-\frac{3}{2}} y^{H-\frac{1}{2}} dy \right) \\
& \times \left(\int_u^s (z-u)^{H-\frac{3}{2}} z^{H-\frac{1}{2}} dz \right) u^{1-2H} du \\
= \; & c_H^2 \beta(2-2H, H-\frac{1}{2}) \int_0^t \int_0^s |y-z|^{2H-2} dz dy \\
= \; & R_H(t,s).
\end{aligned} \tag{6}$$

Property (6) implies that $R_H(t,s)$ is nonnegative definite.

Consider the linear operator K_H^* from $\mathcal{E}$ to $L^2([0,T])$ defined by

$$(K_H^* \varphi)(s) = \int_s^T \varphi_r \frac{\partial K_H}{\partial r}(r,s) dr.$$

Using equations (6) and (3) we have

$$\begin{aligned}
& \langle K_H^* \varphi, K_H^* \psi \rangle_{L^2([0,T])} \\
= \; & \int_0^T \left(\int_s^T \varphi_r \frac{\partial K_H}{\partial r}(r,s) dr \right) \left(\int_s^T \psi_u \frac{\partial K_H}{\partial u}(u,s) du \right) ds \\
= \; & \int_0^T \int_0^T \left(\int_0^{r\wedge u} \frac{\partial K_H}{\partial r}(r,s) \frac{\partial K_H}{\partial u}(u,s) ds \right) \varphi_r \psi_u du dr \\
= \; & \int_0^T \int_0^T \frac{\partial^2 R_H}{\partial r \partial u}(r,u) \varphi_r \psi_u du dr \\
= \; & \alpha_H \int_0^T \int_0^T |r-u|^{2H-2} \varphi_r \psi_u du dr \\
= \; & \langle \varphi, \psi \rangle_{\mathcal{H}}.
\end{aligned} \tag{7}$$

As a consequence, the operator K_H^* provides an isometry between the Hilbert spaces $\mathcal{H}$ and $L^2([0,T])$. Hence, the process $W=\{W_t, t\in[0,T]\}$ defined by

$$W_t = B((K_H^*)^{-1}(\mathbf{1}_{[0,t]}))$$

is a Wiener process, and the process B has an integral representation of the form

$$B_t = \int_0^t K_H(t,s)dW_s, \tag{8}$$

because $\left(K_H^* \mathbf{1}_{[0,t]}\right)(s) = K_H(t,s)$.

The elements of the Hilbert space $\mathcal{H}$ may not be functions but distributions of negative order (see, for instance, [3, Proposition 6] or the recent paper by Taqqu [20]). In fact, from (5) and (7) it follows that $\mathcal{H}$ coincides with the space of distributions f such that $s^{\frac{1}{2}-H} I_{0+}^{H-\frac{1}{2}}(f_{H-\frac{1}{2}})(s)$ is a square integrable function, where $f_{H-\frac{1}{2}}(s) = f(s)s^{H-\frac{1}{2}}$, and $I_{0+}^{H-\frac{1}{2}}$ is the left-sided fractional integral of order $H-\frac{1}{2}$ (see [22]).

We can find a linear space of functions contained in $\mathcal{H}$ in the following way. Let $|\mathcal{H}|$ be the linear space of measurable functions φ on $[0,T]$ such that

$$\|\varphi\|_{|\mathcal{H}|}^2 := \int_0^T \left(\int_s^T |\varphi_r| \frac{\partial K_H}{\partial r}(r,s) dr \right)^2 ds < \infty. \tag{9}$$

From the above computations, it is easy to check that

$$\|\varphi\|_{|\mathcal{H}|}^2 = \alpha_H \int_0^T \int_0^T |\varphi_r| \, |\varphi_u| \, |r-u|^{2H-2} \, drdu. \tag{10}$$

It is not difficult to show that $|\mathcal{H}|$ is a Banach space with the norm $\|\cdot\|_{|\mathcal{H}|}$ and $\mathcal{E}$ is dense in $|\mathcal{H}|$. On the other hand, it has been shown in [20] that the space $|\mathcal{H}|$ equipped with the inner product $\langle \varphi, \psi \rangle_{\mathcal{H}}$ is not complete and it is isometric to a subspace of $\mathcal{H}$. We will identify $|\mathcal{H}|$ with this subspace.

The following estimate has been proved in [16]

$$\|\varphi\|_{|\mathcal{H}|} \le b_H \|\varphi\|_{L^{\frac{1}{H}}([0,T])}, \tag{11}$$

for some constant $b_H > 0$. This estimate implies the inclusion $L^{\frac{1}{H}}([0,T]) \subset |\mathcal{H}|$. For the sake of completeness we reproduce here the main arguments of the proof of (11). Using Hölder's inequality with exponent $q = \frac{1}{H}$ in (10) we get

$$\|\varphi\|_{|\mathcal{H}|}^2 \le \alpha_H \left(\int_0^T |\varphi_r|^{\frac{1}{H}} dr \right)^H \left(\int_0^T \left(\int_0^T |\varphi_u| (r-u)^{2H-2} du \right)^{\frac{1}{1-H}} dr \right)^{1-H}.$$

The second factor in the above expression, up to a multiplicative constant, it is equal to the $\frac{1}{1-H}$ norm of the left-sided fractional integral $I_{0+}^{2H-1}|\varphi|$ (see [22]). Finally is suffices to apply the Hardy-Littlewood inequality (see [24, Theorem 1, p. 119])

$$\left\| I_{0+}^{\alpha} f \right\|_{L^q(0,\infty)} \leq c_{H,p} \left\| f \right\|_{L^p(0,\infty)}, \tag{12}$$

where $0 < \alpha < 1$, $1 < p < q < \infty$ satisfy $\frac{1}{q} = \frac{1}{p} - \alpha$, with the particular values $\alpha = 2H - 1$, $q = \frac{1}{1-H}$, and $p = H$.

3 Malliavin Calculus

The process $B = \{B_t, t \in [0,T]\}$ is Gaussian and, hence, we can develop a stochastic calculus of variations (or Malliavin calculus) with respect to it. Let us recall the basic notions of this calculus.

Let $\mathcal{S}$ be the set of smooth and cylindrical random variables of the form

$$F = f(B(\phi_1), \ldots, B(\phi_n)), \tag{13}$$

where $n \geq 1$, $f \in \mathcal{C}_b^\infty(\mathbb{R}^n)$ (f and all its partial derivatives are bounded), and $\phi_i \in \mathcal{H}$. The derivative operator D of a smooth and cylindrical random variable F of the form (13) is defined as the $\mathcal{H}$-valued random variable

$$DF = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(B(\phi_1), \ldots, B(\phi_n))\phi_i.$$

The derivative operator D is then a closable operator from $L^p(\Omega)$ into $L^p(\Omega; \mathcal{H})$ for any $p \geq 1$. For any integer $k \geq 1$ we denote by D^k the iteration of the derivative operator. For any $p \geq 1$ the Sobolev space $\mathbb{D}^{k,p}$ is the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|_{k,p}^p = E|F|^p + E\sum_{j=1}^{k} \left\| D^j F \right\|_{\mathcal{H}^{\otimes j}}^p.$$

In a similar way, given a Hilbert space V we denote by $\mathbb{D}^{k,p}(V)$ the coresponding Sobolev space of V-valued random variables.

The divergence operator δ is the adjoint of the derivative operator, defined by means of the duality relationship

$$E(F\delta(u)) = E \langle DF, u \rangle_{\mathcal{H}},$$

where u is a random variable in $L^2(\Omega;\mathcal{H})$. We say that u belongs to the domain of the operator δ, denoted by Dom δ, if the above expression is continuous in the L^2 norm of F. A basic result says that the space $\mathbb{D}^{1,2}(\mathcal{H})$ is included in Dom δ.

The following are two basic properties of the divegence operator:

i) For any $u \in \mathbb{D}^{1,2}(\mathcal{H})$

$$E\delta(u)^2 = E\,\|u\|_{\mathcal{H}}^2 + E\,\langle Du, (Du)^*\rangle_{\mathcal{H}\otimes\mathcal{H}}\,, \tag{14}$$

where $(Du)^*$ is the adjoint of (Du) in the Hilbert space $\mathcal{H}\otimes\mathcal{H}$.

ii) For any F in $\mathbb{D}^{1,2}$ and any u in the domain of δ such that Fu and $F\delta(u) + \langle DF, u\rangle_{\mathcal{H}}$ are square integrable, then Fu is in the domain of δ and

$$\delta(Fu) = F\delta(u) + \langle DF, u\rangle_{\mathcal{H}}\,. \tag{15}$$

We denote by $|\mathcal{H}|\otimes|\mathcal{H}|$ the space of measurable functions φ on $[0,T]^2$ such that

$$\|\varphi\|_{|\mathcal{H}|\otimes|\mathcal{H}|}^2 = \alpha_H^2 \int_{[0,T]^4} \left|\varphi_{r,\theta}\right| \left|\varphi_{u,\eta}\right| |r-u|^{2H-2} \left|\theta-\eta\right|^{2H-2} drdud\theta d\eta < \infty.$$

As before, $|\mathcal{H}|\otimes|\mathcal{H}|$ is a Banach space with respect to the norm $\|\cdot\|_{|\mathcal{H}|\otimes|\mathcal{H}|}$. Furthermore, equipped with the inner product

$$\langle\varphi,\psi\rangle_{\mathcal{H}\otimes\mathcal{H}} = \alpha_H^2 \int_{[0,T]^4} \varphi_{r,\theta}\psi_{u,\eta}\, |r-u|^{2H-2} \left|\theta-\eta\right|^{2H-2} drdud\theta d\eta$$

the space $|\mathcal{H}|\otimes|\mathcal{H}|$ is isometric to a subspace of $\mathcal{H}\otimes\mathcal{H}$ and it will be identified with this subspace.

Lemma 1 *If φ belongs to $|\mathcal{H}|\otimes|\mathcal{H}|$ then*

$$\|\varphi\|_{|\mathcal{H}|\otimes|\mathcal{H}|} \le b_H \,\|\varphi\|_{L^{\frac{1}{H}}([0,T]^2)}\,.$$

Proof. A slight extension of (11) implies that

$$\langle|\varphi|,|\psi|\rangle_{\mathcal{H}} \le b_H^2\, \|\varphi\|_{L^{\frac{1}{H}}([0,T])} \|\psi\|_{L^{\frac{1}{H}}([0,T])}\,.$$

for all φ and ψ in $\mathcal{H}$. As a consequence, applying twice this inequality yields the desired result. ∎

For any $p > 1$ we denote by $\mathbb{D}^{1,p}(|\mathcal{H}|)$ the subspace of $\mathbb{D}^{1,p}(\mathcal{H})$ formed by the elements u such that $u \in |\mathcal{H}|$ a.s., $Du \in |\mathcal{H}| \otimes |\mathcal{H}|$ a.s., and

$$E \|u\|_{|\mathcal{H}|}^p + E \|Du\|_{|\mathcal{H}| \otimes |\mathcal{H}|}^p < \infty.$$

Note that the space $\mathbb{D}^{1,2}(|\mathcal{H}|) \subset \mathbb{D}^{1,2}(\mathcal{H})$ is included in the domain of δ, and from (14) we have

$$E\delta(u)^2 \leq E \|u\|_{\mathcal{H}}^2 + E \|Du\|_{|\mathcal{H}| \otimes |\mathcal{H}|}^2 .$$

4 Stochastic integrals with respect to the fractional Brownian motion

In the case of an ordinary Brownian motion, the adapted processes in $L^2([0,T] \times \Omega)$ belong to the domain of the divergence operator, and on this set the divergence operator coincides with the Itô stochastic integral. Actually, the divergence operator coincides with an extension of the Itô stochastic integral introduced by Skorohod in [23]. We can ask in which sense the divergence operator with respect to a fractional Brownian motion B can be interpreted as a stochastic integral. Note that the divergence operator provides an isometry between the Hilbert Space $\mathcal{H}$ associated with the fBm B and the Gaussian space $H_1(B)$, and gives rise to a notion of stochastic integral in the space of determinstic functions $|\mathcal{H}|$ included in $\mathcal{H}$.

Let us recall the definition of the symmetric integral introduced by Russo and Vallois in [21]. By convention we will assume that all processes and functions vanish outside the interval $[0,T]$.

Definition 2 *Let $u = \{u_t, t \in [0,T]\}$ be a stochastic process with integrable trajectories. The symmetric integral of u with respect to the fBm B is defined as the limit in probability as ε tend sto zero of*

$$(2\varepsilon)^{-1} \int_0^T u_s (B_{s+\varepsilon} - B_{s-\varepsilon}) ds,$$

provided this limit exists, and it is denoted by $\int_0^T u_t dB_t$.

Let $\mathcal{S}_T$ be the set of smooth step processes of the form

$$u = \sum_{j=0}^{m-1} F_j \mathbf{1}_{[t_j, t_{j+1}]},$$

where $F_j \in \mathcal{S}$, and $0 = t_1 < ... < t_m = T$. It is not difficult to check that $\mathcal{S}_T$ is dense in $\mathbb{D}^{1,2}(|\mathcal{H}|)$.

The following proposition gives sufficient conditions for the existence of the symmetric integral, and provides a representation of the divergence operator as a stochastic integral.

Proposition 3 *Let $u = \{u_t, t \in [0,T]\}$ be a stochastic process in the space $\mathbb{D}^{1,2}(|\mathcal{H}|)$. Suppose also that a.s.*

$$\int_0^T \int_0^T |D_s u_t| \, |t-s|^{2H-2} \, dsdt < \infty. \tag{16}$$

Then the symmetric integral exists and we have

$$\int_0^T u_t dB_t = \delta(u) + \alpha_H \int_0^T \int_0^T D_s u_t \, |t-s|^{2H-2} \, dsdt. \tag{17}$$

Proof. The proof of this proposition will be decomposed into several steps.

Step 1. Let us define, for all $u \in \mathbb{D}^{1,2}(|\mathcal{H}|)$, the aproximating process u^ε given by

$$u_t^\varepsilon = (2\varepsilon)^{-1} \int_{t-\varepsilon}^{t+\varepsilon} u_s ds.$$

In this first step we will see that

$$\|u^\varepsilon\|^2_{\mathbb{D}^{1,2}(|\mathcal{H}|)} \le d_H \, \|u\|^2_{\mathbb{D}^{1,2}(|\mathcal{H}|)} \,, \tag{18}$$

for some positive constant d_H. In fact, we have that

$$\begin{aligned}
\|u^\varepsilon\|^2_{|\mathcal{H}|} &= \alpha_H \int_0^T \int_0^T |u_r^\varepsilon| \, |u_s^\varepsilon| \, |r-s|^{2H-2} dsdr \\
&\le \frac{\alpha_H}{4\varepsilon^2} \int_0^T \int_0^T \int_{-\varepsilon}^{\varepsilon} \int_{-\varepsilon}^{\varepsilon} |u_{r+y}| \, |u_{s+z}| \, |r-s|^{2H-2} dydzdsdr \\
&\le \frac{\alpha_H}{\varepsilon} \int_0^T \int_0^T \int_{-2\varepsilon}^{2\varepsilon} |u_r| \, |u_s| \, |r-s+x|^{2H-2} dxdsdr.
\end{aligned}$$

There exists a constant $d_H = \max(2^{4-2H}, \frac{2}{2H-1})$ such that

$$\frac{1}{\varepsilon} \int_{-2\varepsilon}^{2\varepsilon} |r-s+x|^{2H-2} \, dx \le d_H |r-s|^{2H-2}. \tag{19}$$

Indeed, we can assume that $r > s$, and if $|r-s| > 4\varepsilon$, then

$$\begin{aligned}\frac{1}{\varepsilon}\int_{-2\varepsilon}^{2\varepsilon}|r-s+x|^{2H-2}\,dx &\leq 4|r-s-2\varepsilon|^{2H-2} \\ &\leq 2^{4-2H}|r-s|^{2H-2}\end{aligned}$$

and for $|t-s| \leq 4\varepsilon$

$$\begin{aligned}\frac{1}{\varepsilon}\int_{-2\varepsilon}^{2\varepsilon}|r-s+x|^{2H-2}\,dx &\leq \frac{2}{\varepsilon}\int_0^{4\varepsilon}x^{2H-2}dx \\ &= \frac{2}{2H-1}(4\varepsilon)^{2H-1} \\ &\leq \frac{2}{2H-1}|t-s|^{2H-2}.\end{aligned}$$

Hence,

$$\|u^\varepsilon\|^2_{|\mathcal{H}|} \leq e_H\,\|u\|^2_{|\mathcal{H}|}\,.$$

In a similar way we can show that

$$\|Du^\varepsilon\|^2_{|\mathcal{H}|\otimes|\mathcal{H}|} \leq e_H\,\|Du\|^2_{|\mathcal{H}|\otimes|\mathcal{H}|}\,,$$

for some constant $e_H > 0$, and now the proof of (18) is complete.

Step 2. Let us see that, for all $\varepsilon > 0$,

$$(2\varepsilon)^{-1}\int_0^T u_s(B_{s+\varepsilon}-B_{s-\varepsilon})ds = \delta(u^\varepsilon) + A^\varepsilon u, \tag{20}$$

where

$$A^\varepsilon u = (2\varepsilon)^{-1}\int_0^T\left\langle Du_s, \mathbf{1}_{[s-\varepsilon,s+\varepsilon]}\right\rangle_{\mathcal{H}\otimes\mathcal{H}}ds.$$

In fact, take a sequence $\{u^n\} \subset \mathcal{S}_T$ such that $u^n \to u$ in the norm of the space $\mathbb{D}^{1,2}(|\mathcal{H}|)$ as n tends to infinity. By formula (15) it follows that

$$(2\varepsilon)^{-1}\int_0^T u_s^n(B_{s+\varepsilon}-B_{s-\varepsilon})ds = (2\varepsilon)^{-1}\int_0^T\delta(u_s^n\mathbf{1}_{[s-\varepsilon,s+\varepsilon]})ds + A^\varepsilon u^n,$$

which implies

$$(2\varepsilon)^{-1}\int_0^T u_s^n(B_{s+\varepsilon}-B_{s-\varepsilon})ds = \delta(u^{n,\varepsilon}) + A^\varepsilon u^n,$$

By Step 1 it follows that $u^{n,\varepsilon}$ converges in $\mathbb{D}^{1,2}(|\mathcal{H}|)$ to the process u^{ε} as n tends to infinity. As a consequence, $\delta(u^{n,\varepsilon})$ will converge in $L^2(\Omega)$ to $\delta(u^{\varepsilon})$ as n tends to infinity. On the other hand,

$$\begin{aligned}
& \left|\int_0^T (u_s^n - u_s)(B_{s+\varepsilon} - B_{s-\varepsilon})ds\right|^2 \\
\leq & \left(\sup_{|r-s|\leq 2\varepsilon} |B_r - B_s|^2\right)\left(\int_0^T |u_s^n - u_s|\, ds\right)^2 \\
\leq & \; T^{2-2H}\left(\sup_{|r-s|\leq 2\varepsilon} |B_r - B_s|^2\right)\int_0^T\int_0^T |u_s^n - u_s|\,|u_r^n - u_r|\,|r-s|^{2H-2}\, ds
\end{aligned}$$

which converges to zero in probability as n tends to infinity. In a similar way we can easily check that $A^{\varepsilon}u^n$ converges in probability to $A^{\varepsilon}u$, which completes the proof of (20).

Step 3. Let us prove that $\delta(u^{\varepsilon})$converges in $L^2(\Omega)$ to $\delta(u)$ as $\varepsilon \to 0$. In fact, take a sequence $\{u^n\} \subset \mathcal{S}_T$ such that $u^n \to u$ in the norm of the space $\mathbb{D}^{1,2}(|\mathcal{H}|)$ as n tends to infinity. Then, for all $\varepsilon > 0, n \geq 1$ we can write

$$\begin{aligned}
& E\,|\delta(u^{\varepsilon}) - \delta(u)|^2 \\
\leq & \; 3\left\{E\,|\delta(u^{\varepsilon}) - \delta(u^{n,\varepsilon})|^2 + E\,|\delta(u^{n,\varepsilon}) - \delta(u^n)|^2 + E\,|\delta(u^n) - \delta(u)|^2\right\}.
\end{aligned}$$

By Step 1 we can easily deduce that for all $\delta > 0$ there exists an integer n_δ such that for all $n \geq n_\delta$,

$$E\,|\delta(u^{\varepsilon}) - \delta(u)|^2 \leq 3\left\{E\,|\delta(u^{n,\varepsilon}) - \delta(u^n)|^2 + \delta\right\}.$$

Letting now $\varepsilon \to 0$ it follows that

$$\lim_{\varepsilon\to 0} E\,|\delta(u^{\varepsilon}) - \delta(u)|^2 \leq 3\delta,$$

which implies the desired convergence.

Step 4. It remains to check the convergence in probability of $A^{\varepsilon}u$ to

$$\alpha_H \int_0^T\int_0^T D_s u_t\,|t-s|^{2H-2}\, dsdt$$

We have

$$\begin{aligned}
A^{\varepsilon}u &= (2\varepsilon)^{-1}\int_0^T \left\langle Du_s, \mathbf{1}_{[s-\varepsilon,s+\varepsilon]}\right\rangle_{\mathcal{H}\otimes\mathcal{H}} ds \\
&= (2\varepsilon)^{-1}\int_0^T\int_0^T D_t u_s\left(\int_{s-\varepsilon}^{s+\varepsilon} |t-r|^{2H-2}\, dr\right) dtds \\
&= (2\varepsilon)^{-1}\int_0^T\int_0^T D_t u_s\left(\int_{-\varepsilon}^{\varepsilon} |t-s+\sigma|^{2H-2}\, d\sigma\right) dtds.
\end{aligned}$$

By Step 1 we know that, for some positive constant d_H,

$$(2\varepsilon)^{-1}\int_{-\varepsilon}^{\varepsilon}|t-s+\sigma|^{2H-2}\,d\sigma \le d_H\,|t-s|^{2H-2}.$$

Applying the dominated convergence theorem we deduce that the term $A^{\varepsilon}u$ converges a.s. to $\alpha_H\int_0^T\int_0^T D_s u_t\,|t-s|^{2H-2}\,dsdt$, as we wanted to prove. ∎

Remark 1 By a similar arguments one could show that the integral $\int_0^T u_t dB_t$ also coincides with the forward and backward integrals, under the assumptions of Proposition 3.

Remark 2 A sufficient condition for (16) is

$$\int_0^T\left(\int_s^T|D_s u_t|^p\,dt\right)^{\frac{1}{p}}ds<\infty,$$

for some $p>\frac{1}{2H-1}$.

Remark 3 Let F be a function of class $C^2(\mathbb{R})$ such that

$$\max\left\{|F(x)|,|F'(x)|,|F''(x)|\right\}\le ce^{\lambda x^2},$$

where c and λ are positive constant such that $\lambda<\frac{1}{2T^{2H}}$. Then (see [3, Theorem 2]) the process $F'(B_t)$ belongs to the space $\mathbb{D}_B^{1,2}(|\mathcal{H}|)$ and the following version of Itô's formula holds

$$F(B_t)=F(0)+\delta\left(F'(B_\cdot)\mathbf{1}_{[0,t]}\right)+H\int_0^t F''(B_s)s^{2H-1}ds.$$

Taking into account that

$$\begin{aligned}
&\int_0^t\int_0^r D_s\left(F'(B_r)\right)(r-s)^{2H-2}dsdr\\
=\;&\int_0^t F''(B_r)\int_0^r(r-s)^{2H-2}dsdr\\
=\;&\frac{1}{2H-1}\int_0^t F''(B_r)r^{2H-1}dr
\end{aligned}$$

we obtain

$$F(B_t)=F(0)+\int_0^t F'(B_s)dB_s.$$

5 Estimates for the stochastic integral

Suppose that $u = \{u_t, t \in [0,T]\}$ is a stochastic process in the space $\mathbb{D}^{1,2}(|\mathcal{H}|)$ such that condition (16) holds. Then, for any $t \in [0,T]$ the process $u\mathbf{1}_{[0,t]}$ also belongs to $\mathbb{D}^{1,2}(|\mathcal{H}|)$ and satisfies (16). Hence, by Proposition 3 we can define the indefinite integral $\int_0^t u_s dB_s = \int_0^T u_s \mathbf{1}_{[0,t]}(s) dB_s$ and the following decomposition holds

$$\int_0^t u_s dB_s = \delta(u\mathbf{1}_{[0,t]}) + \alpha_H \int_0^t \int_0^T D_r u_s \left|s-r\right|^{2H-2} drds.$$

The second summand in this expression is a process with absolutely continuous paths that can be studied by means of usual methods. Therefore, in order to deduce L^p estimates and to study continuity properties of $\int_0^t u_s dB_s$ we can reduce our analysis to the process $\delta(u\mathbf{1}_{[0,t]})$. In this section we will establish L^p maximal estimates for this divergence process. We will make use of the notation $\int_0^t u_s \delta B_s = \delta\ \left(u\mathbf{1}_{[0,t]}\right)$.

For any $p > 1$ we denote by $\mathbb{L}_H^{1,p}$ the set of processes u in $\mathbb{D}^{1,p}(|\mathcal{H}|)$ such that

$$\|u\|_{\mathbb{L}_H^{1,p}}^p =: E\,\|u\|_{L^{1/H}([0,T])}^p + E\,\|Du\|_{L^{1/H}([0,T]^2)}^p < \infty.$$

From (11) and Lemma 1 we obtain

$$\|u\|_{\mathbb{D}^{1,p}(|\mathcal{H}|)} \leq b_H \|u\|_{\mathbb{L}_H^{1,p}}. \tag{21}$$

and, as a consequence, the space $\mathbb{L}_H^{1,2}$ is included in the domain of the divergence operator δ.

By Meyer's inequalities (see for example [18]), if $p > 1$, a process $u \in \mathbb{D}^{1,p}(|\mathcal{H}|)$ belongs to the domain of the divergence in $L^p(\Omega)$, and we have

$$E\left|\delta\left(u\right)\right|^p \leq C_{H,p}\left(\|Eu\|_{|\mathcal{H}|}^p + E\,\|Du\|_{|\mathcal{H}|\otimes|\mathcal{H}|}^p\right).$$

As a consequence, if u belongs to $\mathbb{L}_H^{1,p}$ we can write

$$\begin{aligned} E\left|\delta\left(u\right)\right|^p &\leq C_{H,p}\left(\|Eu\|_{L^{1/H}([0,T])}^p + E\,\|Du\|_{L^{1/H}([0,T]^2)}^p\right) \\ &\leq C_{H,p}\,\|u\|_{\mathbb{L}_H^{1,p}}^p. \end{aligned} \tag{22}$$

The following theorem will give us a maximal L^p-estimate for the indefinite integral $\int_0^t u_s \delta B_s$.

Theorem 4 *Let $p > 1/H$. Let $u = \{u_t,\ t \in [0,T]\}$ be a stochastic process in $\mathbb{L}^{1,p}_{H-\varepsilon}$ for some $0 < \varepsilon < H - \frac{1}{p}$. Then the following inequality holds*

$$\begin{aligned} E\left(\sup_{t\in[0,T]}\left|\int_0^t u_s\delta B_s\right|^p\right) &\leq C\left[\left(\int_0^T |Eu_s|^{\frac{1}{H-\varepsilon}}\,ds\right)^{p(H-\varepsilon)}\right. \\ &\quad \left.+E\left(\int_0^T\left(\int_0^T |D_s u_r|^{\frac{1}{H}}\,dr\right)^{\frac{H}{H-\varepsilon}}ds\right)^{p(H-\varepsilon)}\right], \end{aligned}$$

where the constant C depends on H, ε and T.

Proof. Set $\alpha = 1 - \frac{1}{p} - \varepsilon$. Then $1 - H < \alpha < 1 - \frac{1}{p}$. Using the equality $c_\alpha = \int_r^t (t-\theta)^{-\alpha}(\theta - r)^{\alpha-1}\,d\theta$ we can write

$$\int_0^t u_s\delta B_s = c_\alpha \int_0^t u_s\left(\int_s^t (t-r)^{-\alpha}(r-s)^{\alpha-1}\,dr\right)\delta B_s.$$

Using Fubini's stochastic theorem (see for example [18]) we have that

$$\int_0^t u_s\delta B_s = c_\alpha \int_0^t (t-r)^{-\alpha}\left(\int_0^r u_s(r-s)^{\alpha-1}\,\delta B_s\right)dr.$$

Hölder's inequality and condition $\alpha < 1 - \frac{1}{p}$ yields

$$\left|\int_0^t u_s\delta B_s\right|^p \leq c_{\alpha,p}\int_0^t\left|\int_0^r u_s(r-s)^{\alpha-1}\,\delta B_s\right|^p dr,$$

from where it follows that

$$E\left(\sup_{0\leq t\leq T}\left|\int_0^t u_s\delta B_s\right|^p\right) \leq c_{\alpha,p}E\int_0^T\left|\int_0^r u_s(r-s)^{\alpha-1}\,\delta B_s\right|^p dr.$$

Using now inequality (22) we obtain

$$\begin{aligned} E\left(\sup_{0\leq t\leq T}\left|\int_0^t u_s\delta B_s\right|^p\right) &\leq C_{\alpha,p}\left\{\int_0^T\left(\int_0^r (r-s)^{\frac{\alpha-1}{H}}|Eu_s|^{\frac{1}{H}}\,ds\right)^{pH}dr\right. \\ &\quad \left.+E\int_0^T\left(\int_0^r\int_0^T (r-s)^{\frac{\alpha-1}{H}}|D_\theta u_s|^{\frac{1}{H}}\,d\theta ds\right)^{pH}dr\right\} \\ &= : C_{\alpha,p}(I_1 + I_2). \end{aligned}$$

For the term I_1 we apply the Hardy-Littlewood inequality (12) with $q = pH$ and $p = \frac{H}{H-\varepsilon}$ and we obtain

$$I_1 \leq C_{\alpha,p,H} \left(\int_0^T |Eu_r|^{\frac{1}{H-\varepsilon}} \, dr \right)^{p(H-\varepsilon)} .$$

A similar argument can be used in order to estimate the term I_2 and we obtain

$$I_2 \leq C_{\alpha,p,H} E \left(\int_0^T \left(\int_0^T |D_s u_r|^{\frac{1}{H}} \, ds \right)^{\frac{H}{H-\varepsilon}} dr \right)^{p(H-\varepsilon)} ,$$

which completes the proof. ■

Remark 5 The terms I_1 and I_2 in the proof of Theorem 4 can also be estimated by Hölder's inequality, taking into account that $\alpha > 1 - H$. With this approach we obtain the following maximal inequality, for any $p > \frac{1}{H}$ and any process u in $\mathbb{L}^{1,p}_{1/H}$

$$E \sup_{t \in [0,T]} \left| \int_0^t u_s \delta B_s \right|^p \leq C \left[\int_0^T |Eu_s|^p \, ds + E \int_0^T \left(\int_0^T |D_s u_r|^{\frac{1}{H}} \, ds \right)^{pH} dr \right] ,$$

where the constant $C > 0$ depends on p, H and T.

6 Continuity of the integral process

In this section we will use the preceeding estimates for the divergence operator in order to study the continuity of the integral processes of the form $\int_0^t u_s \delta B_s$.

Theorem 5 *Let $u = \{u_t, \ t \in [0,T]\}$ be a stochastic process in the space $\mathbb{L}^{1,p}_H$, where $pH > 1$ and assume that*

$$\int_0^T |Eu_r|^p \, dr + \int_0^T E \left(\int_0^T |D_\theta u_r|^{\frac{1}{H}} \, d\theta \right)^{pH} dr < \infty.$$

Then the integral process $X_t = \left\{ \int_0^t u_s \delta B_s, \ t \in [0,T] \right\}$ has an a.s. continuous modification. Moreover, for all $\gamma < H - \frac{1}{p}$ there exists a random constant C_γ a.s. finite such that

$$|X_t - X_s| \leq C_\gamma \, |t - s|^\gamma .$$

Proof. Using the estimate (22) we can write

$$
\begin{aligned}
& E\left|X_t - X_s\right|^p \\
\leq\ & C_{H,p}\left\{\left(\int_s^t \left|Eu_r\right|^{\frac{1}{H}} dr\right)^{pH} + E\left(\int_s^t \left(\int_0^T \left|D_\theta u_r\right|^{\frac{1}{H}} d\theta\right) dr\right)^{pH}\right\} \\
\leq\ & C_{H,p}\,(t-s)^{pH-1}\left\{\int_s^t \left|Eu_r\right|^p dr + \int_s^t E\left(\int_0^T \left|D_\theta u_r\right|^{\frac{1}{H}} d\theta\right)^{pH} dr\right\},
\end{aligned}
$$

for some constant $C_{H,p} > 0$. So, there exist a non-negative function $A : [0,T] \to \mathbb{R}_+$ such that $\int_0^T A_r dr < \infty$ and

$$
E\left|X_t - X_s\right|^p \leq (t-s)^{pH-1}\int_s^t A_r dr.
$$

By Fubini's theorem we obtain that, for all $\alpha \in (2, pH+1)$

$$
\begin{aligned}
& E\int_0^T\int_0^T \frac{\left|X_t - X_s\right|^p}{\left|t-s\right|^\alpha} dsdt \\
\leq\ & 2\int_0^T\int_0^t (t-s)^{pH-1-\alpha}\left(\int_s^t A_r dr\right) dsdt \\
=\ & 2\int_0^T A_r\left(\int_0^r\int_r^T (t-s)^{pH-1-\alpha}\, dtds\right) dr \\
=\ & \frac{2}{(pH-\alpha)\,(pH-\alpha+1)}\int_0^T A_r\left[(T-r)^{pH-\alpha+1} - T^{pH-\alpha+1} + r^{pH-\alpha+1}\right] dr.
\end{aligned}
$$

Then the random variable

$$
\Gamma := \int_0^T\int_0^T \frac{\left|X_t - X_s\right|^p}{\left|t-s\right|^\alpha} dsdt
$$

is a.s. finite. Now, by Garsia-Rodemish-Ramsey lemma (see [9]) we deduce that for $\gamma := \frac{\alpha-2}{p}$, there exist a random constant C_γ a.s. finite such that

$$
\left|X_t - X_s\right| \leq C_\gamma \left|t-s\right|^\gamma .
$$

Taking into account that the condition $\alpha < pH+1$ is equivalent to $\gamma < H - \frac{1}{p}$ the proof is complete. ■

Remark 6. The above theorem proves that, for a process $u \in \cap_{p>1}\mathbb{L}_H^{1,p}$, the indefinite integral process $X_t = \left\{\int_0^t u_s \delta B_s,\ t \in [0,T]\right\}$ is γ-Hölder continuous for all $\gamma < H$. .

Remark 7. If we assume also that hypothesis (16) holds, the integral process $\int_0^t\int_0^s D_r u_s (s-r)^{2H-2} drds$ is continuous a.s. in the variable t, which gives us the continuity of the Stratonovich integral process $\int_0^t u_s dB_s$.

7 Itô's formula

The purpose of this section is to prove a change-of-variable formula for the indefinite divergence integral. We recall first the local property of the divergence operator.

Lemma 6 *Let u be an element of $\mathbb{D}^{1,2}(\mathcal{H})$. If $u = 0$ a.s. on a set $A \in \mathcal{F}$, then $\delta(u) = 0$ a.s. on A.*

Given a set L of $\mathcal{H}$-valued random variables we will denote by L_{loc} the set of $\mathcal{H}$-valued random variables u such that there exists a sequence $\{(\Omega^n, u^n)\}, n \geq 1\} \subset \mathcal{F} \times L$ with the following properties:

i) $\Omega^n \uparrow \Omega$ a.s.

ii) $u = u^n$, a.e. on $[0,T] \times \Omega_n$.

We then say that $\{(\Omega^n, u^n)\}$ localizes u in L. If $u \in \mathbb{D}^{1,2}_{loc}(\mathcal{H})$ by Lemma 6 we can define without ambiguity $\delta(u)$ by setting

$$\delta(u)|_{\Omega^n} = \delta(u^n)|_{\Omega^n}$$

for each $n \geq 1$, where $\{(\Omega^n, u^n)\}$ is a localizing sequence for u in L.

The following proposition asserts that the divergence processes have zero quadratic variation.

Proposition 7 *Assume that $u = \{u_t, t \in [0,T]\}$ is a process in the space $\mathbb{D}^{1,2}_{loc}(|\mathcal{H}|)$. Then,*

$$\sum_{i=0}^{n-1} \left(\int_{t_i}^{t_{i+1}} u_s \delta B_s \right)^2 \to 0$$

in probability, as $|\pi| \to 0$, where π denotes a partition $\pi = \{0 = t_0 < ... < t_n = T\}$ of the interval $[0,T]$. Moreover, the convergence is in $L^1(\Omega)$ if $u \in \mathbb{D}^{1,2}(|\mathcal{H}|)$.

Proof. It suffices to assume that $u \in \mathbb{D}^{1,2}(|\mathcal{H}|)$. By the isometry property of the divergence operator we have

$$
\begin{aligned}
& E\sum_{i=0}^{n-1}\left(\int_{t_i}^{t_{i+1}} u_s \delta B_s\right)^2 \\
= \;& \alpha_H E\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}} u_r u_s |r-s|^{2H-2} drds \\
& +\alpha_H^2 E\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}}\int_{[0,T]^2} D_\theta u_r D_\eta u_s \\
& \times |r- \ \eta|^{2H-2}|\theta-s|^{2H-2} \ d\theta d\eta drds,
\end{aligned}
$$

and this expression converges to zero as $|\pi|$ tends to zero because u belongs to $\mathbb{D}^{1,2}(|\mathcal{H}|)$. ■

Now we are in a position to prove the main result of this section.

Theorem 8 *Let F be a function of class $C^2(\mathbb{R})$. Assume that $u = \{u_t, t \in [0,T]\}$ is a process in the space $\mathbb{D}_{loc}^{2,2}(|\mathcal{H}|)$ such that the indefinite integral $X_t = \int_0^t u_s \delta B_s$ is a.s. continuous. Assume that $\|u\|_2$ belongs to $\mathcal{H}$. Then for each $t \in [0,T]$ the following formula holds*

$$
\begin{aligned}
F(X_t) &= F(0) + \int_0^t F'(X_s) u_s \delta B_s \\
&+ \alpha_H \int_0^t F''(X_s)\ u_s \left(\int_0^T |s-\sigma|^{2H-2}\left(\int_0^s D_\sigma u_\theta \delta B_\theta\right) d\sigma\right) ds \\
&+ \alpha_H \int_0^t F''(X_s) u_s \left(\int_0^s u_\theta (s-\theta)^{2H-2} d\theta\right) ds.
\end{aligned} \tag{23}
$$

Proof. uppose that (Ω^n, u^n) is a localizing sequence for u^n in $\mathbb{D}^{2,2}(|\mathcal{H}|)$. For each positive integer k let ψ_k be a smooth function such that $0 \le \psi_k \le 1$, $\psi_k(x) = 0$ if $|x| \ge k+1$, and $\psi_k(x) = 1$ if $|x| < k+1$. Define

$$
u_t^{n,k} = u_t^n \psi_k(\|u^n\|_{|\mathcal{H}|}^2).
$$

Set $X_t^{n,k} = \int_0^t u_s^{n,k} \delta B_s$ and consider the family of sets

$$
G^{n,k} = \Omega^n \cap \left\{\sup_{t\in[0,T]} |X_t| \le k\right\} \cap \left\{\|u^n\|_{|\mathcal{H}|}^2 \le k\right\}.
$$

Define also $F^k = F\psi_k$. Then it suffices to show the result for the process $u^{n,k}$ and the function F^k. In this way we can assume that $u \in \mathbb{D}^{2,2}(|\mathcal{H}|)$, $\|u\|_{|\mathcal{H}|}$ is uniformly bounded and the functions F, F' and F'' are bounded. Moreover we can assume that the process X has a continuous version.

Set $t_i = \frac{it}{n}$, $i = 0, ..., n$. Applying Taylor expansion up to the second order we obtain

$$F(X_t) - F(0) = \sum_{i=0}^{n-1} \left[F'(X_{t_i})\left(X_{t_{i+1}} - X_{t_i}\right) + \frac{1}{2} F''\left(\bar{X}_{t_i}\right)\left(X_{t_{i+1}} - X_{t_i}\right)^2 \right],$$

where $\bar{X}_{t_i}$ denotes a random intermediate point between X_{t_i} and $X_{t_{i+1}}$. Applying now equality (15) it follows that

$$F'(X_{t_i}) \int_{t_i}^{t_{i+1}} u_s \delta B_s = \int_{t_i}^{t_{i+1}} F'(X_{t_i}) u_s \delta B_s + \left\langle D\left(F'(X_{t_i})\right), u\mathbf{1}_{[t_i,t_{i+1}]}\right\rangle_{\mathcal{H}}.$$

Observe that by our assumptions, $F'(X_{t_i})u$ belongs to $\mathbb{D}^{1,2}(|\mathcal{H}|)$, and all the terms in the above equality are square integrable. On the other hand,

$$\begin{aligned}\left\langle D\left(F'(X_{t_i})\right), u\mathbf{1}_{[t_i,t_{i+1}]}\right\rangle_{\mathcal{H}} &= \left\langle F''(X_{t_i}) u\mathbf{1}_{[0,t_i]}, u\mathbf{1}_{[t_i,t_{i+1}]}\right\rangle_{\mathcal{H}} \\ &\quad + F''(X_{t_i}) \left\langle \int_0^{t_i} Du_\theta \delta B_\theta, u\mathbf{1}_{[t_i,t_{i+1}]}\right\rangle_{\mathcal{H}}.\end{aligned}$$

Now the proof will be decomposed into several steps.

Step 1. The term

$$\sum_{i=0}^{n-1} F''\left(\bar{X}_{t_i}\right)\left(X_{t_{i+1}} - X_{t_i}\right)^2$$

converges to zero in $L^1(\Omega)$ as n tends to infinity by Proposition 7.

Step 2. The term

$$\begin{aligned}&\sum_{i=0}^{n-1} \left\langle F''(X_{t_i}) u\mathbf{1}_{[0,t_i]}, u\mathbf{1}_{[t_i,t_{i+1}]}\right\rangle_{\mathcal{H}} \\ = \;& \alpha_H \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} F''(X_{t_i}) u_r \left(\int_0^{t_i} u_s (r-s)^{2H-2} ds\right) dr\end{aligned}$$

converges in $L^1(\Omega)$ to

$$\alpha_H \int_0^t F''(X_s) u_s \left(\int_0^s u_\theta (s-\theta)^{2H-2} d\theta\right) ds$$

as n tends to infinity by the dominated convergence theorem and the continuity of $F(X_t)$.

Step 3. Let us see that

$$\sum_{i=0}^{n-1} F''\left(X_{t_i}\right)\left\langle \int_0^{t_i} Du_\theta \delta B_\theta, u\mathbf{1}_{[t_i,t_{i+1}]}\right\rangle_{\mathcal{H}}$$

converges in $L^1\left(\Omega\right)$ to

$$\alpha_H \int_0^t F''(X_s)\ u_s \left(\int_0^T |s-\sigma|^{2H-2}\left(\int_0^s D_\sigma u_\theta \delta B_\theta\right) d\sigma\right) ds.$$

Notice that

$$\begin{aligned}
&\left\langle \int_0^{t_i} Du_\theta \delta B_\theta, u\mathbf{1}_{[t_i,t_{i+1}]}\right\rangle_{\mathcal{H}} \\
&= \alpha_H \int_0^T \int_{t_i}^{t_{i+1}} \left(\int_0^{t_i} D_\sigma u_\theta \delta B_\theta\right) u_s \left|s-\sigma\right|^{2H-2} ds d\sigma.
\end{aligned}$$

Then we can write

$$\begin{aligned}
&\sum_{i=0}^{n-1} F''\left(X_{t_i}\right)\left\langle \int_0^{t_i} Du_\theta \delta B_\theta, u\mathbf{1}_{[t_i,t_{i+1}]}\right\rangle_{\mathcal{H}} \\
&-\alpha_H \int_0^t F''(X_s)\ u_s \left(\int_0^T |s-\sigma|^{2H-2}\left(\int_0^s D_\sigma u_\theta \delta B_\theta\right) d\sigma\right) ds \\
=\ & \alpha_H \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \left[F''\left(X_{t_i}\right) - F''(X_s)\right] u_s \\
&\times \left(\int_0^T |s-\sigma|^{2H-2}\left(\int_0^s D_\sigma u_\theta \delta B_\theta\right) d\sigma\right) ds \\
&-\alpha_H \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} F''(X_{t_i})\ u_s \left(\int_0^T |s-\sigma|^{2H-2}\left(\int_{t_i}^s D_\sigma u_\theta \delta B_\theta\right) d\sigma\right) ds \\
:\ & = T_1 + T_2.
\end{aligned}$$

Let us first consider the term T_2. We have the following estimate

$$\begin{aligned}
& \alpha_H E \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \left| \int_0^T |s-\sigma|^{2H-2} \; u_s \left(\int_{t_i}^s D_\sigma u_\theta \delta B_\theta \right) d\sigma \right| ds \\
= \; & E \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \left| \int_0^T \int_0^{s\wedge\sigma} \quad u_s \frac{\partial K}{\partial s}(s,r) \right. \\
& \left. \times \left(\int_{t_i}^s D_\sigma u_\theta \delta B_\theta \right) \frac{\partial K}{\partial \sigma}(\sigma,r) dr d\sigma \right| ds \\
\leq \; & \int_0^t \left(\int_r^t \quad \|u_s\|_2 \frac{\partial K}{\partial s}(s,r) ds \right) \\
& \times \sup_i \sup_{s\in[t_i,t_{i+1}]} \left\| \int_r^T \left(\int_{t_i}^s D_\sigma u_\theta \delta B_\theta \right) \frac{\partial K}{\partial \sigma}(\sigma,r) d\sigma \right\|_2 dr.
\end{aligned}$$

Notice that the term

$$\int_0^t \left(\int_r^t \quad \|u_s\|_2 \frac{\partial K}{\partial s}(s,r) ds \right)^2 dr$$

is finite because $\|u\|_2$ belongs to the space $\mathcal{H}$. On the other hand, we can write

$$\begin{aligned}
& E\left[\left(\int_{t_i}^s D_\sigma u_\theta \delta B_\theta \right) \left(\int_{t_i}^s D_{\sigma'} u_\theta \delta B_\theta \right) \right] \\
= \; & \alpha_H \int_{[t_i,s]^2} E(D_\sigma u_\theta D_{\sigma'} u_{\theta'}) |\theta-\theta'|^{2H-2} d\theta d\theta' \\
& + \alpha_H^2 \int_{[0,T]^2} \int_{[t_i,s]^2} E(D_\eta D_\sigma u_\theta D_{\eta'} D_{\sigma'} u_{\theta'}) \\
& \times |\theta-\eta'|^{2H-2} |\theta'-\eta|^{2H-2} d\theta d\theta' d\eta d\eta'.
\end{aligned}$$

Hence,

$$\begin{aligned}
& E \int_0^T \left(\int_r^T \left(\int_{t_i}^s D_\sigma u_\theta \delta B_\theta \right) \frac{\partial K}{\partial \sigma}(\sigma,r) d\sigma \right)^2 dr \\
= \; & \alpha_H E \int_0^T \int_0^T \left(\int_{t_i}^s D_\sigma u_\theta \delta B_\theta \right) \left(\int_{t_i}^s D_{\sigma'} u_\theta \delta B_\theta \right) \\
& \times |\sigma-\sigma'|^{2H-2} d\sigma d\sigma',
\end{aligned}$$

and this converges to zero, uniformly in s and i, because u belongs to the space $\mathbb{D}^{2,2}$.

In a similar way we can show that

$$E\int_0^t \left|\int_0^T u_s\,|s-\sigma|^{2H-2}\left(\int_0^s D_\sigma u_\theta \delta B_\theta\right)d\sigma\right| ds < \infty. \tag{24}$$

(24) which implies that the term T_1 converges to zero in $L^1(\Omega)$.

Step 4. Observe that $F'(X_s)\,u_s$ belongs to $\mathbb{D}^{1,2}(|\mathcal{H}|)$, because $\|u\|_{|\mathcal{H}|}$ is uniformly bounded together with the function F and its first two derivatives. The following convergence

$$\sum_{n=0}^{n-1} F'\left(\bar{X}_{t_i}\right) u\mathbf{1}_{[t_i,t_{i+1}]} \to F'(X)\,u \tag{25}$$

holds in the norm of the space $L^2(\Omega;|\mathcal{H}|)$. In fact,

$$\begin{aligned}
& E\left\|\sum_{n=0}^{n-1} F'\left(\bar{X}_{t_i}\right) u\mathbf{1}_{[t_i,t_{i+1}]} \to F'(X)\,u\right\|_{|\mathcal{H}|}^2 \\
= \;& E\sum_{n=0}^{n-1}\sum_{m=0}^{m-1}\int_{t_i}^{t_{i+1}}\int_{t_j}^{t_{j+1}} \left|\left(F'\left(\bar{X}_{t_i}\right)-F'(X_s)\right)u_s\right| \\
& \times\left|\left(F'\left(\bar{X}_{t_i}\right)-F'(X_r)\right)u_r\right| |r-s|^{2H-2}\,drds,
\end{aligned}$$

which converges to zero as $n\to\infty$, by the dominated convergence theorem. As a consequence, for any smooth and cylindrical random variable G we haveLet us see that

$$\lim_n E\left(G\sum_{n=0}^{n-1}\int_{t_i}^{t_{i+1}} F'\left(\bar{X}_{t_i}\right) u_s\delta B_s\right) = E\left(G\int_0^t F'(X_s)\,u_s\delta B_s\right). \tag{26}$$

On the other hand, by the previous steps, the sequence $\sum_{n=0}^{n-1}\int_{t_i}^{t_{i+1}} F'\left(\bar{X}_{t_i}\right) u_s\delta B_s$ converges in $L^1(\Omega)$, as n tends to infinity to

$$\begin{aligned}
& F(X_t)-F(0)-\alpha_H\int_0^t F''(X_s)\;u_s\left(\int_0^T |s-\sigma|^{2H-2}\left(\int_0^s D_\sigma u_\theta\delta B_\theta\right)d\sigma\right)ds \\
& -\alpha_H\int_0^t F''(X_s)u_s\left(\int_0^s u_\theta\,(s-\theta)^{2H-2}\,d\theta\right)ds.
\end{aligned}$$

This allows us to complete the proof. ■

Remarks: If the process u is adapted, then Itô's formula can be written as

$$F(X_t) = F(0) + \int_0^t F'(X_s) u_s \delta B_s \tag{27}$$
$$+ \alpha_H \int_0^t F''(X_s)\ u_s \left(\int_0^s \left(\int_0^\theta |s-\sigma|^{2H-2} D_\sigma u_\theta d\sigma \right) \delta B_\theta \right) ds$$
$$+ \alpha_H \int_0^t F''(X_s) u_s \left(\int_0^s u_\theta \left(s-\theta\right)^{2H-2} d\theta \right) ds.$$

On the other hand, $2\alpha_H \left(s-\theta\right)^{2H-2} \mathbf{1}_{[0,s]}(\theta)$ is an approximation of the identity as H tends to $\frac{1}{2}$. Therefore, taking the limit as H converges to $\frac{1}{2}$ in equation (27) we recover the usual Itô's formula for the classical Brownian motion, and taking the limit in equation (23) we obtain the Itô's formula for the Skorohod integral proved by Nualart and Pardoux (see, for instance, [18]).

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