

# STOCHASTIC INTERACTING PARTICLE SYSTEMS AND NONLINEAR KINETIC EQUATIONS

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We present the stochastic approach to nonlinear kinetic equations (without gradient terms) in a unifying general framework, which covers many interactions important in applications, such as coagulation, fragmentation, inelastic collisions, as well as source and efflux terms. We provide conditions for the existence of corresponding stochastic particle systems in the sense of regularity (nonexplosion) of a jump process with unbounded intensity. Using an appropriate space of measure-valued functions, we prove relative compactness of the sequence of processes and characterize the weak limits in terms of solutions to the nonlinear equation. As a particular application, we derive existence theorems for Smoluchowski's coagulation equation with fragmentation, efflux and source terms, and for the Boltzmann equation with inelastic collisions.

**1. Introduction.** Studies of the connection between stochastic interacting particle systems and nonlinear kinetic equations have a long history. The earliest references seem to be the papers by Kac [20] and Leontovich [22], where the Boltzmann equation (cf. [6] and [7]) from rarefied gas dynamics was considered. In the simplest (spatially homogeneous) case, this equation describes the time evolution of the velocity distribution of gas molecules that change their velocities during collisions. The stochastic approach to the Boltzmann equation has been further developed in [25] and [32]. The practically relevant (unbounded) hard sphere collision kernel was treated in [31]. We refer to [35] for more comments and references concerning this field. Algorithms based on the corresponding stochastic interacting particle systems are presently the most widely used numerical tools in kinetic theory [5].

Stochastic particle systems related to Smoluchowski's coagulation equation (cf. [8] and [34]) were used in [14], [23] and [24] in the context of various applications. In the spatially homogeneous case, this equation describes the time evolution of the size distribution of particles moving in a physical medium and merging during collisions. The stochastic approach to Smoluchowski's coagulation equation has been reviewed in [1]. We refer to [10] and [12] for comments and references concerning applications of the particle systems in numerics. Note that the coagulation process can be considered as a chemical system with infinitely many species (characterized by size) and simple reactions (merging of two

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partners). The study of the relationship between stochastic and deterministic models for chemical systems with a finite number of species and reactions goes back to [21] (cf. [15] concerning numerical applications).

Developing the stochastic approach to the Boltzmann equation, systems with a general binary interaction between particles and a general (Markovian) single-particle evolution (including spatial motion) were considered in [17] and [26]. Results concerning the approximation of the solution to the corresponding nonlinear kinetic equation by the particle system were obtained in the case of bounded intensities and a constant (in time) number of particles. The weak law of large numbers for stochastic particle systems related to Smoluchowski's coagulation equation with general kernels has attracted attention only recently (cf. [1], Problem 10(a)). Meanwhile, rigorous results of this type are contained, for example, in [11] (continuous coagulation–fragmentation equation), [18] (discrete coagulation–fragmentation equation with bounded kernels), [19] (discrete coagulation–fragmentation equation) and [28] (continuous coagulation equation). Most of the practically relevant coagulation kernels are unbounded. Moreover, if the kernel grows sufficiently fast, solutions to the limiting equation show the so-called gelation effect (loss of mass in finite time). In the stochastic system, this effect corresponds to the fast formation of a very big particle (with size comparable to the size of the whole system). Gelation can also be related to explosion phenomena in specific stochastic models. It has been observed that the stochastic approach provides new existence results for the deterministic limiting equation (cf. the discussion in [11]), besides the approximation results that were the original motivation.

The purpose of this paper is to present the stochastic approach to nonlinear kinetic equations (without gradient terms) in a unifying general framework, which covers the cases mentioned above and allows one to include other effects important in applications. Examples are multiple fragmentation (splitting of particles in more than two pieces), structured clusters (e.g., consisting of several chemical species), inelastic collisions (leading to dissipation of energy), internal degrees of freedom (e.g., rotating gas molecules), sources and efflux (creation and annihilation of particles) and so on (see also [29] and [36], Sections 3.3 and 3.7). To this end, we use an arbitrary locally compact separable metric space as the type space of a single particle and consider rather general multiple interactions with unbounded rates. The state space of the particle system consists of appropriately normalized discrete measures on the type space. The limiting equation is considered in a weak form so that solutions are functions of time taking values in some space of measures on the type space.

The paper is organized as follows. The main results are given in Section 2. The first theorem provides conditions for the existence of the particle system in the sense of regularity (nonexplosion) of a jump process. The second theorem studies the property of relative compactness of the sequence (with respect to the normalization parameter) of processes. The third theorem characterizes the weak

limits of the sequence in terms of solutions to a deterministic nonlinear equation. An existence theorem for the limiting equation is given in the form of a special corollary. In Section 3, the general results are applied to some specific models. First, the coagulation–fragmentation equation with source and efflux is considered. In this case, new existence results are obtained. Second, a generalized Boltzmann equation with inelastic collisions is considered. Such equations have attracted considerable interest in recent years in connection with the study of granular materials (cf., e.g., [4] and [16]). An existence result is obtained that covers the known results in the classical Boltzmann case. The rest of the paper is concerned with the proofs of the main results. In Section 4, we consider the so-called minimal jump process, with a compactly bounded kernel on some locally compact separable metric space, and prove a theorem concerning its regularity (nonexplosion). In Section 5, we give proofs of our main theorems, using the results from the previous section and applying techniques from [13]. Some auxiliary results are collected in the Appendix in order to make the paper self-contained.

In conclusion, we note that convergence of the particle system to the solution of the limiting deterministic equation (weak law of large numbers) is obtained under the assumption of uniqueness of that solution. So far, no general uniqueness results have been obtained (cf. [28] concerning the coagulation case). However, the general framework proposed in this paper provides a basis for the derivation and justification of stochastic algorithms in many fields of application. The results cover both unbounded kernels (thus avoiding any truncation leading to unnecessary numerical errors) and a variable number of particles (possibly unbounded in time). As to concrete applications, we worked out only two specific models in order to keep the length of the paper reasonable. However, any combinations of these interaction models, and many others, can be considered.

**2. Main results.** Let  $E$  and  $E'$  be metric and separable spaces. Let  $M(E)$ ,  $B(E)$ ,  $C(E)$ ,  $C_b(E)$  and  $C_c(E)$  denote the sets of functions on  $E$  that are measurable, bounded measurable, continuous, bounded continuous and continuous with compact support, respectively. For  $E$  locally compact, let  $C_0(E)$  denote the set of continuous functions on  $E$  vanishing at infinity as the closure of  $C_c(E)$  with respect to the sup-norm  $\|\cdot\|$ . Furthermore, the sets of Borel measures, bounded Borel measures and probability measures on the Borel- $\sigma$ -algebra  $\mathcal{B}(E)$  are denoted by  $\mathcal{M}(E)$ ,  $\mathcal{M}_b(E)$  and  $\mathcal{P}(E)$ , respectively. The Dirac measure on  $\xi \in E$  is denoted by  $\delta_\xi$ . Vague and weak convergence of Borel measures are denoted by  $\mu_n \rightarrow_v \mu$  and  $\mu_n \rightarrow_w \mu$ , respectively, whereas the  $\Rightarrow$  sign is used for convergence in distribution. Let  $C([0, \infty), E)$  be the space of continuous paths and  $D([0, \infty), E)$  the Skorohod space of cadlag paths. For  $\varphi \in M(E)$  and  $\mu \in \mathcal{M}(E)$ , we use the notation  $\langle \varphi, \mu \rangle = \int \varphi d\mu$ . Finally, let  $\mathbb{1}_A$  denote the indicator function of a set  $A$ . A kernel from  $E$  to  $E'$  (on  $E$  if  $E = E'$ ) is a function  $\lambda : E \times \mathcal{B}(E') \rightarrow [0, \infty)$  such that

$$\lambda(\cdot, B) \in M(E) \quad \forall B \in \mathcal{B}(E') \quad \text{and} \quad \lambda(\xi, \cdot) \in \mathcal{M}_b(E') \quad \forall \xi \in E.$$

A kernel  $\lambda$  is called *compactly bounded* if

$$\sup_{\xi \in C} \lambda(\xi, E') < \infty \quad \text{for any compact } C \subset E.$$

We consider particles with types from a locally compact separable metric space  $\mathcal{Z}$  and weights  $1/N$ . Define the *state space of the particle system* as

$$(2.1) \quad E^N = \left\{ \frac{1}{N} \sum_{i=1}^n \delta_{x_i} : n \geq 0, x_i \in \mathcal{Z}, i = 1, \dots, n \right\}, \quad N = 1, 2, \dots$$

Any event in the system consists of the interaction of at most  $R$  particles and produces as a result at most  $K$  particles. This includes, for example, the generation of new particles from a source, the extinction or transformation of single particles and the collision of two particles. The admissible subsequent states of  $\mu \in E^N$  are denoted by

$$(2.2) \quad \begin{aligned} J_0(\mu, \xi) &= \mu + \frac{1}{N} \xi, \\ J_1(\mu, i, \xi) &= \mu + \frac{1}{N} [\xi - \delta_{x_i}], \\ J_r(\mu, i_1, \dots, i_r, \xi) &= \mu + \frac{1}{N} [\xi - \delta_{x_{i_1}} - \dots - \delta_{x_{i_r}}], \quad r = 2, \dots, R, \end{aligned}$$

where  $i_1, \dots, i_r$  are pairwise distinct indices from  $\{1, \dots, n\}$  and  $\xi \in E_K$ , with

$$(2.3) \quad E_K = \left\{ \sum_{i=1}^n \delta_{x_i} : 0 \leq n \leq K, x_i \in \mathcal{Z}, i = 1, \dots, n \right\},$$

for some given natural numbers  $R$  and  $K$ . Both spaces  $E^N$  and  $E_K$  are equipped with the weak topology.

The *rates* for the different events are determined by a measure  $q_0$  and kernels  $q_1, \dots, q_R$  such that

$$(2.4) \quad \begin{aligned} q_0 &\in \mathcal{M}_b(E_K) \quad \text{and} \\ q_r &: \mathcal{Z}^r \times \mathcal{B}(E_K) \rightarrow [0, \infty), \quad r = 1, \dots, R, \end{aligned}$$

are compactly bounded. Thus, transitions (jumps) in the system are governed by the kernel

$$(2.5) \quad \begin{aligned} \lambda^N(\mu, B) &= N \int_{E_K} \mathbb{1}_B(J_0(\mu, \xi)) q_0(d\xi) + \sum_{i=1}^n \int_{E_K} \mathbb{1}_B(J_1(\mu, i, \xi)) q_1(x_i, d\xi) \\ &+ \sum_{r=2}^R \frac{1}{N^{r-1}} \sum_{1 \leq i_1, \dots, i_r \leq n} \int_{E_K} \mathbb{1}_B(J_r(\mu, i_1, \dots, i_r, \xi)) \\ &\quad \times q_r(x_{i_1}, \dots, x_{i_r}, d\xi), \quad B \in \mathcal{B}(E^N), \end{aligned}$$

where  $\mu \in E^N$  and  $\tilde{\sum}$  denotes summation over pairwise distinct indices.

We first provide conditions for the regularity (nonexplosion) of the system.

**THEOREM 2.1 (Regularity).** *Consider a locally compact separable metric space  $\mathcal{Z}$  and a function*

$$(2.6) \quad H \in C(\mathcal{Z}) \quad \text{such that } H > 0 \text{ and } \frac{1}{H} \in C_0(\mathcal{Z}).$$

Suppose  $\nu_0^N \in \mathcal{P}(E^N)$  satisfies

$$(2.7) \quad \int_{E^N} \langle H, \mu \rangle \nu_0^N(d\mu) \leq c_0$$

for some  $c_0 \geq 0$ . Suppose  $q_0, q_1, \dots, q_R$  satisfy (2.4) and are such that the kernel (2.5) satisfies

$$(2.8) \quad \int_{E^N} [\langle H, \mu_1 \rangle - \langle H, \mu \rangle] \lambda^N(\mu, d\mu_1) \leq c_1 [\langle H, \mu \rangle + c'_1] \quad \forall \mu \in E^N$$

for some  $c_1, c'_1 \geq 0$ .

Then there exists a random process  $X^N$  with sample paths in  $D([0, \infty), E^N)$  that is indistinguishable from the minimal jump process, corresponding to the kernel  $\lambda^N$  and the initial distribution  $\nu_0^N$ .

Note that, in applications, the function  $H$  is often related to conserved quantities of the particle system, such as mass in the coagulation–fragmentation case or energy in the Boltzmann case.

Next, we study the asymptotic properties of the sequence  $X^N$ . To this end, we construct an appropriate common state space. Consider two functions

$$(2.9) \quad h, H \in C(\mathcal{Z}) : 0 \leq h(x) \leq cH(x) \quad \forall x \in \mathcal{Z} \text{ for some } c > 0,$$

the set

$$(2.10) \quad \mathcal{M}(\mathcal{Z}, H) = \{ \mu \in \mathcal{M}(\mathcal{Z}) : \langle H, \mu \rangle < \infty \}$$

and the metric

$$(2.11) \quad d_h(\mu, \nu) = d_0(\mu, \nu) + \min\{1, |\langle h, \mu \rangle - \langle h, \nu \rangle|\}, \quad \mu, \nu \in \mathcal{M}(\mathcal{Z}, H),$$

where  $d_0$  is a metric generating the vague topology. Introduce the space

$$(2.12) \quad \mathcal{M}(\mathcal{Z}, H, h) = (\mathcal{M}(\mathcal{Z}, H), d_h).$$

Note that  $\mathcal{M}(\mathcal{Z}, 0) = \mathcal{M}(\mathcal{Z})$ ,  $\mathcal{M}(\mathcal{Z}, 1) = \mathcal{M}_b(\mathcal{Z})$  and

$$(2.13) \quad \mathcal{M}(\mathcal{Z}, H) \subset \mathcal{M}_b(\mathcal{Z}) \quad \text{if } \inf_{x \in \mathcal{Z}} H(x) > 0.$$

According to [2], Theorem 45.7, the metric  $d_1$  generates the weak topology on  $\mathcal{M}_b(\mathcal{Z})$ .

**THEOREM 2.2 (Relative compactness).** *Consider a locally compact separable metric space  $\mathcal{Z}$  and functions  $H$  satisfying (2.6) and*

$$(2.14) \quad h \in C(\mathcal{Z}) \quad \text{such that } h \geq 0 \text{ and } \frac{h}{H} \in C_0(\mathcal{Z}).$$

*Suppose  $\nu_0^N \in \mathcal{P}(E^N)$  satisfy (2.7) uniformly in  $N$ . Suppose  $q_0, q_1, \dots, q_R$  are such that (2.8) is satisfied uniformly in  $N$  and*

$$(2.15) \quad q_0(E_K) \leq c_2 \quad \text{and} \quad q_r(x, E_K) \leq c_2 H(x_1) \cdots H(x_r) \\ \forall x = (x_1, \dots, x_r) \in \mathcal{Z}^r$$

*for  $r = 1, \dots, R$  and some  $c_2 \geq 0$ .*

*Then the processes  $X^N$  form a relatively compact sequence of  $D([0, \infty), \mathcal{M}(\mathcal{Z}, H, h))$ -valued random variables.*

**THEOREM 2.3 (Characterization of weak limits).** *Consider a locally compact separable metric space  $\mathcal{Z}$  and functions  $H, h$  satisfying (2.6) and (2.14). Suppose  $\nu_0^N \in \mathcal{P}(E^N)$  satisfy (2.7) uniformly in  $N$  and are such that*

$$(2.16) \quad X^N(0) \Rightarrow \mu_0 \quad \text{for some } \mu_0 \in \mathcal{M}(\mathcal{Z}, H).$$

*Suppose  $q_0, q_1, \dots, q_R$  are such that (2.8) is satisfied uniformly in  $N$ ,*

$$(2.17) \quad q_0(E_K) \leq c_2 \quad \text{and} \quad q_r(x, E_K) \leq c_2 h(x_1) \cdots h(x_r) \\ \forall x = (x_1, \dots, x_r) \in \mathcal{Z}^r$$

*for  $r = 1, \dots, R$  and some  $c_2 \geq 0$ , and*

$$(2.18) \quad q_r(\cdot, E_K) \in C(\mathcal{Z}^r), \quad \int_{E_K} \langle \varphi, \xi \rangle q_r(\cdot, d\xi) \in C(\mathcal{Z}^r), \quad r = 1, \dots, R,$$

*for any  $\varphi \in C_c(\mathcal{Z})$ .*

*Then the processes  $X^N$  form a relatively compact sequence of  $D([0, \infty), \mathcal{M}(\mathcal{Z}, H, h))$ -valued random variables and every weak limit  $X$  satisfies, almost surely,*

$$(2.19) \quad \langle \varphi, X(t) \rangle = \langle \varphi, \mu_0 \rangle + \int_0^t \mathfrak{G}(\varphi, X(s)) ds \quad \forall t \geq 0, \varphi \in C_c(\mathcal{Z}),$$

*where, for  $\mu \in \mathcal{M}(\mathcal{Z}, H)$ ,*

$$(2.20) \quad \mathfrak{G}(\varphi, \mu) = \int_{E_K} \langle \varphi, \xi \rangle q_0(d\xi) \\ + \sum_{r=1}^R \int_{\mathcal{Z}} \cdots \int_{\mathcal{Z}} \int_{E_K} [\langle \varphi, \xi \rangle - \varphi(x_1) - \cdots - \varphi(x_r)] \\ \times q_r(x_1, \dots, x_r, d\xi) \mu(dx_1) \cdots \mu(dx_r).$$

COROLLARY 2.4 (Continuity). *Under the assumptions of Theorem 2.2, every weak limit  $X$  satisfies*

$$(2.21) \quad P(X \in C([0, \infty), \mathcal{M}(\mathcal{Z}, H, h))) = 1.$$

COROLLARY 2.5 (Moments). *Under the assumptions of Theorem 2.2, every weak limit  $X$  satisfies*

$$(2.22) \quad \mathbb{E}\langle H, X(t) \rangle \leq (c_0 + c'_1) \exp(c_1 t) \quad \forall t \geq 0.$$

COROLLARY 2.6 (Existence). *Consider a locally compact separable metric space  $\mathcal{Z}$  and functions  $H, h$  satisfying (2.6) and (2.14). Let  $\mu_0 \in \mathcal{M}(\mathcal{Z}, H)$ . Suppose  $q_0, q_1, \dots, q_R$  are such that (2.8) is satisfied uniformly in  $N$ , and assumptions (2.17) and (2.18) are fulfilled.*

*Then there exists some  $\mu \in C([0, \infty), \mathcal{M}(\mathcal{Z}, H, h))$  solving the macroscopic equation*

$$(2.23) \quad \langle \varphi, \mu(t) \rangle = \langle \varphi, \mu_0 \rangle + \int_0^t \mathcal{G}(\varphi, \mu(s)) ds \quad \forall t \geq 0, \varphi \in C_c(\mathcal{Z}).$$

COROLLARY 2.7 (Convergence). *Let the assumptions of Theorem 2.3 be fulfilled. If there is a unique  $\mu \in C([0, \infty), \mathcal{M}(\mathcal{Z}, H, h))$  satisfying equation (2.23), then the stochastic processes  $X^N$  converge in distribution to  $\mu$ .*

COROLLARY 2.8. *For nonnegative  $g \in C(\mathcal{Z})$  and  $\gamma > 0$ , consider the set*

$$(2.24) \quad \mathcal{M}_\gamma(\mathcal{Z}, g) = \{\mu \in \mathcal{M}(\mathcal{Z}) : \langle g, \mu \rangle \leq \gamma\}.$$

*Suppose  $q_0, q_1, \dots, q_R$  are such that, for all  $N = 1, 2, \dots$ ,*

$$(2.25) \quad \lambda^N(\mu, E^N \cap \mathcal{M}_\gamma(\mathcal{Z}, g)) = \lambda^N(\mu, E^N) \quad \forall \mu \in E^N \cap \mathcal{M}_\gamma(\mathcal{Z}, g).$$

*Then Theorems 2.1–2.3 hold when the spaces  $E^N$  and  $\mathcal{M}(\mathcal{Z}, H, h)$  are replaced by  $E^N \cap \mathcal{M}_\gamma(\mathcal{Z}, g)$  and  $(\mathcal{M}(\mathcal{Z}, H) \cap \mathcal{M}_\gamma(\mathcal{Z}, g), d_h)$ , respectively.*

COROLLARY 2.9. *Consider a locally compact separable metric space  $\mathcal{Z}$  and functions  $H, h$  satisfying (2.6) and (2.14). Let  $\mu_0 \in \mathcal{M}(\mathcal{Z}, H)$  be such that  $\langle g, \mu_0 \rangle < \infty$  for some nonnegative  $g \in C(\mathcal{Z})$  and consider  $\gamma = \langle g, \mu_0 \rangle + 1$ . Suppose  $q_0, q_1, \dots, q_R$  are such that assumptions (2.17), (2.18) and (2.25) are fulfilled and (2.8) is satisfied uniformly in  $N$ , with  $E^N$  replaced by  $E^N \cap \mathcal{M}_\gamma(\mathcal{Z}, g)$ .*

*Then there exists some  $\mu \in C([0, \infty), (\mathcal{M}(\mathcal{Z}, H) \cap \mathcal{M}_\gamma(\mathcal{Z}, g), d_h))$  solving (2.23).*

We finish this section by providing some basic properties of the objects under consideration, which will be used throughout the paper.

REMARK 2.10. Let  $\mu_k = N^{-1} \sum_{i=1}^{n_k} \delta_{x_i^k}$ ,  $k \geq 1$ , and  $\mu = N^{-1} \sum_{i=1}^n \delta_{x_i}$ . Then  $d_1(\mu_k, \mu) \rightarrow 0$  if and only if there are an  $l \geq 1$  and permutations  $\pi_k$  on  $\{1, \dots, n\}$ ,  $k \geq l$ , such that

$$n_k = n, \quad k \geq l \quad \text{and} \quad \lim_{k \rightarrow \infty, k \geq l} x_{\pi_k(i)}^k = x_i, \quad i = 1, \dots, n.$$

Note that the kernel (2.5) satisfies

$$\begin{aligned} & \int_{E^N} \Psi(v) \lambda^N(\mu, dv) \\ &= N \int_{E_K} \Psi(J_0(\mu, \xi)) q_0(d\xi) + \sum_{i=1}^n \int_{E_K} \Psi(J_1(\mu, i, \xi)) q_1(x_i, d\xi) \\ (2.26) \quad &+ \sum_{r=2}^R \frac{1}{N^{r-1}} \sum_{1 \leq i_1, \dots, i_r \leq n} \int_{E_K} \Psi(J_r(\mu, i_1, \dots, i_r, \xi)) \\ & \quad \times q_r(x_{i_1}, \dots, x_{i_r}, d\xi) \end{aligned}$$

for  $\mu \in E^N$  and appropriate test functions, for example,  $\Psi \in C(E^N)$ . In particular, one obtains

$$\begin{aligned} & \lambda^N(\mu, E^N) \\ (2.27) \quad & \leq N \left[ q_0(E_K) + \sum_{r=1}^R \int_{\mathbb{Z}} \dots \int_{\mathbb{Z}} q_r(x_1, \dots, x_r, E_K) \mu(dx_1) \dots \mu(dx_r) \right]. \end{aligned}$$

REMARK 2.11. If  $\varphi \in C(\mathbb{Z})$ , then the function

$$(2.28) \quad \Phi : E^N \rightarrow \mathbb{R}, \quad \Phi(\mu) = \langle \varphi, \mu \rangle,$$

is continuous, according to Remark 2.10.

Using (2.26) and (2.2), one obtains

$$\begin{aligned} & \int_{E^N} [\langle \varphi, v \rangle - \langle \varphi, \mu \rangle]^k \lambda^N(\mu, dv) \\ &= N^{1-k} \left[ \int_{E_K} \langle \varphi, \xi \rangle^k q_0(d\xi) + \frac{1}{N} \sum_{i=1}^n \int_{E_K} [\langle \varphi, \xi \rangle - \varphi(x_i)]^k q_1(x_i, d\xi) \right. \\ (2.29) \quad &+ \sum_{r=2}^R \frac{1}{N^r} \sum_{1 \leq i_1, \dots, i_r \leq n} \int_{E_K} [\langle \varphi, \xi \rangle - \varphi(x_{i_1}) - \dots - \varphi(x_{i_r})]^k \\ & \quad \left. \times q_r(x_{i_1}, \dots, x_{i_r}, d\xi) \right] \end{aligned}$$



for  $\mu \in E^N$ ,  $k = 1, 2$  and  $\varphi \in C(\mathcal{Z})$ . Introduce the notation

$$\begin{aligned}
 Q_0(\varphi) &= \int_{E_K} \langle \varphi, \xi \rangle q_0(d\xi), \\
 (2.30) \quad Q_r(\varphi, x) &= \int_{E_K} [\langle \varphi, \xi \rangle - \varphi(x_1) - \dots - \varphi(x_r)] q_r(x, d\xi), \\
 & \hspace{20em} r = 1, \dots, R,
 \end{aligned}$$

for  $x = (x_1, \dots, x_r) \in \mathcal{Z}^r$  and  $\varphi \in C(\mathcal{Z})$ . Using (2.29) with  $\varphi = H$  and  $k = 1$ , condition (2.8) takes the form

$$\begin{aligned}
 (2.31) \quad Q_0(H) + \int_{\mathcal{Z}} Q_1(H, x) \mu(dx) + \sum_{r=2}^R \frac{1}{N^r} \sum_{1 \leq i_1, \dots, i_r \leq n} \tilde{Q}_r(H, x_{i_1}, \dots, x_{i_r}) \\
 \leq c_1 [\langle H, \mu \rangle + c'_1].
 \end{aligned}$$

It follows from the definition (2.3) that

$$(2.32) \quad |\langle \varphi, \xi \rangle| \leq \|\varphi\| K \quad \forall \xi \in E_K$$

and

$$\begin{aligned}
 (2.33) \quad & \int_{E_K} |\langle \varphi, \xi \rangle|^k q_0(d\xi) \\
 & \leq K^k \|\varphi\|^k q_0(E_K), \\
 & \int_{E_K} |\langle \varphi, \xi \rangle - \varphi(x_1) - \dots - \varphi(x_r)|^k q_r(x, d\xi) \\
 & \leq (K + r)^k \|\varphi\|^k q_r(x, E_K)
 \end{aligned}$$

for  $k = 1, 2$ ,  $x = (x_1, \dots, x_r) \in \mathcal{Z}^r$ ,  $r = 1, \dots, R$  and  $\varphi \in C_b(\mathcal{Z})$ . Using (2.33) with  $k = 1$ , one obtains [cf. (2.30)]

$$\begin{aligned}
 (2.34) \quad |Q_0(\varphi)| &\leq K \|\varphi\| q_0(E_K), \\
 |Q_r(\varphi, x)| &\leq (K + r) \|\varphi\| q_r(x, E_K) \quad \forall x \in \mathcal{Z}^r, r = 1, \dots, R.
 \end{aligned}$$

**3. Applications.** In this section, we apply the general results, in particular Corollaries 2.6 and 2.9, to several special cases. We consider  $R = 2$ . One has to check conditions (2.8), (2.17), (2.18) and (2.25) (in the case of Corollary 2.9) for appropriate functions  $H, h$  satisfying (2.6) and (2.14).

According to (2.31), condition (2.8) is satisfied for all  $\mu \in E^N$  if

$$(3.1) \quad Q_0(H) = \int_{E_K} \langle H, \xi \rangle q_0(d\xi) \leq c_1,$$

$$(3.2) \quad Q_1(H, x) = \int_{E_K} [\langle H, \xi \rangle - H(x)] q_1(x, d\xi) \leq c_1 H(x) \quad \forall x \in \mathcal{Z}$$

and

$$(3.3) \quad \begin{aligned} Q_2(H, x, y) &= \int_{E_K} [\langle H, \xi \rangle - H(x) - H(y)] q_2(x, y, d\xi) \\ &\leq 0 \quad \forall x, y \in \mathcal{Z}. \end{aligned}$$

Using (2.26) with  $\Psi = \mathbb{1}_{E^N \cap \mathcal{M}_\gamma(\mathcal{Z}, g)}$  and (2.2), one observes that condition (2.25) is satisfied provided that

$$(3.4) \quad \begin{aligned} q_0 &= 0, \quad \langle g, \xi \rangle \leq g(x), \quad q_1(x, d\xi) \text{ a.e.}, \\ \langle g, \xi \rangle &\leq g(x) + g(y), \quad q_2(x, y, d\xi) \text{ a.e. } \forall x, y \in \mathcal{Z}. \end{aligned}$$

In this case, condition (2.8) is satisfied for all  $\mu \in E^N \cap \mathcal{M}_\gamma(\mathcal{Z}, g)$  if (3.2) holds and

$$(3.5) \quad Q_2(H, x, y) \leq c_1 [H(x)g(y) + g(x)H(y) + g(x)g(y)] \quad \forall x, y \in \mathcal{Z}.$$

Indeed, (3.5) implies

$$\frac{1}{N^2} \sum_{1 \leq i \neq j \leq n} Q_2(H, x_i, x_j) \leq c_1(\gamma) [\langle H, \mu \rangle + 1] \quad \forall \mu \in \mathcal{M}_\gamma(\mathcal{Z}, g).$$

3.1. *Source and efflux.* Any source term  $q_0 \in \mathcal{M}_b(E_K)$  [cf. condition (2.17)] satisfying (3.1) is covered by the results. In particular, we consider

$$q_0(B) = \int_{\mathcal{Z}} \mathbb{1}_B(\delta_x) S(dx), \quad B \in \mathcal{B}(E_K),$$

where  $S \in \mathcal{M}_b(\mathcal{Z})$ . Condition (3.1) takes the form

$$(3.6) \quad \int_{\mathcal{Z}} H(x) S(dx) < \infty.$$

Note that [cf. (2.30)]

$$(3.7) \quad Q_0(\varphi) = \langle \varphi, S \rangle.$$

The corresponding jump [cf. (2.2) and (2.5)] consists of adding  $(1/N)\delta_x$  to the system, where  $x$  is distributed according to the normalized measure  $S$ .

Next, we consider the *efflux term*

$$q_1(x, B) = \mathbb{1}_B(0) E(x), \quad x \in \mathcal{Z}, B \in \mathcal{B}(E_K),$$

where 0 denotes the zero measure and  $E \in C(\mathcal{Z})$  is such that

$$(3.8) \quad 0 \leq E(x) \leq c_2 h(x) \quad \forall x \in \mathcal{Z}.$$

Conditions (2.17) and (2.18) are satisfied, and condition (3.2) is fulfilled with  $c_1 = 0$ , since the left-hand side is nonpositive. Note that [cf. (2.30)]

$$(3.9) \quad Q_1(\varphi, x) = -\varphi(x) E(x).$$

The corresponding jump [cf. (2.2) and (2.5)] consists of removing  $(1/N)\delta_x$  from the system, where  $E$  determines the intensity of this event.

3.2. *Coagulation and fragmentation.* Let  $\mathcal{Z} = \mathbb{N}$  or  $\mathcal{Z} = (0, \infty)$ . Consider the *coagulation term*

$$q_2(x, y, B) = \mathbb{1}_B(\delta_{x+y})K(x, y), \quad x, y \in \mathcal{Z}, B \in \mathcal{B}(E_K),$$

where  $K \in C(\mathcal{Z} \times \mathcal{Z})$  is nonnegative, and note that [cf. 2.30]

$$(3.10) \quad Q_2(\varphi, x, y) = [\varphi(x + y) - \varphi(x) - \varphi(y)]K(x, y).$$

Consider the *fragmentation term*

$$q_1(x, B) = \int_{\mathcal{Z}} \mathbb{1}_B(\delta_{x-y} + \delta_y)F(x, dy), \quad x \in \mathcal{Z}, B \in \mathcal{B}(E_K),$$

where  $F$  is a kernel on  $\mathcal{Z}$  satisfying

$$(3.11) \quad F(x, [x, \infty)) = 0 \quad \forall x \in \mathcal{Z},$$

and note that [cf. (2.30)]

$$(3.12) \quad Q_1(\varphi, x) = \int_{\mathcal{Z}} [\varphi(x - y) + \varphi(y) - \varphi(x)]F(x, dy).$$

With the terms (3.7), (3.9), (3.10) and (3.12), (2.23) takes the form

$$\begin{aligned} & \int_{\mathcal{Z}} \varphi(x)\mu(t, dx) \\ &= \int_{\mathcal{Z}} \varphi(x) \mu_0(dx) \\ (3.13) \quad & + \int_0^t \left[ \int_{\mathcal{Z}} \varphi(x)S(dx) - \int_{\mathcal{Z}} \varphi(x)E(x)\mu(s, dx) \right. \\ & \quad + \int_{\mathcal{Z}} \int_{\mathcal{Z}} [\varphi(x + y) - \varphi(x) - \varphi(y)]K(x, y)\mu(s, dx)\mu(s, dy) \\ & \quad \left. + \int_{\mathcal{Z}} \int_{\mathcal{Z}} [\varphi(x - y) + \varphi(y) - \varphi(x)]F(x, dy)\mu(s, dx) \right] ds \\ & \quad \forall t \geq 0, \varphi \in C_c(\mathcal{Z}). \end{aligned}$$

**THEOREM 3.1.** *Let  $\mathcal{Z} = \mathbb{N}$  or  $\mathcal{Z} = (0, \infty)$ . Consider functions  $H$  satisfying (2.6) and*

$$(3.14) \quad \frac{H(x)}{x} \geq \frac{H(y)}{y} \quad \forall 0 < x \leq y < \infty,$$

*and  $h$  satisfying (2.14). Let  $\mu_0, S \in \mathcal{M}(\mathcal{Z}, H)$  and  $E \in C(\mathcal{Z})$  such that (3.8) holds. Consider a function  $K \in C(\mathcal{Z} \times \mathcal{Z})$  such that*

$$(3.15) \quad 0 \leq K(x, y) \leq c_2 h(x)h(y) \quad \forall x, y \in \mathcal{Z},$$

*and a kernel  $F$  satisfying (3.11),*

$$(3.16) \quad F(x_n, \cdot) \xrightarrow{w} F(x, \cdot) \quad \text{if } x_n \rightarrow x \in \mathcal{Z},$$

$$(3.17) \quad \int_{\mathcal{Z}} [H(x - y) + H(y) - H(x)]F(x, dy) \leq c_1 H(x) \quad \forall x \in \mathcal{Z},$$

and

$$(3.18) \quad 0 \leq F(x, \mathcal{Z}) \leq c_1 h(x) \quad \forall x \in \mathcal{Z}.$$

Then there exists some  $\mu \in C([0, \infty), \mathcal{M}(\mathcal{Z}, H, h))$  satisfying (3.13).

REMARK 3.2. In the case  $\mathcal{Z} = (0, \infty)$  and  $F \equiv 0$ , any continuous coagulation kernel satisfying (3.15) is covered, provided that  $H$  satisfies (3.14). Note there is no restriction on  $K$  at 0.

COROLLARY 3.3. Let  $\mathcal{Z} = (0, \infty)$ ,

$$H(x) = x^{-\alpha} + x, \quad \alpha \in (0, 1),$$

and

$$h(x) = x^{-\alpha+\varepsilon} + x^{1-\varepsilon}, \quad \varepsilon \in (0, \alpha).$$

Let  $\mu_0, S \in \mathcal{M}(\mathcal{Z}, H)$  and  $E \in C(\mathcal{Z})$  such that (3.8) holds. Consider a function  $K \in C(\mathcal{Z} \times \mathcal{Z})$  satisfying (3.15). Assume  $F$  has the form

$$(3.19) \quad F(x, dy) = \mathbb{1}_{(0,x)}(y) f(x, y) dy,$$

where  $f$  is continuous with respect to the first argument and satisfies

$$0 \leq f(x, y) \leq \frac{R(x)y^{-\beta}}{x^{1-\beta}}, \quad \beta \in [0, 1 - \alpha], R(x) = c(1 + x^{1-\varepsilon}).$$

Then there exists some  $\mu \in C([0, \infty), \mathcal{M}(\mathcal{Z}, H, h))$  satisfying (3.13).

To our knowledge, the most general existence result in the continuous case, including source and efflux terms, is contained in [9], Theorem 2.2. There it is assumed that both  $K$  and  $f$  have compact support and that both the source term and the initial distribution have a finite moment of some order  $r \geq 1$ . Thus, Corollary 3.3 provides a new existence result for unbounded  $K$  and  $f$ .

COROLLARY 3.4. Let  $\mathcal{Z} = \mathbb{N}$  and  $H(x) = x, h(x) = o(x)$ . Consider  $\mu_0, S \in \mathcal{M}(\mathcal{Z}, H)$  and  $E$  such that (3.8) holds. Suppose  $K$  satisfies (3.15) and  $F$  satisfies (3.11) and (3.18). Then there exists some  $\mu \in C([0, \infty), \mathcal{M}(\mathcal{Z}, H, h))$  satisfying (3.13).

To our knowledge, the most general existence result in the discrete case, including source and efflux terms, is contained in [30]. The corresponding assumptions there are  $K(x, y) = o(x)o(y), E(x) = O(x), \sum_{x=1}^{\infty} xS(x) < \infty$  and boundedness of  $F(x, \mathcal{Z})$  in  $x$ . Thus, Corollary 3.4 provides a new existence result for an unbounded total fragmentation rate.

**THEOREM 3.5.** *Let  $\mathcal{Z} = \mathbb{N}$  or  $\mathcal{Z} = (0, \infty)$ . Consider functions  $H$  and  $h$  satisfying (2.6) and (2.14). Let  $\mu_0 \in \mathcal{M}(\mathcal{Z}, H)$  such that*

$$(3.20) \quad \int_{\mathcal{Z}} x \mu_0(dx) < \infty,$$

*$S = 0$  and  $E \in C(\mathcal{Z})$  such that (3.8) holds. Consider a function  $K \in C(\mathcal{Z} \times \mathcal{Z})$  satisfying (3.15) and*

$$(3.21) \quad [H(x + y) - H(x) - H(y)]K(x, y) \leq c_1[H(x)y + xH(y) + xy],$$

*and a kernel  $F$  satisfying (3.11) and (3.16)–(3.18). Then there exists some  $\mu \in C([0, \infty), \mathcal{M}(\mathcal{Z}, H, h))$  satisfying (3.13).*

**COROLLARY 3.6.** *Let  $\mathcal{Z} = (0, \infty)$ ,*

$$H(x) = x^{-\alpha} + x^2, \quad \alpha \in (0, 1),$$

*and*

$$h(x) = x^{-\alpha+\varepsilon} + x^{2-\varepsilon}, \quad \varepsilon \in (0, \alpha].$$

*Let  $\mu_0 \in \mathcal{M}(\mathcal{Z}, H)$ ,  $S = 0$  and  $E \in C(\mathcal{Z})$  such that (3.8) holds. Consider a function  $K \in C(\mathcal{Z} \times \mathcal{Z})$  satisfying*

$$(3.22) \quad K(x, y) \leq c_1(1 + x + y).$$

*Suppose  $F$  has the form (3.19), where  $f$  is continuous with respect to the first argument and satisfies*

$$0 \leq f(x, y) \leq \frac{R(x)y^{-\beta}}{x^{1-\beta}}, \quad \beta \in [0, 1 - \alpha], R(x) = c(1 + x^{2-\varepsilon}).$$

*Then there exists some  $\mu \in C([0, \infty), \mathcal{M}(\mathcal{Z}, H, h))$  satisfying (3.13).*

**COROLLARY 3.7.** *Consider  $\mathcal{Z} = \mathbb{N}$  and the functions  $H(x) = x^r$ ,  $h(x) = x^{r-\varepsilon}$  with some  $r = 2, 3, \dots$  and  $\varepsilon \in (0, 1]$ . Let  $\mu_0 \in \mathcal{M}(\mathcal{Z}, H)$ ,  $S = 0$  and  $E$  such that (3.8) holds. Suppose  $K$  satisfies*

$$(3.23) \quad K(x, y) \leq c_1(x + y),$$

*and  $F$  satisfies (3.11) and (3.18). Then there exists some  $\mu \in C([0, \infty), \mathcal{M}(\mathcal{Z}, H, h))$  satisfying (3.13).*

**REMARK 3.8.** Consider  $\mathcal{Z} = (0, \infty)$ . Suppose  $F$  has the form (3.19) and

$$\mu(t, dx) = c(t, x) dx, \quad S(dx) = S(x) dx.$$

Then, using the identity

$$\int_0^\infty \int_0^\infty \psi(x, y) dy dx = \int_0^\infty \int_0^x \psi(x - y, y) dy dx,$$

equation (3.13) takes the form

$$\begin{aligned}
 & \int_0^\infty \varphi(x)c(t, x) dx \\
 &= \int_0^\infty \varphi(x)c_0(x) dx \\
 &+ \int_0^\infty dx \varphi(x) \int_0^t ds \\
 &\quad \times \left[ S(x) - E(x)c(s, x) \right. \\
 &\quad \left. + 2 \int_0^\infty f(x + y, y)c(s, x + y) dy - \int_0^x f(x, y)c(s, x) dy \right. \\
 &\quad \left. + \int_0^x K(x - y, y)c(s, x - y)c(s, y) dy \right. \\
 &\quad \left. - \int_0^\infty [K(x, y) + K(y, x)]c(s, x)c(s, y) dy \right].
 \end{aligned}$$

Removing the test functions, one obtains

$$\begin{aligned}
 (3.24) \quad & \frac{\partial}{\partial t} c(t, x) = S(x) - E(x)c(t, x) \\
 &+ 2 \int_0^\infty f(x + y, y)c(t, x + y) dy - c(t, x)F(x, \mathcal{Z}) \\
 &+ \int_0^x K(x - y, y)c(t, x - y)c(t, y) dy \\
 &- \int_0^\infty [K(x, y) + K(y, x)]c(t, x)c(t, y) dy.
 \end{aligned}$$

In the discrete case  $\mathcal{Z} = \mathbb{N}$ , analogous transformations of (3.13) lead to the form (3.24), with integrals replaced by sums. In this case, both forms are equivalent, without any additional assumptions.

LEMMA 3.9. *If condition (3.16) holds, then condition (2.18) is satisfied. If  $F$  has the form (3.19), for some nonnegative function  $f$ , which is continuous with respect to the first argument, then condition (3.16) is fulfilled.*

PROOF. Condition (2.18) reduces to

$$(3.25) \quad F(\cdot, \mathcal{Z}) \in C(\mathcal{Z}), \quad \int_{\mathcal{Z}} [\varphi(\cdot - y) + \varphi(y)]F(\cdot, dy) \in C(\mathcal{Z}).$$

Note that

$$\begin{aligned} & \left| \int_{\mathcal{Z}} [\varphi(x-y) + \varphi(y)]F(x, dy) - \int_{\mathcal{Z}} [\varphi(x_n-y) + \varphi(y)]F(x_n, dy) \right| \\ & \leq \left| \int_{\mathcal{Z}} [\varphi(x-y) + \varphi(y)]F(x, dy) - \int_{\mathcal{Z}} [\varphi(x-y) + \varphi(y)]F(x_n, dy) \right| \\ & \quad + \left| \int_{\mathcal{Z}} [\varphi(x-y) + \varphi(y)]F(x_n, dy) - \int_{\mathcal{Z}} [\varphi(x_n-y) + \varphi(y)]F(x_n, dy) \right| \\ & \leq \left| \int_{\mathcal{Z}} [\varphi(x-y) + \varphi(y)]F(x, dy) - \int_{\mathcal{Z}} [\varphi(x-y) + \varphi(y)]F(x_n, dy) \right| \\ & \quad + F(x_n, \mathcal{Z}) \sup_{y \in \mathcal{Z}} |\varphi(x-y) - \varphi(x_n-y)| \end{aligned}$$

and

$$\lim_{n \rightarrow \infty} \sup_{y \in \mathcal{Z}} |\varphi(x-y) - \varphi(x_n-y)| = 0 \quad \forall \varphi \in C_c(\mathcal{Z})$$

if  $x_n \rightarrow x$ . Thus, (3.25) follows from (3.16).

If  $F$  has the form (3.19), then condition (3.16) takes the form

$$(3.26) \quad \int_{\mathcal{Z}} \mathbb{1}_{(0,x)}(y)\varphi(y)f(x, y) dy \in C(\mathcal{Z}) \quad \forall \varphi \in C_b(\mathcal{Z}).$$

Note that

$$\mathbb{1}_{(0,x_n)}(y)\varphi(y)f(x_n, y) \leq \mathbb{1}_{(0,\bar{x})}(y)\|\varphi\| \sup_n \frac{R(x_n)}{x_n^{1-\beta}} y^{-\beta}, \quad \bar{x} = \sup_n x_n,$$

and

$$\lim_{n \rightarrow \infty} \mathbb{1}_{(0,x_n)}(y)\varphi(y)f(x_n, y) = \mathbb{1}_{(0,x)}(y)\varphi(y)f(x, y) \quad \forall y \neq x$$

if  $\lim_{n \rightarrow \infty} x_n = x$ . Thus, (3.26) follows from the dominated convergence theorem, since  $f(x, y)$  is continuous in  $x$ .  $\square$

LEMMA 3.10. *If condition (3.14) holds, then  $H(x) + H(y) \geq H(x + y)$ . If*

$$\frac{H(x)}{x} \leq \frac{H(y)}{y} \quad \forall 0 < x \leq y < \infty,$$

then  $H(x) + H(y) \leq H(x + y)$ .

PROOF. Since

$$\begin{aligned} H(x) + H(y) &= \frac{H(x)}{x}x + \frac{H(y)}{y}y, \\ H(x + y) &= \frac{H(x + y)}{x + y}x + \frac{H(x + y)}{x + y}y, \end{aligned}$$

the assertions follow.  $\square$

LEMMA 3.11. *Let  $\mathcal{Z} = (0, \infty)$ ,  $H_1(x) = x^{-\alpha}$ ,  $\alpha \in (0, 1)$ , and*

$$(3.27) \quad f_1(x, y) = \frac{y^{-\beta}}{x^{1-\beta}}, \quad \beta \in [0, 1 - \alpha].$$

Then

$$(3.28) \quad \int_0^x [H_1(x - y) + H_1(y) - H_1(x)] f_1(x, y) dy \leq c(\alpha, \beta) H_1(x) \quad \forall x > 0.$$

PROOF. Using

$$\begin{aligned} & \int_0^x (x - y)^{-\alpha} y^{-\beta} dy \\ &= \int_0^{x/2} (x - y)^{-\alpha} y^{-\beta} dy + \int_{x/2}^x (x - y)^{-\alpha} y^{-\beta} dy \\ &\leq \int_0^{x/2} \left(\frac{x}{2}\right)^{-\alpha} y^{-\beta} dy + \int_{x/2}^x (x - y)^{-\alpha} \left(\frac{x}{2}\right)^{-\beta} dy \\ &= \left(\frac{x}{2}\right)^{-\alpha} \left(\frac{x}{2}\right)^{1-\beta} \frac{1}{1-\beta} + \left(\frac{x}{2}\right)^{-\beta} \left(\frac{x}{2}\right)^{1-\alpha} \frac{1}{1-\alpha} \\ &= \left(\frac{x}{2}\right)^{1-\alpha-\beta} \left[ \frac{1}{1-\beta} + \frac{1}{1-\alpha} \right] \end{aligned}$$

and (3.27), one obtains

$$\begin{aligned} & \int_0^x [H_1(x - y) + H_1(y) - H_1(x)] f_1(x, y) dy \\ &= \frac{1}{x^{1-\beta}} \int_0^x [(x - y)^{-\alpha} + y^{-\alpha} - x^{-\alpha}] y^{-\beta} dy \\ &\leq x^{-\alpha} \left\{ \frac{1}{2^{1-\alpha-\beta}} \left[ \frac{1}{1-\beta} + \frac{1}{1-\alpha} \right] + \frac{1}{1-\alpha-\beta} - \frac{1}{1-\beta} \right\} \end{aligned}$$

so that (3.28) follows.  $\square$

LEMMA 3.12. *Let  $\mathcal{Z} = (0, \infty)$ ,*

$$H(x) = x^{-\alpha} + x^\gamma, \quad \alpha \in (0, 1), \gamma \geq 1,$$

and

$$h(x) = x^{-\alpha+\varepsilon} + x^{\gamma-\varepsilon}, \quad \varepsilon \in (0, \alpha].$$

Assume  $F$  has the form (3.19), where  $f$  satisfies

$$(3.29) \quad 0 \leq f(x, y) \leq \frac{R(x)y^{-\beta}}{x^{1-\beta}}, \quad \beta \in [0, 1 - \alpha), R(x) = c(1 + x^{\gamma-\varepsilon}).$$

Then conditions (3.17) and (3.18) are fulfilled.



PROOF. Using Lemma 3.10, (3.29) and Lemma 3.11, one obtains

$$\begin{aligned} & \int_{\mathcal{Z}} [H(x - y) + H(y) - H(x)]F(x, dy) \\ & \leq \int_0^x [H_1(x - y) + H_1(y) - H_1(x)]f(x, y) dy \\ & \leq R(x) \int_0^x [H_1(x - y) + H_1(y) - H_1(x)]f_1(x, y) dy \\ & \leq c(\alpha, \beta)R(x)H_1(x), \end{aligned}$$

and condition (3.17) follows. Moreover, (3.29) implies

$$F(x, \mathcal{Z}) \leq \frac{R(x)}{x^{1-\beta}} \int_0^x y^{-\beta} dy = \frac{R(x)}{1-\beta} \leq c_1 h(x)$$

so that condition (3.18) is satisfied.  $\square$

PROOF OF THEOREM 3.1. The statement is a consequence of Corollary 2.6. Indeed, condition (2.17) follows from (3.8), (3.15) and (3.18), and condition (2.18) is satisfied due to the continuity assumptions and Lemma 3.9. Furthermore, condition (3.1) follows from  $S \in \mathcal{M}(\mathcal{Z}, H)$  [cf. (3.6)], condition (3.2) follows from (3.17) and condition (3.3) is fulfilled, according to Lemma 3.10 and (3.14).  $\square$

PROOF OF COROLLARY 3.3. The statement is a consequence of Theorem 3.1. Note that (2.6), (2.14) and condition (3.14) are satisfied. Moreover, condition (3.16) is fulfilled, according to Lemma 3.9, and conditions (3.17) and (3.18) follow from Lemma 3.12, with  $\gamma = 1$ .  $\square$

PROOF OF COROLLARY 3.4. The statement follows immediately from Theorem 3.1.  $\square$

PROOF OF THEOREM 3.5. The statement is a consequence of Corollary 2.9, with  $g(x) = x$ . Note that condition (3.4) is satisfied. Moreover, condition (2.17) follows from (3.8), (3.15) and (3.18), and condition (2.18) is satisfied due to the continuity assumptions and Lemma 3.9. Furthermore, condition (3.2) follows from (3.17), and condition (3.5) is fulfilled, according to (3.21).  $\square$

PROOF OF COROLLARY 3.6. The statement is a consequence of Theorem 3.5. Note that (2.6), (2.14) and (3.20) are satisfied, and (3.15) follows from (3.22). Using Lemma 3.10 and (3.22), one obtains

$$\begin{aligned} & [H(x + y) - H(x) - H(y)]K(x, y) \\ & \leq [(x + y)^2 - x^2 - y^2]K(x, y) \\ & = 2xyK(x, y) \leq 2c_1[xy + x^2y + xy^2] \\ & \leq 2c_1[xy + H(x)y + xH(y)] \end{aligned}$$

so that (3.21) is fulfilled. Moreover, condition (3.16) is fulfilled, according to Lemma 3.9, and conditions (3.17) and (3.18) are consequences of Lemma 3.12, with  $\gamma = 2$ .  $\square$

PROOF OF COROLLARY 3.7. The statement is a consequence of Theorem 3.5. Note that (2.6), (2.14) and (3.20) are satisfied, and (3.15) follows from (3.23). Since

$$x^k y^l = x(x/y)^{k-1} y^{l+k-1} \leq x y^{l+k-1} \quad \text{if } x \leq y,$$

one obtains

$$(3.30) \quad x^k y^l \leq x y^{k+l-1} + y x^{k+l-1} \quad \forall x, y \geq 0, k, l \geq 1.$$

Using (3.23) and (3.30), one obtains

$$\begin{aligned} & [(x + y)^r - x^r - y^r] K(x, y) \\ &= \sum_{l=1}^{r-1} C_l^r x^l y^{r-l} K(x, y) \\ &\leq c_1 \left[ \sum_{l=1}^{r-1} C_l^r x^{l+1} y^{r-l} + \sum_{l=1}^{r-1} C_l^r x^l y^{r-l+1} \right] \\ &\leq 2c_1 \left( \sum_{l=1}^{r-1} C_l^r \right) [x y^r + y x^r], \end{aligned}$$

and condition (3.21) follows. Finally, condition (3.17) is a consequence of Lemma 3.10.  $\square$

3.3. *Inelastic collisions.* Here we consider the case  $\mathcal{Z} = \mathbb{R}^d, d \geq 1$ . Denote

$$(3.31) \quad v'(v, w, e, \theta) = \frac{v + w}{2} + \varepsilon(v, w, \theta) \frac{\|v - w\|e}{2}$$

and

$$(3.32) \quad w'(v, w, e, \theta) = \frac{v + w}{2} - \varepsilon(v, w, \theta) \frac{\|v - w\|e}{2},$$

where  $v, w \in \mathbb{R}^d, e \in \mathbb{S}^{d-1}$  (unit sphere),  $\theta \in \Theta$  for some measurable space  $\Theta$ , and  $\varepsilon$  is some measurable function. Note that

$$(3.33) \quad \|v'\|^2 + \|w'\|^2 = \|v\|^2 + \|w\|^2 - \frac{1 - \varepsilon(v, w, \theta)^2}{2} \|v - w\|^2;$$

that is, energy is dissipated if  $\varepsilon^2 < 1$ , conserved if  $\varepsilon^2 = 1$  and created if  $\varepsilon^2 > 1$ . Transformation (3.31), (3.32) generalizes the one-dimensional model proposed in [33], with

$$\varepsilon(v, w, \theta) = \frac{1}{1 + \theta \|v - w\|^a}, \quad \theta \in [0, \infty), a > 0.$$

In the special case  $\varepsilon \equiv 1$ , we use the notation

$$(3.34) \quad \begin{aligned} v^*(v, w, e) &= \frac{v + w}{2} + \frac{\|v - w\|e}{2}, \\ w^*(v, w, e) &= \frac{v + w}{2} - \frac{\|v - w\|e}{2}. \end{aligned}$$

**THEOREM 3.13.** *Consider the functions*

$$(3.35) \quad H(v) = \|v\|^2 + 1, \quad v \in \mathbb{R}^d,$$

and  $h$  satisfying (2.14). Let  $\beta_1, \beta_2 : \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{B}(\mathbb{S}^{d-1}) \rightarrow [0, \infty)$  be compactly bounded kernels satisfying

$$(3.36) \quad \beta_1(v, w, \mathbb{S}^{d-1}) \leq ch(v)H(w) \quad \forall v, w \in \mathbb{R}^d,$$

$$(3.37) \quad \beta_2(v, w, \mathbb{S}^{d-1}) \leq ch(v)h(w) \quad \forall v, w \in \mathbb{R}^d,$$

$$(3.38) \quad \beta_1(v_n, w, \cdot) \xrightarrow{w} \beta(v, w, \cdot) \quad \forall w \in \mathbb{R}^d$$

and

$$(3.39) \quad \beta_2(v_n, w_n, \cdot) \xrightarrow{w} \beta(v, w, \cdot),$$

when  $v_n \rightarrow v$  and  $w_n \rightarrow w$  in  $\mathbb{R}^d$ . Consider  $\mu_0 \in \mathcal{M}(\mathbb{R}^d, H)$ ,  $\pi \in \mathcal{P}(\Theta)$  and

$$(3.40) \quad M \in \mathcal{P}(\mathbb{R}^d) \quad \text{such that} \quad \int_{\mathbb{R}^d} \|w\|^4 M(dw) < \infty.$$

Suppose

$$(3.41) \quad \int_{\Theta} \varepsilon(v, w, \theta)^2 \pi(d\theta) \leq 1 \quad \forall v, w \in \mathbb{R}^d$$

and

$$(3.42) \quad \varepsilon(\cdot, \cdot, \theta) \in C(\mathbb{R}^d \times \mathbb{R}^d) \quad \forall \theta \in \Theta.$$

Then there exists some  $\mu \in C([0, \infty), \mathcal{M}(\mathbb{R}^d, H, h))$  satisfying the equation

$$(3.43) \quad \begin{aligned} &\langle \varphi, \mu(t) \rangle \\ &= \langle \varphi, \mu_0 \rangle + \int_0^t \left[ \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} [\varphi(v^*(v, w, e)) - \varphi(v)] \right. \\ &\quad \times \beta_1(v, w, de) M(dw) \mu(s, dv) \\ &\quad + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\Theta} \int_{\mathbb{S}^{d-1}} [\varphi(v'(v, w, e, \theta)) \\ &\quad \quad \quad + \varphi(w'(v, w, e, \theta)) \\ &\quad \quad \quad \left. - \varphi(v) - \varphi(w)] \right. \\ &\quad \left. \times \beta_2(v, w, de) \pi(d\theta) \mu(s, dv) \mu(s, dw) \right] ds, \\ &\quad \quad \quad t \geq 0, \end{aligned}$$

for any  $\varphi \in C_c(\mathbb{R}^d)$ .

REMARK 3.14. The probability measure  $\pi \in \mathcal{P}(\Theta)$  introduces some randomness into the collision events. The probability measure  $M \in \mathcal{P}(\mathbb{R}^d)$  represents the influence of some background gas. Condition (3.40) is fulfilled, for example, when  $M$  is a Maxwellian.

REMARK 3.15. Consider the special case

$$\varepsilon \equiv 1, \quad \beta_1(v, w, de) = \|v - w\| de, \quad \beta_2(v, w, de) = \frac{1}{2} \|v - w\| de$$

and suppose  $\mu(t, dv) = f(t, v) dv$ ,  $M(dv) = M(v) dv$ . Then (3.43) takes the form

$$\begin{aligned} & \int_{\mathbb{R}^d} \varphi(v) f(t, v) dv \\ &= \int_{\mathbb{R}^d} \varphi(v) f_0(v) dv \\ &+ \int_{\mathbb{R}^d} dv \varphi(v) \int_0^t ds \\ &\quad \times \left[ \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \|v - w\| [M(w^*) f(s, v^*) - M(w) f(s, v)] de dw \right. \\ &\quad \left. + \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \|v - w\| [f(s, w^*) f(s, v^*) - f(s, w) f(s, v)] de dw \right]. \end{aligned}$$

Removing the test functions, one obtains

$$\begin{aligned} \frac{\partial}{\partial t} f(t, v) &= \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \|v - w\| [M(w^*) f(t, v^*) - M(w) f(t, v)] de dw \\ &+ \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \|v - w\| [f(t, w^*) f(t, v^*) - f(t, w) f(t, v)] de dw. \end{aligned}$$

PROOF OF THEOREM 3.13. Introducing the *background collision term*

$$q_1(v, B) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \mathbb{1}_B(\delta_{v^*(v, w, e)}) \beta_1(v, w, de) M(dw),$$

$v \in \mathbb{R}^d, B \in \mathcal{B}(E_K),$

one obtains [cf. (2.30)]

$$(3.44) \quad Q_1(\varphi, v) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} [\varphi(v^*(v, w, e)) - \varphi(v)] \beta_1(v, w, de) M(dw).$$

With (3.35), condition (3.2) takes the form

$$(3.45) \quad \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} [\|v^*(v, w, e)\|^2 - \|v\|^2] \beta_1(v, w, de) M(dw) \leq c_1 [\|v\|^2 + 1].$$

Note that (3.33) (with  $\varepsilon = 1$ ) implies

$$\|v^*(v, w, e)\|^2 - \|v\|^2 \leq \|w\|^2,$$

so that assumptions (3.36) and (3.40) are sufficient for condition (3.45).

Introducing the *binary collision term*

$$q_2(v, w, B) = \int_{\Theta} \int_{\mathbb{S}^{d-1}} \mathbb{1}_B (\delta_{v'(v, w, e, \theta)} + \delta_{w'(v, w, e, \theta)}) \beta_2(v, w, de) \pi(d\theta),$$

where  $v, w \in \mathbb{R}^d$  and  $B \in \mathcal{B}(E_K)$ , one obtains [cf. (2.30)]

$$(3.46) \quad \begin{aligned} & Q_2(\varphi, v, w) \\ &= \int_{\Theta} \int_{\mathbb{S}^{d-1}} [\varphi(v'(v, w, e, \theta)) + \varphi(w'(v, w, e, \theta)) - \varphi(v) - \varphi(w)] \\ & \quad \times \beta_2(v, w, de) \pi(d\theta). \end{aligned}$$

In view of (3.35) and (3.33), condition (3.3) takes the form

$$\frac{\|v - w\|^2}{2} \left[ \int_{\Theta} \varepsilon(v, w, \theta)^2 \pi(d\theta) - 1 \right] \beta_2(v, w, \mathbb{S}^{d-1}) \leq 0$$

and follows from (3.41).

Condition (2.17) takes the form (3.37) and

$$\int_{\mathbb{R}^d} \beta_1(v, w, \mathbb{S}^{d-1}) M(dw) \leq ch(v) \quad \forall v \in \mathbb{R}^d$$

and follows from (3.36) and (3.40).

Condition (2.18) reduces to  $\beta_2(\cdot, \cdot, \mathbb{S}^{d-1}) \in C(\mathbb{R}^d, \mathbb{R}^d)$ ,

$$(3.47) \quad \begin{aligned} & \int_{\Theta} \int_{\mathbb{S}^{d-1}} [\varphi(v'(\cdot, \cdot, e, \theta)) + \varphi(w'(\cdot, \cdot, e, \theta))] \\ & \quad \times \beta_2(\cdot, \cdot, de) \pi(d\theta) \in C(\mathbb{R}^d, \mathbb{R}^d), \end{aligned}$$

$$(3.48) \quad \begin{aligned} & \int_{\mathbb{R}^d} \beta_1(\cdot, w, \mathbb{S}^{d-1}) M(dw) \quad \text{and} \\ & \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} \varphi(v^*(\cdot, w, e)) \beta_1(\cdot, w, de) M(dw) \in C(\mathbb{R}^d). \end{aligned}$$

Note that [cf. (3.31)]

$$\begin{aligned} & \|v'(v, w, e, \theta) - v'(v_n, w_n, e, \theta)\| \\ & \leq \frac{\|v - v_n\| + \|w - w_n\|}{2} + |\varepsilon(v, w, \theta) - \varepsilon(v_n, w_n, \theta)| \frac{\|v - w\|}{2} \\ & \quad + \frac{\varepsilon(v_n, w_n, \theta)}{2} \left| \|v - w\| - \|v_n - w_n\| \right| \end{aligned}$$

implies

$$\sup_{e \in \mathbb{S}^{d-1}} \|v'(v, w, e, \theta) - v'(v_n, w_n, e, \theta)\| \rightarrow 0 \quad \text{if } (v_n, w_n) \rightarrow (v, w),$$

according to assumption (3.42). Since  $\varphi \in C_c(\mathbb{R}^d)$ , it follows that

$$(3.49) \quad \sup_{e \in \mathbb{S}^{d-1}} |\varphi(v'(v, w, e, \theta)) - \varphi(v'(v_n, w_n, e, \theta))| \rightarrow 0$$

if  $(v_n, w_n) \rightarrow (v, w)$ .

One obtains

$$(3.50) \quad \left| \int_{\mathbb{S}^{d-1}} \varphi(v'(v, w, e, \theta)) \beta_2(v, w, de) - \int_{\mathbb{S}^{d-1}} \varphi(v'(v_n, w_n, e, \theta)) \beta_2(v_n, w_n, de) \right|$$

$$\leq \left| \int_{\mathbb{S}^{d-1}} \varphi(v'(v, w, e, \theta)) \beta_2(v, w, de) - \int_{\mathbb{S}^{d-1}} \varphi(v'(v, w, e, \theta)) \beta_2(v_n, w_n, de) \right|$$

$$+ \sup_{e \in \mathbb{S}^{d-1}} |\varphi(v'(v, w, e, \theta)) - \varphi(v'(v_n, w_n, e, \theta))| \beta_2(v_n, w_n, \mathbb{S}^{d-1}),$$

and the continuity of

$$\int_{\Theta} \int_{\mathbb{S}^{d-1}} \varphi(v'(\cdot, \cdot, e, \theta)) \beta_2(\cdot, \cdot, de) \pi(d\theta)$$

follows from (3.50) and (3.49), assumption (3.39) and the dominated convergence theorem. The other terms in (3.47) and (3.48) are treated in an analogous way, using also assumption (3.38). Finally, the assertion is a consequence of Corollary 2.6, since, with the terms (3.44) and (3.46), (2.23) takes the form (3.43).  $\square$

**4. The minimal jump process.** Let  $\lambda$  be a compactly bounded kernel on a locally compact separable metric space  $E$ . The *minimal jump process*  $X^\Delta$ , corresponding to  $\lambda$  and some initial distribution  $\nu_0 \in \mathcal{P}(E)$ , is constructed on the one-point compactification  $E^\Delta$  in the following way (cf. [13], page 263 and [27], page 69). Let  $Y_0, Y_1, \dots$  be a Markov chain in  $E$  with initial distribution  $\nu_0$  and transition function  $p : E \times \mathcal{B}(E) \rightarrow [0, 1]$  defined by

$$p(\xi, B) = \begin{cases} \frac{\lambda(\xi, B)}{\lambda(\xi, E)}, & \text{if } \lambda(\xi, E) > 0, \\ \mathbb{1}_B(\xi), & \text{if } \lambda(\xi, E) = 0. \end{cases}$$

Let  $T_0, T_1, \dots$  be independent and exponentially distributed random variables with mean 1 that are also independent of  $(Y_k)$ , all defined on some probability space  $(\Omega, \mathcal{F}, P)$ . Introduce the jump and explosion times

$$(4.1) \quad \tau_0 = 0, \quad \tau_l = \sum_{k=0}^{l-1} \frac{T_k}{\lambda(Y_k, E)}, \quad l = 1, 2, \dots, \quad \tau_\infty = \sum_{k=0}^{\infty} \frac{T_k}{\lambda(Y_k, E)},$$

where  $T_k/0 := \infty$ , and define

$$(4.2) \quad X^\Delta(t) = \begin{cases} Y_l, & \text{if } \tau_l \leq t < \tau_{l+1}, \\ \Delta, & \text{if } t \geq \tau_\infty, \end{cases} \quad t \geq 0.$$

Note that  $X^\Delta$  is an  $E^\Delta$ -valued stochastic process and, for every  $A \in \mathcal{B}(E)$ , the random variable

$$(4.3) \quad \sigma_A^\Delta = \inf\{t \geq 0 : X^\Delta(t) \notin A\}$$

is an  $\{\mathcal{F}_t^{X^\Delta}\}$ -stopping time. By construction (4.1), (4.2),  $X^\Delta$  is right-constant, but left limits may not exist at  $\tau_\infty$ . In particular, one obtains

$$(4.4) \quad X^\Delta(t) \in E \quad \forall t \geq 0 \iff \tau_\infty = \infty \iff X^\Delta \in D([0, \infty), E)$$

and

$$(4.5) \quad X^\Delta(t) \in A \quad \forall t < \sigma_A^\Delta, \quad X^\Delta(\sigma_A^\Delta) \notin A.$$

The following theorem provides a sufficient condition for nonexplosion of the minimal jump process, which, then, is equivalent to a cadlag process taking values in  $E$ .

**THEOREM 4.1.** *Let  $\lambda$  be a compactly bounded kernel on a locally compact separable metric space  $E$ , and  $\nu_0 \in \mathcal{P}(E)$ . Suppose there exists a nonnegative continuous function  $\eta$  such that*

$$(4.6) \quad \frac{1}{\eta + 1} \in C_0(E),$$

$$(4.7) \quad \int_E \eta(\xi) \nu_0(d\xi) \leq c_0$$

and

$$(4.8) \quad \int_E [\eta(\xi_1) - \eta(\xi)] \lambda(\xi, d\xi_1) \leq c_1[\eta(\xi) + c'_1] \quad \forall \xi \in E$$

for some  $c_0, c_1, c'_1 \geq 0$ .

Then there exists a  $D([0, \infty), E)$ -valued random variable  $X$  such that

$$(4.9) \quad P(X(t) = X^\Delta(t) \quad \forall t \geq 0) = 1$$

and

$$(4.10) \quad \mathbb{E}\eta(X(t)) \leq (c_0 + c'_1) \exp(c_1 t) \quad \forall t \geq 0.$$

COROLLARY 4.2. *Let  $X$  be given by Theorem 4.1. Then*

$$(4.11) \quad \sigma_{C_m} = \inf\{t \geq 0 : X(t) \notin C_m\},$$

with

$$(4.12) \quad C_m = \{\xi \in E : \eta(\xi) \leq m\},$$

and

$$(4.13) \quad M(\Psi, t) = \Psi(X(t)) - \Psi(X(0)) - \int_0^t \mathcal{A}\Psi(X(s)) ds,$$

with

$$(4.14) \quad \mathcal{A}\Psi(\xi) = \int_E [\Psi(\xi_1) - \Psi(\xi)]\lambda(\xi, d\xi_1), \quad \xi \in E,$$

satisfy

$$(4.15) \quad P(\sigma_{C_m} \leq t) \leq m^{-1}(c_0 + c'_1) \exp(c_1 t)$$

and

$$(4.16) \quad \mathbb{E} \sup_{s \leq t} |M(\Psi, s \wedge \sigma_{C_m})| \leq 2 \left( t \sup_{\xi \in C_m} \int_E [\Psi(\xi_1) - \Psi(\xi)]^2 \lambda(\xi, d\xi_1) \right)^{1/2}$$

for all  $m \geq 1, t \geq 0$  and  $\Psi \in C_b(E)$ .

LEMMA 4.3. *If*

$$(4.17) \quad \sup_{\xi \in A} \lambda(\xi, E) < \infty,$$

then there exists a process  $X_A$  with sample paths in  $D([0, \infty), E)$  such that

$$(4.18) \quad P(X_A(t) = X^\Delta(t \wedge \sigma_A^\Delta) \ \forall t \geq 0) = 1.$$

Moreover, for  $\Psi \in C_b(E)$  and  $t \geq 0$ , it satisfies

$$(4.19) \quad \mathbb{E}\Psi(X_A(t)) = \mathbb{E}\Psi(X_A(0)) + \mathbb{E} \int_0^t \mathcal{A}_A\Psi(X_A(s)) ds$$

and

$$(4.20) \quad \mathbb{E} \sup_{s \leq t} |M_A(\Psi, s)| \leq 2 \left( t \sup_{\xi \in A} \int_E [\Psi(\xi_1) - \Psi(\xi)]^2 \lambda(\xi, d\xi_1) \right)^{1/2},$$

where

$$(4.21) \quad \mathcal{A}_A\Psi(\xi) = \int_E [\Psi(\xi_1) - \Psi(\xi)]\lambda_A(\xi, d\xi_1), \quad \xi \in E,$$

with

$$(4.22) \quad \lambda_A(\xi, B) = \mathbb{1}_A(\xi)\lambda(\xi, B), \quad \xi \in E, B \in \mathcal{B}(E),$$

and

$$(4.23) \quad M_A(\Psi, t) = \Psi(X_A(t)) - \Psi(X_A(0)) - \int_0^t \mathcal{A}_A\Psi(X_A(s)) ds.$$



PROOF. We first check that

$$(4.24) \quad P(\Omega') = 1 \quad \text{where } \Omega' = \{X^\Delta(t \wedge \sigma_A^\Delta) \in E \ \forall t \geq 0\}.$$

If there is a  $k \geq 0$  such that  $T_k(\omega) > 0$  and  $Y_k(\omega) \notin A$ , then  $\sigma_A^\Delta(\omega) \leq \tau_k(\omega)$ ; that is, there are at most  $k$  jumps for this trajectory, and  $\omega \in \Omega'$ . Thus, for  $\omega \notin \Omega'$ , one obtains  $Y_k(\omega) \in A$  whenever  $T_k(\omega) > 0$ , and therefore

$$\tau_\infty(\omega) = \sum_{k=0}^\infty \frac{T_k(\omega)}{\lambda(Y_k(\omega), E)} \geq \frac{1}{\sup_{\xi \in A} \lambda(\xi, E)} \sum_{k=0}^\infty T_k(\omega).$$

Moreover,  $\omega \notin \Omega'$  implies  $\tau_\infty(\omega) < \infty$  [cf. (4.4)] so that

$$P(\Omega \setminus \Omega') \leq P\left\{\sum_{k=0}^\infty T_k < \infty\right\} = 0.$$

Therefore, (4.24) is fulfilled, and a process  $X_A$  satisfying (4.18) is obtained by redefining the process  $X^\Delta(t \wedge \sigma_A^\Delta)$  on the set  $\Omega \setminus \Omega'$  by elements of  $D([0, \infty), E)$ .

It follows from the explicit construction procedure that the process  $X_A$  is equivalent to the minimal jump process corresponding to  $\nu_0$  and the kernel (4.22) and therefore is a Markov process with the bounded generator (4.21) (cf., e.g., [13], page 163). Thus, for  $\Psi \in C_b(E)$ , the processes (4.23) and

$$M_A(\Psi, t)^2 - \int_0^t [\mathcal{A}_A \Psi^2 - 2\Psi \mathcal{A}_A \Psi](X_A(s)) ds$$

are  $\{\mathcal{F}_t^{X_A}\}$ -martingales (cf., e.g., [13], page 93 and Proposition 4.1.7). In particular, one obtains (4.19) and

$$(4.25) \quad \mathbb{E}M_A(\Psi, t)^2 = \mathbb{E} \int_0^t [\mathcal{A}_A \Psi^2 - 2\Psi \mathcal{A}_A \Psi](X_A(s)) ds.$$

Using the identity

$$(4.26) \quad [\mathcal{A}_A \Psi^2 - 2\Psi \mathcal{A}_A \Psi](\xi) = \int_E [\Psi(\xi_1) - \Psi(\xi)]^2 \lambda_A(\xi, d\xi_1), \quad \xi \in E,$$

and Doob's inequality (cf. [13], Corollary 2.2.17)

$$(4.27) \quad \mathbb{E} \sup_{s \leq t} |M_A(\Psi, s)|^2 \leq 4\mathbb{E}M_A(\Psi, t)^2,$$

one obtains (4.20) from (4.27), (4.25), (4.26) and (4.22).  $\square$

LEMMA 4.4. *Let the assumptions of Theorem 4.1 be fulfilled. Then [cf. (4.3) and (4.12)]*

$$(4.28) \quad P(\sigma_{C_m}^\Delta \leq t) \leq m^{-1}(c_0 + c'_1) \exp(c_1 t) \quad \forall t \geq 0.$$

PROOF. Note that

$$C_m \subset O_m := \{\xi \in E : \eta(\xi) < m + 1\} \subset C_{m+1}.$$

According to Lemma A.2, we choose  $e_k \in C_c(E)$ ,  $k \geq 1$ , such that

$$e_k(\xi) = 1, \xi \in C_k, \quad e_k(\xi) = 0, \xi \in C_{k+1}^c, \quad 0 \leq e_k(\xi) \leq 1, \xi \in E.$$

Note that  $e_k(\xi) = 1, k \geq m, \xi \in C_m$ . Thus, the functions  $\Psi_k = e_k[\eta + c'_1] \in C_c(E)$  satisfy [cf. (4.22)]

$$(4.29) \quad \begin{aligned} & \int_E [\Psi_k(\xi_1) - \Psi_k(\xi)] \lambda_{C_m}(\xi, d\xi_1) \\ & \leq \int_E [\eta(\xi_1) - \eta(\xi)] \lambda_{C_m}(\xi, d\xi_1) \quad \forall k \geq m. \end{aligned}$$

It follows from assumption (4.6) and Lemma A.1 that the sets (4.12) are compact. Thus, since  $\lambda$  is compactly bounded, assumption (4.17) is fulfilled so that Lemma 4.3 is applicable. Using (4.19) and (4.29) and assumptions (4.7) and (4.8), one obtains

$$\begin{aligned} & \mathbb{E}\Psi_k(X_{C_m}(t)) \\ & = \mathbb{E}\Psi_k(X(0)) + \mathbb{E} \int_0^t \int_E [\Psi_k(\xi_1) - \Psi_k(X_{C_m}(s))] \lambda_{C_m}(X_{C_m}(s), d\xi_1) ds \\ & \leq \mathbb{E}\Psi_k(X(0)) + \mathbb{E} \int_0^t \int_E [\eta(\xi_1) - \eta(X_{C_m}(s))] \lambda_{C_m}(X_{C_m}(s), d\xi_1) ds \\ & \leq \mathbb{E}\Psi_k(X(0)) + c_1 \mathbb{E} \int_0^t \mathbb{1}_{C_m}(X_{C_m}(s)) [\eta(X_{C_m}(s)) + c'_1] ds \\ & \leq c_0 + c'_1 + c_1 \int_0^t \mathbb{E}[\eta(X_{C_m}(s)) + c'_1] ds \quad \forall t \geq 0, k \geq m, \end{aligned}$$

where  $X_{C_m}$  is a  $D([0, \infty), E)$ -valued random variable such that

$$(4.30) \quad P(X_{C_m}(t) = X^\Delta(t \wedge \sigma_{C_m}^\Delta) \quad \forall t \geq 0) = 1.$$

The monotone convergence theorem (with  $k \rightarrow \infty$ ) implies

$$(4.31) \quad \mathbb{E}\eta(X_{C_m}(t)) + c'_1 \leq c_0 + c'_1 + c_1 \int_0^t \mathbb{E}[\eta(X_{C_m}(s)) + c'_1] ds.$$

Using (4.30), (4.5) and assumption (4.8), one obtains

$$\begin{aligned} \mathbb{E}\eta(X_{C_m}(t)) & = \mathbb{E}\eta(X_{C_m}(t)) \mathbb{1}_{\{\sigma_{C_m}^\Delta > t\}} + \mathbb{E}\eta(X_{C_m}(t)) \mathbb{1}_{\{\sigma_{C_m}^\Delta \leq t\}} \\ & \leq m + \sup_{\xi \in C_m} \int_E \eta(\xi_1) \lambda(\xi, d\xi_1) \\ & \leq m + \sup_{\xi \in C_m} (c_1[\eta(\xi) + c'_1] + \eta(\xi) \lambda(\xi, E)) \\ & \leq m + \left( c_1 [m + c'_1] + m \sup_{\xi \in C_m} \lambda(\xi, E) \right). \end{aligned}$$

An application of Gronwall’s inequality to (4.31) yields

$$(4.32) \quad \mathbb{E}\eta(X_{C_m}(t)) \leq (c_0 + c'_1) \exp(c_1 t) \quad \forall t \geq 0.$$

Using (4.5) and (4.30), one obtains

$$P(\sigma_{C_m}^\Delta \leq t) = P(X_{C_m}(t) \notin C_m) = P(\eta(X_{C_m}(t)) > m) \leq m^{-1} \mathbb{E}\eta(X_{C_m}(t))$$

for all  $m \geq 1$  and  $t \geq 0$ , so that (4.28) follows from (4.32).  $\square$

PROOF OF THEOREM 4.1. Since  $\sigma_{C_m}^\Delta \leq \sigma_{C_{m+1}}^\Delta$ , for all  $m \geq 1$ , Lemma 4.4 implies

$$P\left(\lim_{m \rightarrow \infty} \sigma_{C_m}^\Delta \leq t\right) = 0 \quad \forall t \geq 0,$$

so that

$$(4.33) \quad P\left(\lim_{m \rightarrow \infty} \sigma_{C_m}^\Delta = \infty\right) = 1.$$

Since  $0 \leq \sigma_{C_m}^\Delta \leq \tau_\infty = \sigma_E^\Delta$ , (4.33) implies  $P(\tau_\infty = \infty) = 1$ . Thus, according to (4.4), a process  $X$  with sample paths in  $D([0, \infty), E)$ , satisfying (4.9), is obtained by appropriately redefining  $X^\Delta$  on the set  $\{\tau_\infty < \infty\}$ , for example, by constant paths. This process is a  $D([0, \infty), E)$ -valued random variable according to [13], page 128, since  $E$  is separable.

Using (4.33) and continuity of  $\eta$ , one obtains

$$P\left(\lim_{m \rightarrow \infty} \eta(X(t \wedge \sigma_{C_m})) = \eta(X(t))\right) = 1.$$

Thus, Fatou’s lemma and (4.32) imply (4.10).  $\square$

PROOF OF COROLLARY 4.2. Note that [cf. (4.11) and (4.3)]

$$(4.34) \quad P(\sigma_{C_m} = \sigma_{C_m}^\Delta) = 1,$$

according to (4.9). Thus, (4.15) is a consequence of Lemma 4.4.

Moreover, (4.9) and (4.34) imply

$$P(X^\Delta(t \wedge \sigma_{C_m}^\Delta) = X(t \wedge \sigma_{C_m}) \quad \forall t \geq 0) = 1$$

and [cf. (4.18)]

$$(4.35) \quad P(X_{C_m}(t) = X(t \wedge \sigma_{C_m}) \quad \forall t \geq 0) = 1.$$

Note that [cf. (4.13)]

$$(4.36) \quad \begin{aligned} M(\Psi, t \wedge \sigma_{C_m}) &= \Psi(X(t \wedge \sigma_{C_m})) - \Psi(X(0)) \\ &\quad - \int_0^{t \wedge \sigma_{C_m}} \mathcal{A}\Psi(X(s)) ds \end{aligned}$$

and [cf. (4.23), (4.21), (4.22) and (4.14)]

$$(4.37) \quad \begin{aligned} M_{C_m}(\Psi, t) &= \Psi(X_{C_m}(t)) - \Psi(X_{C_m}(0)) \\ &\quad - \int_0^t \mathbb{1}_{C_m}(X_{C_m}(s)) \mathcal{A}\Psi(X_{C_m}(s)) ds. \end{aligned}$$

Since  $X_{C_m}(s) \notin C_m$  a.e., for  $s \geq \sigma_{C_m}$ , one obtains, using (4.35)–(4.37),

$$P(M(\Psi, t \wedge \sigma_{C_m}) = M_{C_m}(\Psi, t) \quad \forall t \geq 0) = 1,$$

so that (4.16) follows from (4.20), with  $A = C_m$ .  $\square$

**5. Proofs of the main results.**

5.1. *Regularity.*

LEMMA 5.1. *Let  $\mathcal{Z}$  be a locally compact separable metric space. Then the space  $(E^N, d_1)$  [cf. (2.1) and (2.11)] is separable and locally compact. If, in addition, assumption (2.4) holds, then the kernel  $\lambda^N$  [cf. (2.5)] on  $(E^N, d_1)$  is compactly bounded.*

PROOF. Let  $\mathcal{Z}'$  be a countable dense set in  $\mathcal{Z}$ . Then the set

$$\left\{ \frac{1}{N} \sum_{i=1}^n \delta_{x_i} : n \geq 0, x_i \in \mathcal{Z}', i = 1, \dots, n \right\}$$

is countable and dense in  $(E^N, d_1)$ , according to Remark 2.10. Choose compact and open sets  $\Gamma_m, \Omega_m, m \geq 1$ , according to Lemma A.3. Then the sets

$$(5.1) \quad G_m = \left\{ \frac{1}{N} \sum_{i=1}^n \delta_{x_i} : 0 \leq n \leq m, x_i \in \Gamma_m, i = 1, \dots, n \right\}$$

and

$$O_m = \left\{ \frac{1}{N} \sum_{i=1}^n \delta_{x_i} : 0 \leq n \leq m, x_i \in \Omega_m, i = 1, \dots, n \right\}$$

are, respectively, compact and open (this is easily established using Remark 2.10). They satisfy

$$(5.2) \quad G_m \subset O_m \subset G_{m+1} \quad \text{and} \quad E^N = \bigcup_m G_m.$$

In particular, every  $\mu \in E^N$  has a compact neighborhood, which proves local compactness of the space  $E^N$ . Any compact set  $C \subset E^N$  is covered by a finite number of sets  $O_m$ , according to (5.2). Thus, it is contained in some  $G_m$ , and (2.27) implies

$$(5.3) \quad \sup_{\mu \in C} \lambda^N(\mu, E^N) \leq N \left[ q_0(E_K) + \sum_{r=1}^R \sup_{x \in \Gamma_m^r} q_r(x, E_K) \left( \frac{m}{N} \right)^r \right].$$

The right-hand side of (5.3) is finite by assumption (2.4), so that the kernel is compactly bounded.  $\square$

REMARK 5.2. For any  $\mu = (1/N) \sum_{i=1}^n \delta_{x_i} \in E^N$ , one obtains

$$(5.4) \quad \frac{n}{N} = \mu(E^N) \leq \frac{\langle H, \mu \rangle}{\inf H} \quad \text{and} \quad H(x_i) \leq N \langle H, \mu \rangle \quad \forall i = 1, \dots, n.$$

LEMMA 5.3. If assumption (2.6) holds, then the function  $\Phi$  defined in (2.28), with  $\varphi = H$ , satisfies

$$(5.5) \quad \frac{1}{\Phi + 1} \in C_0(E^N).$$

PROOF. It follows from assumption (2.6) that the set  $\Gamma = \{x \in \mathcal{Z} : H(x) \leq mN\}$  is compact, according to Lemma A.1, and that  $\inf_{x \in \mathcal{Z}} H(x) > 0$ . As a consequence of Remark 5.2, the set

$$(5.6) \quad \{\mu \in E^N : \Phi(\mu) \leq m\}, \quad m > 0,$$

is contained in the set

$$\left\{ \frac{1}{N} \sum_{i=1}^n \delta_{x_i} : 0 \leq n \leq \frac{mN}{\inf H}, x_i \in \Gamma \right\},$$

which is compact [cf. (5.1)]. Since the function  $\Phi$  is continuous, according to assumption (2.6) and Remark 2.11, the set (5.6) is closed and therefore compact. Finally, property (5.5) follows from Lemma A.1 and the first part of Lemma 5.1.  $\square$

PROOF OF THEOREM 2.1. According to assumption (2.4) and Lemma 5.1, the kernel  $\lambda^N$  is compactly bounded on the separable and locally compact space  $(E^N, d_1)$ . Condition (4.6), with  $\eta = \Phi$ , follows from assumption (2.6) and Lemma 5.3. Assumptions (2.7) and (2.8) take the form (4.7) and (4.8). Thus, Theorem 2.1 follows from Theorem 4.1.  $\square$

We finish this section by providing further consequences of the assumptions of Theorem 2.1, which will be used later.

REMARK 5.4. The process  $X^N$  provided by Theorem 4.1 is a  $D([0, \infty), E^N)$ -valued random variable. Note that  $E^N \subset \mathcal{M}(\mathcal{Z}, H)$  and the embedding  $(E^N, d_1) \rightarrow \mathcal{M}(\mathcal{Z}, H, h)$  is continuous (cf. Remark 2.10). Thus, in view of Lemma A.4, the process  $X^N$  can be considered as a  $D([0, \infty), \mathcal{M}(\mathcal{Z}, H, h))$ -valued random variable.

Using (4.10), one obtains

$$(5.7) \quad \mathbb{E}\langle H, X^N(t) \rangle \leq (c_0 + c'_1) \exp(c_1 t) \quad \forall t \geq 0.$$

Moreover, Corollary 4.2 implies

$$(5.8) \quad P(\sigma_m^N > T) \geq 1 - m^{-1}(c_0 + c'_1) \exp(c_1 T) \quad \forall T \geq 0, m \geq 1,$$

and

$$(5.9) \quad \begin{aligned} & \mathbb{E} \sup_{t \leq T} |M^N(\Psi, t \wedge \sigma_m^N)| \\ & \leq 2 \left( T \sup_{\mu \in C_m^N} \int_{E^N} [\Psi(\nu) - \Psi(\mu)]^2 \lambda^N(\mu, d\nu) \right)^{1/2} \end{aligned}$$

for all  $m \geq 1, T \geq 0$  and  $\Psi \in C_b(E^N)$ , where

$$(5.10) \quad \sigma_m^N = \inf\{t \geq 0 : X^N(t) \notin C_m^N\},$$

$$(5.11) \quad C_m^N := \{\mu \in E^N : \langle H, \mu \rangle \leq m\}, \quad m > 0,$$

$$(5.12) \quad \begin{aligned} M^N(\Psi, t) &= \Psi(X^N(t)) - \Psi(X^N(0)) \\ &\quad - \int_0^t \mathcal{A}^N \Psi(X^N(s)) ds, \quad t \geq 0, \end{aligned}$$

and

$$(5.13) \quad \mathcal{A}^N \Psi(\mu) = \int_{E^N} [\Psi(\nu) - \Psi(\mu)] \lambda^N(\mu, d\nu), \quad \mu \in E^N.$$

LEMMA 5.5. *Suppose the assumptions of Theorem 2.1 are satisfied. Let  $\varphi \in C_b(\mathcal{Z})$  and  $\Phi$  be defined in (2.28). Then [cf. (5.10), (5.11)]*

$$(5.14) \quad \begin{aligned} & \mathbb{E} \sup_{t \leq T} |M^N(\Phi, t \wedge \sigma_m^N)| \\ & \leq 2 \left( T \sup_{\mu \in C_m^N} \int_{E^N} [\langle \varphi, \nu \rangle - \langle \varphi, \mu \rangle]^2 \lambda^N(\mu, d\nu) \right)^{1/2} \end{aligned}$$

for any  $T \geq 0$  and  $m \geq 1$ .

PROOF. The function (2.28) is continuous but, in general, unbounded. However, the stopped process reaches only a set, on which the function is bounded. Indeed, introduce the sets  $\hat{C}_m^N$  of all  $\nu \in E^N$  such that  $\nu = J_0(\mu, \xi)$  or  $\nu = J_r(\mu, i_1, \dots, i_r, \xi)$ , for some  $\mu \in C_m^N$ , pairwise distinct indices  $i_1, \dots, i_r$  from  $\{1, \dots, n\}, r = 1, \dots, R$  and  $\xi \in E_K$ . Using

$$|\Phi(\mu)| \leq \|\varphi\| \mu(E^N) \leq \|\varphi\| \frac{\langle H, \mu \rangle}{\inf H} \quad \forall \mu \in E^N$$

and (2.32), one obtains [cf. (2.2)]

$$(5.15) \quad |\Phi(v)| \leq \|\varphi\| \left( \frac{m}{\inf H} + \frac{K}{N} \right) \quad \forall v \in C_m^N \cup \hat{C}_m^N.$$

Note that

$$P(X^N(t \wedge \sigma_m^N) \in C_m^N \cup \hat{C}_m^N \quad \forall t \geq 0) = 1.$$

Consequently, (5.15) implies

$$(5.16) \quad P(\Phi(X^N(t \wedge \sigma_m^N)) = \Phi_m(X^N(t \wedge \sigma_m^N)) \quad \forall t \geq 0) = 1,$$

where

$$\Phi_m(\mu) := \Phi(\mu) \wedge \|\varphi\| \left( \frac{m}{\inf H} + \frac{K}{N} \right), \quad \mu \in E^N,$$

is a bounded function. Moreover, (2.26) and (5.15) imply, for  $\mu \in C_m^N$ ,

$$(5.17) \quad \begin{aligned} & \int_{E^N} [\Phi(v) - \Phi(\mu)]^k \lambda^N(\mu, dv) \\ &= \int_{E^N} [\Phi_m(v) - \Phi_m(\mu)]^k \lambda^N(\mu, dv), \quad k = 1, 2. \end{aligned}$$

It follows from (5.16) and (5.17), with  $k = 1$ , that

$$(5.18) \quad P(M^N(\Phi, t \wedge \sigma_m^N) = M^N(\Phi_m, t \wedge \sigma_m^N) \quad \forall t \geq 0) = 1.$$

Finally, (5.14) follows from (5.9), (5.18) and (5.17), with  $k = 2$ .  $\square$

### 5.2. Relative compactness.

LEMMA 5.6. *Suppose assumption (2.15) is satisfied. Then [cf. (5.11)]*

$$\begin{aligned} & \sup_{\mu \in C_m^N} \left| \int_{E^N} [\langle \varphi, v \rangle - \langle \varphi, \mu \rangle]^k \lambda^N(\mu, dv) \right| \\ & \leq c_2 N^{1-k} (K + R)^k \|\varphi\|^k \sum_{r=0}^R m^r \end{aligned}$$

for any  $\varphi \in C_b(\mathcal{Z})$ ,  $k = 1, 2$  and  $m \geq 1$ .

PROOF. Using (2.29) and (2.33) and assumption (2.15), one obtains

$$\begin{aligned} & \left| \int_{E^N} [\langle \varphi, v \rangle - \langle \varphi, \mu \rangle]^k \lambda^N(\mu, dv) \right| \\ & \leq N^{1-k} (K + R)^k \|\varphi\|^k \\ & \quad \times \left[ q_0(E_K) + \sum_{r=1}^R \int_{\mathcal{Z}} \cdots \int_{\mathcal{Z}} q_r(x_1, \dots, x_r, E_K) \mu(dx_1) \cdots \mu(dx_r) \right] \\ & \leq c_2 N^{1-k} (K + R)^k \|\varphi\|^k [1 + \langle H, \mu \rangle + \cdots + \langle H, \mu \rangle^R], \end{aligned}$$

and the assertion follows from the definition (5.11).  $\square$

**COROLLARY 5.7.** *Suppose the assumptions (2.6)–(2.8) and (2.15) are satisfied. Let  $\varphi \in C_b(\mathcal{Z})$  and  $\Phi$  be defined in (2.28). Then [cf. (5.10)]*

$$(5.19) \quad \mathbb{E} \sup_{t \leq T} |M^N(\Phi, t \wedge \sigma_m^N)| \leq 2\|\varphi\|(K + R) \left( \frac{c_2 T}{N} \sum_{r=0}^R m^r \right)^{1/2}$$

for any  $T \geq 0$  and  $m \geq 1$ .

**PROOF.** Property (5.19) follows from Lemmas 5.5 and 5.6, with  $k = 2$ .  $\square$

**LEMMA 5.8.** *Suppose the assumptions (2.6) and (2.15) are satisfied, and the assumptions (2.7) and (2.8) hold uniformly in  $N$ . Consider*

$$(5.20) \quad \varphi \in C(\mathcal{Z}) \quad \text{such that} \quad \frac{\varphi}{H} \in C_0(\mathcal{Z}).$$

Then, for any  $T > 0$  and  $\varepsilon > 0$ ,

$$(5.21) \quad \exists \Delta t, N_0 > 0: \sup_{N \geq N_0} P \left( \sup_{|s-t| \leq \Delta t, t \leq T} |\langle \varphi, X^N(s) \rangle - \langle \varphi, X^N(t) \rangle| \geq \varepsilon \right) \leq \varepsilon.$$

**PROOF.** Consider  $T > 0$  and  $\varepsilon > 0$  fixed. According to Lemma A.1, the set

$$(5.22) \quad \Gamma = \left\{ x \in \mathcal{Z}: \frac{|\varphi(x)|}{H(x)} \geq \frac{\varepsilon}{8m} \right\}$$

is compact, for any  $m \geq 1$ . Choosing  $\psi$  according to Lemma A.2, we find  $\tilde{\varphi} = \varphi \psi \in C_c(\mathcal{Z})$  such that

$$(5.23) \quad \tilde{\varphi}(x) = \varphi(x), \quad x \in \Gamma \quad \text{and} \quad |\tilde{\varphi}(x)| \leq |\varphi(x)|, \quad x \in \mathcal{Z}.$$

Consider [cf. (5.10) and (5.11)]

$$(5.24) \quad 0 \leq s \leq t < \sigma_m^N$$

so that  $X^N(s) \in C_m^N$  and  $\langle H, X^N(s) \rangle \leq m$ . Using (5.23) and (5.22), we obtain

$$(5.25) \quad \begin{aligned} & |\langle \varphi, X^N(s) \rangle - \langle \varphi, X^N(t) \rangle| \\ & \leq |\langle \varphi - \tilde{\varphi}, X^N(s) \rangle| + |\langle \varphi - \tilde{\varphi}, X^N(t) \rangle| + |\langle \tilde{\varphi}, X^N(s) \rangle - \langle \tilde{\varphi}, X^N(t) \rangle| \\ & \leq \int_{\Gamma^c} \frac{2|\varphi(z)|}{H(z)} H(z) X^N(s, dz) + \int_{\Gamma^c} \frac{2|\varphi(z)|}{H(z)} H(z) X^N(t, dz) \\ & \quad + |\langle \tilde{\varphi}, X^N(s) \rangle - \langle \tilde{\varphi}, X^N(t) \rangle| \\ & \leq \frac{\varepsilon}{2} + |\langle \tilde{\varphi}, X^N(s) \rangle - \langle \tilde{\varphi}, X^N(t) \rangle|. \end{aligned}$$



Furthermore, setting  $\tilde{\Phi}(\mu) = \langle \tilde{\varphi}, \mu \rangle$ , we obtain [cf. (5.24), (5.12) and (5.13)]

$$\begin{aligned}
 & |\langle \tilde{\varphi}, X^N(s) \rangle - \langle \tilde{\varphi}, X^N(t) \rangle| \\
 (5.26) \quad & \leq |M^N(\tilde{\Phi}, s) - M^N(\tilde{\Phi}, t)| + \int_s^t |\mathcal{A}^N \tilde{\Phi}(X^N(r))| dr \\
 & \leq |M^N(\tilde{\Phi}, s) - M^N(\tilde{\Phi}, t)| + c(t - s),
 \end{aligned}$$

where  $c = c(m, \varphi, K, R)$ , according to Lemma 5.6, with  $k = 1$ . Choose  $m$  according to (5.8) such that

$$(5.27) \quad \inf_N P(\sigma_m^N > T + 1) \geq 1 - \frac{\varepsilon}{2},$$

and let  $0 < \Delta t < \varepsilon/4c \wedge 1$ . Using (5.25), (5.26) [which hold under condition (5.24)], (5.27) and the Chebyshev inequality, we obtain

$$\begin{aligned}
 & P\left(\sup_{|s-t| \leq \Delta t, t \leq T} |\langle \varphi, X^N(s) \rangle - \langle \varphi, X^N(t) \rangle| \geq \varepsilon\right) \\
 & \leq P\left(\sup_{|s-t| \leq \Delta t, t \leq T} |\langle \varphi, X^N(s) \rangle - \langle \varphi, X^N(t) \rangle| \geq \varepsilon, \sigma_m^N > T + 1\right) + \frac{\varepsilon}{2} \\
 & \leq P\left(\frac{3\varepsilon}{4} + \sup_{|s-t| \leq \Delta t, t \leq T} |M^N(\tilde{\Phi}, s) - M^N(\tilde{\Phi}, t)| \geq \varepsilon, \sigma_m^N > T + 1\right) + \frac{\varepsilon}{2} \\
 & \leq P\left(\sup_{t \leq T+1} |M^N(\tilde{\Phi}, t)| \geq \frac{\varepsilon}{8}, \sigma_m^N > T + 1\right) + \frac{\varepsilon}{2} \\
 & \leq \frac{8}{\varepsilon} \mathbb{E} \sup_{t \leq T+1} |M^N(\tilde{\Phi}, t \wedge \sigma_m^N)| + \frac{\varepsilon}{2}.
 \end{aligned}$$

By Corollary 5.7, the mean value of the right-hand side becomes smaller than  $\varepsilon^2/16$  for sufficiently large  $N$  and thus (5.21) is satisfied.  $\square$

LEMMA 5.9. *Let  $\mathcal{Z}$  be a locally compact separable metric space. Consider*

$$(5.28) \quad H \in C(\mathcal{Z}) \quad \text{such that } H > 0,$$

and  $h$  satisfying (2.14). Then the sets

$$(5.29) \quad C_\varepsilon = \{\mu \in \mathcal{M}(\mathcal{Z}) : \langle H, \mu \rangle \leq \varepsilon\}, \quad \varepsilon > 0,$$

are compact subsets of the space  $\mathcal{M}(\mathcal{Z}, H, h)$ .

PROOF. By [3], Corollary 31.3, the set  $\tilde{C} = \{\nu \in \mathcal{M}(\mathcal{Z}) : \nu(\mathcal{Z}) \leq \varepsilon\}$  is vaguely compact. Consider the mapping [cf. (2.11)]

$$T : (\tilde{C}, d_0) \rightarrow (C_\varepsilon, d_h), \quad T(\nu)(B) = \int_B \frac{1}{H} d\nu, \quad B \in \mathcal{B}(\mathcal{Z}),$$

which is invertible and continuous. Indeed, according to assumption (2.14) and [3], Theorem 30.6, one obtains

$$\langle h, T(v_k) \rangle = \left\langle \frac{h}{H}, v_k \right\rangle \longrightarrow \left\langle \frac{h}{H}, v \right\rangle = \langle h, T(v) \rangle,$$

if  $v_k, v \in \tilde{C}$  are such that  $d_0(v_k, v) \rightarrow 0$ . Hence,  $C_\varepsilon = T(\tilde{C})$  is compact w.r.t. the topology generated by  $d_h$ .  $\square$

REMARK 5.10. Since  $\mathcal{Z}$  is locally compact and separable, there is a countable subset  $\{\psi_k\}_{k=1}^\infty$  of  $C_c(\mathcal{Z})$  which is dense in  $C_c(\mathcal{Z})$  w.r.t. uniform convergence (cf. [3], Lemma 31.4). According to Lemmas A.3 and A.2, choose compact and open sets satisfying (A.1), and localizing functions  $e_m \in C_c(\mathcal{Z}), m \geq 1$ , such that

$$(5.30) \quad e_m(z) = 1, z \in \Gamma_m, \quad e_m(z) = 0, z \notin \Omega_m, \quad 0 \leq e_m(z) \leq 1, z \in \mathcal{Z}.$$

Reorder the elements of the countable set

$$(5.31) \quad \{\psi_k : k \geq 1\} \cup \{\psi_k \cdot e_m : k, m \geq 1\} \cup \{e_m : m \geq 1\}$$

and denote them by  $\{\varphi_k\}_{k=1}^\infty$ . According to [3], Proof of Theorem 31.5, we introduce the metric

$$(5.32) \quad d_0(\mu, \nu) = \sum_{k=1}^\infty \frac{1}{2^k} \min\{1, |\langle \varphi_k, \mu \rangle - \langle \varphi_k, \nu \rangle|\}, \quad \mu, \nu \in \mathcal{M}(\mathcal{Z}),$$

generating the vague topology.

PROOF OF THEOREM 2.2. Note that assumption (2.15) implies (2.4) so that Theorem 2.1 is applicable. To prove relative compactness of the sequence  $(X^N)$ , we apply [13], Theorem 3.7.6, with  $E = \mathcal{M}(\mathcal{Z}, H, h)$  (cf. Remark 5.4).

The first condition to be checked is the compact containment condition

$$(5.33) \quad \forall T, \varepsilon > 0, \quad \exists \text{ compact } C \subset E : \inf_N P(X^N(t) \in C, 0 \leq t \leq T) \geq 1 - \varepsilon.$$

Choose  $m$  according to (5.8) such that  $\inf_N P(\sigma_m^N > T) \geq 1 - \varepsilon$ . Note that the set  $C_m$  [cf. (5.29)] is compact, according to Lemma 5.9. One obtains [cf. (5.11) and (5.10)]

$$\inf_N P(X^N(t) \in C_m, 0 \leq t \leq T) = \inf_N P(\sigma_m^N > T) \geq 1 - \varepsilon;$$

that is, (5.33) is satisfied.

The second condition to be checked is

$$(5.34) \quad \forall T, \varepsilon > 0, \quad \exists \delta > 0 : \sup_N P(w(X^N, \delta, T) \geq \varepsilon) \leq \varepsilon,$$

where the modulus of continuity

$$(5.35) \quad w(\mu, \delta, T) = \inf_{\{t_i\}} \max_i \sup_{s, t \in [t_{i-1}, t_i]} d_h(\mu(s), \mu(t))$$

is defined for  $\delta, T > 0$  and  $\mu \in D([0, \infty), E)$ . Here  $\{t_i\}$  ranges over all partitions of the form  $0 = t_0 < t_1 < \dots < t_{n-1} < T \leq t_n$  with  $\min_{1 \leq i \leq n} (t_i - t_{i-1}) > \delta$  and  $n \geq 1$ . Recall the definition (2.11) of the metric  $d_h$  and Remark 5.10. Let  $T, \varepsilon > 0$  and choose  $L \geq 0$  such that  $\sum_{k=L+1}^\infty 1/2^k \leq \varepsilon/2$ . With the notation  $\varphi_0 = h$  [cf. (5.32)], we obtain

$$\begin{aligned}
 & P\left(\sup_{|s-t| \leq \Delta t, t \leq T} d_h(X^N(s), X^N(t)) \geq \varepsilon\right) \\
 (5.36) \quad & \leq P\left(\sup_{|s-t| \leq \Delta t, t \leq T} \sum_{k=0}^L |\langle \varphi_k, X^N(s) \rangle - \langle \varphi_k, X^N(t) \rangle| \geq \frac{\varepsilon}{2}\right) \\
 & \leq \sum_{k=0}^L P\left(\sup_{|s-t| \leq \Delta t, t \leq T} |\langle \varphi_k, X^N(s) \rangle - \langle \varphi_k, X^N(t) \rangle| \geq \frac{\varepsilon}{2(L+1)}\right).
 \end{aligned}$$

Taking into account that (5.20) is fulfilled for  $\varphi = h$ , according to assumption (2.14), we apply Lemma 5.8 to the right-hand side of (5.36). Thus, there are  $\Delta t, N_0 > 0$  such that

$$(5.37) \quad \sup_{N \geq N_0} P\left(\sup_{|s-t| \leq \Delta t, t \leq T} d_h(X^N(s), X^N(t)) \geq \varepsilon\right) \leq \varepsilon.$$

Since [cf. (5.35)]

$$w(\mu, \delta, T) \leq \sup_{|s-t| \leq \Delta t, s \leq T} d_h(\mu(s), \mu(t)), \quad \delta < \Delta t,$$

we obtain from (5.37) that

$$(5.38) \quad \sup_{N \geq N_0} P(w(X^N, \delta, T) \geq \varepsilon) \leq \varepsilon, \quad \delta < \Delta t.$$

For any  $N$ , there exists  $\delta_N > 0$  such that  $P(w(X^N, \delta_N, T) \geq \varepsilon) \leq \varepsilon$ , according to [13], Lemma 3.6.2(a). Thus, for  $0 < \delta < \min\{\Delta t, \delta_1, \dots, \delta_{N_0-1}\}$ , (5.38) implies

$$\sup_N P(w(X^N, \delta, T) \geq \varepsilon) \leq \varepsilon;$$

that is, condition (5.34) is satisfied.  $\square$

### 5.3. Characterization of weak limits.

LEMMA 5.11. Consider  $\mu, \mu_n \in \mathcal{M}(\mathcal{Z}, H)$  such that

$$(5.39) \quad \lim_{n \rightarrow \infty} d_h(\mu_n, \mu) = 0.$$

Let  $\mu_n^{(k)}, \mu^{(k)}, h^{(k)}, k = 1, 2, \dots$ , denote the  $k$ -fold products of  $\mu_n, \mu, h$ , respectively. Then

$$(5.40) \quad \lim_{n \rightarrow \infty} \langle \psi, \mu_n^{(k)} \rangle = \langle \psi, \mu^{(k)} \rangle$$

for any  $\psi \in C(\mathcal{Z}^k)$  such that

$$(5.41) \quad |\psi(x)| \leq ch^{(k)}(x) \quad \forall x \in \mathcal{Z}^k \text{ for some } c > 0.$$

PROOF. Define the measures  $\nu_n, \nu \in \mathcal{M}_b(\mathcal{Z})$  by

$$\begin{aligned} \nu_n(B) &= \int_B h^{(k)}(x)\mu_n^{(k)}(dx), \\ \nu(B) &= \int_B h^{(k)}(x)\mu^{(k)}(dx), \quad B \in \mathcal{B}(\mathcal{Z}^k). \end{aligned}$$

Since  $\varphi h^{(k)} \in C_c(\mathcal{Z}^k)$  for every  $\varphi \in C_c(\mathcal{Z}^k)$  and since  $\mu_n^{(k)}$  converges vaguely to  $\mu^{(k)}$ , one obtains

$$\langle \varphi, \nu_n \rangle = \langle \varphi h^{(k)}, \mu_n^{(k)} \rangle \rightarrow \langle \varphi h^{(k)}, \mu^{(k)} \rangle = \langle \varphi, \nu \rangle;$$

that is,  $\nu_n$  converges vaguely to  $\nu$ . Using (5.39), one obtains

$$\nu_n(\mathcal{Z}^k) = \langle h^{(k)}, \mu_n^{(k)} \rangle = [\langle h, \mu_n \rangle]^k \quad \longrightarrow \quad \langle h^{(k)}, \mu^{(k)} \rangle = \nu(\mathcal{Z}^k),$$

so that  $\nu_n$  converges weakly to  $\nu$ . Note that  $\mathcal{Z}^k$  is a locally compact separable metric space and therefore complete according to [2], Theorem 44.1. Using Lemma A.6, we find, for any  $\varepsilon > 0$ , a compact  $\Gamma \subset \mathcal{Z}^k$  satisfying

$$\nu_n(\Gamma^c) \leq \frac{\varepsilon}{3c}, n \geq 1, \quad \text{and} \quad \nu(\Gamma^c) \leq \frac{\varepsilon}{3c}.$$

Choose  $f \in C_c(\mathcal{Z}^k)$  according to Lemma A.2 and consider  $\psi \in C(\mathcal{Z}^k)$  satisfying (5.41). Then, for sufficiently large  $n$ , one obtains

$$\begin{aligned} &| \langle \psi, \mu_n^{(k)} \rangle - \langle \psi, \mu^{(k)} \rangle | \\ &\leq | \langle \psi(1-f), \mu_n^{(k)} \rangle - \langle \psi(1-f), \mu^{(k)} \rangle | + | \langle \psi f, \mu_n^{(k)} \rangle - \langle \psi f, \mu^{(k)} \rangle | \\ &\leq c \int_{\Gamma^c} h^{(k)}(x)\mu_n^{(k)}(dx) + c \int_{\Gamma^c} h^{(k)}(x)\mu^{(k)}(dx) + | \langle \psi f, \mu_n^{(k)} \rangle - \langle \psi f, \mu^{(k)} \rangle | \\ &\leq \varepsilon, \end{aligned}$$

which proves (5.40.)  $\square$

LEMMA 5.12. *Suppose assumptions (2.17) and (2.18) are satisfied. Then the mapping*

$$M_\varphi : D([0, \infty), \mathcal{M}(\mathcal{Z}, H, h)) \rightarrow D([0, \infty), \mathbb{R})$$

defined by [cf. (2.20)]

$$(5.42) \quad M_\varphi(\mu)(t) = \langle \varphi, \mu(t) \rangle - \langle \varphi, \mu(0) \rangle - \int_0^t \mathcal{G}(\varphi, \mu(s)) ds, \quad t \geq 0,$$

is continuous, for any  $\varphi \in C_c(\mathcal{Z})$ .

PROOF. According to Lemma A.5, the mapping

$$F_1 : D([0, \infty), \mathcal{M}(\mathcal{Z}, H, h)) \rightarrow D([0, \infty), \mathbb{R}), \quad F_1(\mu)(t) = \langle \varphi, \mu(0) \rangle,$$

is continuous. In view of Lemma A.4 and since (cf. [13], page 153, (11.10)) the mapping

$$F_2 : D([0, \infty), \mathbb{R}) \rightarrow D([0, \infty), \mathbb{R}), \quad F_2(\xi)(t) = \int_0^t \xi(s) ds,$$

is continuous, it remains to show that the mappings  $f_1(v) = \langle \varphi, v \rangle$  and  $f_2(v) = \mathcal{G}(\varphi, v)$  from  $\mathcal{M}(\mathcal{Z}, H, h)$  into  $\mathbb{R}$  are continuous. For  $f_1$ , this is obvious, since convergence in  $\mathcal{M}(\mathcal{Z}, H, h)$  implies vague convergence. Using (2.34), the continuity of  $f_2$  follows from assumptions (2.17) and (2.18) and Lemma 5.11.  $\square$

LEMMA 5.13. *Suppose assumption (2.17) is satisfied. Let  $\psi_n, \psi \in C_c(\mathcal{Z})$  be such that  $\lim_{n \rightarrow \infty} \|\psi - \psi_n\| = 0$  and*

$$(5.43) \quad \{x \in \mathcal{Z} : |\psi_n(x)| > 0\} \subset \Gamma, \quad n \geq 1,$$

for some compact  $\Gamma \subset \mathcal{Z}$ . Then [cf. (5.42)]

$$\lim_{n \rightarrow \infty} M_{\psi_n}(\mu)(t) = M_{\psi}(\mu)(t) \quad \forall t \geq 0,$$

for any  $\mu \in D([0, \infty), \mathcal{M}(\mathcal{Z}, H, h))$ .

PROOF. Note that

$$(5.44) \quad \sup_n \|\psi_n\| < \infty \quad \text{and} \quad \langle \psi_n, \xi \rangle \rightarrow \langle \psi, \xi \rangle \quad \forall \xi \in E_K.$$

The dominated convergence theorem implies, using (2.32) and (5.44),

$$Q_0(\psi_n) \rightarrow Q_0(\psi), \quad Q_r(\psi_n, x) \rightarrow Q_r(\psi, x) \quad \forall x \in \mathcal{Z}^r, r = 1, \dots, R,$$

and, using (2.34) and assumption (2.17),

$$(5.45) \quad \lim_{n \rightarrow \infty} \mathcal{G}(\psi_n, \mu) = \mathcal{G}(\psi, \mu) \quad \forall \mu \in \mathcal{M}(\mathcal{Z}, H).$$

Using (5.43), one obtains

$$(5.46) \quad \lim_{n \rightarrow \infty} \langle \psi_n, \mu \rangle = \langle \psi, \mu \rangle \quad \forall \mu \in \mathcal{M}(\mathcal{Z}, H).$$

Moreover, it follows from (2.34) and assumption (2.17) that [cf. (2.20)]

$$(5.47) \quad \begin{aligned} & \sup_n \sup_{s \leq t} |\mathcal{G}(\psi_n, \mu(s))| \\ & \leq c_2(K + R) \sup_n \|\psi_n\| \sup_{s \leq t} \left[ 1 + \sum_{r=1}^R \langle h, \mu(s) \rangle^r \right] < \infty \end{aligned}$$

for any  $\mu \in D([0, \infty), \mathcal{M}(\mathcal{Z}, H, h))$ . Using (5.45)–(5.47), a further application of the dominated convergence theorem completes the proof.  $\square$

LEMMA 5.14. *Suppose assumptions (2.14) and (2.17) are satisfied. Let  $\varphi \in C_b(\mathbb{Z})$  and  $\Phi$  be defined in (2.28). Then [cf. (5.13) and (2.20)]*

$$(5.48) \quad \lim_{N \rightarrow \infty} \sup_{\mu \in C_m^N} |\mathcal{A}^N \Phi(\mu) - \mathfrak{G}(\varphi, \mu)| = 0 \quad \forall m \geq 1.$$

PROOF. Using (5.13), (2.29), (2.30) and (2.20), one obtains

$$(5.49) \quad \mathcal{A}^N \Phi(\mu) = \mathfrak{G}(\varphi, \mu) - \sum_{r=2}^R \frac{1}{N^r} \sum_{1 \leq i_1, \dots, i_r \leq n} \mathcal{Q}_r(\varphi, x_{i_1}, \dots, x_{i_r}),$$

where  $\mu \in E^N$  and  $\hat{\sum}$  denotes summation over those indices, at least two of which are equal. According to (2.34) and assumption (2.17), (5.49) implies

$$\begin{aligned} & |\mathcal{A}^N \Phi(\mu) - \mathfrak{G}(\varphi, \mu)| \\ & \leq \sum_{r=2}^R \frac{c_2(K+r)\|\varphi\|}{N^r} \sum_{1 \leq i_1, \dots, i_r \leq n} h(x_{i_1}) \cdots h(x_{i_r}) \\ & \leq \sum_{r=2}^R \frac{c_2(K+r)\|\varphi\| r(r-1)}{2N} \langle h^2, \mu \rangle \langle h, \mu \rangle^{r-2} \end{aligned}$$

and

$$(5.50) \quad \begin{aligned} & \sup_{\mu \in C_m^N} |\mathcal{A}^N \Phi(\mu) - \mathfrak{G}(\varphi, \mu)| \\ & \leq \frac{1}{N} \sup_{\mu \in C_m^N} \langle h^2, \mu \rangle \frac{c_2\|\varphi\|}{2} \sum_{r=2}^R (K+r)r(r-1)(cm)^{r-2}. \end{aligned}$$

According to assumption (2.14) and Lemma A.1, the set

$$\Gamma(\varepsilon) = \left\{ x \in \mathbb{Z} : \frac{h(x)}{H(x)} \geq \varepsilon \right\}$$

is compact, for any  $\varepsilon > 0$ . Using (5.4), one obtains

$$(5.51) \quad \begin{aligned} \langle h^2, \mu \rangle &= \int_{\Gamma(\varepsilon)} h^2(x) \mu(dx) + \int_{\Gamma(\varepsilon)^c} h^2(x) \mu(dx) \\ &\leq m \sup_{x \in \Gamma(\varepsilon)} \frac{h^2(x)}{H(x)} + \varepsilon^2 \frac{1}{N} \sum_{i=1}^n H^2(x_i) \\ &\leq m \sup_{x \in \Gamma(\varepsilon)} \frac{h^2(x)}{H(x)} + \varepsilon^2 m^2 N \end{aligned}$$

for any  $\mu \in C_m^N$ . Finally, (5.50) and (5.51) imply (5.48).  $\square$

PROOF OF THEOREM 2.3. Using (5.42) and (5.12), one obtains

$$(5.52) \quad M_\varphi(X^N)(t) = M^N(\Phi, t) + \int_0^t [\mathcal{A}^N \Phi(X^N(s)) - \mathcal{G}(\varphi, X^N(s))] ds.$$

For  $\varepsilon > 0$  and  $t \geq 0$ , choose  $m \geq 1$  according to (5.8) such that

$$(5.53) \quad \inf_N P(\sigma_m^N > t) \geq 1 - \frac{\varepsilon}{2}.$$

By (5.53), (5.52), Lemma 5.14 and the Chebyshev inequality, one obtains

$$(5.54) \quad \begin{aligned} & P\left(\sup_{s \leq t} |M_\varphi(X^N)(s)| \geq \varepsilon\right) \\ & \leq P\left(\sup_{s \leq t} |M_\varphi(X^N)(s)| \geq \varepsilon, \sigma_m^N > t\right) + \frac{\varepsilon}{2} \\ & \leq P\left(\frac{\varepsilon}{2} + \sup_{s \leq t} |M^N(\Phi, s)| \geq \varepsilon, \sigma_m^N > t\right) + \frac{\varepsilon}{2} \\ & \leq P\left(\sup_{s \leq t} |M^N(\Phi, s \wedge \sigma_m^N)| \geq \frac{\varepsilon}{2}\right) + \frac{\varepsilon}{2} \\ & \leq \frac{2}{\varepsilon} \mathbb{E} \sup_{s \leq t} |M^N(\Phi, s \wedge \sigma_m^N)| + \frac{\varepsilon}{2} \end{aligned}$$

for sufficiently large  $N$ . By Corollary 5.7, the right-hand side of (5.54) becomes smaller than  $\varepsilon$  for sufficiently large  $N$ , that is,

$$\limsup_N P\left(\sup_{s \leq t} |M_\varphi(X^N)(s)| \geq \varepsilon\right) \leq \varepsilon \quad \forall \varepsilon > 0, t > 0.$$

This implies

$$\sup_{s \leq t} |M_\varphi(X^N)(s)| \Rightarrow 0 \quad \forall t > 0,$$

and, recalling the definition of the Skorohod metric  $d$  ([13], page 117),

$$(5.55) \quad d(M_\varphi(X^N), 0) \Rightarrow 0.$$

Suppose  $X^{N_l} \Rightarrow X$  for some subsequence  $N_l$ . According to Lemma 5.12, the mapping  $M_\varphi$  is continuous so that  $M_\varphi(X^{N_l}) \Rightarrow M_\varphi(X)$  and  $d(M_\varphi(X^{N_l}), 0) \Rightarrow d(M_\varphi(X), 0)$ . Thus, (5.55) implies

$$M_\varphi(X) = 0 \quad \text{a.e. for any fixed } \varphi.$$

Using Remark 5.10, Lemma 5.13 and right-continuity of the trajectories, one obtains

$$(5.56) \quad M_\varphi(X) = 0 \quad \forall \varphi \in C_c(\mathbb{Z}) \text{ a.e.}$$

Moreover, it follows from Lemma A.5 that  $X^{N_l}(0) \Rightarrow X(0)$  so that assumption (2.16) implies

$$(5.57) \quad X(0) = \mu_0 \quad \text{a.e.}$$

According to (5.56) and (5.57),  $X$  satisfies (2.19) almost everywhere. Note that, by Theorem 2.2, the sequence of  $D([0, \infty), \mathcal{M}(\mathbb{Z}, H, h))$ -valued random variables  $X^N$  is relatively compact.  $\square$

5.4. Corollaries.

PROOF OF COROLLARY 2.4. Note that

$$\sup_{t \leq T} d_h(X^N(t), X^N(t-)) \leq \sup_{|s-t| \leq \Delta t, t \leq T} d_h(X^N(s), X^N(t)) \quad \forall \Delta t > 0.$$

Thus, (5.37) implies

$$\sup_{N \geq N_0} P\left(\sup_{t \leq T} d_h(X^N(t), X^N(t-)) \geq \varepsilon\right) \leq \varepsilon,$$

so that

$$\sup_{t \leq T} d_h(X^N(t), X^N(t-)) \Rightarrow 0, \quad T > 0,$$

as  $N \rightarrow \infty$ . An application of [13], Theorem 3.10.2(a), gives (2.21), for every weak limit  $X$  of the sequence  $(X^N)$ .  $\square$

PROOF OF COROLLARY 2.5. Suppose  $X^{N_l} \Rightarrow X$  for some subsequence  $N_l$ . Then Corollary 2.4 and [13], Proposition 3.5.2, imply

$$(5.58) \quad X^{N_l}(t) \Rightarrow X(t) \quad \forall t \geq 0.$$

For any  $\psi \in C_c(\mathbb{Z})$ , the mapping  $\Psi(\mu) = \langle \psi, \mu \rangle$  from  $\mathcal{M}(\mathbb{Z}, H, h)$  into  $\mathbb{R}$  is continuous, and

$$(5.59) \quad \Psi(X^{N_l}(t)) \Rightarrow \Psi(X(t)) \quad \forall t \geq 0,$$

as a consequence of (5.58).

Consider a sequence of localizing functions  $e_k \in C_c(\mathbb{Z})$  satisfying (5.30). Then (5.59), with  $\psi = He_k$ , implies

$$(5.60) \quad \langle He_k, X^{N_l}(t) \rangle \Rightarrow \langle He_k, X(t) \rangle \quad \forall t \geq 0, k = 1, 2, \dots$$

Since, according to (5.7),

$$\mathbb{E}\langle He_k, X^N(t) \rangle \leq \mathbb{E}\langle H, X^N(t) \rangle \leq (c_0 + c'_1) \exp(c_1 t) \quad \forall N,$$

Fatou's lemma and (5.60) imply

$$\mathbb{E}\langle He_k, X(t) \rangle \leq (c_0 + c'_1) \exp(c_1 t) \quad \forall t \geq 0, k = 1, 2, \dots$$



Thus, (2.22) follows from the monotone convergence theorem.  $\square$

PROOF OF COROLLARY 2.6. Consider  $\mu_0 \in \mathcal{M}(\mathcal{Z}, H)$  and note that  $\mu_0(\mathcal{Z}) < \infty$ , according to assumption (2.6) and (2.13). Let  $y_1, y_2, \dots$  be i.i.d. random variables with distribution  $(1/\mu_0(\mathcal{Z}))\mu_0(dx)$ , and

$$(5.61) \quad \lim_{N \rightarrow \infty} \frac{n_N}{N} = \mu_0(\mathcal{Z}).$$

Then

$$(5.62) \quad Y^N = \frac{1}{N} \sum_{i=1}^{n_N} \delta_{y_i} \in E^N$$

and, by (5.61) and the law of large numbers,

$$(5.63) \quad \begin{aligned} \langle \varphi, Y^N \rangle &= \frac{n_N}{N} \frac{1}{n_N} \sum_{i=1}^{n_N} \varphi(y_i) \\ &\rightarrow \mu_0(\mathcal{Z}) \int_{\mathcal{Z}} \varphi(x) \frac{1}{\mu_0(\mathcal{Z})} \mu_0(dx) = \langle \varphi, \mu_0 \rangle \quad \text{a.e.} \end{aligned}$$

for all nonnegative  $\varphi \in M(\mathcal{Z})$  such that  $\langle \varphi, \mu_0 \rangle < \infty$ . Thus,  $\lim_{N \rightarrow \infty} d_h(Y^N, \mu_0) = 0$  a.e., and  $Y^N \Rightarrow \mu_0$ . Moreover,

$$(5.64) \quad \begin{aligned} \int_{E^N} \langle H, \mu \rangle v_0^N(d\mu) &= \mathbb{E} \langle H, Y^N \rangle \\ &= \frac{n_N}{N} \int_{\mathcal{Z}} H(x) \frac{1}{\mu_0(\mathcal{Z})} \mu_0(dx), \end{aligned}$$

where  $v_0^N \in \mathcal{P}(E^N)$  denotes the distribution of  $Y^N$ . Thus,  $v_0^N$  satisfies assumptions (2.7) and (2.16). By Theorem 2.3 and Corollary 2.4, there is at least one  $\mu \in C([0, \infty), \mathcal{M}(\mathcal{Z}, H, h))$  satisfying (2.23).  $\square$

PROOF OF COROLLARY 2.7. By Theorem 2.3, any weak limit is concentrated on the set of solutions, which now consists only of one element. Thus, all weak limits are the same, and the assertion follows.  $\square$

PROOF OF COROLLARY 2.8. Assumption (2.25) assures that the processes remain in the restricted space, once they have started there. According to Remark 2.10, the mapping  $\langle g, \mu \rangle$  is continuous so that the subset  $E^N \cap \mathcal{M}_\gamma(\mathcal{Z}, g)$  is closed in  $E^N$ . According to Lemma A.7, the subset  $\mathcal{M}(\mathcal{Z}, H) \cap \mathcal{M}_\gamma(\mathcal{Z}, g)$  is closed in  $\mathcal{M}(\mathcal{Z}, H, h)$ . Thus, all statements about compact sets remain true for the restricted spaces.  $\square$

PROOF OF COROLLARY 2.9. Define [cf. (5.62)]

$$\tilde{Y}^N(\omega) = \begin{cases} Y^N(\omega), & \text{if } \langle g, Y^N \rangle \leq \langle g, \mu_0 \rangle + 1, \\ 0, & \text{otherwise,} \end{cases}$$

and let  $\tilde{\nu}_0^N$  denote the distribution of  $\tilde{Y}^N$ . By definition, one obtains  $\langle g, \tilde{Y}^N \rangle \leq \langle g, \mu_0 \rangle + 1$  and  $\mathbb{E}\langle H, \tilde{Y}^N \rangle \leq \mathbb{E}\langle H, Y^N \rangle$ . This implies  $\tilde{\nu}_0^N \in \mathcal{P}(E^N \cap \mathcal{M}_\gamma(\mathcal{Z}, g))$ , with  $\gamma = \langle g, \mu_0 \rangle + 1$ , and (2.7), according to (5.64). Moreover, it follows from (5.63) that  $\langle g, Y^N \rangle \rightarrow \langle g, \mu_0 \rangle$  a.e., so that (5.63) holds for  $\tilde{Y}^N$ . Consequently,  $\tilde{Y}^N \Rightarrow \mu_0$ , that is, (2.16) is fulfilled, and the assertion follows from Corollaries 2.8 and 2.4.  $\square$

APPENDIX

LEMMA A.1 (cf. [3], Theorem 27.6). *Let  $E$  be a locally compact space. Then  $\Psi \in C_0(E)$  iff*

$$\Psi \in C(E) \quad \text{and} \quad \{\xi \in E : |\Psi(\xi)| \geq \varepsilon\} \text{ is compact for every } \varepsilon > 0.$$

LEMMA A.2 (cf. [3], Corollary 27.3). *Let  $E$  be a locally compact space and  $C$  and  $O$  be compact and open subsets such that  $C \subset O$ . Then there is a  $\Psi \in C_c(E)$  such that*

$$\Psi(\xi) = 1, \xi \in C, \quad \Psi(\xi) = 0, \xi \notin O, \quad 0 \leq \Psi(\xi) \leq 1, \xi \in E.$$

LEMMA A.3. *Let  $\mathcal{Z}$  be a locally compact separable metric space. Then there are compact and open subsets  $\Gamma_m$  and  $\Omega_m$  such that*

$$(A.1) \quad \mathcal{Z} = \bigcup_{m=1}^{\infty} \Gamma_m \quad \text{and} \quad \Gamma_m \subset \Omega_m \subset \Gamma_{m+1}, \quad m \geq 1.$$

PROOF. Since  $\mathcal{Z}$  is  $\sigma$ -compact, the statement is given by [3], Lemma 29.8.  $\square$

LEMMA A.4 (cf. [13], page 151). *Let  $E, E_1$  be metric spaces. If  $f : E \rightarrow E_1$  is continuous, then the mapping*

$$F : D([0, \infty), E) \rightarrow D([0, \infty), E_1), \quad F(\xi)(t) = f(\xi(t)), \quad t \geq 0,$$

*is continuous.*

LEMMA A.5. *Let  $E$  be a metric space. If  $\lim_{n \rightarrow \infty} \mu_n = \mu$  in  $D([0, \infty), E)$ , then  $\lim_{n \rightarrow \infty} \mu_n(0) = \mu(0)$  in  $E$ .*

PROOF. The assertion follows from [13], Chapter 3, Proposition 5.2, since 0 is a continuity point for any  $\mu \in D([0, \infty), E)$ .  $\square$

LEMMA A.6. *Let  $E$  be a complete separable metric space, and  $\mu, \mu_n \in \mathcal{M}_b(E)$  such that  $\mu_n \rightarrow_w \mu$ . Then, for each  $\varepsilon > 0$ , there exists a compact  $K_\varepsilon$  such that*

$$\mu_n(E \setminus K_\varepsilon) \leq \varepsilon \quad \forall n, \quad \mu(E \setminus K_\varepsilon) \leq \varepsilon.$$

PROOF. Introduce the measures

$$\begin{aligned}
 \nu_n(B) &= \begin{cases} \frac{1}{\mu_n(E)}\mu_n(B), & \text{if } \mu_n(E) > 0, \\ 0, & \text{otherwise,} \end{cases} \\
 \nu(B) &= \begin{cases} \frac{1}{\mu(E)}\mu(B), & \text{if } \mu(E) > 0, \\ 0, & \text{otherwise.} \end{cases}
 \end{aligned}$$

Note that, if  $\mu(E) = 0$ , then  $\mu_n(E) \leq \varepsilon$  for all  $n$  except a finite number. For those, one finds the corresponding compact. If  $\mu(E) > 0$ , then  $\nu_n \rightarrow \nu$  weakly, and the statement follows from Prohorov's theorem and the boundedness of  $\mu_n(E)$ .  $\square$

LEMMA A.7. *Let  $\mathcal{Z}$  be a locally compact space, and  $\mu, \mu_n \in \mathcal{M}(\mathcal{Z})$  such that  $\mu_n \rightarrow \nu\mu$ . Then*

$$\langle H, \mu \rangle \leq \liminf_{n \rightarrow \infty} \langle H, \mu_n \rangle \quad \text{for any nonnegative } H \in C(\mathcal{Z}).$$

PROOF. Note that  $\nu_n \rightarrow \nu$ , where

$$\nu_n(B) = \int_B H(x)\mu_n(dx), \quad \nu(B) = \int_B H(x)\mu(dx), \quad B \in \mathcal{B}(\mathcal{Z}).$$

Thus, the assertion follows from [3], Lemma 30.3.  $\square$

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