MATHEMATICAL COMMUNICATIONS
Math. Commun. **26**(2021), 77–99

Stochastic models with mixtures of tempered stable subordinators

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Received April 30, 2020; accepted December 28, 2020

Abstract. In this article, we introduce mixtures of tempered stable subordinators. These mixtures define a class of subordinators which generalize tempered stable subordinators (TSS). The main properties like the probability density function (pdf), Lévy density, moments, governing Fokker-Planck-Kolmogorov (FPK) type equations and the asymptotic form of potential density are derived. Further, the governing FPK type equation and the asymptotic form of the renewal function for the corresponding inverse subordinator are discussed. We generalize these results to n-th order mixtures of TSS. The governing fractional difference and differential equations of the time-changed Poisson process and Brownian motion are also discussed.
AMS subject classifications: 60G10, 60G51

Key words: Tempered stable subordinator, mixture, Lévy density, Fokker-Planck-Kolmogorov equations

1. Introduction

In recent years, the subordinated stochastic processes have found many interesting real-life applications, see [6, 17, 19, 23, 24, 25, 26, 39] and references therein. In general, a subordinated process is defined by taking superposition of two independent stochastic processes. In a subordinated process, the time of a process called a parent process (or outer process) is replaced by another independent stochastic process called an inner process or subordinator. A subordinator is a non-decreasing Lévy process [3]. Note that subordinated processes are a convenient way to develop a stochastic model, where it is required to keep some properties of the parent process and at the same time some characteristics need to be altered. Some well-known subordinators include the gamma process, the Poisson process, a one-sided stable process with index $\alpha \in (0,1)$ or an α-stable subordinator, tempered stable subordinators, geometric stable subordinators, iterated geometric stable subordinators and Bessel subordinators [12].

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In this article, we introduce a class of subordinators which generalize the class of tempered stable subordinators and α-stable subordinators. This class of subordinators can be used as a time change to define another subordinated process instead of the tempered stable subordinator or the α-stable subordinator. We have discussed the main properties of the introduced subordinator.

The rest of the paper is organized as follows. In Section 2, we introduce an α-stable subordinator, a tempered stable subordinator (TSS), and also the mixtures of TSS and the inverse of mixtures of TSS. In Section 3, the distributional properties of mixtures of TSS are discussed. Section 4 deals with the asymptotic forms of potential density and renewal function. The n-th order mixtures of TSS are also discussed in this section. In the last section, as an application, we introduce a time-changed Poisson process and Brownian motion by considering the mixtures of the tempered stable subordinator and its inverse as time changes.

2. Tempered stable subordinators and their Mixtures

In this section, we recall the definitions of the α-stable subordinator, the tempered stable subordinator, as well as the mixtures of tempered stable subordinators and the inverse of mixtures of tempered stable subordinators.

2.1. Tempered stable subordinators

In this subsection, we present the main properties of the α-stable subordinator and the tempered stable subordinator. The class of stable distributions is denoted by $S(\alpha,\beta,\mu,\sigma)$, where parameter $\alpha \in (0,2]$ is the stability index, $\beta \in [-1,1]$ is the skewness parameter, $\mu \in \mathbb{R}$ is the location parameter, and $\sigma > 0$ is the shape parameter. The stable class probability density functions do not possess a closed form except for three cases (Gaussian ($\alpha = 2$), Cauchy ($\alpha = 1$), and Lévy ($\alpha = 1/2$)). Generally, stable distributions are represented in terms of their characteristic functions or Laplace transforms. Stable distributions are infinitely divisible and hence generate a class of continuous time Lévy processes. The one-sided stable Lévy process $S_\alpha(t)$ with the Laplace transform (see e.g. [34])

$$\mathbb{E}(e^{-sS_\alpha(t)}) = e^{-ts^\alpha},\ s > 0,\ \alpha \in (0,1), \tag{1}$$

is called an α-stable subordinator. The α-stable subordinator $S_\alpha(t)$ has stationary independent increments. The right tail of the α-stable subordinator behaves [34]

$$\mathbb{P}(S_\alpha(t) > x) \sim \frac{tx^{-\alpha}}{\Gamma(1-\alpha)},\ \text{as } x \to \infty. \tag{2}$$

Next, we introduce a tempered stable subordinator (TSS). The TSS $S_{\alpha,\lambda}(t)$ with the tempering parameter $\lambda > 0$ and the stability index $\alpha \in (0,1)$ is the Lévy process with the Laplace transform (LT) [26]

$$\mathbb{E}(e^{-sS_{\alpha,\lambda}(t)}) = e^{-t\left((s+\lambda)^\alpha - \lambda^\alpha\right)}. \tag{3}$$

Note that TSS are obtained by exponential tempering in the distributions of α-stable subordinators [32]. The advantage of a tempered stable subordinator over an α-stable subordinator is that it has finite moments of all orders and its density is also infinitely divisible. However, in the process of tempering it ceases to be self-similar. The probability density function for $S_{\alpha,\lambda}(t)$ is given by

$$f_{\alpha,\lambda}(x,t) = e^{-\lambda x+\lambda^\alpha t} f_\alpha(x,t), \quad \lambda > 0,\ \alpha \in (0,1), \tag{4}$$

where $f_\alpha(x,t)$ is the PDF of an α-stable subordinator [40]. The Lévy density corresponding to a TSS is given by [15]

$$\pi_{S_{\alpha,\lambda}}(x) = \frac{\alpha}{\Gamma(1-\alpha)} \frac{e^{-\lambda x}}{x^{\alpha+1}},\ x > 0. \tag{5}$$

The sample paths of the α-stable subordinator and the TSS are strictly increasing with jumps by applying Theorem 21.3 of Sato [35]. The tail probability of the TSS has the following asymptotic behavior:

$$\mathbb{P}(S_{\alpha,\lambda}(t) > x) \sim c_{\alpha,\lambda,t} \frac{e^{-\lambda x}}{x^\alpha},\ \text{as } x \to \infty, \tag{6}$$

where $c_{\alpha,\lambda,t} = \frac{t}{\alpha\pi}\Gamma(1+\alpha)\sin(\pi\alpha)e^{\lambda^\alpha t}$. The first two moments and covariance of the TSS are given by

$$\mathbb{E}(S_{\alpha,\lambda}(t)) = \alpha\lambda^{\alpha-1}t,\ \ \mathbb{E}(S_{\alpha,\lambda}(t))^2 = \alpha(1-\alpha)\lambda^{\alpha-2}t + (\alpha\lambda^{\alpha-1}t)^2, \tag{7}$$
$$\mathrm{Cov}(S_{\alpha,\lambda}(t), S_{\alpha,\lambda}(s)) = \alpha(1-\alpha)\lambda^{\alpha-2}\min(t,s),\ t,s \geq 0.$$

2.2. Mixtures of TSS and the inverse of mixtures of TSS

In this subsection, the mixtures of TSS (MTSS) are introduced and their governing fractional FPK type differential equations are discussed. Further, the inverse of mixtures of TSS (IMTSS) are also introduced. Mixtures of inverse stable subordinators have been considered in [4].

Definition 1 (Mixture tempered stable subordinator)**.** *We define an MTSS denoted by $S_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(t),\ t \geq 0$, as a Lévy process with the Laplace transform*

$$\mathbb{E}\left(e^{-sS_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(t)}\right) = e^{-t\left(c_1\left((s+\lambda_1)^{\alpha_1}-\lambda_1^{\alpha_1}\right)+c_2\left((s+\lambda_2)^{\alpha_2}-\lambda_2^{\alpha_2}\right)\right)},\ s > 0, \tag{8}$$

where $c_1 + c_2 = 1$ and $c_1,\ c_2 \geq 0$. An alternative representation of an MTSS can be given as a sum of two independent tempered stable subordinators $S_{\alpha_1,\lambda_1}(t)$ and $S_{\alpha_2,\lambda_2}(t)$ with time scaling and the condition $c_1 + c_2 = 1$, such that

$$S_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(t) = S_{\alpha_1,\lambda_1}(c_1 t) + S_{\alpha_2,\lambda_2}(c_2 t),\ c_1,\ c_2 \geq 0. \tag{9}$$

Representation (9) directly follows from (8), and using the Laplace transforms of the tempered stable subordinators $S_{\alpha_1,\lambda_1}(t)$ and $S_{\alpha_2,\lambda_2}(t)$ and the fact that both processes in the LHS and RHS are Lévy processes and hence the equivalence of their one-dimensional distributions lead to the equivalence of two processes. Further, the sample paths of MTSS are strictly increasing since sample paths of independent TSS used in (9) are strictly increasing. Next, the governing fractional Fokker-Planck-Kolmogorov (FPK) type equation for MTSS is discussed. We recall the LT denoted by $\mathcal{L}_t$ with respect to the time variable t of shifted fractional Riemann-Liouville (RL) derivatives, which is given by [8, 18],

$$\mathcal{L}_t\left(c+\frac{\partial}{\partial t}\right)^{\nu} f(x,t) = (c+s)^{\nu}\mathcal{L}_t f(x,t) - (c+s)^{\nu-1} f(x,0),\ s>0. \tag{10}$$

The shifted fractional RL derivative can be defined as in [8], see also the approach discussed in [20]. We also recall the definition of generalized Mittag-Leffler function [31],

$$M^r_{p,q}(z) = \sum_{k=0}^{\infty} \frac{(r)_n}{\Gamma(pn+q)}\frac{z^n}{n!},$$

where $p,q,r \in \mathbb{C}$ with $\mathcal{R}(q) > 0$ and $(r)_n = \frac{\Gamma(r+n)}{\Gamma(r)}$ is a Pochhammer symbol. Let us recall that the following LT formula $F(s) = \mathcal{L}[t^{q-1}M^r_{p,q}(-at^p)] = \frac{s^{pr-q}}{(s^p+a)^r}$ has the inverse LT in [29]

$$\mathcal{L}^{-1}[F(s)] = t^{q-1}M^r_{p,q}(-at^p). \tag{11}$$

Proposition 1. *The pdf $g_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(x,t) \equiv G(x,t)$ of the MTSS satisfies the following fractional partial differential equation (FPDE):*

$$\begin{aligned}\frac{\partial}{\partial t}G(x,t) = &- c_1\left(\lambda_1+\frac{\partial}{\partial x}\right)^{\alpha_1} G(x,t) - c_2\left(\lambda_2+\frac{\partial}{\partial x}\right)^{\alpha_2} G(x,t)\\ &+ \lambda_1^{\alpha_1}c_1G(x,t) + \lambda_2^{\alpha_1}c_2G(x,t),\end{aligned} \tag{12}$$

with initial conditions

$$\begin{cases} G(x,0) = \delta(x) \\ G(0,t) = 0. \end{cases} \tag{13}$$

Proof. Using (21),

$$\begin{aligned}\mathcal{L}_x\left(g_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(x,t)\right) =& \mathcal{L}_x\left(G(x,t)\right)\\ =& e^{-t\left(c_1\left((s+\lambda_1)^{\alpha_1}-\lambda_1^{\alpha_1}\right)+c_2\left((s+\lambda_2)^{\alpha_2}-\lambda_2^{\alpha_2}\right)\right)} = \overline{G}(s,t).\end{aligned}$$

Differentiating with respect to t yields

$$\begin{aligned}\frac{\partial}{\partial t}\overline{G}(s,t) =& -\left[c_1\left((s+\lambda_1)^{\alpha_1}-\lambda_1^{\alpha_1}\right)+c_2\left((s+\lambda_2)^{\alpha_2}-\lambda_2^{\alpha_2}\right)\right]\overline{G}(s,t)\\ =& -\left[c_1(s+\lambda_1)^{\alpha_1}\overline{G}(s,t) - c_1(s+\lambda_1)^{\alpha_1-1}G(0,t)\right]\\ & -\left[c_2(s+\lambda_2)^{\alpha_2}\overline{G}(s,t) - c_1(s+\lambda_1)^{\alpha_1-1}G(0,t)\right]\\ & + c_1{\lambda_1}^{\alpha_1}\overline{G}(s,t) + c_2{\lambda_2}^{\alpha_2}\overline{G}(s,t)\\ & - c_1(s+\lambda_1)^{\alpha_1-1}G(0,t) - c_1(s+\lambda_1)^{\alpha_1-1}G(0,t).\end{aligned}$$

Taking the inverse LT on both sides, using equation (10) and applying the initial conditions, we obtain the desired result. □

Remark 1. *For the density of the α-stable subordinator, the time derivative is equal to the negative of the fractional RL derivative. However, for the TSS density the time derivative is equal to the negative of the shifted fractional RL space derivative with an extra term. The density of MTSS involves two shifted fractional RL space derivatives.*

Next, we define the IMTSS and derive the fractional FPK type differential equation for its pdf. Let $E_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(t)$ be the right continuous inverse of MTSS $S_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(t)$, defined by

$$E_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(t) = \inf\{u > 0 : \ S_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(u) > t\}.$$

The process $E_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(t)$ is called the inverse of mixture tempered stable (IMTS) subordinator. This is also called the first-exist time. Since MTSS is a strictly increasing Lévy process, the sample paths of $E_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(t)$ are almost surely continuous and constant over the intervals where $S_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(t)$ have jumps. Let $h_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(t)$ be the pdf of IMTSS; then the Laplace transform $\tilde{h}_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(x,s)$ of the density $h_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(t)$ with respect to the time variable t is given by [28],

$$\tilde{h}_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(x,s) = \frac{\phi(s)}{s} e^{-x\phi(s)}, \tag{14}$$

where

$$\phi(s) = c_1\left((s+\lambda_1)^{\alpha_1} - \lambda_1^{\alpha_1}\right) + c_2\left((s+\lambda_2)^{\alpha_2} - \lambda_2^{\alpha_2}\right). \tag{15}$$

The Laplace transform in (14) has a simple pole at $s = 0$ and branch points at $s = -\lambda_1$ and $s = -\lambda_2$, and hence using the contour in Figure 1, the density function $h_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(t)$ can be found using complex inversion of the Laplace transform, see [22] for a similar approach.

Proposition 2. *The pdf $h_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(x,t) \equiv H(x,t)$ of IMTSS governs the following time-fractional differential equation:*

$$\begin{aligned}\frac{\partial}{\partial x}H(x,t) =& - c_1\left(\lambda_1 + \frac{\partial}{\partial t}\right)^{\alpha_1} H(x,t) - c_2\left(\lambda_2 + \frac{\partial}{\partial t}\right)^{\alpha_2} H(x,t) + \lambda_1^{\alpha_1}c_1 H(x,t)\\ &+ \lambda_2^{\alpha_1}c_2H(x,t) - c_1t^{-\alpha_1}M^{1-\alpha_1}_{1,1-\alpha_1}(-\lambda_1 t)\delta(x) - c_2t^{-\alpha_2}M^{1-\alpha_2}_{1,1-\alpha_2}(-\lambda_2 t)\delta(x),\end{aligned} \tag{16}$$

with $H(x,0) = \delta(x)$.

Proof. Using (14),

$$\begin{aligned}\mathcal{L}_t\left(h_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(x,t)\right) =& \frac{c_1\left((s+\lambda_1)^{\alpha_1} - \lambda_1^{\alpha_1}\right) + c_2\left((s+\lambda_2)^{\alpha_2} - \lambda_2^{\alpha_2}\right)}{s}\\ &\times e^{-t\left(c_1\left((s+\lambda_1)^{\alpha_1} - \lambda_1^{\alpha_1}\right) + c_2\left((s+\lambda_2)^{\alpha_2} - \lambda_2^{\alpha_2}\right)\right)}\\ =&\overline{H}(x,s),\end{aligned}$$

which implies

$$\begin{aligned}\frac{\partial}{\partial x}\overline{H}(x,s) =& -\left[c_1\left((s+\lambda_1)^{\alpha_1}-\lambda_1^{\alpha_1}\right)+c_2\left((s+\lambda_2)^{\alpha_2}-\lambda_2^{\alpha_2}\right)\right]\overline{H}(x,s)\\ =& -\left[c_1(s+\lambda_1)^{\alpha_1}\overline{H}(x,s)-c_1(s+\lambda_1)^{\alpha_1-1}H(x,0)\right]\\ &-\left[c_2(s+\lambda_2)^{\alpha_2}\overline{H}(x,t)-c_1(s+\lambda_1)^{\alpha_1-1}H(x,0)\right]\\ &+c_1{\lambda_1}^{\alpha_1}\overline{H}(x,s)+c_2{\lambda_2}^{\alpha_2}\overline{H}(x,s)-c_1(s+\lambda_1)^{\alpha_1-1}H(x,0)\\ &-c_1(s+\lambda_1)^{\alpha_1-1}H(x,0).\end{aligned}$$

Taking the inverse LT on both sides and using equation (10), we obtain

$$\begin{aligned}\frac{\partial}{\partial x}H(x,t) =& -c_1\left(\lambda_1+\frac{\partial}{\partial t}\right)^{\alpha_1}H(x,t)-c_2\left(\lambda_2+\frac{\partial}{\partial t}\right)^{\alpha_2}H(x,t)+c_1{\lambda_1}^{\alpha_1}H(x,t)\\ &+c_2{\lambda_2}^{\alpha_2}H(x,t)-\mathcal{L}^{-1}\left[c_1(s+\lambda_1)^{\alpha_1-1}\right]H(x,0)\\ &-\mathcal{L}^{-1}\left[c_1(s+\lambda_1)^{\alpha_1-1}\right]H(x,0). \end{aligned}\tag{17}$$

In (11), taking $p=1, q=1-\alpha, r=1$ and $a=\lambda$ yields

$$\mathcal{L}^{-1}\left[\frac{1}{(s+\lambda)^{1-\alpha}}\right]=t^{-\alpha}M_{1,1-\alpha}^{1-\alpha}(-\lambda t).\tag{18}$$

Using (17) and (18) yields the desired result. □

3. Distributional properties of MTSS

The distributional properties like pdf, Lévy density and moments of MTSS are discussed in this section.

3.1. Probability density function (pdf)

We discuss the pdf $g_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(x,t)$ of the introduced strictly increasing Lévy process $S_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(t)$. Here we use the technique of complex inversion of the Laplace Transform (LT) for finding the pdf of MTSS.

Proposition 3. *The pdf $g_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(x,t)$ of MTSS defined in (8) is given by the following integral representation, if $\lambda_1\neq\lambda_2$,*

$$\begin{aligned}g_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(x,t) =& \frac{1}{\pi}\int_0^\infty e^{-x\lambda_2}e^{-wx}e^{t(c_1\lambda_1^{\alpha_1}+c_2\lambda_2^{\alpha_2})}\\ &\times e^{-t\left(c_1(\lambda_1-\lambda_2)^{\alpha_1}\sum_{k=0}^\infty\binom{\alpha_1}{k}\frac{w^k}{(\lambda_1-\lambda_2)^k}\cos\pi k+c_2w^{\alpha_2}\cos\pi\alpha_2\right)}\\ &\times\sin\left(c_1t(\lambda_1-\lambda_2)^{\alpha_1}\sum_{k=0}^\infty\binom{\alpha_1}{k}\frac{w^k}{(\lambda_1-\lambda_2)^k}\sin(\pi k)\right.\end{aligned}$$

$$
\begin{aligned}
&+c_2tw^{\alpha_2}\sin(\pi\alpha_2))dw\\
&+\frac{1}{\pi}\int_0^{\lambda_2-\lambda_1} e^{-x\lambda_1}e^{-wx}e^{t(c_1\lambda_1^{\alpha_1}+c_2\lambda_2^{\alpha_2})}\\
&\times e^{-t\left(c_1w^{\alpha_1}\cos(\pi\alpha_1)+c_2(\lambda_1-\lambda_2)^{\alpha_2}\sum_{k=0}^{\infty}\binom{\alpha_2}{k}\frac{w^k}{(\lambda_1-\lambda_2)^k}\cos(\pi k)\right)}\\
&\times\sin(c_1tw^{\alpha_1}\sin(\pi\alpha_1)\\
&+c_1t(\lambda_1-\lambda_2)^{\alpha_2}\sum_{k=0}^{\infty}\binom{\alpha_2}{k}\frac{w^k}{(\lambda_1-\lambda_2)^k}\sin(\pi k)\Bigg)dw,
\end{aligned}
\tag{19}
$$

and if $\lambda_1,\lambda_2=\lambda$*:*

$$
\begin{aligned}
g_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(x,t)=&\frac{1}{\pi}\int_0^{\infty} e^{-x\lambda}e^{-wx}e^{t(c_1\lambda^{\alpha_1}+c_2\lambda^{\alpha_2})}\\
&\times e^{-t(c_1w^{\alpha_1}\cos(\pi\alpha_1)+c_2w^{\alpha_2}\cos(\pi\alpha_2))}\\
&\times\sin(t(c_1w^{\alpha_1}\sin(\pi\alpha_1)+c_2w^{\alpha_2}\sin(\pi\alpha_2)))dw,
\end{aligned}
\tag{20}
$$

where $c_1+c_2=1$ *and* $c_1,c_2\geq 0$.

Proof. Let $\mathcal{L}_x(f(x,t))$ be the LT of the function $f(x,t)$ with respect to the x variable. Then for $g_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(x,t)$ from (8), we have

$$
\mathcal{L}_x\left(g_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(x,t)\right)=e^{-t\left(c_1\left((s+\lambda_1)^{\alpha_1}-\lambda_1^{\alpha_1}\right)+c_2\left((s+\lambda_2)^{\alpha_2}-\lambda_2^{\alpha_2}\right)\right)}\doteq\overline{G}(s,t). \tag{21}
$$

The pdf $g_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(x,t)$ can be obtained by using the complex inversion formula for the Laplace transform, namely [36]

$$
g_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(x,t)=\frac{1}{2\pi i}\int_{x_0-i\infty}^{x_0+i\infty} e^{sx}\overline{G}(s,t)ds. \tag{22}
$$

Here the integration is to be performed along a vertical line at $x_0>a$ for some a such that the integrand is analytic for $\mathcal{R}e(s)>a$. Note that the function $e^{sx}\bar{G}(s,t)$ is an exponential function which is analytic in the whole complex plane. However, due to fractional power in the exponent, the integrand $e^{sx}\bar{G}(s,t)$ has branch points at $s=-\lambda_1$ and $s=-\lambda_2$. Thus we take a branch cut along the non-positive real axis and consider a (single-valued) analytic branch of the integrand. We assume here $\lambda_1<\lambda_2$; for calculating the integral in (22), consider a closed double-key-hole contour C: ABCDEFGHIJA (Figure 1) with branch points at $P=(-\lambda_1,0)$ and $Q=(-\lambda_2,0)$. In the contour, AB and IJ are arcs of radius R with the center at $O=(0,0)$, BC, DE, FG and HI are line segments parallel to the x-axis, CD, EF and GH are arcs of circles with radius r, and JA is the line segment from x_0-iy to x_0+iy (see Figure 1). The integrand is analytic within and on the contour C so that by Cauchys residue theorem [36]

$$
\frac{1}{2\pi i}\int_C e^{sx}\overline{G}(s,t)ds=0. \tag{23}
$$

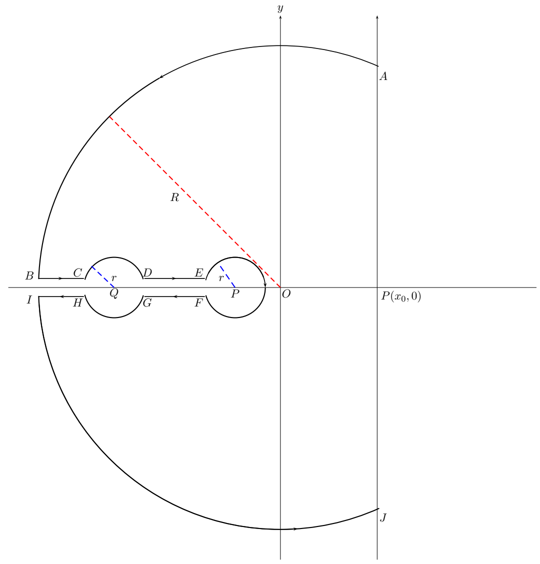

Figure 1: *Contour ABCDEFGHIJA*

Along CD, we have $s = -\lambda_2 + re^{i\theta}, \epsilon < \theta < \pi - \epsilon$, which implies $ds = ire^{i\theta}$ and

$$\begin{aligned}\left|\int_{CD} e^{sx}\overline{G}(s,t)ds\right| =& e^{t(c_1\lambda_1^{\alpha_1}+c_2\lambda_2^{\alpha_2})}\left|\int_{CD} e^{sx}e^{-t(c_1((s+\lambda_1)^{\alpha_1})+c_2((s+\lambda_2)^{\alpha_2}))}\right|ds\\ \leq& re^{t(c_1\lambda_1^{\alpha_1}+c_2\lambda_2^{\alpha_2})}\int_{\pi-\epsilon}^{\epsilon}\left|e^{-t(c_1(-\lambda_2+\lambda_1+re^{i\theta})^{\alpha_1}+c_2(r^{\alpha_1}e^{i\theta\alpha_2})}\right|\\ &\times\left|e^{(-\lambda_2+re^{i\theta})x}\right|\left|ie^{i\theta}\right|d\theta \to 0, \end{aligned} \tag{24}$$

as $r \to 0$, since the integrand is bounded. Similarly, for the arcs EF and GH, the integrals tend to zero as the radius r goes to zero. We have $e^{-by^{\alpha}} < y^{-\alpha}/b$ for b, y and $\alpha > 0$. Thus

$$\begin{aligned}|\overline{G}(s,t)| &= e^{t(c_1\lambda_1^{\alpha_1}+c_2\lambda_2^{\alpha_2})}|e^{-tc_1(s+\lambda_1)^{\alpha_1}}||e^{-tc_2(s+\lambda_2)^{\alpha_2}}|\\ &\leq \frac{e^{t(c_1\lambda_1^{\alpha_1}+c_2\lambda_2^{\alpha_2})}}{t^2c_1c_2}|(s+\lambda_1)^{-\alpha_1}||(s+\lambda_2)^{-\alpha_2}| \leq \frac{e^{t(c_1\lambda_1^{\alpha_1}+c_2\lambda_2^{\alpha_2})}}{t^2c_1c_2}|s|^{-(\alpha_1+\alpha_2)}.\end{aligned}$$

Hence applying Lemma 4.1 to the circular arc AB (see [36, p. 154]) gives

$$\lim_{R\to\infty}\int_{AB} e^{sx}\overline{G}(s,t)ds = 0.$$

Similarly, for the circular arc IJ, the integral vanishes as the radius R goes to ∞. Along BC, we have $s = -\lambda_2 + we^{i\pi}$, which implies $ds = -dw$ and

$$\begin{aligned}\int_{BC} e^{sx}\overline{G}(s,t)ds = \int_{r}^{-\lambda_2+R} & e^{-x\lambda_2}e^{-wx}e^{t(c_1\lambda^{\alpha_1}+c_2\lambda_2^{\alpha_2})}\\ &\times e^{-t[c_1(\lambda_1-\lambda_2+we^{i\pi})^{\alpha_1}+c_2(w^{\alpha_2}e^{i\pi\alpha_2})]}dw.\end{aligned} \tag{25}$$

Similarly, along DE, we have, $s = -\lambda_1 + we^{i\pi}$, which implies $ds = -dw$. Further,

$$\int_{DE} e^{sx}\overline{G}(s,t)ds = \int_{r}^{-r+\lambda_2-\lambda_1} e^{-x\lambda_1}e^{-wx}e^{t(c_1\lambda_1^{\alpha_1}+c_2\lambda_2^{\alpha_2})} \times e^{-t[c_1(w^{\alpha_1}e^{i\pi\alpha_1})+c_2(\lambda_2-\lambda_1+we^{i\pi})^{\alpha_2}]}dw. \tag{26}$$

Along FG, take $s = -\lambda_1 + we^{-i\pi}$, which implies $ds = -dw$ and leads to

$$\int_{FG} e^{sx}\overline{G}(s,t)ds = -\int_{r}^{-r+\lambda_2-\lambda_1} e^{-x\lambda_1}e^{-wx}e^{t(c_1\lambda_1^{\alpha_1}+c_2\lambda_2^{\alpha_2})} \times e^{-t[c_1(w^{\alpha_1}e^{-i\pi\alpha_1})+c_2(\lambda_2-\lambda_1+we^{-i\pi})^{\alpha_2}]}dw. \tag{27}$$

Along HI, take $s = -\lambda_2 + we^{-i\pi}$, which implies $ds = -dw$. Hence,

$$\int_{HI} e^{sx}\overline{G}(s,t)ds = -\int_{r}^{-\lambda_2+R} e^{-x\lambda_2}e^{-wx}e^{t(c_1\lambda^{\alpha_1}+c_2\lambda_2^{\alpha_2})} \times e^{-t[c_1(\lambda_1-\lambda_2+we^{-i\pi})^{\alpha_1}+c_2(w^{\alpha_2}e^{-i\pi\alpha_2})]}dw. \tag{28}$$

Thus,

$$\int_{DE} e^{sx}\overline{G}(s,t)ds + \int_{FG} e^{sx}\overline{G}(s,t)ds = -\int_{r}^{-r+\lambda_2-\lambda_1} e^{-x\lambda_1}e^{-wx}e^{t(c_1\lambda_1^{\alpha_1}+c_2\lambda_2^{\alpha_2})} \times \left[e^{-t(c_1w^{\alpha_1}\cos(\pi\alpha_1)+c_2(\lambda_2-\lambda_1+we^{-i\pi})^{\alpha_2})}2i\sin(tc_1w^{\alpha_1}\sin(\pi\alpha_1))\right]dw. \tag{29}$$

Similarly,

$$\int_{BC} e^{sx}\overline{G}(s,t)ds + \int_{HI} e^{sx}\overline{G}(s,t)ds = \int_{r}^{-\lambda_2+R} e^{-x\lambda_2-wx+t(c_1\lambda^{\alpha_1}+c_2\lambda_2^{\alpha_2})} \times \left[e^{-t(c_1(\lambda_1-\lambda_2+we^{i\pi})^{\alpha_1}+c_2(w^{\alpha_2}e^{i\pi\alpha_2}))} - e^{-t(c_1(\lambda_1-\lambda_2+we^{-i\pi})^{\alpha_1}+c_2(w^{\alpha_2}e^{-i\pi\alpha_2}))}dw\right]. \tag{30}$$

For $R \to \infty$ and $r \to 0$, using (21), (30), (29) and (23), we obtain the desired result. $\square$

Remark 2. *For the special case, $\alpha_1 = \alpha_2 = \alpha$ and $\lambda_1 = 0, \lambda_2 = 0$ with condition $c_1 + c_2 = 1$, (3) reduces to*

$$g_{\alpha,0,\alpha,0}(x,t) = \frac{1}{\pi}\int_0^\infty e^{-wx}e^{-tw^\alpha\cos(\pi\alpha)}\sin(tw^\alpha\sin(\pi\alpha))dw, \tag{31}$$

which is the pdf of α-stable subordinator [22].

Remark 3. *Substituting $\alpha_1 = \alpha_2 = \alpha$, $\lambda_1 = 0$, $\lambda_2 = \lambda > 0$, $c_1 = 0$ and $c_2 = 1$ in the equation, then we obtain the PDF of TSS with tempering parameter λ*

$$g_{\alpha,0,\alpha,\lambda}(x,t) = e^{-\lambda x+\lambda^\alpha t}\frac{1}{\pi}\int_0^\infty e^{-wx}e^{-tw^\alpha\cos(\pi\alpha)}\sin(tw^\alpha sin(\pi\alpha))dw \tag{32}$$

3.2. Lévy density

In this subsection, we discuss the Lévy density for MTSS. Here, we apply the result discussed in [6] for strictly increasing Lévy processes.

Proposition 4. *The Lévy density denoted by ν_S for MTSS $S_{\alpha_1,\lambda_1\alpha_2,\lambda_2}(t)$ has the following forms. When $\lambda_1 \neq \lambda_2$,*

$$\begin{aligned}
&\nu_S(dx)\\
&= \frac{1}{\pi}\int_0^\infty e^{-x\lambda - wx}\left[c_1(\lambda_1-\lambda_2)^{\alpha_1}\sum_{k=0}^{\infty}\binom{\alpha_1}{k}\frac{w^k}{(\lambda_1-\lambda_2)^k}\sin(\pi k) + c_2 w^{\alpha_2}\sin(\pi\alpha_2)\right]dw\\
&\quad + \frac{1}{\pi}\int_0^{\lambda_2-\lambda_1} e^{-x\lambda-wx}\Big[c_1 w^{\alpha_1}\sin(\pi\alpha_1)\\
&\quad + c_1(\lambda_1-\lambda_2)^{\alpha_2}\sum_{k=0}^{\infty}\binom{\alpha_2}{k}\frac{w^k}{(\lambda_1-\lambda_2)^k}\sin(\pi k)\Big]dw. \qquad (33)
\end{aligned}$$

When $\lambda_1 = \lambda_2$,

$$\nu_S(dx) = \frac{1}{\pi}\int_0^\infty e^{-x\lambda-wx}\left(c_1 w^{\alpha_1}\sin(\pi\alpha_1) + c_2 w^{\alpha_2}\sin(\pi\alpha_2)\right)dw. \qquad (34)$$

Proof. Let $f(x,t)$ be the pdf for a strictly increasing Lévy process; then the Lévy density $\nu(dx)$ is given by [6]:

$$\nu(dx) = \lim_{t\downarrow 0}\frac{1}{t}f(x,t).$$

Using the above result in (19) and (3) with the help of $\lim_{t\to 0}\frac{\sin(at)}{t} \to a$, $a \neq 0$, gives the desired result. □

Remark 4. *Substituting $\alpha_1 = \alpha_2 = \alpha$ with the condition $c_1 + c_2 = 1$ in (34), we obtain the Lévy density of tempered stable subordinator, which is given by [15]*

$$\nu_S(dx) = \frac{\alpha e^{-\lambda x}}{\Gamma(1-\alpha)x^{1+\alpha}},\ x > 0.$$

Further, for $\lambda = 0$ and $\alpha_1 = \alpha_2 = \alpha$ in (34), the Lévy density corresponds to that of the α-stable subordinator, and is given by [3]:

$$\nu_S(dx) = \frac{\alpha}{\Gamma(1-\alpha)x^{1+\alpha}}$$

Proof. By putting $\alpha_1 = \alpha_2 = \alpha$ in (34), we obtain

$$\nu_S(dx) = \frac{1}{\pi}\int_0^\infty e^{-x\lambda-wx}w^\alpha\sin(\pi\alpha)dw = \frac{\alpha e^{-\lambda x}}{\Gamma(1-\alpha)x^{1+\alpha}},\ x > 0.$$

Using Euler's identity $\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin(\pi\alpha)}$, $\forall\ \alpha \in (0,1)$, we obtain the Lévy density of TSS. □

3.3. Moments

In this subsection, we discuss the moments of MTSS. We also discuss the asymptotic forms of the moments for large t. The n-th order moment of MTSS is obtained by using the n-th order cumulant such that

$$k_n = \frac{d^n}{ds^n} K(s)|_{s=0},$$

where $K(s) = -t\left(c_1\left((-s+\lambda_1)^{\alpha_1} - \lambda_1^{\alpha_1}\right) + c_2\left((-s+\lambda_2)^{\alpha_2} - \lambda_2^{\alpha_2}\right)\right)$ is obtained from (21). The first moment and variance are $k_1 = \mathbb{E}(S_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(t)) = t(c_1\alpha_1{\lambda_1}^{\alpha_1-1} + c_2\alpha_2{\lambda_2}^{\alpha_2-1})$ and $k_2 = \mathrm{Var}(S_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(t)) = t(c_1\alpha_1(1-\alpha_1){\lambda_1}^{\alpha_1-2} + c_2\alpha_2(1-\alpha_2){\lambda_2}^{\alpha_2-2})$. The n-th order cumulant is

$$\begin{aligned} k_n =&(-1)^n t[c_1\alpha_1(\alpha_1-1)(\alpha_1-2)\cdots(\alpha_1-n+1){\lambda_1}^{\alpha_1-n} + c_2\alpha_2(\alpha_2-1)(\alpha_2-2) \\ &\cdots(\alpha_2-n+1){\lambda_2}^{\alpha_1-n}]. \end{aligned} \tag{35}$$

Moments and cumulants can be expressed in terms of each other by using Bell polynomials [9].
Next, we discuss the asymptotic behavior of the p-th order moments $\mathbb{E}(S_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(t)^p)$ of MTSS $S_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(t)$, where $0<p<1$.

Proposition 5. *For $0<p<1$, the asymptotic behavior of the p-th order moments of MTSS is given by*

$$\mathbb{E}(S_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(t)^p) \sim (c_1\alpha_1{\lambda_1}^{\alpha_1-1} + c_2\alpha_2{\lambda_2}^{\alpha_2-1})^p t^p, \text{ as } t\to\infty, \tag{36}$$

with condition $c_1+c_2=1$, $c_1,c_2\geq 0$.

Proof. Using the result in [21],

$$\begin{aligned} \mathbb{E}(S_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(t)^p) =& \frac{(-1)}{\Gamma(1-p)}\int_0^\infty \frac{d}{ds} e^{-t[c_1((s+\lambda_1)^{\alpha_1}-\lambda_1^{\alpha_1})+c_2((s+\lambda_2)^{\alpha_2}-\lambda_2^{\alpha_2})]} s^{-p} ds \\ =& \frac{te^{t(c_1\lambda_1^{\alpha_1}+c_2\lambda_2^{\alpha_2})}}{\Gamma(1-p)}\int_0^\infty s^{-p}(c_1\alpha_1(s+\lambda_1)^{\alpha_1-1} + c_2\alpha_2(s+\lambda_2)^{\alpha_2-1}) \\ &\times e^{-t[c_1(s+\lambda_1)^{\alpha_1}+c_2(s+\lambda_2)^{\alpha_2}]} ds \end{aligned}$$

By choosing $f(s) = c_1(s+\lambda_1)^{\alpha_1} + c_2(s+\lambda_2)^{\alpha_2}$ and $g(s) = s^{-p}(c_1\alpha_1(s+\lambda_1)^{\alpha_1-1} + c_2\alpha_2(s+\lambda_2)^{\alpha_2-1})$, it follows

$$\begin{aligned} f(s) &= (c_1\lambda_1^{\alpha_1} + c_2\lambda_2^{\alpha_2}) + \left(c_1\alpha_1\lambda^{\alpha_1-1} + c_2\alpha_2\lambda^{\alpha_2-1}\right)s + \cdots \\ &= f(0) + \sum_{k=0}^{\infty} a_k s^{k+\beta}, \end{aligned}$$

where $f(0) = c_1\lambda_1^{\alpha_1} + c_2\lambda_2^{\alpha_2}$, $a_0 = c_1\alpha_1\lambda^{\alpha_1-1} + c_2\alpha_2\lambda^{\alpha_2-1}$ and $\beta = 1$. Further,

$$\begin{aligned} g(s) &= \left(c_1\alpha_1\lambda_1^{\alpha_1-1} + c_2\alpha_2\lambda_2^{\alpha_2-1}\right)s^{-p} + \left(c_1(\alpha_1-1)\lambda_1^{\alpha_1-2} + c_2(\alpha_2-1)\lambda^{\alpha_2-2}\right)s^{p-1} + \cdots \\ &= \sum_{k=0}^{\infty} b_k s^{k+\gamma+1}, \end{aligned}$$

where $b_0 = c_1\alpha_1\lambda_1^{\alpha_1-1} + c_2\alpha_2\lambda_2^{\alpha_2-1}$, $b_1 = c_1(\alpha_1-1)\lambda_1^{\alpha_1-2} + c_2(\alpha_2-1)\lambda^{\alpha_2-2}$ and $\gamma = 1-p$. Using the Laplace-Erdelyi theorem from [16], we have

$$\mathbb{E}(S_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(t)^p) \sim \frac{t}{\Gamma(1-p)} \sum_{k=0}^{\infty} \Gamma(k+1-p) \frac{D_k}{t^{k+1-p}}, \tag{37}$$

where D_k in terms of coefficients a_k and b_k is given by

$$D_k = \frac{1}{a_0^{(k+\gamma)/\beta}} \sum_{j=0}^{k} b_{k-j} \sum_{i=0}^{j} \binom{-\frac{k+\gamma}{\beta}}{i} \frac{1}{a_0^i} \overline{B}_{j,i}(a_1, a_2, \ldots, a_{j-i+1}), \tag{38}$$

where $\overline{B}_{j,i}$ are partial ordinary Bell polynomial (see e.g. [2]). The Bell polynomials arises naturally from differentiating a composite function n times and exhibits important applications in combinatorics, statistics, numerical solutions of non-linear differential equations and other fields, and is given by (see e.g. [2, 14, 9]):

$$\overline{B}_{j,i}(a_1, a_2, \ldots, a_{j-i+1}) = \sum \frac{i!}{m_1! m_2! \cdots m_{j-i+1}} {a_1}^{m_1} {a_2}^{m_2} \cdots {a_{j-i+1}}^{m_{j-i+1}},$$

where the sum is taken over the sequences satisfying

$$m_1 + m_2 + \cdots + m_{j-i+1} = i, m_1 + 2m_2 + \cdots + (j-i+1)m_{j-i+1} = j,$$

where $m_1, m_2, \ldots, m_{j-i+1} \geq 0$. For large t, the dominating term is the first one in the series given in (37), which implies

$$\mathbb{E}(S_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(t)^p) \sim D_0 t^p, \tag{39}$$

where $D_0 = (c_1\alpha_1{\lambda_1}^{\alpha_1-1} + c_2\alpha_2{\lambda_2}^{\alpha_2-1})^p$. □

Remark 5. *For a positive integer n, the n-th order moments of MTSS satisfy*

$$\begin{aligned}\mathbb{E}(S_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(t))^n &= \sum_{m=1}^{n} B_{n,m}(k_1, k_2, \ldots, k_{n-m+1}) \\ &\sim (k_1)^n \sim (c_1\alpha_1{\lambda_1}^{\alpha_1-1} + c_2\alpha_2{\lambda_2}^{\alpha_2-1})^n t^n \ \text{ as } t \to \infty,\end{aligned}$$

where $B_{n,m}$ are partial (or incomplete) exponential Bell polynomials [2]. For more information on cumulants, Bell polynomials and moments, see [33, 38].

Remark 6. *By taking $\alpha_1,\ \alpha_2 = \alpha,\ \ \lambda_1,\ \lambda_2 = \lambda$ with condition $c_1+c_2 = 1$, $c_1, c_2 \geq 0$ in* (53)*, we obtain the asymptotic behaviour of the p-th* $(0 < p < 1)$ *order moment of TSS $S_{\alpha,\lambda}$ given by*

$$\mathbb{E}(S_{\alpha,\lambda}(t))^p \sim (\alpha\lambda^{\alpha-1}t)^p, \quad \text{as } t \to \infty.$$

Similarly, the asymptotic behavior of the n-th order moment for TSS is

$$\mathbb{E}(S_{\alpha,\lambda}(t))^n \sim (\alpha\lambda^{\alpha-1}t)^n, \quad \text{as } t \to \infty.$$

Next, we discuss the algorithm to simulate the sample trajectories of MTSS and its inverse.

Example 1 (Simulation of MTSS sample trajectories and its inverse)**.** *The algorithm for generating the sample trajectories of MTSS are as follows:*
Step 1*: fix the values of parameters; generate independent and uniformly distributed in* $[0,1]$ *rvs* U, V*;*
Step 2*: generate the increments of the* α*-stable subordinator* $S_\alpha(t)$ *from [13] with pdf* $f_\alpha(x,t)$ *using the relationship* $S_\alpha(t+dt) - S_\alpha(t) \stackrel{d}{=} S_\alpha(dt) \stackrel{d}{=} (dt)^{1/\alpha} S_\alpha(1)$*, where*

$$S_\alpha(1) \stackrel{d}{=} \frac{\sin(\alpha\pi U)[\sin((1-\alpha)\pi U)]^{1/\alpha-1}}{[\sin(\pi U)]^{1/\alpha}|\ln V|^{1/\alpha-1}};$$

Step 3*: for generating the increments of TSS* $S_{\alpha,\lambda}(t)$ *with pdf* $f_{\alpha,\lambda}(x,t)$*, we use the following steps called "acceptance-rejection method",*

(a) generate the stable random variable $S_\alpha(dt)$*;*

(b) generate uniform $(0,1)$ *rv* W *(independent of* S_α*);*

(c) if $W \le e^{-\lambda S_\alpha(dt)}$*, then* $S_{\alpha,\lambda}(dt) = S_\alpha(dt)$ *("accept"); otherwise go back to* (a) *("reject").*
Note that here we used (4)*, which implies* $\frac{f_{\alpha,\lambda}(x,dt)}{c f_\alpha(x,dt)} = e^{-\lambda x}$ *for* $c = e^{\lambda^\alpha dt}$*, and the ratio is bounded between 0 and 1;*

Step 4*: cumulative sum of increments gives the TSS* $S_{\alpha,\lambda}(t)$ *sample trajectories;*
Step 5*: generate* $S_{\alpha_1,\lambda_1}(c_1 t)$*,* $S_{\alpha_2,\lambda_2}(c_2 t)$ *and add these to get the MTSS, see* (9)*. The inverse MTSS sample trajectories are obtained by reversing the axis.*

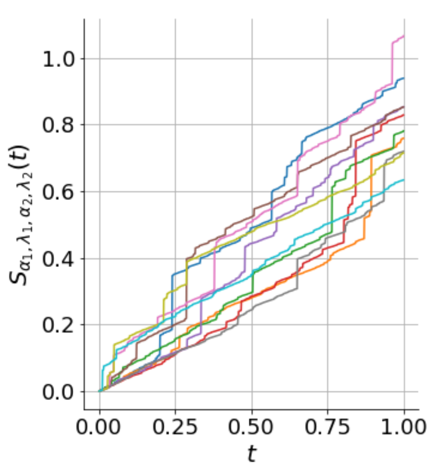

Figure 2: *MTSS*

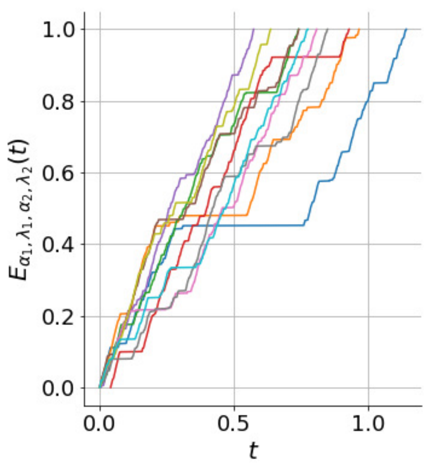

Figure 3: *Inverse MTSS*

4. Asymptotic forms of potential density and renewal function

In this section, we discuss the asymptotic behavior of the potential density at 0 (respectively at ∞) for MTSS and the asymptotic form of the renewal function for the IMTSS. The potential measure of a subordinator $S(t)$ is defined by [37]:

$$V(A) = \mathbb{E}\int_0^\infty 1_{(S(t)\in A)}dt. \tag{40}$$

The LT of the measure V is given by

$$\overline{V}(s) = \mathbb{E}\int_0^\infty \exp(-sS(t))dt = \frac{1}{\phi(s)}. \tag{41}$$

Note that the potential measure represents the expected time the subordinator spent in the set A.

Proposition 6. *Let v be the potential density of the MTSS. For any $\alpha_1, \alpha_2 \in (0,1]$, we have*

$$v(x) \sim \begin{cases} \frac{x^{\alpha_1+\alpha_2-\min(\alpha_1,\alpha_2)-1}}{\Gamma(\min(\alpha_1,\alpha_2))(c_1x^{\alpha_2-\min(\alpha_1,\alpha_2)}+c_2x^{\alpha_1-\min(\alpha_1,\alpha_2)})}, & \textit{as } x\to 0,\\ \frac{1}{c_1\alpha_1{\lambda_1}^{\alpha_1-1}+c_2\alpha_2{\lambda_2}^{\alpha_2-1}},\ \lambda_1,\lambda_2>0, & \textit{as } x\to\infty,\\ \frac{x^{\alpha_1+\alpha_2-\min(\alpha_1,\alpha_2)-1}}{\Gamma(\min(\alpha_1,\alpha_2))(c_1x^{\alpha_2-\min(\alpha_1,\alpha_2)}+c_2x^{\alpha_1-\min(\alpha_1,\alpha_2)})},\ \lambda_1,\lambda_2=0 & \textit{as } x\to\infty. \end{cases} \tag{42}$$

Proof. We apply the Tauberian theorem [10], which connects the asymptotic form of a Laplace transform with its inverse Laplace transform for a function. We have

$$\overline{V}(s) = \frac{1}{\phi(s)} \sim \frac{s^{-1}}{(\alpha_1{\lambda_1}^{\alpha_1-1}+\alpha_2{\lambda_2}^{\alpha_2-1})},\ \text{as } s\to 0.$$

Similarly, for $\lambda_1,\lambda_2>0$,

$$\overline{V}(s) \sim \frac{1}{c_1s^{\alpha_1}+c_2s^{\alpha_2}},\ \text{as } s\to\infty.$$

Applying the Tauberian theorem at $x\to 0$ (respectively at $x\to\infty$) gives the desired result. $\square$

Remark 7. *By substituting $\alpha_1=\alpha_2=\alpha$ and $\lambda_1=\lambda_2=\lambda$, with the condition $c_1+c_2=1$ in* (42)*, we obtain the asymptotic behavior of the potential density for TSS such that*

$$v(x) \sim \begin{cases} \frac{\lambda^{1-\alpha}}{\alpha}, & \textit{as } x\to\infty,\\ \frac{x^{\alpha-1}}{\Gamma(\alpha)} & \textit{as } x\to 0. \end{cases} \tag{43}$$

Here, we discuss the asymptotic form of the renewal function for IMTSS. The renewal function is given by $U(t) = \mathbb{E}(E_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(t))$. The Laplace transform (LT) of $U(t)$ is $\overline{U}(s) = \frac{1}{s\phi(s)}$, see [23].

Proposition 7. *The renewal function $U(t)$ has following asymptotic form,*

$$U(t)\sim\begin{cases}\frac{t^{\alpha_1+\alpha_2-\min(\alpha_1,\alpha_2)}}{\Gamma(1+\min(\alpha_1,\alpha_2))(c_1t^{\alpha_2-\min(\alpha_1,\alpha_2)}+c_2t^{\alpha_1-\min(\alpha_1,\alpha_2)})}, & as\ t\to 0,\\ \frac{t}{(c_1\alpha_1{\lambda_1}^{\alpha_1-1}+c_2\alpha_2{\lambda_2}^{\alpha_2-1})},\ \lambda_1,\lambda_2>0, & as\ t\to\infty,\\ \frac{t^{\alpha_1+\alpha_2-\min(\alpha_1,\alpha_2)}}{\Gamma(1+\min(\alpha_1,\alpha_2))(c_1t^{\alpha_2-\min(\alpha_1,\alpha_2)}+c_2t^{\alpha_1-\min(\alpha_1,\alpha_2)})},\ \lambda_1=\lambda_2=0, & as\ t\to\infty.\end{cases}$$

Proof. Using the Tauberian theorem [10], which says that $U(t)\sim t^p\frac{l(t)}{\Gamma(p+1)}$ as $t\to\infty$ (respectively to 0) is equivalent to $\overline{U}(s)\sim s^{-1-p}l(\frac{1}{s})$ as $s\to 0$ (respectively at ∞), where $l:(0,\infty)\to(0.\infty)$ is a slowly varying function at 0 (respectively to ∞). When $s\to 0$, the Laplace exponent behaves as

$$\phi(s)\sim s(c_1\alpha_1{\lambda_1}^{\alpha_1-1}+c_2\alpha_2{\lambda_2}^{\alpha_2-1}), \tag{44}$$

and hence

$$\begin{aligned}\overline{U}(s)=&\frac{1}{s\phi(s)}=\frac{1}{s(c_1\left((s+\lambda_1)^{\alpha_1}-\lambda_1^{\alpha_1}\right)+c_2\left((s+\lambda_2)^{\alpha_2}-\lambda_2^{\alpha_2}\right))}\\ &\sim\frac{s^{-2}}{(\alpha_1{\lambda_1}^{\alpha_1-1}+\alpha_2{\lambda_2}^{\alpha_2-1})},\end{aligned}$$

which further implies that the renewal function has the asymptotic form

$$U(t)\sim\frac{t}{(c_1\alpha_1{\lambda_1}^{\alpha_1-1}+c_2\alpha_2{\lambda_2}^{\alpha_2-1})},\ \lambda_1,\lambda_2>0,\ \text{as}\ t\to\infty. \tag{45}$$

For $\lambda_1=\lambda_2=0$, we have

$$U(t)\sim\frac{t^{\alpha_1+\alpha_2-\min(\alpha_1,\alpha_2)}}{\Gamma(1+\min(\alpha_1,\alpha_2))(c_1t^{\alpha_2-\min(\alpha_1,\alpha_2)}+c_2t^{\alpha_1-\min(\alpha_1,\alpha_2)})},\ \text{as}\ t\to\infty.$$

Moreover,

$$\phi(s)\sim c_1s^{\alpha_1}+c_2s^{\alpha_2}\ \text{as}\ s\to\infty, \tag{46}$$

and hence

$$\overline{U}(s)\sim\frac{1}{s^{1+\min(\alpha_1.\alpha_2)}(c_2s^{\alpha_2-\min(\alpha_1.\alpha_2)}+c_1s^{\alpha_1-\min(\alpha_1.\alpha_2)})},\ \text{as}\ s\to\infty, \tag{47}$$

where $l(s)=\frac{s^{\alpha_1+\alpha_2-2\min(\alpha_1.\alpha_2)}}{(c_2s^{\alpha_1-\min(\alpha_1.\alpha_2)}+c_1s^{\alpha_2-\min(\alpha_1.\alpha_2)})}$ is a slowly varying function at ∞ and hence the renewal function

$$U(t)\sim\frac{t^{\alpha_1+\alpha_2-\min(\alpha_1,\alpha_2)}}{\Gamma(1+\min(\alpha_1,\alpha_2))(c_1t^{\alpha_2-\min(\alpha_1,\alpha_2)}+c_2t^{\alpha_1-\min(\alpha_1,\alpha_2)})},\ \text{as}\ t\to 0. \tag{48}$$

$\square$

Remark 8. *Substitute $\alpha_1 = \alpha_2 = \alpha$ and $\lambda_1 = \lambda_2 = \lambda$ with condition $c_1 + c_2 = 1$ in* (50)*, which gives the asymptotic behaviour of the renewal function corresponding to TSS $S_{\alpha,\lambda}(t)$, see [23],*

$$U(t) \sim \begin{cases} t\frac{\lambda^{\alpha-1}}{\alpha}, & \textit{as } t \to \infty \\ \frac{t^{\alpha}}{\Gamma(1+\alpha)} & \textit{as } t \to 0. \end{cases} \tag{49}$$

Next, we discuss the asymptotic behavior of the q-th order moments $M_q(t) = \mathbb{E}(E_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(t))^q, q > 0$, of $E_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(t)$. The LT of $M_q(t)$ is given by $\overline{M}_q(s) = \frac{\Gamma(1+q)}{s(\phi(s))^q}$, see [21, 41, 42], where $\phi(s)$ is the Laplace exponent given in (15). Again using the Tauberian theorem, we have the following asymptotic behavior for $M_q(t)$:

$$M_q(t) = \begin{cases} \frac{t^{q(\alpha_1+\alpha_2-\min(\alpha_1,\alpha_2))}\Gamma(1+q)}{\Gamma(1+q\min(\alpha_1,\alpha_2))(c_1t^{\alpha_2-\min(\alpha_1,\alpha_2)}+c_2t^{\alpha_1-\min(\alpha_1,\alpha_2)})^q}, & \text{as } t \to 0, \\ \frac{t^q}{(c_1\alpha_1\lambda_1{}^{\alpha_1-1}+c_2\alpha_2\lambda_2{}^{\alpha_2-1})^q}, \ \lambda_1,\lambda_2 > 0, & \text{as } t \to \infty, \\ \frac{t^{q(\alpha_1+\alpha_2-\min(\alpha_1,\alpha_2))}\Gamma(1+q)}{\Gamma(1+q\min(\alpha_1,\alpha_2))(c_1t^{\alpha_2-\min(\alpha_1,\alpha_2)}+c_2t^{\alpha_1-\min(\alpha_1,\alpha_2)})^q}, \ \lambda_1 = \lambda_2 = 0, & \text{as } t \to \infty. \end{cases} \tag{50}$$

Remark 9. *$\alpha_1 = \alpha_2 = \alpha$ and $\lambda_1 = \lambda_2 = \lambda$ with condition $c_1 + c_2 = 1$, $c_1, c_2 \geq 0$ in* (50) *gives the asymptotic behavior of $M_q(t)$ for TSS $S_{\alpha,\lambda}(t)$ such that*

$$M_q(t) \sim \begin{cases} \frac{\Gamma(1+q)}{\Gamma(1+q\alpha)}t^{q\alpha}, & \textit{as } t \to 0, \\ \frac{\lambda^{q(1-\alpha)}}{\alpha^q}t^q, \ \lambda > 0, & \textit{as } t \to \infty, \\ \frac{\Gamma(1+q)}{\Gamma(1+q\alpha)}t^{q\alpha}, \ \lambda = 0, & \textit{as } t \to \infty. \end{cases} \tag{51}$$

In the next remark, n-th order mixtures of TSS are discussed.

Remark 10. *We define the n-th order mixtures of TSS as a Lévy process with LT:*

$$\mathbb{E}\left(e^{-sS_{\alpha_1,\lambda_1,\alpha_2,\lambda_2,\ldots,\alpha_n,\lambda_n}(t)}\right) = e^{-t\sum_{i=1}^{n} c_i((s+\lambda_i)^{\alpha_i}-\lambda_i{}^{\alpha_i})}, \ s > 0, \tag{52}$$

where $c_i \geq 0$ and $\sum_{i=1}^{n} c_i = 1$. The alternative representation of the n-th order MTSS is given by

$$S_{\alpha_1,\lambda_1,\alpha_2,\lambda_2,\ldots,\alpha_n,\lambda_n}(t) = \sum_{i=1}^{n} S_{\alpha_i,\lambda_i}(c_i t),$$

with the conditions $c_i \geq 0$, $\sum_{i=1}^{n} c_i = 1$. Using similar approaches as in previous subsections, we can obtain analogue results for the n-th order mixtures of TSS. The pdf of the n-th order mixtures of TSS is difficult to obtain using complex inversion. For $0 < p < 1$, the asymptotic behavior of the p-th order moments of the n-th order mixtures of TSS is given by

$$\mathbb{E}\left(S_{\alpha_1,\lambda_1,\alpha_2,\lambda_2,\ldots,\alpha_n,\lambda_n}(t)^p\right) \sim \left(\sum_{i=1}^{n} (c_i\alpha_i\lambda_i{}^{\alpha_i-1})\right)^p t^p, \textit{ as } t \to \infty. \tag{53}$$

The generalized pdf $g_{\alpha_1,\lambda_1,\alpha_2,\lambda_2\ldots,\alpha_n,\lambda_n}(x,t)$ for the n-th order mixtures of TSS satisfies the following FPDE with condition $g_{\alpha_1,\lambda_1,\alpha_2,\lambda_2\ldots,\alpha_n,\lambda_n}(0,t)=0$,

$$\frac{\partial}{\partial t}g_{\alpha_1,\lambda_1,\alpha_2,\lambda_2\ldots,\alpha_n,\lambda_n}(x,t) = -\sum_{i=1}^{n} c_i\left(\lambda_i+\frac{\partial}{\partial x}\right)^{\alpha_i} g_{\alpha_1,\lambda_1,\alpha_2,\lambda_2\ldots,\alpha_n,\lambda_n}(x,t) + \left(\sum_{i=1}^{n}\lambda_i^{\alpha_i}c_i\right)g_{\alpha_1,\lambda_1,\alpha_2,\lambda_2\ldots,\alpha_n,\lambda_n}(x,t).$$

Further, the asymptotic behaviour of $v(x)$ for the n-th order mixtures of TSS is obtained in the same manner and given by

$$v(x)\sim\begin{cases} \frac{x^{\sum_{i=1}^n \alpha_i-(n-1)\min(\alpha_1,\alpha_2,\ldots,\alpha_n)-1}}{\Gamma(\min(\alpha_1,\alpha_2,\ldots,\alpha_n))\left(\sum_{i=1}^n c_i x^{\sum_{j\neq i}^n \alpha_i-(n-1)\min(\alpha_1,\alpha_2,\ldots,\alpha_n)}\right)}, & as\ x\to 0,\\ \frac{1}{\left(\sum_{i=0}^n c_i\alpha_i{\lambda_i}^{\alpha_i-1}\right)},\ \lambda_1,\lambda_2,\ldots,\lambda_n>0, & as\ x\to\infty,\\ \frac{x^{\sum_{i=1}^n \alpha_i-(n-1)\min(\alpha_1,\alpha_2,\ldots,\alpha_n)-1}}{\Gamma(\min(\alpha_1,\alpha_2,\ldots,\alpha_n))\left(\sum_{i=1}^n c_i x^{\sum_{j\neq i}^n \alpha_i-(n-1)\min(\alpha_1,\alpha_2,\ldots,\alpha_n)}\right)}, & \\ \qquad \lambda_1,\lambda_2,\ldots,\lambda_n=0, & as\ x\to\infty. \end{cases}$$

The asymptotic behavior of the renewal function corresponding to the n-th order compositions of TSS is given by

$$U(t)\sim\begin{cases} \frac{t^{\left(\sum_{i=1}^n \alpha_i-(n-1)\min(\alpha_1,\alpha_2,\ldots,\alpha_n)\right)}}{\Gamma(1+\min(\alpha_1,\alpha_2,\ldots,\alpha_n))\left(\sum_{i=1}^n c_i t^{\sum_{j\neq i}^n \alpha_i-(n-1)\min(\alpha_1,\alpha_2,\ldots,\alpha_n)}\right)}, & \text{as } t\to 0,\\ \frac{t}{\left(\sum_{i=0}^n c_i\alpha_i{\lambda_i}^{\alpha_i-1}\right)},\ \lambda_1,\lambda_2,\ldots,\lambda_n>0, & \text{as } t\to\infty,\\ \frac{t^{\left(\sum_{i=1}^n \alpha_i-(n-1)\min(\alpha_1,\alpha_2,\ldots,\alpha_n)\right)}}{\Gamma(1+min(\alpha_1,\alpha_2,\ldots,\alpha_n))\left(\sum_{i=1}^n c_i t^{\sum_{j\neq i}^n \alpha_i-(n-1)\min(\alpha_1,\alpha_2,\ldots,\alpha_n)}\right)}, & \\ \qquad \lambda_1,\lambda_2,\ldots,\lambda_n=0, & \text{as } t\to\infty. \end{cases}$$

Further, the corresponding $M_q(t)$ has the following asymptotic form:

$$M_q(t)\sim\begin{cases} \frac{t^{q\left(\sum_{i=1}^n \alpha_i-(n-1)\min(\alpha_1,\alpha_2,\ldots,\alpha_n)\right)}\Gamma(1+q)}{\Gamma(1+q\min(\alpha_1,\alpha_2,\ldots,\alpha_n))\left(\sum_{i=1}^n c_i t^{\sum_{j\neq i}^n \alpha_i-(n-1)\min(\alpha_1,\alpha_2,\ldots,\alpha_n)}\right)^q}, & as\ t\to 0,\\ \frac{t^q}{\left(\sum_{i=0}^n c_i\alpha_i{\lambda_i}^{\alpha_i-1}\right)^q},\ \lambda_1,\lambda_2,\ldots,\lambda_n>0, & as\ t\to\infty,\\ \frac{t^{q\left(\sum_{i=1}^n \alpha_i-(n-1)\min(\alpha_1,\alpha_2,\ldots,\alpha_n)\right)}\Gamma(1+q)}{\Gamma(1+q\min(\alpha_1,\alpha_2,\ldots,\alpha_n))\left(\sum_{i=1}^n c_i t^{\sum_{j\neq i}^n \alpha_i-(n-1)\min(\alpha_1,\alpha_2,\ldots,\alpha_n)}\right)^q}, & \\ \qquad \lambda_1,\lambda_2,\ldots,\lambda_n=0, & as\ t\to\infty. \end{cases}$$

5. Applications of MTSS and IMTSS as time changes

In this section, we introduce a time-changed Poisson process and Brownian motion by considering MTSS and IMTSS as time-changes. Note that a Poisson process time-changed by MTSS generalizes the space-fractional Poisson process [30] and the Poisson process time-changed by IMTSS generalize the time-fractional Poisson process ([25] and references therein). Further, Brownian motion time-changed by IMTSS generalize Brownian motion time-changed by the inverse stable subordinator model which is the scaling limit of continuous time random walk with the infinite mean waiting time [27]. It is worth mentioning here that the governing equation of Brownian motion time-changed by the inverse stable subordinator is a fractional analogue of the heat equation which involves a fractional derivative in a time variable. We discuss the governing fractional differential equations of these time-changed processes.

5.1. The mixture tempered-space fractional Poisson process (MTSFPP)

In this section, we introduce and give the governing fractional difference-differential equation of the mixture tempered space-fractional Poisson process (MTSFPP). A subordination representation of the MTSFPP can be written as

$$X(t) = N(S_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(t)), \tag{54}$$

where a homogeneous Poisson process $N(t)$ with intensity $\mu > 0$ is independent of $S_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(t)$. The main purpose is to generalize a homogeneous Poisson process in fraction sense by introducing a fractional difference operator in the governing equation in the state space. The PMF $r(k,t) = P(X(t) = k)$ of the MTSFPP can be easily obtained in an infinite series form by the standard conditioning argument. The probability generating function (PGF) $G(z,t) = \mathbb{E}[z^{X(t)}]$ for $X(t)$ is given by

$$G(z,t) = e^{-t(c_1\{(\lambda_1+\mu(1-z))^{\alpha_1}-\lambda_1{}^{\alpha_1}\}+c_2\{(\lambda_2+\mu(1-z))^{\alpha_2}-\lambda_2{}^{\alpha_2}\})},\ |z| \leq 1, \mu \leq \frac{\lambda_i}{2}, i = 1,2. \tag{55}$$

Proposition 8. *The marginal distribution $r(k,t) = \mathbb{P}(X(t) = k)$ satisfies the following fractional difference differential equation:*

$$\frac{d}{dt}r(k,t) = -\sum_{i=1}^{2} c_i\{(\lambda_i + \mu(1-B))^{\alpha_i} - \lambda_i^{\alpha_i}\}r(k,t),\ \alpha_i \in (0,1), \tag{56}$$

with the conditions $r(0,0) = 1$ and $r(k,0) = 0$ for $k \neq 0$, where B is the backward shift operator.

Proof. Using the PGF, it follows

$$\begin{aligned}\frac{\partial}{\partial t}G(z,t) &= -\sum_{i=1}^{2} c_i \left[\sum_{l=0}^{\infty}\binom{\alpha_i}{l}\lambda_i{}^{\alpha_i-l}\mu^l\sum_{m=0}^{\infty}\binom{l}{m}(-1)^m\sum_{k=0}^{\infty}z^k r_{k-m}(t)\right.\\ &\qquad \left. -\lambda_i{}^{\alpha_i}\sum_{k=0}^{\infty}z^k r_k(t)\right]\\ &= -\sum_{i=1}^{2} c_i\left[\sum_{l=0}^{\infty}\binom{\alpha_i}{l}\lambda_i{}^{\alpha_i-l}\mu^l\sum_{m=0}^{\infty}\binom{l}{m}(-z)^m - \lambda_i{}^{\alpha_i}\right]G(z,t)\end{aligned}$$

$$\frac{\partial}{\partial t}G(z,t) = -\,G(z,t)\sum_{i=1}^{2}c_i\left[(\lambda_i+\mu(1-z))^{\alpha_i} - \lambda_i{}^{\alpha_i}\right].$$

The result follows by using $G(z,0) = 1$ and (55). $\square$

5.2. The mixture tempered time-fractional Poisson process (MTTFPP)

One can also define a mixture tempered time-fractional Poisson process (MTTFPP) by subordinating homogeneous Poisson process $N(t)$ with the IMTSS process $E_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(t)$ such as

$$Y(t) = N(E_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(t)), \tag{57}$$

where $N(t)$ and $E_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(t)$ are independent. Next, we derive the governing fractional difference differential equation for the marginal distribution of MTTFPP $Y(t)$.

Proposition 9. *The marginal PMF $p_\mu(k,t)$ of $Y(t)$ satisfies the following governing equation:*

$$\begin{aligned}
&\left[c_1\left(\lambda_1+\frac{\partial}{\partial t}\right)^{\alpha_1}+c_2\left(\lambda_2+\frac{\partial}{\partial t}\right)^{\alpha_2}\right]p_\mu(k,t)\\
&\quad= -\mu[p_\mu(k,t)-p_\mu(k-1,t)]+[\lambda_1^{\alpha_1}c_1+\lambda_2^{\alpha_1}c_2]p_\mu(k,t)\\
&\quad\quad - c_1t^{-\alpha_1}M^{1-\alpha_1}_{1,1-\alpha_1}(-\lambda_1 t)\delta(x) - c_2t^{-\alpha_2}M^{1-\alpha_2}_{1,1-\alpha_2}(-\lambda_2 t)\delta(x), \quad k\geq 1.
\end{aligned}$$

Proof. Let $p(k,t)$ be the PMF of the standard Poisson process. By a standard conditioning argument and using (16), we have

$$\begin{aligned}
&\left[c_1\left(\lambda_1+\frac{\partial}{\partial t}\right)^{\alpha_1}+c_2\left(\lambda_2+\frac{\partial}{\partial t}\right)^{\alpha_2}\right]p_\mu(k,t)\\
&\quad= \int_0^\infty p(k,u)\left[c_1\left(\lambda_1+\frac{\partial}{\partial t}\right)^{\alpha_1}+c_2\left(\lambda_2+\frac{\partial}{\partial t}\right)^{\alpha_2}\right]H(u,t)du\\
&\quad= -\int_0^\infty p(k,u)\frac{\partial}{\partial u}H(u,t)du+[\lambda_1c_1+\lambda_2c_2]p_\mu(k,t)\\
&\quad\quad - c_1t^{-\alpha_1}M^{1-\alpha_1}_{1,1-\alpha_1}(-\lambda_1 t)\delta(x) - c_2t^{-\alpha_2}M^{1-\alpha_2}_{1,1-\alpha_2}(-\lambda_2 t)\delta(x),
\end{aligned}$$

and finally, integration by parts yields the desired result. $\square$

5.3. Time-changed Brownian motion

In this section, we introduce time-changed processes $Z(t)$ and $W(t)$ as Brownian motion $B(t)$ time-changed by MTSS $S_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(t)$ and IMTSS $E_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(t)$, respectively, i.e.

$$\begin{aligned}
Z(t) &= B(S_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(t)), \\
W(t) &= B(E_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(t)), \quad t>0.
\end{aligned} \tag{58}$$

By applying the previous results, we can find the governing equations for the pdf of $Z(t)$ and $W(t)$ and the same can be generalized for the N-th order mixtures of TSS.

Proposition 10. *The pdf $r_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(x,t) = \mathbb{P}(Z(t) \in dx)$ of the time-changed Brownian motion $Z(t)$ defined in (58) satisfies the following space-fractional differential equation:*

$$\begin{aligned} \frac{\partial}{\partial t} r_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(x,t) =& - c_1 \left(\lambda_1 - \frac{\partial^2}{\partial x^2}\right)^{\alpha_1} r_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(x,t) \\ & - c_2 \left(\lambda_2 - \frac{\partial^2}{\partial x^2}\right)^{\alpha_2} r_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(x,t) \\ & + \lambda_1^{\alpha_1} c_1 r_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(x,t) + \lambda_2^{\alpha_1} c_2 r_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(x,t), \end{aligned} \tag{59}$$

with initial and boundary conditions

$$\begin{cases} r_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(x,0) = \delta(x) \\ \lim_{|x|\to\infty} r_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(x,t) = 0, \\ \lim_{|x|\to\infty} \frac{\partial}{\partial x} r_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(x,t) = 0. \end{cases} \tag{60}$$

Proof. We will use Proposition 1 to prove this result. One can write

$$r_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(x,t) = \int_0^\infty q(x,u) G(u,t) du,$$

where $q(x,t)$ is the pdf of the standard Brownian motion $B(t)$. Further,

$$\begin{aligned} \frac{\partial}{\partial t} r_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(x,t) =& \int_0^\infty q(x,u) \frac{\partial}{\partial t} G(u,t) du \\ =& (\lambda_1^{\alpha_1} c_1 + \lambda_2^{\alpha_1} c_2) r_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(x,t) \\ & - c_1 \sum_{i=0}^{\infty} \binom{\alpha_1}{i} \lambda_1^{\alpha_1 - i} \int_0^\infty q(x,u) \frac{\partial^i}{\partial u^i} G(u,t) du \\ & - c_1 \sum_{j=0}^{\infty} \binom{\alpha_2}{j} \lambda_2^{\alpha_2 - j} \int_0^\infty q(x,u) \frac{\partial^j}{\partial u^j} G(u,t) du \\ =& (\lambda_1^{\alpha_1} c_1 + \lambda_2^{\alpha_1} c_2) r_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(x,t) \\ & - c_1 \sum_{i=0}^{\infty} (-1)^i \binom{\alpha_1}{i} \lambda_1^{\alpha_1 - i} \int_0^\infty \frac{\partial^i}{\partial u^i} q(x,u) G(u,t) du \\ & - c_1 \sum_{j=0}^{\infty} \binom{\alpha_2}{j} (-1)^j \lambda_2^{\alpha_2 - j} \int_0^\infty \frac{\partial^j}{\partial u^j} q(x,u) G(u,t) du \\ =& (\lambda_1^{\alpha_1} c_1 + \lambda_2^{\alpha_1} c_2) r_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(x,t) \\ & - \left[c_1 \sum_{i=0}^{\infty} \binom{\alpha_1}{i} \lambda_1^{\alpha_1 - i} \left(-\frac{\partial^2}{\partial x^2}\right)^i \right. \\ & \left. + c_2 \sum_{j=0}^{\infty} \binom{\alpha_2}{j} \lambda_2^{\alpha_1 - j} \left(-\frac{\partial^2}{\partial x^2}\right)^j \right] r_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(x,t), \end{aligned}$$

hence proved. ∎

Proposition 11. *The density* $w_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(x,t)$ *of the process* $W(t)$ *defined in* (58) *satisfies the following fractional differential equation:*

$$\begin{aligned}\frac{\partial^2}{\partial x^2} w_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(x,t) = &- \left[c_1\left(\lambda_1 + \frac{\partial}{\partial t}\right)^{\alpha_1} + c_2\left(\lambda_2 + \frac{\partial}{\partial t}\right)^{\alpha_2}\right] w_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(x,t)\\ &+ \left[\lambda_1^{\alpha_1} c_1 + \lambda_2^{\alpha_1} c_2\right] w_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(x,t)\\ &- c_1 t^{-\alpha_1} M^{1-\alpha_1}_{1,1-\alpha_1}(-\lambda_1 t)\delta(x) - c_2 t^{-\alpha_2} M^{1-\alpha_2}_{1,1-\alpha_2}(-\lambda_2 t)\delta(x),\end{aligned}$$

with initial and boundary conditions

$$\begin{cases} w_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(x,0) = \delta(x)\\ \lim_{|x|\to\infty} w_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(x,t) = 0,\\ \lim_{|x|\to\infty} \frac{\partial}{\partial x} w_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(x,t) = 0.\end{cases}$$

Proof. The result can be proved by using similar argument as given in Proposition 10 and with the help of (16). $\square$

Remark 11. *The demeaned and rescaled standard Poisson process converges in* $D([0,\infty))$ *with respect to* J_1*-topology to a standard Brownian motion* $\{B(t),\ t \geq 0\}$*, which follows by the functional central limit theorem, i.e.*

$$\left(\frac{N(t) - \lambda t}{\sqrt{\lambda}}\right)_{t\geq 0} \xrightarrow[\lambda\to\infty]{J_1} B(t).$$

Using Theorem 13.2.2 *of Whitt [43], since* $B(t)$ *has continuous paths and* $E_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(t)$ *has strictly increasing paths; then*

$$\left(\frac{N(E_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(t)) - \lambda E_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(t)}{\sqrt{\lambda}}\right)_{t\geq 0} \xrightarrow[\lambda\to\infty]{J_1} B(E_{\alpha_1,\lambda_1,\alpha_2,\lambda_2}(t)).$$

Hence the process $Y(t)$ *defined in* (57) *and the process* $Z(t)$ *defined in* (58) *are connected through the scaling limit.*

Acknowledgements

The authors are grateful to the reviewers for several helpful suggestions and comments which improved the quality and presentation of the paper. N.G. would like to thank the Council of Scientific and Industrial Research (CSIR), India, for supporting her research under the fellowship award number 09/1005(0021)2018-EMR-I. Further, A.K. would like to express his gratitude to the Science and Engineering Research Board (SERB), India, for the financial support under the MATRICS research grant MTR/2019/000286.

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