STOCHASTIC PROCESSES AND POINT PROCESSES OF EXCURSIONS

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PROEFSCHRIFT

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S T E L L I N G E N

bij het proefschrift

STOCHASTIC PROCESSES AND POINT PROCESSES OF EXCURSIONS

van

J.A.M.van der Weide

1. Zij P een Markov-kern op een meetbare ruimte $(E,\mathcal{E})$. Definieer de Markov-kern $\tilde{P}$ op de meetbare ruimte $(E \times [0,1], \mathcal{E} \otimes \mathcal{B}([0,1]))$ door

$$\tilde{P}f(x,\alpha) = \int_E P(x,dy_1) \int_0^1 f(y_1,y_2)dy_2$$

waarbij $f : E \times [0,1] \to \mathbb{R}$ een begrensde, meetbare functie is en $(x,\alpha) \in E \times [0,1]$. De in [1] voor meetbare functies $h : E \to [0,1]$ gedefinieerde potentiaalkern U_h heeft de volgende stochastische interpretatie. Voor $x \in E$ en $A \in \mathcal{E}$ is

$$U_h(x,A) = E_{(x,\alpha)}\left[\sum_{n=1}^{\tau_h} 1_{A \times [0,1]}(Y_n)\right]$$

waarbij $(Y_n)_{n \geq 0}$ de Markovketen op $E \times [0,1]$ is met overgangskern $\tilde{P}$ en waarbij τ_h de eerste terugkeertijd is van $\Lambda_h = \{(x,y) \in E \times [0,1] : y \leq h(x)\}$.

[1] Revuz, D. : Markov Chains. North-Holland, Publ.Comp. Amsterdam 1975.

2. Zij $X = (X_t)_{t \geq 0}$ standaard Brownse beweging startend vanuit 0 en zij $A = (A_t)_{t \geq 0}$ het additief functionaal gedefinieerd door

$$A_t = \int_0^t 1_{[0,\infty[}(X_s)ds.$$

Zij $\tau = (\tau_t)_{t \geq 0}$ de van rechts-continue inverse van A. Het stochastisch proces $\tilde{X} = (\tilde{X}_t)_{t \geq 0}$ gegeven door $\tilde{X}_t = X \circ \tau_t$ kan direct geconstrueerd worden uit het Itô-Poisson puntproces van excursies vanuit 0 van de Brownse beweging X. Uit een eenvoudige berekening volgt dat $\tilde{X}$ gereflecteerde Brownse beweging is.

Williams,D : Diffusions, Markov Processes and Martingales, Volume 1 : Foundations. ch. III, section 38, Wiley, New York 1979.

3. In [1] wordt een Suslin ruimte gedefinieerd als een Hausdorff topologische ruimte E waarvoor een Suslin-metriseerbare ruimte P bestaat en een continue, surjectieve afbeelding van P op E. De opmerking dat dit Bourbaki's definitie van een Suslin ruimte is als aangenomen wordt dat P een Poolse ruimte is, is onjuist.

[1] Dellacherie, C. and Meyer, P.A. : Probabilities and Potential. North-Holland, Amsterdam 1978.

4. Als de kansverdeling $\{p_k : k \in \mathbb{Z}\}$ in sectie (4.2) van dit proefschrift een eindig tweede moment bezit, dan convergeert de rij kansmaten $(P_{\nu_n}^{(n)})_{n \geq 1}$ in stelling (4.2.4) zwak naar de kansmaat P_m^{α}.

5. Zij $Y_\gamma = (Y_\gamma(t))_{t \geq 0}$ het Markov proces geconstrueerd uit het Itô-Poisson puntproces van excursies vanuit 0 van standaard Brownse beweging waarbij de term $\gamma\tau$, $\gamma \geq 0$, is opgeteld bij de som $(\tau_a + A(\tau))$ van de lengtes van de excursies tot en met tijd τ (zie sectie (4.3) van dit proefschrift). Zij verder L de Blumenthal-Getoor locale tijd in 0 van proces Y_γ. Dan is voor $\lambda, \mu > 0$ en $0 < \alpha < \frac{1}{2}\lambda^2$

$$\int_0^\infty e^{-\alpha t}\, E_0(\exp[-\lambda Y_\gamma(t) - \mu L(t)])dt = \frac{2\sqrt{2\alpha}}{(2\alpha-\lambda^2)(\mu(\sqrt{2}+\gamma)+\alpha\gamma+\sqrt{2\alpha})} .$$

6. Laat het Markov proces Y_γ gedefinieerd zijn als in stelling 5 en laat $m_a = \min(t : Y_\gamma(t) = a)$ het eerste treffen van toestand $a \in \mathbb{R}$ zijn. Dan is voor $\lambda > 0$

$$E_0(e^{-\lambda m_a}) = \sqrt{2\lambda}\, e^{-a\sqrt{2\lambda}}\, (\lambda\gamma+\sqrt{2\lambda} - \lambda\gamma e^{-2a\sqrt{2\lambda}})^{-1} .$$

7. Het verdient aanbeveling om ook bij het onderwijs aan de TU Delft in de vakken Analyse, Lineaire Algebra, Kansrekening en Statistiek meer gebruik te maken van computers.

8. In de Breitkopf uitgave van Johann Sebastian Bach's Matthaeus Passion is in de aria "Gebt mir meinen Jesum wieder" de tweede tel van de vierde maat niet in overeenstemming met het handschrift van Bach. Bij uitvoeringen van deze aria verdient de oorspronkelijke, door Bach aangeduide stokvoering de voorkeur.

Voor Els,
Mark, Jorrit en Hiske

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CONTENTS

INTRODUCTION AND SUMMARY

In his studies [35] and [36] of the sample paths of Brownian motion, Lévy developed the idea to decompose the time set $[0,\infty[$ in a part Z at which the process is in state 0 and intervals of time spent in $\mathbb{R}\backslash\{0\}$. Throughout the years this has proved to be a very fruitful idea. On one hand the study of the set of zeros Z led Lévy to the description of local time as an occupation density (Lévy used in [35] the term "mesure du voisinage" . See for occupation densities the survey article of Geman and Horowitz [14], who discuss connections between the behaviour of a (non-random) real-valued Borel function and the behaviour of its occupation density. Local times for general Markov processes were introduced by Blumenthal and Getoor in [3]). On the other hand Lévy's study of the behaviour of Brownian motion on zero-free intervals was the starting point of excursion theory. Lévy's theory was extended in Itô-McKean [27], (2.9) and (2.10). See also Chung's article [6], in which elementary derivations are given of a number of Lévy's results. This research led to many deep theorems about the behaviour of the paths of diffusions, see for instance Williams [58] and Walsh's discussion of Williams' results in [53]. Another important application of excursion theory can be found in the construction of those strong Markov processes, which behave outside a fixed state (or more generally outside a set D) as a given Markov process X. In this area the works of Dynkin [9], [10] and Watanabe [54], [55] are important. For excursions from a subset S, see the works of Maisonneuve [38], [39] and Getoor [16]. Getoor gives also an application to invariant measures, see also Kaspi [30] and [31]. Unlike occupation densities, which are also useful in the study of non-random functions, excursion theory takes its use from the Markov character of the random process. To make clear the ideas behind excursion theory, let $X=(X_n)_{n\geq 0}$ be a homogeneous Markov chain with state space E. Let $a \in E$ be a given state. Denote for $k=1,2,\ldots$ by ν_a^k the time at which the Markov chain X visits the state a for the k^{th} time; $\nu_a^k = \infty$ if there are less than k visits to a. Suppose that a is a recurrent state, i.e. $\mathbb{P}_a[\nu_a^1 < \infty] = 1$. Then the k^{th} excursion $V_k = (V_k(n))_{n\geq 0}$ from a of the Markov Chain X is defined as follows

$$V_k(n) = \begin{cases} X_{\nu_a^k+n} \ldots & \text{for } 0 \leq n < \nu_a^{k+1}-\nu_a^k \\ a & \text{for } n \geq \nu_a^{k+1}-\nu_a^k \end{cases}.$$

Let $V_o = (V_o(n))_{n\geq 0}$ be defined by

$$V_o(n) = \begin{cases} X_n & \text{for } n < \nu_a^1 \\ a & \text{for } n \geq \nu_a^1 \end{cases}.$$

It follows from the strong Markov property that the sequence of excursions $(V_k)_{k\geq 1}$ is independent and identically distributed. It is clear that the process X can be reconstructed pathwise from the sequence $(V_k)_{k\geq 0}$.

For Markov processes with continuous time parameter the situation is more complicated. As an example take standard Brownian motion $B = (B_t)_{t\geq 0}$ and consider the excursions from state 0. Let Z be the set of zeros of B. The component intervals of $[0,\infty[\setminus Z$ are called excursion intervals. Since Z is a topological Cantor set of Lebesgue measure 0 (see Itô-McKean [27], problem 5, p.29), with probability 1 there is no first excursion interval. Let $I=]\alpha,\beta[$ be an excursion interval. The map $V_I : [0,\infty[\to \mathbb{R}$ defined by

$$V_I(t) = \begin{cases} B_{\alpha+t} & \text{for } 0 \leq t < \beta-\alpha \\ 0 & \text{for } t \geq \beta-\alpha \end{cases}$$

is the excursion made by B from 0 corresponding to the excursion interval I; $\zeta=\beta-\alpha$ is called the length of the excursion. Put $\tau_I=\varphi(\alpha)$, where φ is the local time of B at zero. Itô proved in [25] (see also Meyer [41]) that the random distribution of points (τ_I,V_I) in $[0,\infty[\times U$, I running through the excursion intervals and U being the space of excursions from 0, is a Poisson point process on $[0,\infty[\times U$ whose intensity measure is the product of Lebesgue measure λ on $[0,\infty[$ and some σ-finite measure ν on U. This means that the number of points (τ_I, V_I) in a subset $[u,v[\times U_0$ of $[0,\infty[\times U$ is Poisson distributed with expectation $(v-u)\nu(U_0)$ whilst the numbers of points (τ_I, V_I) in disjoint subsets of $[0,\infty[\times U$ are independent. Itô proved this result actually for excursions of a standard Markov process X from a regular point a, and he gave a characterization of the excursion law ν of a recurrent extension of X, i.e. a strong Markov process, which behaves as X until the first hitting of state a.

It is interesting to look at Itô's definition of a point process. Let $(S,\mathcal{S})$ be a measurable space. A point function $p :]0,\infty[\to U$ is defined to be a map from a countable set $D_p \subset]0,\infty[$ into U. Meyer in [41] considers a point function p as a map defined for all points in $]0,\infty[$ by putting $p(x)=\partial$ for $x \in]0,\infty[\backslash D_p$ where ∂ is an extra point added to U. Let now Π be the space of all point functions: $]0,\infty[\to U$. Denote for $p \in \Pi$ and for $E \in \mathcal{B}(]0,\infty[) \otimes \mathcal{S}$ by $N(E,p)$ the number of the time points $t \in D_p$ for which $(t,p(t)) \in E$. The Borel σ-algebra $\mathcal{B}(\Pi)$ on Π is defined as the σ-algebra generated by the sets $\{p \in \Pi : N(E,p) = k\}$, $E \in \mathcal{B}(]0,\infty[) \otimes \mathcal{S}$, $k=0,1,2,\ldots$ Itô defined a point process as a $(\Pi, \mathcal{B}(\Pi))$-valued random variable. For instance the point process of excursions from 0 of Brownian motion is the (stochastic) point function p defined by

$$D_p = \{\tau_I : \text{I an excursion interval}\} \quad \text{and}$$
$$p(t) = V_I, \quad t = \tau_I \in D_p.$$

This definition gives a clear picture of point processes such as they appear in excursion theory and that is presumably the reason why in studies about excursion theory this definition is always used, see for instance Watanabe [54] and Greenwood and Pitman [17]. Beside this definition of a point process as a stochastic point function, there exists a fairly general theory of point processes which views a point process as a discrete random measure. See Neveu [44], Jagers [29] and Krickeberg [34] for point processes on a locally compact space and Matthes, Kerstan and Mecke [40] for point processes on a complete, separable metric space. This measure-theoretical approach to excursions makes it possible to use some important results from this theory, such as e.g. the Palm-formula, which were up to now not used in the literature about excursion theory. An example of the use of the Palm-formula can be found in the construction of a Markov process from a Poisson point process of excursions. Itô only remarks in [25] that this can be done by reversing the procedure of deriving the excursion process from a Markov process. In Ikeda & Watanabe [22] Brownian motion is constructed from its excursion process using the general theory of stochastic processes (compensators and stochastic integrals). And in [2] Blumenthal gives a construction of which he claims that it is the construction Itô had in mind; this contruction consists of a pathwise approximation of the Markov process. The most recent and complete work along these lines can be found in Salisbury [47] and [48]. The construction that we will give is based on an application of the Palm formula and on the so-called renewal property of a Poisson point process of excursions. This construction has in our opinion the advantage that it makes clear why the constructed process has the Markov property and it displays the role of local time in the construction. The same method can be used to write down a formula for the resolvent of the constructed process.

We continue with the definition of a point process as a discrete random measure. Let X be a topological space with Borel σ-algebra $\mathcal{B}(X)$. Roughly stated, a point process on X is a probability measure on the space of locally finite point measures on (X, $\mathcal{B}(X)$), or a random variable with values in the space of locally finite point measures on

$(X,\mathscr{B}(X))$ by which we identify a random variable with its distribution. From now on we will use the word point process only in this sense. In excursion theory the topological space X is the (topological) product of the set of nonnegative reals $[0,\infty[$ with the usual topology and the space of excursions U endowed with the Skorohod topology. For example the point process of excursions from 0 of Brownian motion is the random measure $\Sigma\{\delta_{(\tau_I,V_I)}$:I an excursion interval} where δ_x is the notation for the Dirac measure in x. Note that $X = [0,\infty[\times U$ is a polish space. The main reference on point processes on polish spaces is the book [40] of Matthes, Kerstan and Mecke. The theory which they develop depends essentially on a fixed metric d on X, chosen in advance, such that the metric *topology coincides with the topology of* X and (X,d) is a complete, separable metric space. A nonnegative Borel measure on X is locally finite if it is finite on the sets in $\mathscr{B}(X)$, which are bounded in the sense of the metric d. This theory is not directly applicable to excursion theory. The point measures which arise in excursion theory are finite on the sets $[a,b[\times [\zeta > l]$, $l > 0$, (remember that ζ is the length of the excursion) and the most interesting cases are those where the set $[a,b[\times U$ has infinite mass. Note that the set $[\zeta > l]$ is dense in U. Thus it is not clear how to choose a metric d on X for which the set of locally finite measures contains this family of point measures. Instead of trying to find such a metric, it seems more natural to define local finiteness directly in terms of the sets $[a,b[\times [\zeta > l]$. More general, let $\mathscr{S}$ be a family of Borel subsets of X. A nonnegative Borel measure μ on X is called $\mathscr{S}$-finite if $\mu(A) < \infty$ for every $A \in \mathscr{S}$ and a point process P is an $\mathscr{S}$-finite point process if the probability measure P is concentrated on the space of $\mathscr{S}$-finite measures. The set of locally finite measures in the sense of Matthes, Kerstan and Mecke coincides then with the $\mathscr{S}$-finite measures, $\mathscr{S}$ being the family of all open balls with finite radius. Point processes on locally compact spaces are probability measures on the set of Radon measures, which is the same as the set of $\mathcal{S}$-finite measures with $\mathcal{S}$ consisting of the compact subsets.

So far we did not discuss a measurable structure on the set $\mathcal{M}^+(\mathcal{S})$ of $\mathcal{S}$-finite measures, which is of course necessary for the definition of probability measures on $\mathcal{M}^+(\mathcal{S})$. A σ-algebra on $\mathcal{M}^+(\mathcal{S})$ should at least measure the maps $\mu \in \mathcal{M}^+(\mathcal{S}) \to \mu(A)$, $A \in \mathcal{B}(X)$. In Matthes et al. [40] the σ-algebra on $\mathcal{M}^+(\mathcal{S})$ ($\mathcal{S}$ being the family of open balls of finite radius) is defined in an abstract way as the σ-algebra $\mathcal{A}$ generated by these maps. In the literature about point processes on locally compact spaces, on the other hand a σ-algebra on the set of Radon measures is introduced in a topological way as the Borel σ-algebra $\mathcal{B}(\mathcal{M}^+)$ corresponding to the vague topology on $\mathcal{M}^+(\mathcal{S})$. It turns out that $\mathcal{B}(\mathcal{M}^+)=\mathcal{A}$ in this case, so we have a definition of $\mathcal{A}$ as a Borel σ-algebra corresponding to a nice topology on $\mathcal{M}^+(\mathcal{S})$, which makes it possible to use the apparatus of topological measure theory. In section (1.1) we will define a topology on the set $\mathcal{M}^+(\mathcal{S})$ of $\mathcal{S}$-finite measures on an arbitrary polish space X. Let $\mathcal{K}(\mathcal{S}) = \{f \in C_b(X) : \exists A \in \mathcal{S} : \text{supp}(f) \subset A\}$ and let $\tau(\mathcal{S})$ be the topology $\sigma(\mathcal{M}^+(\mathcal{S}), \mathcal{K}(\mathcal{S}))$ of pointwise convergence on $\mathcal{K}(\mathcal{S})$. If $\mathcal{S}$ is a family of open subsets of X filtering to the right such that $\mathcal{S}$ covers X and such that $\mathcal{S}$ contains a countable, cofinal subset, then it will turn out that $(\mathcal{M}^+(\mathcal{S}), \tau(\mathcal{S}))$ is a Suslin space whilst the Borel σ-algebra on $\mathcal{M}^+(\mathcal{S})$ coincides with $\mathcal{A}$. At the end of the section we compare our results with the results of Harris in [19] and [20], who also defines a topology on some family of nonnegative Borel measures on a complete, separable metric space.

Section (1.2) contains standard results for $\mathcal{S}$-finite point processes, in particular the Palm-formula which is now a direct consequence of a general theorem on disintegrations of measures from topological measure theory. Further $\mathcal{S}$-finite Poisson point processes and Cox processes are discussed. Section (1.3) is devoted to the study of a special class of

$\mathcal{S}$-finite Poisson point processes on $X=[0,\infty[\times U$, $\mathcal{S}$ being the family of subsets $I \times U_n$ of X where I is a bounded, open sub-interval of $[0,\infty[$ and $(U_n)_{n\geq 1}$ is a sequence of open subsets of U, increasing to U. It is clear that $\mathcal{S}$ is a filtering family of open subsets of X which covers X and has a countable, cofinal subsequence. Denote by $\mathcal{M}_1^{\bullet}(\mathcal{S})$ the set of $\mathcal{S}$-finite point measures μ for which $\mu(\{t\} \times U) \leq 1$, $t \geq 0$. An Itô-Poisson point process is a Poisson point process P on X with intensity measure $\lambda \otimes \nu$, λ denoting Lebesgue measure on $[0,\infty[$ and ν a σ-finite measure on U satisfying $\nu(U_n) < \infty$, $n \geq 1$. We choose the name Itô-Poisson point process, because the point process of excursions, as constructed by Itô, is of this type. Following Itô, the measure ν is called the characteristic measure of P. Further $P(\mathcal{M}_1^{\bullet}(\mathcal{S})) = 1$ for an Itô-Poisson point process P. The first important property of Itô-Poisson point processes is the renewal property which is treated here as a generalization of the property that a Poisson process is free from after-effects. The renewal property was already mentioned in Itô [25], but without a proof. We continue with Itô's characterization of Itô-Poisson point processes with a proof using "point process techniques". We end section (1.3) with a beautiful theorem of Greenwood and Pitman [17], which states that an Itô-Poisson point process P has an intrinsic time clock in the following sense: if $\mu \in \mathcal{M}_1^{\bullet}(\mathcal{S})$, denote by $\xi_{k1}(\mu)$, $\xi_{k2}(\mu)$, ... the U_k-sequence of μ, i.e.
$\mathrm{supp}(\mu) \cap ([0,\infty[\times U_k) = (\tau_{ki}(\mu), \xi_{ki}(\mu))_{i\geq 1}$ where the enumeration is such that the sequence $(\tau_{ki}(\mu))_{i\geq 1}$ is increasing in the order of $\mathbb{R}$. The sequence $\xi_k = (\xi_{ki})_{i\geq 1}$ is an i.i.d. sequence on the probability space $(\mathcal{M}_1^{\bullet}(\mathcal{S}), P)$. The theorem of Greenwood and Pitman states that the time coordinates τ_{ki} can be reconstructed from the sequence $\xi_{k1}, \xi_{k2}, \ldots$ if $\nu(U)=+\infty$. We give a complete proof of a slightly more general version of this theorem, which was formulated in [17] as a theorem on stochastic point functions.

In chapter 2 excursion theory is treated for Ray processes. We have chosen to treat excursion theory for Ray processes, since this class is in some sense the most general class of strong Markov processes, see Getoor [15] and Williams [59]. After a brief survey of Ray processes in section (2.1), we construct in section (2.2) the Itô-Poisson point process of excursions from a given state a of a Ray process Y. Since we want to include branchpoints in our discussion, we use a definition for excursions which differs a bit from Itô's definition, see also Rogers [45] who uses the same definition. We call excursion intervals the connected components of the complement in $[0,\infty[$ of the closed set of time points where the process hits or approaches the state a. Let $(r_k)_{k\geq 1}$ be a decreasing sequence of positive real numbers and let $U_k=\{u \in U : \zeta_u > r_k\}$. Denote by V_{kn} the n^{th} excursion of Y with length exceeding r_k. The strong Markov property implies that the sequence $(V_{kn})_{n\geq 1}$ is an independent, identically distributed sequence. Let $\tau_a=\inf\{t > 0 : Y_t=a \text{ or } Y_{t-}=a\}$. An application of the theorem of Greenwood and Pitman yields:

- If $P_a[\tau_a=0]=1$ there exists an $\mathcal{S}$-finite Itô-Poisson point process N defined on $(\Omega,\mathcal{F},P_a)$ whose $[\zeta > l]$-subsequence is the sequence of excursions of Y of length greater than l. The characteristic measure ν of N is the unique (modulo a multiplicative constant) measure on U of which the conditional distribution $\nu|_{U_j}$ is the probability distribution of V_{j1}; ν is a σ-finite measure with total mass $\nu(U)=+\infty$. The Markovian properties which ν inherits from the process Y are described in theorems (2.2.3) and (2.2.4).
- In the remaining case where $P_a[\tau_a=0] = 0$ there exists an i.i.d. sequence $(\xi_n)_{n\geq 1}$ of U-valued random variables on $(\Omega,\mathcal{F},P_a)$ whose

$[\zeta > l]$-subsequence is the sequence of excursions of Y of length greater than l.

Note that it was not necessary for this construction to introduce explicitly a local time at state a. Local time at state a will be discussed in section (2.3), in which we construct Markov processes from an $\mathcal{S}$-finite Itô-Poisson point process. The basic idea is the following. If $\mu \in \mathcal{M}_1^{\bullet}(\mathcal{S})$, then $\text{supp}(\mu)$ can be considered as a countable, ordered subset $(u_\sigma)_{\sigma \in J(\mu)}$ of U where $J(\mu)$ denotes the projection on $[0,\infty[$ of $\text{supp}(\mu)$ and where $u_\sigma = u$ iff $(\sigma,u) \in \text{supp}(\mu)$. Note that $(u_\sigma)_\sigma \in J(\mu)$ is not necessarily a totally ordered subset of U. Let $L : U \to [0,\infty[$ be a given, measurable function on U. Define for $\sigma \in [0,\infty[$

$$B(\sigma,\mu) = \Sigma\{L(u_\tau) : \tau \in J(\mu) \cap [0,\sigma]\}$$

$$= \int \mu(d\tau du) 1_{[0,\sigma]}(\tau) L(u)$$

and

$$C(\mu) = \bigcup_{\sigma \in J(\mu)} [B(\sigma-,\mu), B(\sigma,\mu)[.$$

If $T=C(\mu)$ then denote. by $\tilde{\mu}$ the concatenation of the functions $u_\sigma|[0,L(u_\sigma)[$, $\sigma \in J(\mu)$, that is

$$\tilde{\mu} : [0,\infty[\to E$$

$$\tilde{\mu}(s) = u_\sigma(s-B(\sigma-,\mu)) = \int \mu(d\tau dv)(v \cdot 1_{[0,L(v)[})(s-B(\tau-,\mu))$$

where $\sigma \in J(\mu)$ such that $s \in [B(\sigma-,\mu), B(\sigma,\mu)[$. In general we do not have that $[0,\infty[=C(\mu)$. If $B(\sigma,\mu)$ is strictly increasing as a function of σ, then $[0,\infty[$ is the disjoint union of $C(u)$ and the range R of $B(.,\mu)$. Let now P be an $\mathcal{S}$-finite Itô-Poisson point process with characteristic measure ν. We want to construct Markov processes, so we have to assume that ν satisfies the properties of the characteristic measures which arose by the construction of the Itô-Poisson point processes of excursions in section (2.2). But it is not necessary to assume that $\nu(U)=+\infty$. In this context it is more natural to consider a family $(P_x)_{x \in E}$ of point processes, where P_x is the $\mathcal{S}$-finite Itô-Poisson point process P to which is added a first excursion corresponding to a start from x, taking in account the transition mechanism which is contained in the measure υ. For our construction we will follow the above described basic idea with the lifetime ζ in the role of L. Considered as a function of $\mu \in \mathcal{M}_1^{\bullet}(\mathcal{S})$, $B(\tau,\mu)$ is a random variable on the probability space $(\mathcal{M}_1^{\bullet}(\mathcal{S}), P_x)$. The Poisson-property of the point process P_x implies that the stochastic process $(B(\tau))_{\tau\geq 0}$ is a subordinator (i.e. the process $(B(\tau))_{\tau\geq 0}$ has nondecreasing càdlàg realizations and stationary independent increments). In our construction we add a linear term stationary $\gamma\tau$ to B(t), with γ a nonnegative real parameter, which gives us the general form of a subordinator with the same Lévy measure as $B(\tau)$. An interpretation of the parameter γ will be given in chapter 4. The simple Markov property for the constructed process is proved in theorem (2.3.6). In theorem (2.3.8) we give an expression for the resolvent and in theorem (2.3.9) the strong Markov property is proved under a weak extra condition. In theorem (2.3.10) we give an explicit formula for the Blumenthal-Getoor local time at state a. We end this section with an example of the construction of a stochastic process from a more general point process than an Itô-Poisson point process. This construction is based on a Cox process and leads to a strong Markov process which is killed exponentially in the local time at a.

In chapter 3 we give some applications of excursion theory. In the first two sections we derive explicit expressions for the characteristic measures of the Itô-Poisson point processes of excursions from 0 attached to standard Brownian motion and Brownian motion with constant drift. A natural problem is to describe all strong Markov processes which behave like a given Ray process Y until the first hitting or approach of a given state a. As far as we know the only complete solution for this problem is given in Itô and McKean [26] for the case of reflecting Brownian motion on

$[0,\infty[$. In section (3.3) we given an interpretation in terms of excursion theory of the parameters which appear in its description. In section (3.4) we will construct a model for random motion on an n-pod E_n, that is a tree with one single vertex 0 and with n legs having infinite length. This is the most simple example of random motion on a graph. In defining the process on E_n we should like it to be Markovian with stationary transition probabilities. We should also like to have the process to behave like standard Brownian motion restricted to a half line, when restricted to a single leg. Using the results for reflecting Brownian motion from section (3.3) we are able to characterize all strong Markov processes which satisfy this description. Frank and Durham present in [12] for the first time an intuitive description of such a process for the case n=3. They considered the case of continuous entering from 0 in a leg, which was chosen according to some given probability distribution. The difficulty which arises in the construction of this process is that the process, when starting from 0, will visit 0 infinitely many times in a finite time interval. It is therefore not possible to indicate the leg which is visited first starting from 0. In section (4.2) we will explain what is meant with choosing a leg according to some given probability distribution with the help of a random walk approximation. The construction that we will give is based on section (2.3); our model allows also jumping in a leg, stickiness at 0 and killing with a rate proportional to local time at 0. In section (3.5) we show how theory of section (2.3) can be applied to the construction of certain Markov processes which Blumenthal uses in [2] and for the construction of which he refers to Meyer [42]. As already mentioned above, chapter 4 contains random walk approximations. Let $S_n=(S_{nk})_{k\in\mathbb{N}}$, n=1,2,... be a sequence of Markov chains on $\mathbb{Z}$ with transition matrices P_n and initial distributions ν_n. Define for $n \geq 1$ the process $X_n=(X_n(t))_{t\geq 0}$ with continuous time parameter by

$$X_n(t) = n^{-1/2} S_{n,[nt]}$$

and let $P^{(n)}_{\mu_n}$ be the distribution of X_n where μ_n is the distribution of $X_n(0)$. In section (4.2) we consider the case where $P_n=P$ does not depend on n where P is given by

$$P(m,m-1) = P(m,m+1) = \frac{1}{2} \quad \text{for } m \neq 0$$
$$P(0,k) = p_k \quad \text{for } k \in \mathbb{Z}$$

in which $\{p_k : k \in \mathbb{Z}\}$ is a probability distribution on $\mathbb{Z}$. Harrison and Shepp proved in [21] that for the special case $p_1=\alpha$, $p_{-1}=\beta$, $\alpha+\beta=1$ the sequence of probability measures $(P^{(n)}_o)$ converges weakly to the distribution of skew Brownian motion starting from 0. Skew Brownian motion was introduced in Itô-McKean [27] as an example of a diffusion process. In the terminology of section (3.4) skew Brownian motion can be considered as a random motion on a 2-pod which behaves like standard Brownian motion outside 0 and which enters continuously in a leg chosen with probability distribution $(\alpha,1-\alpha)$. We will prove a part of a more general result which was stated without proof in Harrison and Shepp. If the probability distribution (p_k) has a finite first moment and if the sequence of probability measures $(\nu_n)_{n\geq1}$ on $\mathbb{R}$ with $\text{supp}(\nu_n) \subset n^{-1/2}\mathbb{Z}$ converges weakly to a probability measure m on $\mathbb{R}$, then the finite dimensional distributions of the sequence $(P^{(n)}_{\nu_n})$ converge weakly to the finite dimensional distributions of skew Brownian motion with initial distribution m and with parameter $\alpha = \dfrac{\Sigma k^+ p_k}{\Sigma |k| p_k}$.

Section (4.3) gives a random approximation for the stochastic process Y_γ constructed from the Itô-Poisson point process of excursions from 0 of standard Brownian motion, where we have added to the total excursion length $B(\tau)$ up to time τ the term $\gamma\tau$. Let the transition matrix P_n of the Markov chain S_n be defined by

$$P_n(i,i+1) = P_n(i,i-1) = \frac{1}{2} \quad \text{for } i \neq 0$$

$$P_n(0,0) = \alpha_n$$

$$P_n(0,1) = P_n(0,-1) = \frac{1}{2}(1-\alpha_n)$$

where $\alpha_n = \dfrac{\gamma\cdot n^{1/2}}{1+\gamma\cdot n^{1/2}}$. It turns out that the distributions of the processes X_n defined as above converge weakly to the distribution of Y_γ.

In this thesis only excursions from a single state a are treated. It looks as if it is not too difficult to generalize this approach to the description of excursions from a finite set of states. It seems that one will need Cox processes to describe the excursions from a finite set of states. These Cox processes will not satisfy the renewal property, which will be replaced by some kind of Markov property. See also Itô [25], who proposes to call the excursion point process Markov in this case. However these Markov excursion point processes are not discussed by Itô.

CHAPTER 1

POINT PROCESSES

A point process is a random distribution of points in some space X. The case where X is the real line, more generally a locally compact, second countable Hausdorff space or a separable metric space, has been studied extensively. One always assumes that there is a family $\mathcal{S}$ of subsets of X, each of which can contain only a finite number of points. $\mathcal{S}$ is the family of compact subsets if X is locally compact and if X has a metric structure then $\mathcal{S}$ is the family of bounded subsets of X.

Mathematically the concept of a point process is formalized as follows. Let X be a topological space and let $\mathcal{S}$ be a family of open subsets of X. To a distribution Z of points in X we assign the point measure $\sum_{z \in Z} \delta_z$, where δ_z is the Dirac measure in z. The description with measures on X has greater flexibility than the description with subsets of X and is mathematically more convenient because of the richer structure of the linear topological nature of the space of measures. Moreover in the case of point processes with multiple points the approach via measures is more natural. So let $\mathcal{M}^+ = \mathcal{M}^+(\mathcal{S})$ be the set of all nonnegative Borel measures on X which are finite on the elements of $\mathcal{S}$. Denote by $\mathcal{A}$ the smallest σ-algebra on $\mathcal{M}^+$ which measures the maps $\mu \in \mathcal{M}^+ \rightarrow \mu(A)$, $A \in \mathcal{B}(X)$. Let $\mathcal{M}'' = \mathcal{M}''(\mathcal{S})$ be the subset of $\mathcal{M}^+$ consisting of the point measures on X. An $\mathcal{S}$-finite point process on X is a probability measure on $(\mathcal{M}^+, \mathcal{A})$ which is concentrated on $\mathcal{M}''$, or an $\mathcal{M}''$-valued random variable where we identify a random variable with its distribution. However, the measure-theoretic introduction of the σ-algebra $\mathcal{A}$ is not quite satisfactory. There are several reasons to prefer a definition of $\mathcal{A}$ as the Borel σ-algebra corresponding to some topological structure on $\mathcal{M}^+$: a topology on $\mathcal{M}^+$, which induces the corresponding narrow topology on the space of measures on $\mathcal{M}^+$, makes it possible to discuss weak convergence of point processes. Further, measurability properties of subsets of $\mathcal{M}^+$ (for instance $\mathcal{M}''$) can be derived from topological properties and there is a powerful disintegration theorem for measures on topological spaces.

As an example, consider briefly the set of nonnegative Borel measures on a locally compact, second countable Hausdorff space X. Then $\mathcal{M}^+ = \mathcal{M}^+(\mathcal{S})$ is the set of all Radon measures on X ($\mathcal{S}$ is the family of compact subsets of X). Let $\mathcal{K} = \mathcal{K}(\mathcal{S})$ be the set of all continuous functions on X with compact support. Endow $\mathcal{M}^+$ with the vague topology $\tau = \sigma(\mathcal{M}^+,\mathcal{K})$ of pointwise convergence on the elements of $\mathcal{K}$. A net (μ_α) in $\mathcal{M}^+$ converges vaguely to $\mu \in \mathcal{M}$ iff $\mu_\alpha(f) \to \mu(f)$ for each $f \in \mathcal{K}$, where $\mu(f)$ is the functional-analytic notation for the integral of f with respect to μ. The vague topology renders $\mathcal{M}^+$ a polish space, i.e. $\mathcal{M}^+$ is metrizable with a complete metric. The Borel σ-algebra $\mathcal{B}$ on $(\mathcal{M}^+,\tau)$ coincides with the σ-algebra $\mathcal{A}$ generated by the maps $\mu \in \mathcal{M}^+ \to \mu(A)$, $A \in \mathcal{B}(X)$. The basic result on weak convergence is Prohorov's theorem, which gives a characterization of the relative compact subsets of $(\mathcal{M}^+,\tau)$. The set of point measures $\mathcal{M}''$ is a vaguely closed subset of $\mathcal{M}^+$. See for proofs Bourbaki [5] and Krickeberg [34].

In the literature about point processes on complete, separable metric spaces (X,ρ) one studies always $\mathcal{S}$-finite point processes, where $\mathcal{S}$ is the family of bounded Borel subsets of X. The point processes which arise in excursion theory turn out to be $\mathcal{S}$-finite point processes on a polish space U, where $\mathcal{S}$ is some family of open subsets of U. The theory of point processes on complete, separable metric spaces is not applicable in this case, since it is not clear whether there exists a complete metric d for U such that $\mathcal{S}$ coincides with the family of d-bounded subsets of U. So, before we can study excursion theory we have to study the set $\mathcal{M}^+(\mathcal{S})$ of $\mathcal{S}$-finite Borel measures on a polish space U for some family $\mathcal{S}$ of open subsets of U.

Let X be a completely regular Suslin space (for example a polish space) and let $\mathcal{S}$ be a family of open subsets of X such that

(i) $\mathcal{S}$ is filtering to the right with respect to inclusion,

(ii) $\mathcal{S}$ has a countable cofinal subset, and

(iii) $\mathcal{S}$ covers X.

We will construct in section (1.1) a topology on the set $\mathcal{M}^+ = \mathcal{M}^+(\mathcal{S})$ of nonnegative, $\mathcal{S}$-finite Borel measures on X. Let $\mathcal{H} = \mathcal{H}(\mathcal{S})$ be the set of all bounded, continuous functions on X with support contained in an element of $\mathcal{S}$. Equipped with the topology $\tau = \sigma(\mathcal{M}^+, \mathcal{H})$ of pointwise convergence on the elements of $\mathcal{H}$, $\mathcal{M}^+$ turns to be a Suslin space whilst the Borel σ-algebra on $(\mathcal{M}^+, \tau)$ is identical to the σ-algebra $\mathcal{A}$ generated by the maps $\mu \in \mathcal{M}^+ \to \mu(A)$, $A \in \mathcal{B}(X)$. For polish spaces X we have the stronger result that $(\mathcal{M}^+, \tau)$ is a Lusin space. Our treatment is based on the results for polish, Lusin and Suslin topological spaces in Schwartz [49]. The set of $\mathcal{S}$-finite point measures $\mathcal{M}''(\mathcal{S})$ turns out to be a closed subset of $\mathcal{M}^+$, as it is for the vague topology on the set of Radon measures on a locally compact, second countable Hausdorff space. In section (1.2) we will discuss $\mathcal{S}$-finite point processes on a polish space X. The existence of Palm measures, which are, loosely stated, the conditional distributions of the point process if it is known that a certain element $x \in X$ occurs (see Jagers [28] for the case that X is locally compact), follows now from a general disintegration theorem from topological measure theory. Further Poisson point processes and Cox processes (so called doubly stochastic point processes) are discussed. It will be shown that for every $\mathcal{S}$-finite measure ν on X there exists an $\mathcal{S}$-finite Poisson point process with intensity measure ν. The point processes, which arise in excursion theory are $\mathcal{S}$-finite

Poisson point processes on a polish space X, which is the topological product of the half line $[0,\infty[$ and a polish space U, whilst $\mathcal{S}$ consists of the sets $I \times U_k$ where I is an open, bounded sub-interval of $[0,\infty[$ and $(U_n)_{n\geq 1}$ is an increasing sequence of open subsets of U which covers U; the intensity measure is the product of the Lebesgue measure λ and a Borel measure ν on U which is finite on the sequence $(U_n)_{n\geq 1}$. Itô was the first who studied these processes as stochastic point functions and that is the reason why we call them Itô-Poisson point processes. In section (1.3) we discuss the renewal property for Itô-Poisson point processes and the characterizations of these processes which were given in Itô [25] and in Greenwood & Pitman [17]. We will give full proofs of slightly more general versions of these theorems which were originally formulated in terms of stochastic point functions; the renewal property was stated in Itô [25] without proof.

1.1 Topological spaces of Borel measures.

Let X be a Suslin space with Borel σ-algebra $\mathcal{B}(X)$. A Suslin space is a Hausdorff topological space for which there exists a polish space Y and a continuous surjection from Y to X, see Schwartz [49], p. 96. To have enough continuous functions on X we will assume that X is a completely regular space, i.e. for each $x \in X$ and each open neighbourhood U of x there is a continuous function f on X to the closed unit interval such that $f(x)=1$ and f is identically zero on $X\backslash U$.

Let G be an open subset of X. Equipped with the relative topology, G is a completely regular Suslin space (Schwartz [49] theorem 3, p.96).

Denote by $C_b(G)$ the space of bounded continuous functions on G and by $\mathcal{K}(G)$ the subspace of $C_b(G)$ consisting of restrictions to G of bounded continuous functions on X with support contained in G

$$\mathcal{K}(G) = \{f|_G : f \in C_b(X),\ \mathrm{supp}(f) \subset G\}.$$

The space of nonnegative, bounded Borel measures on G will be denoted

by $\mathcal{M}_b^+(G)$. Endowed with the narrow topology, that is the topology $\tau_1(G) = \sigma(\mathcal{M}_b^+(G), C_b(G))$ of pointwise convergence on $C_b(G)$, $\mathcal{M}_b^+(G)$ is a Suslin space, see Bourbaki [5], p.6. Denote by $\tau_2(G)$ the topology $\sigma(\mathcal{M}_b^+(G), \mathcal{K}(G))$ on $\mathcal{M}_b^+(G)$. It is clear that $\tau_2(G) \subset \tau_1(G)$.
Note that $\tau_2(G) \neq \tau_1(G)$. Indeed if (x_n) is a sequence in G converging for $n\to\infty$ to a point x in the boundary of G, then the sequence of Dirac measures (δ_{x_n}) converges in the space $(\mathcal{M}_b^+(G), \tau_2(G))$ and diverges in the space $(\mathcal{M}_b^+(G), \tau_1(G))$.

1.1.1 Proposition. Let X be a completely regular Suslin space. If G is an open subset of X, then $(\mathcal{M}_b^+(G), \tau_2(G))$ is a Suslin space.

Proof. Since $\tau_2(G) \subset \tau_1(G)$ and $(\mathcal{M}_b^+(G), \tau_1(G))$ is a Suslin space, it is sufficient to prove that $(\mathcal{M}_b^+(G), \tau_2(G))$ is a Hausdorff space. Let $O \subset G$ be an open subset of G and let $x \in O$. X is a completely regular Hausdorff topological space, so there is an open neighbourhood V of x such that $\bar{V} \subset O$ and there is a continuous function f_x on X to the closed unit interval such that $f_x(x)=1$ and f_x is zero on $X\setminus V$. It is clear that

$$\text{supp}(f_x) \subset \bar{V} \subset O \subset G \text{ and } 1_O = \sup\{f_x : x \in O\}.$$

Any Suslin space is a Lindelöf space, so that the family $\{f_x: x \in O\}$ has a countable subfamily $(f_n)_{n\geq 1}$ with the same upper envelope 1_O, see Schwartz [49], p. 103 and 104. Define for $n \geq 1$ the function g_n as the pointwise supremum of f_1 up to and including f_n. The sequence (g_n) is an increasing sequence of functions on X with support contained in G and with supremum 1_O. Hence for $\nu,\mu \in \mathcal{M}_b^+(G)$ we have

$$\forall\, f \in \mathcal{K}(G) : \nu(f) = \mu(f) \Rightarrow \nu(O) = \mu(O)$$

for any open O in G. Let

$$\mathcal{S} = \{A : A \in \mathcal{B}(G),\ \nu(A) = \mu(A)\}.$$

$\mathcal{S}$ is a d-system containing the open subsets of G. By the monotone class theorem it follows that $\mathcal{S} = \mathcal{B}(G)$, see Williams [59], p.40. So $\mathcal{K}(G)$

separates the points of $\mathcal{M}_b^+(G)$ and this implies that $\tau_2(G)$ is a Hausdorff topology on $\mathcal{M}_b^+(G)$. □

1.1.2 Remark. A Hausdorff topological space is said to be a Lusin space if there exists a polish space Y and a continuous bijective mapping from Y to X, see Schwartz [49], p.94. It is clear that any Lusin space is a Suslin space. If we assume that X is a polish space, then the topological space $(\mathcal{M}_b^+(G), \tau_1(G))$ is also a polish space (see Bourbaki [5], p.62) and we may conclude that $(\mathcal{M}_b^+(G), \tau_2(G))$ is a Lusin space.

Let $\mathcal{G}$ be a family of open subsets of X, which is filtering to the right with respect to inclusion, i.e. $\forall\ A,B \in \mathcal{G}\ \ \exists\ C \in \mathcal{G} : A \subset C$ and $B \subset C$. For all pairs $A,B \in \mathcal{G}$, $A \subset B$, we define the map π_{AB} by

$$\pi_{AB} : \mathcal{M}_b^+(B) \to \mathcal{M}_b^+(A),\ \pi_{AB}(\mu) = {}_A\mu$$

where ${}_A\mu$ denotes the restriction of μ to A: ${}_A\mu(G) = \mu(G)$, $G \in \mathcal{B}(A)$. It is clear that $((\mathcal{M}_b^+(A), \tau_2(A)), \pi_{AB})$ is a projective system of Suslin spaces. Note that $((\mathcal{M}_b^+(A), \tau_1(A)), \pi_{AB})$ is not a projective system of topological spaces, as the π_{AB} are not τ_1-continuous. The projective limit $M = M(\mathcal{G}) = \underleftarrow{\lim}\ \pi_{AB}\ \mathcal{M}_b^+(B)$ is the subspace of the product $\prod_{A \in \mathcal{G}} \mathcal{M}_b^+(A)$ whose elements $\mu = (\mu_A)$ satisfy the relation $\mu_A = \pi_{AB}(\mu_B)$ whenever $A \subset B$. The projective topology on M is the coarsest topology which makes the projections

$$\pi_B : M \to \mathcal{M}_b^+(B),\ \pi_B((\mu_A)) = \mu_B$$

continuous and is therefore the trace on M of the product topology on $\prod \mathcal{M}_b^+(A)$. Let $\mathcal{M}^+ = \mathcal{M}^+(\mathcal{G})$ be the space of nonnegative Borel measures on $(X, \mathcal{B}(X))$, which are finite on $\mathcal{G}$. Elements of $\mathcal{M}^+$ are called $\mathcal{G}$-finite measures.

1.1.3 Proposition. If $\mathcal{G}$ covers X, then the map

$$\varphi : \mu \in \mathcal{M}^+ \ \to\ ({}_A\mu) \in M$$

is a bijection of $\mathcal{M}^+$ onto M.

Proof. It is clear that φ maps $\mathcal{M}^+$ into M.

Let $(\mu_A) \in M$ and $G \in \mathcal{B}(A) \cap \mathcal{B}(B)$ for some $A,B \in \mathcal{S}$. Since $\mathcal{S}$ is filtering, there exists a $C \in \mathcal{S}$ such that $A,B \subset C$.

Then $\mu_A(G) = (\pi_{AC}\,\mu_C)(G) = \mu_C(A \cap G) = \mu_C(G)$ and in the same way $\mu_B(G) = \mu_C(G)$. Therefore

$$\mu : \bigcup_{A\in\mathcal{S}} \mathcal{B}(A) \to \mathbb{R}\ ,\ \mu(G) = \mu_A(G) \text{ if } G \in \mathcal{B}(A)$$

is an unambiguously defined setfunction on the ring $\bigcup\mathcal{B}(A)$.

If $(G_n)_{n\geq 1}$ is a pairwise disjoint sequence in $\bigcup\mathcal{B}(A)$ with union ΣG_n contained in $\bigcup\mathcal{B}(A)$, then $G_n, \Sigma G_n \in \mathcal{B}(C)$ for some $C \in \mathcal{S}$.

It follows that

$$\mu(\Sigma G_n) = \mu_C(\Sigma G_n) = \Sigma\mu_C(G_n) = \Sigma\mu(G_n).$$

So μ is a finite, σ-additive measure on $(X, \bigcup_{A\in\mathcal{S}} B(A))$.

Being an open cover of a Lindelöf space, $\mathcal{S}$ has a countable subcover. It follows that $\mathcal{B}(X)$ is the σ-ring generated by the ring $\bigcup\mathcal{B}(A)$ and μ has a unique extension to a measure $\bar{\mu} \in \mathcal{M}^+$. See Halmos [18], p.54. It is clear that $\varphi(\bar{\mu}) = (\mu_A)$.

This proves that φ is a bijection of $\mathcal{M}^+$ onto M. □

Assume that $\mathcal{S}$ covers X. Denote by $\tau = \tau(\mathcal{S})$ the coarsest topology on $\mathcal{M}^+$ which makes the bijection $\varphi : \mathcal{M}^+ \to M$ continuous.

Defining

$$\mathcal{K}(\mathcal{S}) = \{f \in C_b(X) : \exists A \in \mathcal{S} : \text{supp}(f) \subset A\},$$

τ is equal to the topology $\sigma(\mathcal{M}^+, \mathcal{K}(\mathcal{S}))$ of pointwise convergence on $\mathcal{K}(\mathcal{S})$. If $\mathcal{D}$ is a cofinal subset of $\mathcal{S}$ (i.e. for each $A \in \mathcal{S}$ there is a $D \in \mathcal{D}$ such that $A \subset D$) then $\mathcal{K}(\mathcal{S}) = \mathcal{K}(\mathcal{D})$ and $\sigma(\mathcal{M}^+, \mathcal{K}(\mathcal{S})) = \sigma(\mathcal{M}^+, \mathcal{K}(\mathcal{D}))$.

Denote the Borel σ-algebra on $(\mathcal{M}^+,\tau)$ by $\mathcal{B} \doteq \mathcal{B}(\mathcal{M}^+(\mathcal{S}))$ and define the σ-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ on $\mathcal{M}^+$ by

$$\mathcal{A}_1 = \sigma(\mu \in \mathcal{M}^+ \to \mu(B),\ B \in \mathcal{B}(X))$$

and

$$\mathcal{A}_2 = \sigma(\mu \in \mathcal{M}^+ \to \mu(f),\ f \in \mathcal{K}(\mathcal{S})).$$

1.1.4 Theorem. Let X be a completely regular Suslin space and let $\mathcal{S}$ be a family of open subsets of X filtering to the right. $\mathcal{S}$ is a cover of X and if $\mathcal{S}$ contains a countable cofinal subset, then

(i) $(\mathcal{M}^+(\mathcal{S}),\ \tau(\mathcal{S}))$ is a Suslin space and

(ii) $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{B}$.

Proof. Let $\mathcal{D}$ be a countable cofinal subset of $\mathcal{S}$. Being a countable projective limit of Suslin spaces, $M(\mathcal{D})$ is a Suslin space, see Schwartz [49],p.111. Hence $(\mathcal{M}^+(\mathcal{D}),\ \tau(\mathcal{D}))$ is a Suslin space. It is clear that $\mathcal{M}^+(\mathcal{S}) = \mathcal{M}^+(\mathcal{D})$ and $\tau(\mathcal{D}) = \tau(\mathcal{S})$, which proves (i).

The family of continuous maps $\mu \in \mathcal{M}^+ \to \mu(f)$, $f \in \mathcal{K}(\mathcal{S})$, separates the points of $\mathcal{M}^+(\mathcal{S})$. Indeed, let $\mu, \nu \in \mathcal{M}^+$ so that $\mu(f) = \nu(f)$ for every $f \in \mathcal{K}(\mathcal{S})$ and let $A \in \mathcal{S}$. Every $f \in \mathcal{K}(A)$ being the restriction to A of a function $g \in \mathcal{K}(\mathcal{S})$ with support contained in A,

$$_A\mu(f) = \mu(g) = \nu(g) = {}_A\nu(f).$$

So $_A\mu = {}_A\nu$ by the proof of proposition (1.1.1) and it follows that $\mu = \nu$ since A was arbitrarily chosen in $\mathcal{S}$. Since $(\mathcal{M}^+(\mathcal{S}),\ \tau(\mathcal{S}))$ is a Suslin space, there is a countable subfamily $(f_n)_{n\geq 1}$ of $\mathcal{K}(\mathcal{S})$ such that the points of $\mathcal{M}^+(\mathcal{S})$ are separated by the maps $\psi_n : \mu \in \mathcal{M}^+(\mathcal{S}) \to \mu(f_n)$. By Fernique's lemma, the sequence (ψ_n) generates $\mathcal{B}$. See Schwartz [49], p. 104, p.105 and p. 108.

So $\mathcal{B} \subset \mathcal{A}_2$.

Let now $f \in \mathcal{K}(\mathcal{S})$ and let $A \in \mathcal{S}$ be such that supp $f \subset A$. Since f is a continuous function, there exists a sequence of $\mathcal{B}(X)$-stepfunctions, zero outside A, converging uniformly to f. It follows that the map $\mu \in \mathcal{M}^+(\mathcal{S}) \to \mu(f)$ is $\mathcal{A}_1$-measurable.

So $\mathcal{A}_2 \subset \mathcal{A}_1$.

Let $A \in \mathcal{S}$. Let $O \subset A$ be an open subset of G. As in the proof of proposition (1.1.1) we can construct an increasing sequence $(f_n)_{n\geq 1}$ in $\mathcal{K}(\mathcal{S})$ with $\mathrm{supp}(f_n) \subset A$ and with supremum 1_O.

It follows that the map $\mu \in \mathcal{M}^+(\mathcal{S}) \to \mu(O)$ is $\mathcal{B}$-measurable. A monotone class argument gives the $\mathcal{B}$-measurability of the maps $\mu \in \mathcal{M}^+(\mathcal{S}) \to \mu(G)$, $G \in \bigcup_{A\in\mathcal{S}} \mathcal{B}(A)$. Since $\mathcal{S}$ has a countable cofinal subset, every Borel set in X can be written as a countable union of elements of $\cup\, \mathcal{B}(A)$.

So $\mathcal{A}_1 \subset \mathcal{B}$.

It follows that $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{B}$. □

From now on we will assume that the space X is a polish space. Let d be a metric on X such that the metric topology is the topology of X and (X,d) is a complete metric space. $\mathcal{S}$ will be a fixed family of open subsets of X satisfying the conditions of theorem (1.1.4). By $\mathcal{S}'$ we will denote the family of all Borel subsets of X contained in some element of $\mathcal{S}$. Note that the space $(\mathcal{M}^+,\tau)$ of $\mathcal{S}$-finite measures is a Lusin space in this case, see remark (1.1.2).

1.1.5 Remark. Even for polish spaces it need not be true that a filtering family of open subsets, which covers the space, has a countable cofinal subset. For example, let X be the space of all pairs of non-negative integers with the discrete topology. X is a polish space. A set A is member of the family $\mathcal{S}$ iff for all except a finite number of integers m the set $\{n : (m,n) \in A\}$ is finite. $\mathcal{S}$ is a filtering family of open subsets of X, which covers X. But $\mathcal{S}$ does not have a countable cofinal subset. Indeed, let $(A_k)_{k\geq 1}$ be a sequence of subsets of X contained in $\mathcal{S}$. For every $k \geq 1$ we can choose an element $x_k = (m,n) \in X$ such that $n \geq k$ and $x_k \notin A_k$. The set $B = \{x_i : i\geq 1\}$ is an element of $\mathcal{S}$ and there is no A_k such that $B \subset A_k$.

The following proposition provides a condition equivalent to convergence of a net in $(\mathcal{M}^+,\tau)$.

1.1.6 Proposition. Let X be a polish space and let $\mathcal{S}$ be a family of open subsets of X satisfying the conditions of theorem (1.1.4) and let (μ_α) be a net in $\mathcal{M}^+$ and $\mu \in \mathcal{M}^+$.

Then the following statements are equivalent:

(i) $\mu_\alpha \to \mu$ in $(\mathcal{M}^+, \tau)$,

(ii) $\limsup \mu_\alpha(F) \leq \mu(F)$ for all closed $F \in \mathcal{S}'$ and

$\liminf \mu_\alpha(O) \geq \mu(O)$ for all open $O \in \mathcal{S}'$.

Proof.

(i) $\Rightarrow$ (ii)

Let F be a closed subset of X, $F \subset A$ for some $A \in \mathcal{S}$ and let $\mathcal{D}$ be a countable dense subset of X. Define

$$I = \{(x,q) : x \in \mathcal{D},\ q \in \mathbb{Q}_+,\ B_x(q) \cap F = \emptyset\}$$

where $B_x(q) = \{y : y \in X,\ d(x,y) \leq q\}$. I is a countable set. For $i = (x,q) \in I$, the sets F and $A^* \cup B_x(q)$ are disjoint closed sets, where A^* denotes the complement of A. Since X is a normal topological space, there are disjoint open sets U and V such that $F \subset U$ and $A^* \cup B_x(q) \subset V$. By Urysohn's lemma there is a continuous function f_i on X to the interval $[0,1]$ such that f_i is zero on U^* and one on F. It is clear that $\text{supp}\,(f_i) \subset U \cap V^* \subset A$, so $f_i \in \mathcal{K}$. If $y \notin F$, then there is an element $(x,q) \in I$ such that $y \in B_x(q)$.

It follows that

$$1_F = \inf\,\{f_i : i \in I\}.$$

Define $g_n = \inf\{f_{i_1},\dots, f_{i_n}\}$, $n \geq 1$, where $(i_n)_{n\geq 1}$ is an enumeration of I. It is clear that (g_n) is a sequence in $\mathcal{K}$ converging pointwise to 1_F.

So for each $n \geq 1$

$$\lim_{\alpha} \mu_\alpha(g_n) = \mu(g_n) \text{ and}$$

$$\limsup_{\alpha} \mu_a(F) \leq \limsup_{\alpha} \mu_\alpha(g_n) = \mu(g_n).$$

It follows that

$$\limsup \mu_\alpha(F) \leq \mu(F).$$

Let O be an open subset of X, $O \subset A$ for some $A \in \mathcal{S}$. Let (g_n) be an increasing sequence of bounded continuous functions such that $\text{supp}(g_n) \subset A$ and $1_O = \sup g_n$. (See the proof of proposition (1.1.1).)

For each $n \geq 1$

$$\lim_{\alpha} \mu_\alpha(g_n) = \mu(g_n) \text{ and}$$

$$\liminf_{\alpha} \mu_\alpha(O) \geq \liminf_{\alpha} \mu_\alpha(g_n) = \mu(g_n).$$

It follows that

$$\liminf \mu_\alpha(O) \geq \mu(O).$$

This completes the proof of (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (i)

Let f be a bounded nonnegative continuous function on X with $\text{supp}(f) \subset A$ for some $A \in \mathcal{S}$.

Define for $k \geq 1$ the functions u_k, $v_k : X \to \mathbb{R}$ by

$$u_k = \sum_{i \geq 1} \frac{1}{k} 1_{[f < \frac{i}{k}] \cap A}$$

$$v_k = \frac{1}{k} 1_{\text{supp } f} + \sum_{i \geq 1} \frac{1}{k} 1_{[f < \frac{i}{k}]},$$

the summations being finite summations since f is bounded.

It is clear that $u_k \leq f \leq v_k$, for all $k \geq 1$ and that $u_k \uparrow f$ and $v_k \downarrow f$.

Hence

$$\begin{aligned}\liminf_{\alpha} \mu_\alpha(u_k) &\geq \sum_{i \geq 1} \frac{1}{k} \liminf_{\alpha} \mu_\alpha([f < \frac{1}{k}] \cap A)\\ &\geq \sum_{i \geq 1} \frac{1}{k} \mu([f < \frac{1}{k}] \cap A) \quad \text{by (ii)}\\ &= \mu(u_k)\end{aligned}$$

and analogously

$$\limsup_{\alpha} \mu_\alpha(v_k) \leq \mu(v_k).$$

It follows that

$$\mu(u_k) \leq \liminf \mu_\alpha(f) \leq \limsup \mu_\alpha(f) \leq \mu(v_k).$$

By taking limits for $k \to \infty$ we get

$$\lim \mu_\alpha(f) = \mu(f)$$

which completes the proof of (ii) ⇒ (i).□

A measure $\mu \in \mathcal{M}^+$ is called an $\mathcal{G}$-finite point measure if

$$\forall\ G \in \mathcal{G}' : \mu(G) \in \mathbb{N}.$$

An $\mathcal{G}$-finite point measure is called simple if

$$\forall x \in X,\ \mu_x = \mu(\{x\}) \in \{0,1\}.$$

The set of $\mathcal{G}$-finite point measures will be denoted by $\mathcal{M}'' = \mathcal{M}''(\mathcal{G})$ and the set of simple $\mathcal{G}$-finite point measures by $\mathcal{M}' = \mathcal{M}'(\mathcal{G})$.

1.1.7 Proposition. Let X be a polish space and let $\mathcal{G}$ be a family of open subsets of X satisfying the conditions of theorem (1.1.4).

Then $\mathcal{M}''$ is a closed subset of $(\mathcal{M}^+, \tau)$.

Proof. Let (μ_α) be a net in $\mathcal{M}''$ converging to $\mu \in \mathcal{M}^+$. Take $x \in \mathrm{supp}(\mu)$ and let U be an open neighbourhood of x, $U \in \mathcal{G}'$.

Then by proposition (1.1.6)

$$0 < \mu(U) \leq \liminf \mu_\alpha(U).$$

Since $\mu_\alpha(U) \in \mathbb{N}$, it follows that

$$\liminf \mu_\alpha(U) \geq 1.$$

Consider now a decreasing sequence $(U_n)_{n\geq 1}$ of open neighbourhoods of x in $\mathcal{G}'$ such that $U_n \downarrow \{x\}$ and $\bar{U}_{n+1} \subset U_n$ for every $n \geq 1$.

From Urysohn's lemma follows the existence of a sequence (h_n) in $\mathcal{K}(\mathcal{G})$ such that $1_{\bar{U}_{n+1}} \leq h_n \leq 1_{U_n}$ for every $n \geq 1$.

Then

$$\mu(h_n) = \lim_{\alpha} \mu_\alpha(h_n) \geq \limsup_{\alpha} \mu_\alpha(\bar{U}_{n+1})$$
$$\geq \liminf_{\alpha} \mu_\alpha(U_{n+2}) \geq 1$$

and

$$\mu_x = \lim_{n\to\infty} \mu(h_n) \geq 1.$$

It follows that supp(μ) is a discrete set and therefore for n sufficiently large

$$\mu_x = \mu(U_n) = \mu(\bar{U}_n)\,..$$

Proposition (1.1.6) implies that

$$\mu(\bar{U}_n) \geq \limsup_{\alpha} \mu_\alpha(\bar{U}_n)$$
$$\geq \limsup_{\alpha} \mu_\alpha(U_n) \geq \liminf_{\alpha} \mu_\alpha(U_n) \geq \mu(U_n).$$

Hence for n sufficiently large

$$\mu_x = \lim_{\alpha} \mu_\alpha(U_n) \in \mathbb{N},$$

and it follows that $\mu \in \mathcal{M}''$. □

Let $U_1,\ldots,U_n$ be a finite sequence of open subsets of X such that $U_1,\ldots,U_n \in \mathcal{S}'$ and let $k_1,\ldots, k_n \in \mathbb{N}$.

Define

$$V_{U_1,\ldots,U_n;k_1,\ldots,k_n} = \{\mu \in \mathcal{M}'' : \mu(U_i) = \mu(\bar{U}_i) = k_i, i=1,\ldots,n\}.$$

It follows from proposition (1.1.6) that the map

$$\mu \in \mathcal{M}^+ \to \mu(G)$$

is lower semicontinuous (resp. upper semicontinuous) for each open (resp. closed) subset $G \subset X$ in $\mathcal{S}'$. Hence

$$V_{U_1,\ldots,U_n;k_1,\ldots,k_n} = \mathcal{M}'' \cap \{\mu : \mu(U_i) > k_i - \frac{1}{2} \text{ and } \mu(\bar{U}_i) < k_i + \frac{1}{2},\ i=1,\ldots,n\}$$

is open in $\mathcal{M}''$.

Let $\mathcal{U}$ be a countable base for the topology of X consisting of open subsets with closure in $\mathcal{S}'$ (see appendix A1) and let $A_1, A_2,\ldots$ be an increasing, countable cofinal subfamily of $\mathcal{S}$.

Define for k, n $\geq$ 1

$$O_{k,n} = \cup\, V_{U_1,\ldots,U_n;1,\ldots,1}$$

where the union is taken over all finite sequences $U_1,\ldots,U_n$ in $\mathcal{U}$ whose elements are contained in A_k. It follows that $O_{k,n}$ is open in $\mathcal{M}''$ and that the set

$$\{\mu\in\mathcal{M}'' : \mu(A_k)=n\} = \{\mu\in\mathcal{M}'' : \mu(A_k)>n-1\} \setminus \{\mu\in\mathcal{M}'' : \mu(A_k)>n\}$$

is a Borel subset of $\mathcal{M}''$.

So

$$\mathcal{M}' = \bigcap_{k=1}^{\infty} \bigcup_{n=1}^{\infty} \{\mu \in \mathcal{M}'' : \mu(A_k) = n\} \cap O_{k,n}$$

is a Borel subset of $\mathcal{M}''$.

So we have derived the following proposition:

1.1.8 Proposition. Let X be a polish space and let $\mathcal{S}$ be a family of open subsets of X satisfying the conditions of theorem (1.1.4). Then $\mathcal{M}'$ is a Borel subset of $(\mathcal{M}^+,\tau)$.

In chapter 2 we will be interested in a special class of point measures on a product space.

Let X be the product T × U of the halfline $T = [0,\infty[$ with the usual topology and a polish space U. The space X with the product topology is a polish space. Let $(U_k)_{k\geq 1}$ be an increasing sequence of open subsets of U, $U_k \uparrow U$. Define

$$\mathcal{S} = \{I \times G : I \subset T \text{ open and bounded}, G \subset U \text{ open and } G \subset U_k \text{ for some } k \geq 1\}.$$

$\mathcal{S}$ is a filtering family of open subsets of X which satisfies the conditions of theorem (1.1.4). Denote by $\mathcal{M}^+$ the set of $\mathcal{S}$-finite measures on $(X,\mathfrak{B}(X))$ and by $\mathcal{M}_1^\bullet$ the set of simple $\mathcal{S}$-finite point measures μ satisfying the condition

$$\forall t \in T : \mu(\{t\} \times U) \leq 1.$$

1.1.9 Proposition. Let X and $\mathcal{S}$ be defined as above.

Then $\mathcal{M}_1^\bullet$ is a Borel subset of $(\mathcal{M}^+,\tau)$.

Proof. The proof is analogous to the proof of proposition (1.1.7) and is therefore omitted. □

1.1.10 Remarks.

(i) If X is a locally compact, second countable Hausdorff space and $\mathcal{S}$ the family of compact subsets of X, then $\mathcal{M}'$ is a dense G_δ set in $\mathcal{M}''$.

(ii) Let (X,d) be a complete, separable metric space and let $\mathcal{S}$ be the family of all bounded open subsets of X. The family $\mathcal{S}$ satisfies the conditions of theorem (1.1.4); a countable cofinal subset of $\mathcal{S}$ is the sequence of open balls $(B_n(z))_{n\geq 1}$ with radius $n \in \mathbb{N}$ and center a fixed point $z \in X$. Matthes, Kerstan and Mecke define in [40], section (1.15) a metric ρ on $\mathcal{M}''$. It turns out that $(\mathcal{M}'',\rho)$ is a complete, separable metric space and the metric topology on $\mathcal{M}''$ coincides with the relative topology on $\mathcal{M}''$ as a subspace of $(\mathcal{M}^+,\tau)$.

(iii) Let (X,d) be a complete, separable metric space and x_∞ be a fixed point of X. Harris calls in [19] a (nonnegative Borel) measure μ on X x_∞-finite if

(a) $\mu(X\backslash V) < \infty$ for each open set V containing x_∞ and

(b) $\mu(\{x_\infty\}) = 0$.

Let M be the class of x_∞-finite measures and let

$E_t = \{x : x \in X,\ d(x,x_\infty) \geq \frac{1}{t}\}$, $t > 0$. The sets E_t are closed and have disjoint boundaries.It is clear that

$$\mu \in M \Leftrightarrow \mu(E_t) < \infty \text{ for all } t > 0.$$

Harris introduced in [19] a topology on M, which we will describe now. Denote for $t > 0$ by L_t^* the Levy-Prohorov distance on $\mathcal{M}_b^+(E_t)$, that is a metric on $\mathcal{M}_b^+(E_t)$ such that $(\mathcal{M}_b^+(E_t),\ L_t^*)$ is a complete, separable metric space and the L_t^*-topology on $\mathcal{M}_b^+(E_t)$ is the narrow topology $\sigma(\mathcal{M}_b^+(E_t),\ C_b(E_t))$.

If μ, $\nu \in M$ put

$$L(\mu,\nu) = \int_0^\infty \frac{e^{-t}L_t(\mu,\nu)}{1+L_t(\mu,\nu)}\,dt,$$

where $L_t(\mu,\nu)$ denotes the L_t^* distance of the restrictions of μ and ν to E_t. L is well-defined and is a metric for M such that (M,L) is a complete, separable metric space.

Consider now the polish space $X\setminus\{x_\infty\}$. Let $\mathcal{S}$ be the family of open subsets $A_t = \{x \in X\setminus\{x_\infty\} : d(x,x_\infty) > \frac{1}{t}\}$, $t > 0$.

$\mathcal{S}$ satisfies the conditions of theorem (1.1.4). Denote by $\bar{\mu}$ the restriction of the measure $\mu \in M$ to $X\setminus\{x_\infty\}$. $\bar{\mu}$ is a $\mathcal{S}$-finite measure on $X\setminus\{x_\infty\}$.

<u>Proposition</u>. The map $\chi : \mu \in M \to \bar{\mu} \in \mathcal{M}^+$ is a continuous bijection from (M,L) onto $(\mathcal{M}^+,\tau)$.

<u>Proof</u>. It is clear that χ is a bijection. To see that χ is continuous, let $f \in \mathcal{H}$ and let (μ_n) be a sequence in M converging to μ, i.e. $\lim_{n\to\infty} L(\mu_n,\mu)=0$. Then $\mathrm{supp}(f) \subset A_t$ for all t sufficiently small. It follows that there exists $t > 0$ such that $\mathrm{supp}(f) \subset A_t$ and $\mu(\{x : x \in X,\ d(x,x_\infty) = \frac{1}{t}\}) = 0$. From Harris [19], theorem (2.2) we conclude that

$$\lim_{n\to\infty} \mu_n(f) = \lim_{n\to\infty} {}_{A_t}\mu_n(f|_{A_t})$$
$$= {}_{A_t}\mu(f|_{A_t}) = \mu(f).$$

So the maps $\mu \in M \to (\chi(\mu))(f)$, $f \in \mathcal{H}$ are continuous, which implies the continuity of χ.□

If $\mu,\nu \in \mathcal{M}^+$ put

$$d(\mu,\nu) = L(\chi^{-1}(\mu),\ \chi^{-1}(\nu)).$$

$(\mathcal{M}^+,d)$ is a complete, separable metric space. Let τ_d denote the d-topology on $\mathcal{M}^+$. From the foregoing proposition it follows that $\tau \subset \tau_d$.

Proposition. Let (μ_n) be a sequence in $\mathcal{M}^+$ and $\mu \in \mathcal{M}^+$. Then

$$\tau\text{-}\lim_{n\to\infty} \mu_n = \mu \iff \text{d-}\lim_{n\to\infty} \mu_n = \mu.$$

Proof. The implication ($\Leftarrow$) holds since $\tau \subset \tau_d$. So assume that $\tau\text{-}\lim_{n\to\infty} \mu_n = \mu$. If $\mu(\{x : x \in X,\ d(x,x_\infty) = \frac{1}{t}\}) = 0$, then $\mu(\delta E_t) = 0$, where δE_t denotes the boundary of E_t. By proposition (1.1.6) we have that $\lim_{n\to\infty} \mu_n(E_t) = \mu(E_t)$. Identifying μ and $\gamma(\mu)$, it follows that the restrictions of (μ_n) to E_t converge in $(\mathcal{M}_b^+(E_t),\ L_t^*)$ to μ, see Topsoe [50], p.40. From Harris [19], theorem 2.2 we may conclude that d-lim $\mu_n = \mu$. □

So for the topologies τ and τ_d on $\mathcal{M}^+$ we have:

$(\mathcal{M}^+, \tau_d)$ is a polish space,

$\tau \subset \tau_d$,

$\mu_n \xrightarrow{\tau} \mu$ iff $\mu_n \xrightarrow{\tau_d} \mu$.

One cannot conclude from this that $\tau = \tau_d$. Take for instance (X,τ) as in example E of Kelley [33], p.77 and take for τ_d the discrete topology on X. It is clear that τ and τ_d satisfy the above conditions and that $\tau \neq \tau_d$.

1.2 Poisson point processes.

Let X be a polish space and let $\mathcal{S}$ be a family of open subsets of X which is filtering to the right with respect to inclusion. Assume that $\mathcal{S}$ has a countable cofinal subset and that $\mathcal{S}$ filters to X. Denote by $\mathcal{M}^+$ the Lusin space of nonnegative Borel measures on X which are finite on $\mathcal{S}$. See section (1.1). Let P be a probability measure on $(\mathcal{M}^+, \mathcal{B}(\mathcal{M}^+))$.

For a finite sequence $B_1,\dots,B_m$ in $\mathcal{B}(X)$ the finite-dimensional distribution $P_{B_1\dots B_m}$ is defined as the image of P under the map

$$\mu \in \mathcal{M}^+ \to [\mu(B_1),\dots,\mu(B_m)] \in ([0,\infty])^m.$$

Note that it is a consequence of theorem (1.1.4) that probability measures on $\mathcal{M}^+$ with the same finite-dimensional distributions are identical.

The Laplace transform $\hat{P}$ of P is defined by

$$\hat{P}(f) = \int_{\mathcal{M}^+} P(d\mu) \exp [- \int_X f(x)\mu(dx)]$$

where f runs through the cone $\mathcal{B}(X)_+$ of nonnegative measurable functions. The moment generating functions of the finite-dimensional distributions $P_{B_1\dots B_m}$ are determined by $\hat{P}$ as follows from

$$\int u_1^{x_1}\dots u_m^{x_m} P_{B_1\dots B_m}(dx_1\dots dx_m) = \hat{P}(\sum_1^m \ln (\frac{1}{u_i})1_{B_i}),$$

where $0 < u_i \leq 1$, $i=1,\dots,m$. So P is uniquely determined by its Laplace transform. The intensity measure $i=i_P$ of P is the Borel measure on X defined by

$$i(B) = \int_{\mathcal{M}^+} P(d\mu)\mu(B), \ B \in \mathcal{B}(X).$$

We say that P has $\mathcal{G}$-finite intensity if $i \in \mathcal{M}^+$. Denote by ρ the Campbell measure of P, that is the measure on $\mathcal{M}^+ \times X$ defined by

$$\int_{\mathcal{M}^+\times X} F(\mu,x)\rho(d\mu,dx) = \int_{\mathcal{M}^+} P(d\mu) \int_X \mu(dx) F(\mu,x).$$

It is clear that ρ is a σ-finite measure if the intensity measure i of P is $\mathcal{G}$-finite. The projection $\rho(\mathcal{M}^+ \times .)$ of ρ on X is the intensity measure i of P. If the intensity measure i_P of P is $\mathcal{G}$-finite, then a general theorem on disintegrations of measures (see Bourbaki [5], section 2.7) implies the existence of a measurable family of probability measures $(P_x)_{x\in X}$ on $\mathcal{M}^+$ such that

$$\int_{\mathcal{M}^+\times X} F \, d\rho$$

$$= \int_{\mathcal{M}^+} P(d\mu) \int_X \mu(dx) F(\mu,x) = \int_X i(dx) \int_{\mathcal{M}^+} P_x(d\mu)F(\mu,x) \qquad (*)$$

for every measurable nonnegative function $F : \mathcal{M}^+ \times X \to \mathbb{R}$. The probability measures P_x on $\mathcal{M}^+$ are called Palm measures of P and the formula (*) is the so-called Palm formula for P. Note that P is completely determined by the intensity measure i_P and the Palm measures $(P_x)_{x\in X}$. A straightforward calculation gives a formula for the Laplace transforms of the Palm measures P_x. Let $f,g \in \mathcal{B}(X)_+$, then

$$\int i_P(dx)\ \hat{P}_x(f)\cdot g(x) = -\frac{d}{dt}\hat{P}(f + tg)\big|_{t=0}.$$

A probability measure P on $\mathcal{M}^+$ will be called an $\mathcal{S}$-finite point process with phase space X or an $\mathcal{S}$-finite point process on X if $P(\mathcal{M}''(\mathcal{S}))=1$. If the phase space and the family $\mathcal{S}$ are clear from the context we will speak of a point process. A point process P will be called a simple point process if $P(\mathcal{M}^{\bullet})=1$. As usual in probability theory, an $\mathcal{M}^+$-valued random variable N will also be called a (simple) point process if its distribution on $\mathcal{M}^+$ is so. A point process is said to be free from after-effects if its finite-dimensional distributions satisfy the relation

$$P_{B_1 \ldots B_m} = P_{B_1} \otimes \ldots \otimes P_{B_m}$$

where $P_{B_1} \otimes \ldots \otimes P_{B_m}$ is the product of the measures P_{B_i} on $\bar{\mathbb{N}}$. A point process P will be called a Poisson point process with intensity measure ν if P is free from after-effects and if the one-dimensional distributions P_B, $B \in \mathcal{B}(x)$, are Poisson distributions with expectation $\nu(B)$, i.e.

$$P_B(\{k\}) = \frac{[\nu(B)]^k}{k!}\, e^{-\nu(B)},\quad k=0,1,2,\ldots \quad \text{if } \nu(B) < \infty,$$

$$P_B(\{\infty\}) = 1 \quad \text{if } \nu(B) = \infty.$$

1.2.1 <u>Proposition</u>. Let P be a Poisson point process with $\mathcal{S}$-finite intensity measure ν. Then the Laplace transform $\hat{P}$ and the Palm-measures P_x are given by

$$\hat{P}(f) = \exp\Big[-\int_X \nu(dx)(1-e^{-f(x)})\Big] \quad ,\ f \in \mathcal{B}(X)_+.$$

$P_x = \tilde{\delta}_x * P$, $x \in X$,

where $\tilde{\delta}_x$ denotes the Dirac measure on $\mathcal{M}^+$ in the point δ_x.

Proof. Follows from standard calculations.□

1.2.2 Proposition. For every $\nu \in \mathcal{M}^+(\mathcal{S})$ there exists a unique $\mathcal{S}$-finite Poisson point process on X with intensity measure ν.

Proof. Let $A \in \mathcal{S}$. The restriction ${}_A\nu$ of ν to A is a finite measure on $(A, \mathcal{B}(A))$. So there is a unique $\{A\}$-finite Poisson point process ${}_AP$ on A with intensity measure ${}_A\nu$, see Matthes et al [40], section 1.7. ${}_AP$ is a probability measure on $(\mathcal{M}_b^+(A), \mathcal{B}(\mathcal{M}_b^+(A)))$, where $\mathcal{B}(\mathcal{M}_b^+(A))$ is the Borel σ-algebra on $(\mathcal{M}_b^+(A), \tau_2(A))$, see for $\tau_2(A)$ the definitions preceeding proposition (1.1.1). Let $A, B \in \mathcal{S}$, $A \subset B$ and let π_{AB} be the projection of $\mathcal{M}_b^+(B)$ on $\mathcal{M}_b^+(A)$ as defined in section (1.1). The image $\pi_{AB}({}_BP)$ of ${}_BP$ is a probability measure on $(\mathcal{M}_b^+(A), \mathcal{B}(\mathcal{M}_b^+(A)))$. A straightforward calculation gives $(\pi_{AB}({}_BP))^\wedge = ({}_AP)^\wedge$. So $\pi_{AB}({}_BP) = {}_AP$ and it follows that $(\mathcal{M}_b^+(A), \mathcal{B}(\mathcal{M}_b^+(A)), {}_AP, \pi_{AB})$ is a projective system of probability spaces. Since $\mathcal{S}$ has a countable cofinal subset, it is a consequence of Bochner's theorem (see Bochner [4], p. 120) that there exists a projective limit P, which is a probability measure on $(\mathcal{M}^+, \mathcal{B}(\mathcal{M}^+))$. An easy calculation yields that P is the $\mathcal{S}$-finite Poisson point process on X with intensity measure ν.□

Denote by P_ν the $\mathcal{S}$-finite Poisson point-process on X with intensity measure $\nu \in \mathcal{M}^+$. Note that P_ν is a simple point process iff the intensity measure ν is a diffuse measure (i.e. $\nu(\{x\}) = 0$ for every $x \in X$). The family of point processes $\{P_\nu: \nu \in \mathcal{M}^+\}$ is a measurable family, i.e. for every $G \in \mathcal{B}(\mathcal{M}^+)$ the map $\nu \in \mathcal{M}^+ \to P_\nu(G)$ is measurable. Let V be a probability measure on $(\mathcal{M}^+, \mathcal{B}(\mathcal{M}^+))$ and let Q be the probability measure on $(\mathcal{M}^+, \mathcal{B}(\mathcal{M}^+))$ defined by

$$Q = \int_{\mathcal{M}^+} V(d\nu)P_\nu.$$

It is clear that Q is a point process on X, which is simple iff V is concentrated on the diffuse measures in $\mathcal{M}^+$. Such a process Q is called a Cox process.

1.2.3 Proposition. Let Q be a Cox process as defined above. The intensity measure i_Q, the Laplace transform $\hat{Q}$ and the Palm measures $(Q_x)_{x\in X}$ of Q are given by

$$\begin{aligned} i_Q(B) &= i_V(B) && , B \in \mathcal{B}(X), \\ \hat{Q}(f) &= \hat{V}(1-e^{-f}) && , f \in \mathcal{B}(X)_+, \\ Q_x &= \tilde{\delta}_x * \int V_x(d\nu)P_\nu && , x \in X, \end{aligned}$$

where $\tilde{\delta}_x$ denotes the Dirac measure on $\mathcal{M}^+$ in the point δ_x.

Proof. The formulas for i_Q and $\hat{Q}$ follow directly from the definitions. To prove the formula for Q_x, let $F : \mathcal{M}^+ \times X \to \mathbb{R}$ be a measurable, non-negative function. Then

$$\begin{aligned} \int Q(d\mu) \int \mu(dx)F(\mu,x) &= \int V(d\nu) \int P_\nu(d\mu) \int \mu(dx)F(\mu,x) \\ &= \int V(d\nu) \int \nu(dx) \int (\tilde{\delta}_x * P_\nu)(d\mu)F(\mu,x) \\ &= \int i_V(dx) \int V_x(d\nu) \int P_\nu(d\mu)F(\mu+\delta_x,x) \\ &= \int i_Q(dx) \int (\tilde{\delta}_x * \int V_x(d\nu)P_\nu)(d\mu)F(\mu,x) \end{aligned}$$

from which the result follows. □

1.3 Itô-Poisson point processes.

Let X be the product T × U of the halfline T = $[0,\infty[$ with the usual topology and a polish space U. The Borel σ-algebras on T and U will be denoted by $\mathcal{B}_T$ and $\mathcal{U}$. Endowed with the product topology X is a polish space and its Borel σ-algebra $\mathcal{B}(X)$ is identical to the product σ-algebra $\mathcal{B}_T \otimes \mathcal{U}$. Let $(U_k)_{k\geq 1}$ be an increasing sequence of open subsets of U which covers U. The family $\mathcal{S}$ of open subsets of X defined by

$$\mathcal{S} = \{A : A = I \times G,\ I \subset T \text{ open and bounded}, \quad G \subset U \text{ open and } G \subset U_k \text{ for some } k \geq 1\}$$

is filtering with respect to inclusion and contains a countable, cofinal subset. The topological space of $\mathcal{S}$-finite measures on X will be denoted by $(\mathcal{M}^+,\tau)$, see section (1.1). The Borel σ-algebra $\mathcal{G}$ on $\mathcal{M}^+$ is identical to the σ-algebra generated by the family of maps $\{p_A: A\in\mathcal{B}(X)\}$, where p_A is defined by $p_A : \nu \in \mathcal{M}^+ \to \nu(A)$. The family $(\mathcal{G}_t)_{t\geq 0}$ of sub σ-algebras of $\mathcal{G}$ defined by

$$\mathcal{G}_t = \sigma(p_A,\ A \in \mathcal{B}(X),\ A \subset [0,t] \times U)$$

is a filtration on $(\mathcal{M}^+,\mathcal{G})$. A measurable map $\psi : \mathcal{M}^+ \to T$ is called $(\mathcal{G}_t)$-adapted if $[\psi \leq t] \in \mathcal{G}_t$ for every $t \in T$. An Itô-Poisson point process on U is an $\mathcal{S}$-finite Poisson point process P with phase space X whose intensity measure μ is the product of the Lebesgue measure λ on T and a nonnegative Borel measure ν on U, which is finite on the sequence $(U_k)_{k\geq 1}$. Following Itô [25], ν is called the characteristic measure of the Itô-Poisson point process P.

1.3.1 Proposition. Let P be an $\mathcal{S}$-finite Itô-Poisson process on U, then $P(\mathcal{M}_1^{\cdot})=1$.

Proof. Since the intensity measure $\mu = \lambda\otimes\nu$ of P is diffuse, P is a simple point process on X. Define the mappings π_k $(k\geq 1)$ by

$$\pi_k : \mathcal{M}^{\cdot} \to \mathcal{M}_T^{\cdot\cdot},\quad \pi_k(\mu) = [B \in \mathcal{B}_T \to \mu(B \times U_k)],$$

where $\mathcal{M}_T^{\cdot\cdot}$ is the space of point measures on T which are finite on all bounded subintervals of T. The map π_k is a P-a.e. defined, measurable map on $\mathcal{M}^+$. The measure $P_k = \pi_k(P)$ is the Poisson point process on T with intensity measure $i_k = \nu(U_k)\cdot\lambda$. Since the intensity measure i_k is diffuse, the point process P_k is a simple point process. It follows that

$$P(\pi_k^{-1}(\mathcal{M}_T^{\cdot})) = P_k(\mathcal{M}_T^{\cdot}) = 1$$

and

$$P(\mathcal{M}_1^{\cdot}) = P(\bigcap_{k\geq 1} \pi_k^{-1}(\mathcal{M}_T^{\cdot})) = 1,$$

which completes the proof of the proposition.□

Let $\varphi : \mathcal{M}^+ \to T$ be a measurable map. Define the transformation R_φ by

$$R_\varphi : \mathcal{M}^+ \to \mathcal{M}^+,$$
$$\int R_\varphi(\mu)(d\sigma du)f(\sigma,u) = \int \mu(d\sigma du)f(\sigma,u)1_{[0,\varphi(\mu)]}(\sigma), \quad f \in \mathcal{B}(x)_+.$$

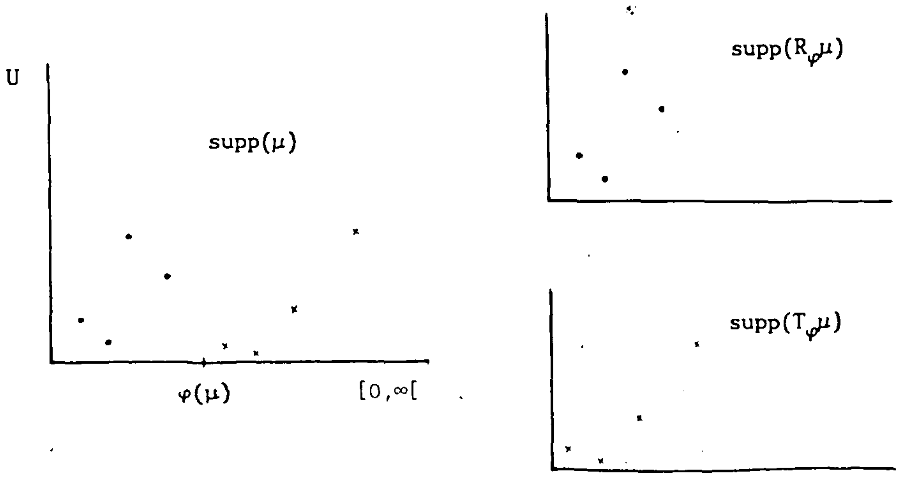
Let for $\sigma \in T$ the map t_σ be defined by

$$t_\sigma : (\tau,v) \in \,]\sigma,\infty[\, \times U \to (\tau-\sigma,v) \in X.$$

Define the transformation T_φ by

$$T_\varphi : \mathcal{M}^+ \to \mathcal{M}^+,$$
$$\int T_\varphi(\mu)(d\sigma du)f(\sigma,u) = \int \mu(d\sigma du)f\circ t_{\varphi(\mu)}(\sigma,u)1_{]\varphi(\mu),\infty[}(\sigma)$$
$$= \int \mu(d\sigma du)f(\sigma-\varphi(\mu),u)1_{]\varphi(\mu),\infty[}(\sigma).$$

We will write simply R_s and T_s if φ is the constant map $\mu \in \mathcal{M}^+ \to s$. The following picture illustrates these definitions. The picture shows $\mathrm{supp}(\mu)$, $\mathrm{supp}(R_\varphi\mu)$ and $\mathrm{supp}(T_\varphi\mu)$ for a simple point measure μ.



1.3.2 Lemma. Let $\varphi : \mathcal{M}^+ \to T$ be a measurable map. The above defined transformations R_φ and T_φ are measurable.

Proof. Define for k,n = 1,2,...

$$A_{kn} = \{\mu \in \mathcal{M}^+ : k\cdot 2^{-n} \leq \varphi(\mu) < (k+1)\cdot 2^{-n}\}$$

and

$$\varphi_n = \Sigma_k (k+1)\cdot 2^{-n} 1_{A_{kn}}.$$

The sequence of measurable stepfunctions $(\varphi_n)_{n\geq 1}$ is a strictly decreasing sequence, which converges pointwise to φ. It is clear that for every bounded continuous function $f : X \to \mathbb{R}$ with support contained in some element of $\mathcal{S}$ and for every $\mu \in \mathcal{M}^+$

$$\lim_{n\to\infty} (T_{\varphi_n}\mu)(f) = (T_\varphi\mu)(f)$$

and

$$\lim_{n\to\infty} (R_{\varphi_n}\mu)(f) = (R_\varphi\mu)(f).$$

It follows that the sequences $(T_{\varphi_n})_{n\geq 1}$ and $(R_{\varphi_n})_{n\geq 1}$ converge pointwise to T_φ and R_φ. So it is sufficient to prove the measurability of T_{φ_n} and R_{φ_n}. Let $A \in \mathcal{B}(X)$ and $\mu \in \mathcal{M}^+$. Since

$$(R_{\varphi_n}\mu)(A) = \sum_k 1_{A_{kn}}(\mu)\ \mu(A \cap [0,(k+1)\cdot 2^{-n}[\times U)$$

and

$$(T_{\varphi_n}\mu)(A) = \sum_k 1_{A_{kn}}(\mu)\ \mu((t_{(k+1)\cdot 2^{-n}})^{-1}(A)),$$

it is clear that the maps $\mu \in \mathcal{M}^+ \to (R_{\varphi_n}\mu)(A)$ and $\mu \in \mathcal{M}^+ \to (T_{\varphi_n}\mu)(A)$ are measurable maps which implies measurability of the transformations R_{φ_n} and T_{φ_n}.□

1.3.3 Theorem (Renewal property). Let $\varphi : \mathcal{M}^+ \to T$ be a measurable map and let P be an Itô-Poisson point process. If φ is $(\mathcal{G}_t)$-adapted, then R_φ and T_φ are independent $\mathcal{M}^+$-valued random variables on $(\mathcal{M}^+,\mathcal{G},P)$ and $T_\varphi(P)=P$.

Proof. Consider first the case that φ is a stepfunction, say

$$\varphi = \sum_{i\geq 1} s_i 1_{A_i}, \quad A_i \in \mathcal{G}_{s_i} \quad (i\geq 1).$$

Since P is free from after-effects

$$\int P(d\mu) e^{-(R_\varphi\mu)(f)-(T_\varphi\mu)(g)}$$

$$= \sum_i \int P(d\mu) 1_{A_i}(\mu)\ e^{-(R_{s_i}\mu)(f)-(T_{s_i}\mu)(g)}$$

$$= \sum_i \int P(d\mu) 1_{A_i}(\mu)\ e^{-(R_{s_i}\mu)(f)} \cdot \int P(d\mu) e^{-(T_{s_i}\mu)(g)}$$

for f,g $\in \mathcal{B}(X)_+$. Let υ be the characteristic measure of P. Then for every s$\geq$0:

$$\int P(d\mu)\ e^{-(T_s\mu)(g)}$$

$$= \exp\,[- \int_s^\infty d\sigma \int \upsilon(d\mu)(1-e^{-g(\sigma-s,\mu)})]$$

$$= \exp\,[- \int_0^\infty d\sigma \int \upsilon(d\mu)(1-e^{-g(\sigma,\mu)})]$$

$$= \hat{P}(g).$$

Hence

$$\int P(d\mu)\ e^{-(R_\varphi\mu)(f)-(T_\varphi\mu)(g)} = \int P(d\mu) e^{-(R_\varphi\mu)(f)} \cdot \hat{P}(g)$$

which completes the proof of the theorem for stepfunctions. The general case will follow by approximating φ from above by a sequence of step-functions as in the proof of lemma (1.3.2).□

<u>Remark</u> Without further assumptions, it is not possible to say more about $R_\varphi(P)$. As an example, let P be an Itô-Poisson point process on X with $\mathcal{S}$-finite characteristic measure υ. Let $U_o \in \mathcal{S}$ be a subset of U such that $\upsilon(U_o) > 0$. Define the map $\varphi : \mathcal{M}_1^{\cdot} \to T$ by

$$\varphi(\mu) = \min\,\{t \in T : \mu(\{t\} \times U_o) = 1\}.$$

Since

$$\{\mu \in \mathcal{M}_1^{\cdot} : \varphi(\mu) \leq t\} = \{\mu \in \mathcal{M}_1^{\cdot} : \mu([0,t] \times U_o) \geq 1\},$$

μ is a $(\mathcal{G}_t)$-adapted map on $\mathcal{M}_1^{\cdot}$. For f $\in \mathcal{B}(X)_+$

$$\begin{aligned}
&(R_\varphi(P))^\wedge(f) \\
&= \int P(d\mu)\ e^{-(R_\varphi\mu)(f)} \\
&= \int P(d\mu) \int \mu(d\tau dv) 1_{[\varphi(\mu)=\tau]}\ e^{-(R_\tau\mu)(f)} \\
&= \int_0^\infty d\tau \int_U \upsilon(dv) \int P(d\mu)\ 1_{[\varphi(\mu+\delta_{(\tau,v)})=\tau]} e^{-(R_\tau(\mu+\delta_{(\tau,v)}))(f)} \\
&= \int_0^\infty d\tau \int_U \upsilon(dv) \int P(d\mu) [1_{[\varphi(\mu)=\tau]} + 1_{[\varphi(\mu)>\tau]} 1_{U_o}(v)] e^{-(R_\tau\mu)(f)-f(\tau,v)} \\
&= \int_0^\infty d\tau \int_{U_o} \upsilon(dv) \int P(d\mu) [1_{[\mu([0,\tau]\times U_o)=0]}\ e^{-\mu(1_{[0,\tau]\times U\setminus U_o}\cdot f)-f(\tau,v)} \\
&= \int_0^\infty d\tau \int_{U_o} \upsilon(dv)\ e^{-\tau\upsilon(U_o)-f(\tau,v)} \int P(d\mu)\ e^{-\mu(1_{[0,\tau]\times U\setminus U_o}\cdot f)}.
\end{aligned}$$

Let Q_τ be the image of the probability measure $\upsilon(.|U_o)$ under the map $u \in U \to \delta_{(\tau,u)} \in \mathcal{M}^+$. It is clear that for $f \in \mathcal{B}(X)_+$

$$\hat{Q}_\tau(f) = \frac{1}{\upsilon(U_o)} \int_{U_o} \upsilon(dv) e^{-f(\tau,v)}.$$

Let S_τ be the image of the probability measure P under the map $\mu \in \mathcal{M}^+ \to 1_{[0,\tau]\times U\setminus U_o}\cdot\mu \in \mathcal{M}^+$. It is easy to see that S_τ is the Poisson point process with intensity measure $1_{[0,\tau]\times U\setminus U_o}\cdot(\lambda\otimes\upsilon)$. It follows that

$$\begin{aligned}
(R_\varphi(P))^\wedge(f) &= \int_0^\infty d\tau\ \upsilon(U_o) e^{-\tau\upsilon(U_o)}\ \hat{Q}_\tau(f)\cdot\hat{S}_\tau(f) \\
&= \int_0^\infty d\tau\ \upsilon(U_o) e^{-\tau\upsilon(U_o)} (Q_\tau * S_\tau)^\wedge(f)
\end{aligned}$$

and

$$R_\varphi(P) = \int_0^\infty d\tau\ \upsilon(U_o)\ e^{-\tau\upsilon(U_o)} (Q_\tau * S_\tau).$$

A measure $\mu \in \mathcal{M}^+$ is called recurrent if

$$\mu([t,\infty[\times U_k) > 0 \text{ for every } t > 0 \text{ and } k \geq 1.$$

Denote by $\mathcal{M}_r$ the set of recurrent measures. A point process will be called recurrent if $P(\mathcal{M}_r)=1$. An Itô-Poisson point process with characteristic measure ν is recurrent if $\nu(U_k) > 0$ for every $k \geq 1$.

Let $\mu \in \mathcal{M}_1^{\cdot} \cap \mathcal{M}_r$. For every $k \geq 1$ the support of the restriction ${}_k\mu$ of μ to $T \times U_k$ is a countable infinite set whose projection on T has finite intersections with bounded subintervals of T. So we can write

$\mathrm{supp}({}_k\mu) = ((t_{ki}, u_{ki}))_{i\geq 1}$ where $t_{ki} \in T, t_{ki} < t_{k,i+1}$ and $u_{ki} \in U_k$, $i\geq 1$.

For $i,k=1,2,\ldots$ define the maps τ_{ki}, ξ_{ki} and σ_{ki} on $\mathcal{M}_1^{\cdot} \cap \mathcal{M}_r$ by putting

$$\tau_{ki}(\mu) = t_{ki},$$

$$\xi_{ki}(\mu) = u_{ki},$$

$$\sigma_{ki}(\mu) = \begin{cases} \tau_{k1}(\mu) & \text{if } i=1 \\ \tau_{ki}(\mu) - \tau_{k,i-1}(\mu) & \text{if } i > 1 \end{cases}.$$

All these maps are measurable. Denote by σ_k and ξ_k the vectors $(\sigma_{k1}, \sigma_{k2}, \ldots)$ and $(\xi_{k1}, \xi_{k2}, \ldots)$, $k\geq 1$.

1.3.4 <u>Theorem</u> (Itô). Let P be an $\mathcal{S}$-finite recurrent, simple point process on X and let ν be a measure on $(U, \mathcal{U})$ such that $0 < \nu(U_k) < \infty$ $(k\geq 1)$. Then, P is the Itô-Poisson point process on U with characteristic measure ν iff for each $k\geq 1$

(i) $(\xi_{ki})_{i\geq 1}$ is a sequence of independent and identically distributed random variables, with distribution

$$P[\xi_{ki} \in A] = \frac{\nu(A\cap U_k)}{\nu(U_k)} \quad (A \in \mathcal{U});$$

(ii) $(\sigma_{ki})_{i\geq 1}$ is a sequence of independent and identically distributed random variables, with distribution

$$P[\sigma_{ki} > t] = e^{-t\nu(U_k)}, \quad (t > 0);$$

(iii) σ_k and ξ_k are independent vectors.

<u>Proof</u>. Let P be the Itô-Poisson process on U with characteristic

measure ν. Then for $k \geq 1$, $\lambda > 0$ and $A \in \mathcal{U}$ we have

$$\int P(d\mu)(e^{-\lambda\sigma_{k1}} 1_A(\xi_{k1}))(\mu)$$

$$= \int P(d\mu) \int \mu(d\sigma du)\ 1_{\{\sigma_{k1}(\mu)\}\times(A\cap U_k)}(\sigma,u)e^{-\lambda\sigma}$$

$$= \int_0^\infty d\sigma \int \nu(du) \int P(d\mu)\ 1_{\{\sigma_{k1}(\mu+\delta_{(\sigma,u)})\}\times(A\cap U_k)}(\sigma,u)e^{-\lambda\sigma}$$

by an application of the Palm formula and proposition (1.2.1)

$$= \int_0^\infty d\sigma \int \nu(du) \int P(d\mu)\ 1_{A\cap U_k}(u)1_{\{\sigma_{k1}>\sigma\}}e^{-\lambda\sigma}$$

$$= \nu(A\cap U_k) \int_0^\infty d\sigma\ e^{-\sigma\nu(U_k)}e^{-\lambda\sigma}$$

$$= \frac{\nu(A\cap U_k)}{\nu(U_k)} \cdot \frac{\nu(U_k)}{\lambda + \nu(U_k)} .$$

Let now $k,n \geq 1$, $\lambda_1,\dots, \lambda_n > 0$ and $A_1,\dots,A_n \in \mathcal{U}$.

Then

$$\int P(d\mu) \prod_{i=1}^{n} (e^{-\lambda_i\sigma_{ki}}\ 1_{A_i}(\xi_{ki}))(\mu)$$

$$= \int P(d\mu)(e^{-\lambda_1\sigma_{k1}}\ 1_{A_1}(\xi_{k1}))(R_{\sigma_{k1}}\mu)\cdot \prod_{i=2}^{n} (e^{-\lambda_i\sigma_{k,i-1}}\ 1_{A_i}(\xi_{k,i-1}))(T_{\sigma_{k1}}\mu)$$

$$= \int P(d\mu)(e^{-\lambda_1\sigma_{k1}}\ 1_{A_1}(\xi_{k1}))(\mu)\cdot \int P(d\mu)\prod_{i=1}^{n-1} (e^{-\lambda_{i+1}\sigma_{ki}}\ 1_{A_{i+1}}(\xi_{ki}))(\mu)$$

by an application of theorem (1.3.3)

$$= \prod_{i=1}^{n} \int P(d\mu)(e^{-\lambda_i\sigma_{k1}}\ 1_{A_i}(\xi_{k1}))(\mu)$$

by mathematical induction

$$= \prod_{i=1}^{n} \frac{\nu(A\cap U_k)}{\nu(U_k)} \cdot \frac{\nu(U_k)}{\lambda_i + \nu(U_k)} .$$

It follows that (i), (ii) and (iii) hold. To prove the converse, let $f \in \mathcal{B}(X)_+$.

$$\hat{P}(f) = \int P(d\mu)\ e^{-\int f d\mu}$$

$$= \lim_{\substack{k\to\infty \\ \tau\to\infty}} \int P(d\mu)\ e^{-\int_{[0,\tau]\times U_k} f d\mu}$$

$$= \lim \{\sum_{n=1}^{\infty} \int \exp(-\sum_{1}^{n} f(\tau_i,u_i))$$
$$P[\xi_{ki} \in du_i,\ t_{ki} \in d\tau_i,\ i=1,\ldots,n \mid t_{kn} \leq \tau < t_{k,n+1}] P[t_{kn} \leq \tau < t_{k,n+1}]$$
$$+ P[t_{k1} > \tau]\}$$

$$= \lim \{\sum_{n=1}^{\infty} e^{-\nu(U_k)\tau} \int_0^{\tau} d\tau_1 \int_{\tau_1}^{\tau} d\tau_2 \ldots \int_{\tau_{n-1}}^{\tau} d\tau_n \int_{U_k} \nu(du_1)\ldots\int_{U_k} \nu(du_n)$$
$$\exp[-\sum_{1}^{n} f(\tau_i,u_i)] + e^{-\nu(U_k)\tau}\}$$

$$= \lim \sum_{n=0}^{\infty} \frac{1}{n!} \{ \int_0^{\tau} d\sigma \int_{U_k} \nu(du) e^{-f(\sigma,u)}\}^n .\ e^{-\nu(U_k)\tau}$$

$$= \exp\ (-\int_0^{\infty} d\sigma \int \nu(du)(1-e^{-f(\sigma,u)})).$$

Hence P is the Itô-Poisson point process with characteristic measure ν.□

1.3.5 <u>Theorem</u>. (Greenwood & Pitman). Let N be an $\mathcal{M}^+$-valued random variable defined on some probability space $(\Omega,\mathcal{F},P)$. If N is an Itô-Poisson point process with characteristic measure ν and if $0 < \nu(U_n) < \infty$ $(n \geq 1)$ and $\nu(U) = \infty$ then

$$\lim_{n\to\infty} \frac{1}{\nu(U_n)} N_\omega([0,t] \times U_n) = t$$

uniformly on bounded t-intervals for P-a.e. $\omega \in \Omega$.

<u>Proof</u>. Fix $t \geq 0$ and define for $n \geq 1$

$$\mathcal{F}_n = \sigma(\omega \in \Omega \to N_\omega([0,t] \times U_k),\ k \geq n).$$

The random variable $N([0,t] \times U_k)$ is integrable with expectation given by

$$E\ N([0,t] \times U_k) = t\nu(U_k).$$

A straightforward calculation yields

$$E^{\mathcal{F}_{n+1}} N([0,t] \times U_n) = \frac{\nu(U_n)}{\nu(U_{n+1})} N([0,t] \times U_{n+1}).$$

Therefore $(\frac{1}{\nu(U_n)} N([0,t] \times U_n), \mathcal{F}_n)_{n\geq 1}$ is a reversed martingale on $(\Omega,\mathcal{F},P)$. It follows that $\frac{1}{\nu(U_n)} N([0,t] \times U_n)$ converges a.s. and in L^1 to a limit which is a random variable measurable with respect to $\mathcal{F}_\infty = \bigcap_{n\geq 1} \mathcal{F}_n$. See Neveu [43]. Since for $\lambda > 0$

$$\exp(-\lambda \frac{1}{\nu(U_n)} N([0,t] \times U_n))$$

$$= \exp\{t\nu(U_n)(1-e^{-\frac{\lambda}{\nu(U_n)}})\} \to e^{-\lambda t} \text{ as } n \to \infty,$$

we may conclude

$$\lim_{n\to\infty} \frac{1}{\nu(U_n)} N([0,t] \times U_n) = t \quad \text{a.s. and in } L^1.$$

The statement of the theorem now follows from a general lemma on the convergence of positive non-decreasing functions for which we refer to Appendix A3.1□

Let N be as in theorem (1.3.5). For $\omega \in \Omega$ such that $N_\omega \in \mathcal{M}_1^{\cdot} \cap \mathcal{M}_r$ we write $\tau_{ki}(\omega)$, $\xi_{ki}(\omega)$ and $\sigma_{ki}(\omega)$ for $\tau_{ki}(N_\omega)$, $\xi_{ki}(N_\omega)$ and $\sigma_{ki}(N_\omega)$, see the definitions preceeding theorem (1.3.4). Denote for k>j by $T_{kji}(\omega)$ the index at which the i^{th} point of type U_j appears in the vector $\xi_k(\omega) = \xi_k(N_\omega)$.

If N is an Itô-Poisson point process, then τ_{ki}, ξ_{ki} and σ_{ki} are P-a.e. defined random variables.

1.3.6 Corollary. Under the assumptions of theorem (1.3.5) we have

$$\lim_{k\to\infty} \frac{1}{\nu(U_k)} T_{kji}(\omega) = \tau_{ji}(\omega) \text{ P-a.s. }.$$

Proof. By definition of $T_{kji}(\omega)$ we have

$$N_\omega([0,\tau_{ji}(\omega)] \times U_k) = T_{kji}(\omega).\square$$

The corollary implies that an Itô-Poisson point process N can be constructed from (U_k,ξ_k) whenever $\nu(U)=+\infty$; the times τ_{ki} at which the ξ_{ki} occur, are already determined by (U_k,ξ_k). We will put this in a more general framework.

Let for $k=1,2,\dots$ $V_k = (V_{k1},V_{k2},\dots)$ be a sequence of U_k-valued random variables on $(\Omega,\mathcal{F},P)$.

$(U_k,V_k)_{k\geq 1}$ is called a nested array if

(i) V_k is a sequence of independent and identically distributed random variables, and

(ii) for $j < k$, V_j is the U_j-subsequence of V_k consisting of those terms which are in U_j.

See Greenwood & Pitman [17].

From theorem (1.3.4) it is clear that $(U_k, \xi_k)_{k\geq 1}$ is an example of a nested array.

If μ is a measure on $(U,\mathcal{U})$ and if $E \in \mathcal{U}$ is such that $0 < \mu(E) < \infty$, then $\mu|_E$ denotes the measure on U defined by

$$\mu|_E(A) = \frac{\mu(A\cap E)}{\mu(E)} \qquad (A \in \mathcal{U}).$$

1.3.7 <u>Proposition</u>. If $(U_k,V_k)_{k\geq 1}$ is a nested array on the probability space $(\Omega,\mathcal{F},P)$, then there exists a unique measure ν on $(U,\mathcal{U})$ such that $\nu(U_1)=1$ and $\nu|_{U_j}=\nu_j$, where ν_j denotes the probability distribution of V_{j1}.

<u>Proof</u>. Let $j < k$. Define S_{kji} as the index at which the i^{th} point of type U_j occurs in the sequence V_k. For $A \in \mathcal{U}$

$$\begin{aligned}\nu_j(A) &= P(V_{j1} \in A)\\ &= \sum_{i=1}^{\infty} P(V_{ki} \in A,\ S_{kj1}=i)\\ &= \sum_{i=1}^{\infty} \{P\ (V_{k1} \notin U_j)\}^{i-1}\ P(V_{k1} \in A\cap U_j)\\ &= \nu_k|_{U_j}(A).\end{aligned}$$

Substitution of $A=U_1$ yields $\nu_j(U_1) = \dfrac{\nu_k(U_1)}{\nu_k(U_j)}$. It follows that for $j < k$ and $A \in \mathcal{U}$

$$\frac{\nu_j(A\cap U_j)}{\nu_j(U_1)} = \frac{\nu_k(A\cap U_j)}{\nu_k(U_1)} .$$

Define

$$\mathcal{R} = \bigcup_j \{A : A \in \mathcal{U} \text{ and } A \subset U_j\}.$$

$\mathcal{R}$ is a ring of subsets of U, and the σ-ring generated by $\mathcal{R}$ is $\mathcal{U}$. From the above it follows that we can define consistently a setfunction ν on $\mathcal{R}$ by putting

$$\nu(A) = \frac{\nu_j(A)}{\nu_j(U_1)} \quad \text{for } A \in \mathcal{R},\ A \subset U_j \ (j \geq 1).$$

ν is a σ-finite measure on $\mathcal{R}$. So ν has a unique extension to a measure on $(U,\mathcal{U})$, see Halmos [18]. From the construction follows that ν has the desired properties.□

Define for $k \geq 1$

$$p_k = P(V_{k1} \in U_1).$$

It follows from the proof of proposition (1.3.7) that $p_k = \dfrac{1}{\nu(U_k)}$. Hence $(p_k)_{k\geq 1}$ is a decreasing sequence of positive real numbers and $\lim_{k\to\infty} p_k$ exists:

$$\nu(U) = +\infty \quad \Leftrightarrow \quad \lim_{k\to\infty} p_k = 0.$$

In the next theorem we associate an Itô-Poisson point process N to a nested array (U_k,V_k) such that the U_k-subsequence $(\xi_{ki})_{i\geq 1}$ of N is V_k.

1.3.8 Theorem. (Greenwood & Pitman). Let $(U_k,V_k)_{k\geq 1}$ be a nested array on the probability space $(\Omega,\mathcal{F},P)$. If

$$\lim_{k\to\infty} p_k = 0,$$

then for P-a.e. ω the limits

$$t_{jn}(\omega) = \lim_{k\to\infty} p_k\, S_{kjn}(\omega)$$

exist for all $n,j = 1,2,\ldots$ where $S_{kjn}(\omega)$ is the index at which the

n^{th} point of type U_j occurs in the sequence V_k. The sequence of limitpoints $(t_{jn}(\omega))_{n\geq 1}$ is a strictly increasing sequence of positive real numbers.

The P-a.e. defined random variable

$$\omega \in \Omega \to \sum_{j=1}^{\infty} \sum_{n: V_{jn}(\omega) \notin U_{j-1}} \delta_{(t_{jn}(\omega), V_{jn}(\omega))}$$

is an Itô-Poisson point process on U with characteristic measure υ, where υ is the measure defined in proposition (1.3.7).

Proof. Define for $k > j$, $n \geq 1$,

$$D_{kj1} = S_{kj1},$$

$$D_{kj,n+1} = S_{kj,n+1} - S_{kjn},$$

and

$$D_{kj} = (D_{kj1}, D_{kj2}, \ldots).$$

Then for measurable sets $A_1, \ldots, A_m \subset U_j$ and $d_1, \ldots, d_m \in \mathbb{N}$:

$$P(D_{kji} = d_i + 1, V_{ji} \in A_i, i=1,\ldots,m)$$

$$= \{P(V_{k1} \notin U_j)\}^{\sum_1^m d_i} \cdot \prod_{i=1}^m P(V_{k1} \in A_i)$$

$$= (1-\upsilon_k(U_j))^{\Sigma d_i} \upsilon_k(A_1)\ldots\upsilon_k(A_m)$$

$$= (1 - \frac{p_k}{p_j})^{\Sigma d_i} (\frac{p_k}{p_j})^m \upsilon_j(A_1)\ldots\upsilon_j(A_m)$$

since $\upsilon_j = \upsilon_k|_{U_j}$; see the proof of proposition (1.3.7).

So D_{kj} and V_j are independent sequences of random variables. The random variables $(D_{kji})_{i\geq 1}$ form an i.i.d. sequence and are geometrically distributed with expectation $\frac{p_j}{p_k}$.

For $\mathcal{F}_j = \sigma(V_i, i \leq j)$ $(j \geq 1)$, $k' > k > j$ and $n \geq 1$ we have

$$E(S_{k'jn}|\mathcal{F}_k) = E(\sum_{l=1}^{S_{kjn}} D_{k'kl}|\mathcal{F}_k) = \frac{p_k}{p_{k'}} S_{kjn}.$$

So $(p_k S_{kjn}, \mathcal{F}_k)_{k\geq j}$ is a martingale on $(\Omega, \mathcal{F}, P)$, and it follows that outside a set of P-measure 0 the sequence $(p_k S_{kjn})_{k\geq j}$ converges for all $n, j \geq 1$ to a finite limit, say t_{jn}.

$$t_{jn}(\omega) = \lim_{k\to\infty} p_k S_{kjn}(\omega).$$

It is clear that $0 \leq t_{j1}(\omega) \leq t_{j2}(\omega) \leq \ldots$. Since $p_k \to 0$,

$(e^{-\lambda p_k D_{kji}})$

$$= \frac{\frac{p_k}{p_j} e^{-\lambda p_k}}{1-(1-\frac{p_k}{p_j})e^{-\lambda p_k}} \to \frac{\frac{1}{p_j}}{\frac{1}{p_j}+\lambda} \text{ as } k \to \infty,$$

for $i,j \geq 1$. It follows that the random variables t_{j1}, $t_{j,n+1}-t_{j,n}$ $(n\geq 1)$ are independent and exponentially distributed with expectation p_j. So P-a.s. the sequence $(t_{jn}(\omega))_{n\geq 1}$ is strictly increasing. Theorem (1.3.8) implies that

$$\omega \in \Omega \to \sum_{j=1}^{\infty} \sum_{n:V_{jn}(\omega)\notin U_{j-1}} \delta_{(t_{jn}(\omega),V_{jn}(\omega))}$$

is a Poisson point process on $T \times U$ with intensity measure $\lambda \otimes \upsilon$.□

Let $N' : \Omega \to \mathcal{M}^+$ be an Itô-Poisson point process with characteristic measure υ' such that the U_k subsequence $(\xi'_{ki})_{i\geq 1}$ of N' is V_k $(k \geq 1)$. Using the notations and definitions of theorem (1.3.8) we get:

1.3.9 Corollary. There exists a positive constant c such that for $j,n \geq 1$

$$\upsilon' = c\upsilon \text{ and } \tau'_{jn} = \frac{1}{c} t_{jn}.$$

Proof. $P(\xi'_{k1} \in U_1) = p_k$, but also $P(\xi'_{k1} \in U_1) = \dfrac{\upsilon'(U_1)}{\upsilon'(U_k)}$.

Hence $\upsilon'(U_k) = \dfrac{1}{p_k} \upsilon'(U_1)$.

It follows that for $A \in \mathcal{U}$, $A \subset U_k$, $k \geq 1$

$$\upsilon'(A) = \upsilon'(U_k)\cdot P(\xi'_{k1} \in A) = \frac{\upsilon'(U_1)}{p_k}\frac{\upsilon(A)}{\upsilon(U_k)} = \upsilon'(U_1)\cdot\upsilon(A).$$

So $\upsilon' = c\upsilon$ on the ring $\cup \mathcal{B}(U_k)$, where $c = \upsilon'(U_1)$.

Finally for $j,n \geq 1$,

$$\tau'_{jn} = \lim_{k\to\infty} \frac{1}{\nu'(U_k)} S'_{kjn} = \frac{1}{c} \lim_{k\to\infty} \frac{1}{p_k} S_{kjn} = \frac{1}{c} t_{jn}. \square$$

1.3.10 <u>Corollary</u>. Let $(U_k, V_k)_{k\geq 1}$ be a nested array on the probability space $(\Omega, \mathcal{F}, P)$. If

$$\lim_{k\to\infty} p_k > 0$$

then for P-a.e. ω the limits

$$\xi_n(\omega) = \lim_{k\to\infty} V_{kn}(\omega)$$

exist for every $n \geq 1$. The sequence $(\xi_n)_{n\geq 1}$ is an i.i.d. sequence of random variables,

$$P(\xi_n \in A) = \frac{\nu(A)}{\nu(U)}, \quad A \in \mathcal{U}.$$

Further the (U_k)-subsequence of the sequence $(\xi_n)_{n\geq 1}$ is P-a.s. equal to $(V_{ki})_{i\geq 1}$.

<u>Proof</u>. Since $\lim_{k\to\infty} p_k > 0$, it follows from the proof of theorem (1.3.8) that $\lim_{k\to\infty} S_{kjn}(\omega)$ exists P-a.s. So for every $n \geq 1$, S_{k1n} is constant for k sufficiently large, say $k > K_n$. It follows that $(V_{k1}, \ldots, V_{kM_n})$ is constant for $k > K_n$, where $M_n = \lim_{k\to\infty} S_{k1n} \geq n$.
Define for $n \geq 1$: $\xi_n = \lim_{k\to\infty} V_{kn}$.
It is clear that the sequence $(\xi_n)_{n\geq 1}$ is an i.i.d. sequence of random variables. Let $A \in \mathcal{U}$, then

$$P(\xi_n \in A) = \lim_{k\to\infty} P(V_{kn} \in A) = \lim_{k\to\infty} \frac{\nu(A\cap U_k)}{\nu(U_k)} = \frac{\nu(A)}{\nu(U)}.$$

Finally, it is clear that the (U_k)-subsequence of the sequence $(\xi_n)_{n\geq 1}$ is P-a.s. equal to $(V_{ki})_{i\geq 1}$. $\square$

CHAPTER 2

EXCURSION THEORY

Let Y be a standard Markov process with state space S and let $a \in S$ be a given state, which is recurrent for Y. In [25], Itô defined the excursion process of Y with respect to P_a in the following way: let $S(t)$ be the inverse local time of Y at a. If t is such that $S(t-) < S(t)$, then the function u_t defined by

$$u_t(s) = \begin{cases} Y(S(t-)+s) & , 0 \leq s < S(t)-S(t-) \\ a & , s \geq S(t) - S(t-) \end{cases}$$

is called the excursion of Y in $]S(t-), S(t)[$. Itô proved that the random distribution of the points (t,u_t), $t \in \{s : S(s-) < S(s)\}$, is a Poisson point process on $[0,\infty[\times U$, where U denotes the space of all excursions. In the first part of this chapter we will study the excursions of a Ray process Y in the maximal components of $[0,\infty[\backslash$, where $Z = \{t \in [0,\infty[: Y_t = a \text{ or } Y_{t-} = a\}$. The strong Markov property implies that the sequence of excursions in the intervals of length greater than a given positive real number r, is an i.i.d. sequence with respect to P_a. Using the theorem of Greenwood & Pitman (see section (1.3)), one can construct the Itô-Poisson point process of excursions without the explicit introduction of local time; the characteristic measure υ is determined by the sub-Markov semigroup $(K_t)_{t \geq 0}$ defined by $K_t(x,dy) = P_x[Y_t \in dy;\ Y_s,\ Y_{s-} \neq a \text{ for all } s \leq t]$ and an entrance law for the semigroup (K_t). If the state a is regular for Y, the total mass of υ is $+\infty$.

In the last part of the chapter we construct stochastic processes from Itô-Poisson point processes. Let P be an Itô-Poisson point process with characteristic measure ν determined by the semigroup (K_t) and an entrance law $(\eta_s)_{s>0}$ for the semigroup (K_t) such that $\eta_s(\{a\}) = 0$ for every $s > 0$, such that

$$\int_U (1-e^{-\zeta_u})\nu(du) < \infty.$$

The stochastic process Y constructed by concatenation of the excursions of P, will turn out to have the simple Markov property: the proof of this property is based on an application of the renewal property of Itô-Poisson point processes and an application of the Palm formula. Of course, the assumptions about ν are necessary to get a Markov evolution of the process inside an excursion. We will give sufficient conditions for the process Y to be a Ray process. A simple calculation based on the Palm formula will give a formula for the resolvent of Y. Further we will give a formula for the Blumenthal-Getoor local time of Y at state a. Finally, we will give an example of the construction of a Markov process from a Cox point process.

2.1 Ray processes.

This section contains a summary of some important of properties of Ray processes. For proofs we refer to the books of Getoor [15] and Williams [59]. Let E denote a compact metric space with Borel σ-algebra $\mathcal{E}$. C(E) is the space of continuous functions on E.

A family $(R_\lambda)_{\lambda>0}$ of kernels on $(E,\mathcal{E})$ is called a Ray resolvent on E if

(i) $\lambda R_\lambda 1 \le 1$, $(\lambda > 0)$, (sub-Markov property),

(ii) $R_\lambda - R_\mu + (\lambda-\mu)R_\lambda R_\mu = 0$, $(\lambda,\mu > 0)$, (resolvent equation),

(iii) $R_\lambda(C(E)) \subset C(E)$, $(\lambda > 0)$,

(iv) $\bigcup_{\alpha\geq 0} CSM^\alpha$ separates the points of E, (Ray property),

where CSM^α is the family of continuous α-supermedian functions relative to (R_λ). There is a standard construction to change a Ray resolvent (R_λ) into a Markov Ray resolvent (i.e. $\lambda R_\lambda 1 = 1$ for every $\lambda > 0$) on a space

E', which arises from E by adjoining an isolated point. So we may suppose that (R_λ) is a Markov Ray resolvent on E. The construction of the Ray process with resolvent (R_λ) goes via a Markov semigroup $(P_t)_{t\geq 0}$ whose existence and uniqueness is guaranteed by a theorem of Ray.

2.1.1 Theorem (Ray). Let (R_λ) be a Markov Ray resolvent on a compact metric space E. Then there exists a unique Markov semigroup $(P_t)_{t\geq 0}$ satisfying for $f \in C(E)$ and $x \in E$

(i) $t \to P_t f(x)$ is right continuous on $[0,\infty[$,

(ii) $$R_\lambda f(x) = \int_0^\infty e^{-\lambda t} P_t f(x)dt \qquad (\lambda>0).$$

Proof. See Williams [59], p. 187. □

Let $(R_\lambda)_{\lambda>0}$ be a Markov Ray resolvent on E and let $(P_t)_{t\geq 0}$ be the semigroup determined by (R_λ). We continue with a brief description of the construction of the canonical realization of the Markov process with transition semigroup (P_t).

The sample space Ω is the space $D_{[0,\infty[}(E)$ of càdlàg functions from $[0,\infty[$ to E. Let $Y=(Y_t)_{t\geq 0}$ be the coordinate process on Ω:

$$Y_t(\omega)=\omega(t) \quad \text{for } \omega \in \Omega,\ t\geq 0.$$

Let $\mathcal{F}_t^o$ be the σ-algebra $\sigma(Y_s, 0\leq s\leq t)$ on Ω generated by the maps Y_s, $s\leq t$, and $\mathcal{F}^o = \sigma(Y_t, t\geq 0)$. For every probability μ on $(E,\mathcal{E})$, there exists a unique probability measure P_μ on $(\Omega,\mathcal{F}^o)$ such that for $0\leq t_1\leq \ldots\leq t_n$

$$P_\mu([Y_o \in dx_o,\ Y_{t_i} \in dx_i,\ i=1,\ldots,n])$$
$$= \int \mu(dx)\ P_o(x,dx_o)P_{t_1}(x_o,dx_1)P_{t_2-t_1}(x_1,dx_2)\ldots P_{t_n-t_{n-1}}(x_{n-1},dx_n).$$

Let $(\Omega, \mathcal{F}^\mu, \{\mathcal{F}_t^\mu\})$ be the usual P_μ-augmentation of $(\Omega, \mathcal{F}^o, \{\mathcal{F}_t^o\})$, that is $\mathcal{F}^\mu$ is the P_μ-completion of $\mathcal{F}^o$ and $\mathcal{F}_t^\mu$ is the σ-algebra generated by $\mathcal{F}_{t+}^o = \bigcap_{s>t} \mathcal{F}_s^o$ and the class of all P_μ-null sets in $\mathcal{F}^\mu$. Then $Y = (\Omega, \mathcal{F}^\mu, \{\mathcal{F}_t^\mu\}, (Y_t)_{t\geq 0}, P_\mu)$ is a strong Markov process, this means that for every $\{\mathcal{F}_t^\mu\}$-stopping time T and every bounded $\mathcal{F}^\mu$-measurable random variable η on Ω

$$E_\mu\, [\eta\circ\theta_T \cdot 1_{[T<\infty]} | \mathcal{F}_T^\mu] = E_{Y(T)}[\eta]\cdot 1_{[T<\infty]} \quad P_\mu\text{-a.s.},$$

see Getoor [15], p.24. Let D be the set of $x \in E$ such that $P_0(x,\cdot) = \delta_x$. The set D is a Borel subset of E. Points in $B = E\setminus D$ are called branchpoints.

$$P_\mu[Y_t \in D,\ \forall t \geq 0] = 1 \quad \text{for every } \mu.$$

So the paths $t \in [0,\infty[\to Y_t(\omega)$ are a.s. right continuous functions with values in D and left limits in E.

2.1.2 Theorem. Let μ be a probability measure on E and let $Y = (\Omega, \mathcal{F}^\mu, \{\mathcal{F}_t^\mu\}, (Y_t)_{t\geq 0}, P_\mu)$ we as above. Let (τ_n) be an increasing sequence of $\{\mathcal{F}_t^\mu\}$-stopping times. Let $\tau = \sup \tau_n$ and $\Lambda = [\tau < \infty\ ;\ \tau_n<\tau \text{ for all } n]$. If f is a bounded universally measurable function on E, then P_μ-a.s. we have

$$E_\mu[f\circ Y_\tau\cdot 1_{[\tau<\infty]} | \bigvee_{n\geq 1} \mathcal{F}_{\tau_n}^\mu] = f\circ Y_\tau\cdot 1_{[\tau<\infty]}\cdot 1_{\Lambda^*} + P_0 f(Y_{\tau-})\cdot 1_\Lambda.$$

Proof. See Getoor [15], p.25. □

$Y = (\Omega, \mathcal{F}^\mu, \{\mathcal{F}_t^\mu\}, (Y_t)_{t\geq 0}, P_\mu)$, where μ runs through the set of probability measures on $(E,\mathcal{E})$ will be called the canonical realization of the Ray process associated with the resolvent (R_λ). Define $\mathcal{F}$ (resp. $\mathcal{F}_t$) as the intersection of the σ-algebras $\mathcal{F}^\mu$ (resp. $\mathcal{F}_t^\mu$) where μ runs through the set of all probability measures on $(E,\mathcal{E})$. Y also has the strong Markov property with respect to the filtration $\{\mathcal{F}_t\}$.

2.2 Point processes of excursions of a Ray-process from a given state.

Let Y be the (canonical realization of the) Ray process with Ray resolvent $(R_\lambda)_{\lambda>0}$ on the compact metric state space E. We will use the notations of section (2.1). Let $a \in E$ be a given state. The polish space of càdlàg functions $f : [0,\infty[\to E$ will be denoted by U. See appendix A2. The Borel σ-algebra $\mathcal{U}$ on U is equal to the σ-algebra generated by the coordinate evaluations. For $f \in U$, let $Z(f)$ be the closed set of points at which f approaches or hits the state a:

$$Z(f) = \{t \in [0,\infty[\ : \ f(t) = a \text{ or } f(t-) = a\}.$$

The connected components of $[0,\infty[\backslash Z(f)$ are called excursion intervals from a of f. Let $I=]D,T[$ be an excursion interval of f. The map $V_I(f)\colon [0,\infty[\to E$ defined by

$$V_I(f)(t) = \begin{cases} f(D+t) & \text{if } 0 \leq t < T-D \\ a & \text{if } t \geq T-D \end{cases},$$

is called the excursion of the function f from point a on the excursion interval I. If it is clear from which point the excursions are considered, we will speak simply of excursions and excursion intervals of f. The map $\zeta : f \in U \to \zeta_f \in [0,\infty[$ defined by

$$\zeta_f = \inf(Z(f)\backslash\{0\})$$

is a lower semi-continuous function, see appendix A2. If $\inf(Z(f)\backslash\{0\}) > 0$, then ζ_f is the first time after zero at which f hits or approaches the state a. $\zeta(V_I(f)) = T-D$ is called the length (or the duration) of the excursion $V_I(f)$. We will now study the excursions of length greater than r, for a fixed positive r, of the realizations of the Ray process Y. Denote by $(]D_n(\omega), T_n(\omega)[)_{n\geq 1}$ the sequence of all excursion intervals of $Y(\omega)$ with length exceeding r, enumerated in such a way that $D_n(\omega) < D_{n+1}(\omega)$, $n \geq 1$. This sequence is at most countable, and it is also possible that there are only finitely many excursion intervals of length exceeding r. The excursion corresponding to the excursion interval $]D_n(\omega), T_n(\omega)[$ will be denoted by $V_n(\omega)$.
Note that $V_n : \omega \in \Omega \to V_n(\omega) \in U$ is a partially defined, U-valued random variable. Define $T_n(\omega) = +\infty$ if there are less than n excursion intervals of length exceeding r. Note that $T_n\colon \omega \in \Omega \to T_n(\omega) \in [0,\infty]$ is an $(\mathcal{F}_t)$-stopping time for each $n\geq 1$.

With the above introduced definitions and notations we have the following theorem.

2.2.1 Theorem. Let Y be a Ray process on E, $a \in E$ and $r > 0$.

(i) The sequence of excursion intervals from a of length greater than r is P_a-a.s. an infinite sequence if and only if $P_a[T_1 < \infty] = 1$.

(ii) If $P_a[T_1 < \infty] = 1$, then the sequence $(V_n)_{n\geq 1}$ of excursions from a of length greater than r is a sequence of independent, identically distributed U-valued random variables.

Proof. We start with the construction of a sequence of $(\mathcal{F}_t^o)$-stopping times which increases to T_1. Let $(\delta_n)_{n\geq 1}$ be a strictly decreasing sequence of positive real numbers, $\delta_n \downarrow 0$ $(n \to \infty)$. The sequence $(\tau_n)_{n\geq 1}$ defined by

$$\tau_n = \inf \{t : t > r,\ Y_t \in B_a(\delta_n) \text{ and } Y_s, Y_{s-} \neq a,\ s \in [t-r, t[\}$$

where $B_a(\delta_n)$ is the open ball in E with center a and radius δ_n, is an increasing sequence of $(\mathcal{F}_t^o)$-stopping times. Denote as in theorem (2.1.2)

$$\tau = \sup \tau_n \text{ and } \Lambda = [\tau < \infty;\ \tau_n < \tau,\ n \geq 1].$$

Let $\omega \in \Omega$ and let t be such that $Y_t(\omega)$ or $Y_{t-}(\omega) = a$ and $Y_s(\omega)$, $Y_{s-}(\omega) \neq a$ for all $s \in [t-r, t[$.

If $Y_t(\omega) = a$, then $\tau_n(\omega) \leq t$ for all $n \geq 1$.

If $Y_{t-}(\omega) = a$, then for every $n \geq 1$ there exists a positive real number ϵ_n such that $Y_r(\omega) \in B_a(\delta_n)$ for all $r \in]t-\epsilon_n, t[$. Since $Y_{(t-r)-}(\omega) \neq a$, there exists an $\eta > 0$ such that $Y_s(\omega), Y_{s-}(\omega) \neq a$ for all $s \in [t-r-\eta, t-r]$. Hence $\tau_n(\omega) \leq t - \min(\epsilon_n, \eta) \leq t$. Consequently in both cases $\tau_n(\omega) \leq t$ for all $n \geq 1$.

It follows that $\tau(\omega) \leq T_1(\omega)$. (1)

To prove the converse inequality $\tau(\omega) \geq T_1(\omega)$, we consider first the case that $\omega \in \Lambda$. Since $Y_{\tau_n}(\omega) \in \overline{B_a(\delta_n)}$, it follows that $Y_{\tau-}(\omega)=a$. Further, it is clear that $Y_s(\omega)$, $Y_{s-}(\omega) \neq a$ for all $s \in [\tau(\omega)-r, \tau(\omega)[$. Hence $\tau(\omega)$ is the right-hand endpoint of an excursion interval of length greater than r and it follows that $T_1(\omega) \leq \tau(\omega)$ for $\omega \in \Lambda$.

Suppose now that $\omega \notin \Lambda$ and $\tau(\omega) < \infty$. For k sufficiently large, $\tau_k(\omega)=\tau(\omega)$. This implies

$$Y_{\tau(\omega)}(\omega) = a \text{ and } \forall s \in [\tau(\omega)-r, \tau(\omega)[\ : Y_s(\omega), Y_{s-}(\omega) \neq a.$$

It follows that $T_1(\omega) \leq \tau(\omega)$. (2)

From (1) and (2) we may conclude that $T_1=\tau$.

To prove (i) let $A_n = [T_n < \infty]$ be the event that there are at least n excursions of length greater than r and let A be the event that there is an infinite sequence of excursions of length greater than r.

$$\begin{aligned} P_a(A_{n+1}) &= E_a[1_{[T_1<\infty]} \cdot 1_{A_n} \circ \theta_{T_1}] \\ &= E_a[1_{[T_1<\infty]} P_{Y(T_1)}(A_n)] \text{ (strong Markov property)} \\ &= E_a[\ a(1_{[T_1<\infty]} P_{Y(T_1)}(A_n) \mid \vee \mathcal{F}_{\tau_n})] \\ &= E_a[1_{[T_1<\infty]} P_{Y(T_1)}(A_n) 1_{\Lambda^*} + 1_\Lambda \cdot \int P_o(Y_{T_1-}, dy) P_y(A_n)] \end{aligned}$$

by an application of theorem (2.1.2). It is clear that $Y_{T_1} = a$ on $[T_1<\infty] \cap \Lambda^*$ and $Y_{T_1-} = a$ on Λ.

Hence

$$P_a(A_{n+1}) = E_a[1_{[T_1<\infty]} P_a(A_n)] = P_a(A_1) P_a(A_n)$$

and by mathematical induction

$$P_a(A_n) = (P_a[T_1<\infty])^n.$$

It follows that

$$P_a(A) = \lim_{n\to\infty} (P_a[T_1<\infty])^n = 1 \text{ iff } P_a[T_1<\infty] = 1.$$

This completes the proof of (i).

To prove (ii) let $n \geq 2$ and let $(f_i)_{1\leq i\leq n}$ be a finite sequence of bounded measurable functions on U. An application of the strong Markov property yields

$$E_a[\prod_{i=1}^{n} f_i(V_i)]$$
$$= E_a[f_1(V_1)\cdot f_2(V_1\circ \theta_{T_1}) \cdot \ldots \cdot f_n(V_{n-1} \circ \theta_{T_1})]$$
$$= E_a[f_1(V_1)\, E_{Y(T_1)}(f_2(V_1)\cdot \ldots \cdot f_n(V_{n-1}))] = (*).$$

Consider first the case that the state a is a branch point. Then

$$P_a(\Lambda^*) \le P_a[Y_\tau = a] = 0.$$

Hence $Y_{\tau-}= a$ (P_a-a.s.) and the sequence (τ_n) foretells $\tau=T_1$ and T_1 is predictable. It follows that $\mathcal{F}_{T_1-} = \vee\, \mathcal{F}_{\tau_n}$. An application of Galmarino's test (see Dellacherie & Meyer [7], p. 149) yields the $\mathcal{F}_{T_1-}$-measurability of $f_1(V_1)$. So

$$(*) = E_a[f_1(V_1)\, E_a(\ {Y(\tau)}[f_2(V_1)\cdot\ldots\cdot f_n(V_{n-1}) \mid \vee\, \mathcal{F}_{\tau_n}])]$$
$$= E_a[f_1(V_1) \int P_o(Y_{\tau-},dy)\, E_y(f_2(V_1)\cdot\ldots\cdot f_n(V_{n-1}))]$$
$$= E_a[f_1(V_1) \int P_o(a,dy)\, E_y(f_2(V_1)\cdot\ldots\cdot f_n(V_{n-1}))]$$
$$= E_a[f_1(V_1)]\, E_a[f_2(V_1)\cdot\ldots\cdot f_n(V_{n-1})].$$

If a is not a branch point, then by theorem (2.1.2)

$$P_a[Y(T_1)=a]$$
$$= E_a[1_{\{a\}}\circ Y_{T_1}\cdot 1_{\Lambda^*}] + E_a(1_\Lambda \int P_o(Y_{T_1-},dy) 1_{\{a\}}(y)) = 1.$$

So in both cases we have

$$E_a[\prod_{i=1}^{n} f_i(V_i)] = E_a f_1(V_1)\cdot E_a[f_2(V_1)\cdot \ldots \cdot f_n(V_{n-1})].$$

An induction argument completes the proof of (ii). □

Let Y, r, a, D_n, T_n and V_n be defined as in the beginning of the section. Define the maps

$$\rho_a : \Omega \to [0,\infty],\ \rho_a(\omega) = \sup\{t \ge 0 : Y_s(\omega)=a \text{ for every } s\le t\},$$
$$\sigma_a : \Omega \to [0,\infty],\ \sigma_a(\omega) = \inf\{t \ge 0 : Y_t(\omega)=a \text{ or } Y_{t-}(\omega)=a\},$$
$$\tau_a : \Omega \to [0,\infty],\ \tau_a(\omega) = \inf\{t > 0 : Y_t(\omega)=a \text{ or } Y_{t-}(\omega)=a\}.$$

The maps σ_a and τ_a are stopping times. The point a is called a holding point for Y if $P_a[\rho_a>0] > 0$. Define for $t\ge 0$ the kernel K_t on $(E,\mathcal{E})$ by

$$K_t(x,A) = P_x[Y_t \in A,\ \sigma_a > t],\ x \in E,\ A \in \mathcal{E}.$$

The family $(K_t)_{t\ge 0}$ is a sub-Markov semigroup on $(E,\mathcal{E})$.

Define for $s \geq r$ the measure e_s on $(E,\mathcal{E})$ by

$$e_s(A) = P_a[Y_{D_1+s} \in A,\ D_1+s < T_1 < \infty],\ A \in \mathcal{E}.$$

Let μ be the P_a-distribution of V_1; μ is a finite measure on $(U,\mathcal{U})$. With the above introduced notations and definitions we have the following lemma.

2.2.2 Lemma. (i) $e_s K_t = e_{s+t},\ s \geq r,\ t \geq 0$;

(ii) For $0 \leq t_1 \leq \dots \leq t_n$ and $x_1,\dots,x_n \in E\setminus\{a\}$

$$\mu[u(r+t_i) \in dx_i,\ i=1,\dots,n]$$
$$= e_{r+t_1}(dx_1)K_{t_2-t_1}(x_1,dx_2)\dots K_{t_n-t_{n-1}}(x_{n-1},dx_n);$$

(iii) $P_a[\tau_a=0] = 0$ or 1.

Proof. Note that D_1+s is a stopping time for $s \geq r$. Let $A \in \mathcal{E}$, $s \geq r$ and $t \geq 0$, then

$$e_s K_t(A) = \int P_a[Y_{D_1+s} \in dx,\ D_1+s<T_1<\infty]\ P_x[Y_t \in A,\ \sigma_a > t]$$
$$= E_a(1_{[D_1+s<T_1<\infty]} E_{X_{D_1+s}}[1_A(Y_t);\ \sigma_a > t])$$
$$= E_a[1_{[D_1+s<T_1<\infty]} 1_{[\sigma_a \circ \theta_{D_1+s} > t]} 1_A(Y_{D_1+s+t})]$$

(strong Markov property)

$$= e_{s+t}(A).$$

This completes the proof of the first part of the lemma. Let $0 \leq t_1 \leq \dots \leq t_n$ and $x_1,\dots,x_n \in E\setminus\{a\}$,

$$\mu[u(r+t_i) \in dx_i,\ i=1,\dots,n]$$
$$= P_a[D_1+r<T_1<\infty,\ \sigma_a \circ \theta_{D_1+r} > t_n,\ Y_{t_i} \circ \theta_{D_1+r} \in dx_i,\ i=1,\dots,n]$$
$$= \int e_r(dx)\ P_x[Y_{t_i} \in dx_i,\ i=1,\dots,n,\ \sigma_a > t_n],$$

by an application of the strong Markov property on stopping time D_1+r. A repeated application of the simple Markov property yields

$$P_x[Y_{t_i} \in dx_i,\ i=1,\dots,n,\ \sigma_a > t_n]$$
$$= K_{t_1}(x,dx_1)K_{t_2-t_1}(x_1,dx_2)\dots K_{t_n-t_{n-1}}(x_{n-1},dx_n).$$

An application of statement (i) completes the proof of (ii).

The third statement is a consequence of Blumenthal's 01 law, see Williams [59], p. 126. □

Let X be the product T × U of the halfline $T = [0,\infty[$ with the usual topology and the polish space U of càdlàg functions $f : [0,\infty[\to E$.
Let $a \in E$ and let $\mathcal{S}$ be the family of open subsets of X defined by

$\mathcal{S} = \{A \subset X : A = I \times [\zeta > t],\ I \subset T \text{ open and bounded},\ t > 0\}$,

where ζ is the first time after zero at which f hits or approaches a. The elements of $U_\infty = \{u : u \in U,\ \zeta_u > 0\}$ will be called excursions. Let Y, σ_a and K_t be defined as above and let $\bar{Y}$ be the process defined by

$$\bar{Y}_t = \begin{cases} Y_t & \text{if } t < \sigma_a \\ a & \text{if } t \geq \sigma_a \end{cases}.$$

Denote by α_x, $x \in E$, the P_x-distribution of $\bar{Y}$. Then α_x is a probability measure on U and the finite-dimensional distributions of α_x are given by

$$\alpha_x([u(t_i) \in dx_i,\ i=1,\dots,n])$$
$$= K_{t_1}(x,dx_1)K_{t_2-t_1}(x_1,dx_2)\dots K_{t_n-t_{n-1}}(x_{n-1},dx_n),$$

where $0<t_1<\dots< t_n$ and $x_1,\dots,x_n \in E\setminus\{a\}$. The shift operator on U will also be denoted by θ_t, so for $t\geq 0$ and $u \in U$, $\theta_t u$ is the element of U defined by

$$(\theta_t u)(s) = u(s+t),\ s\geq 0.$$

With the above introduced notations and definitions we have the following theorem.

2.2.3 Theorem. Let Y be a Ray process and $a \in E$, which is not a holding point for Y and let $P_a[T_1^r < \infty] = 1$ for some $r > 0$ where T_1^r denotes the endpoint of the first excursion of length greater than r.

- If $P_a[\tau_a=0] = 0$ there exists an i.i.d. sequence $(\xi_n)_{n\geq 1}$ on $(\Omega,\mathcal{F},P_a)$ of U_∞-valued random variables whose $[\zeta>l]$-subsequence is the sequence of excursions of Y of length exceeding l.

- If $P_a[\tau_a=0]=1$ there exists an $\mathcal{S}$-finite Itô-Poisson point process N defined on $(\Omega,\mathcal{F},P_a)$ whose $[\zeta > l]$-subsequence is the sequence of excursions of Y of length exceeding l.

The characteristic measure ν of N is a σ-finite measure on U_∞ having the following properties,

(i) ν is concentrated on $\{u : u \in U_\infty \text{ and } u(s)=a \text{ for every } s \geq \zeta_u\}$,

(ii) for each $f \in b\mathcal{U}_\infty$, $t > 0$ and $\Lambda \in \sigma(u \in U_\infty \to u(r),\ r \leq t)$ we have

$$\int_{\Lambda\cap[\zeta>t]} f(\theta_t u)\nu(du) = \int_{\Lambda\cap[\zeta>t]} \alpha_{u(t)}(f)\ \nu(du),$$

(iii) $\nu([\zeta > t]) < \infty$ for every $t > 0$.

Proof. Let $(r_k)_{k\geq 1}$ be a strictly decreasing sequence of positive real numbers, such that $\lim_{k\to\infty} r_k=0$ and $P_a[T_{11}<\infty]=1$, where T_{11} is the endpoint of the first excursion of length exceeding r_1. For $k,n = 1,2,\ldots$ denote by $]D_{kn}(\omega),\ T_{kn}(\omega)[$ resp. $V_{kn}(\omega)$ the n^{th} excursion interval resp. excursion of the realization $Y_\cdot(\omega)$ with length exceeding r_k. Since $T_{k1} \leq T_{11}$, it follows that $P_a[T_{k1}<\infty]=1$ and the sequence $(V_{kn})_{n\geq 1}$ of excursions of length exceeding r_k is an i.i.d. sequence of U-valued random variables, see theorem (2.2.1).

Define for $k=1,2,\ldots$

$$U_k = \{u : u \in U,\ \zeta_u > r_k\},$$
$$V_k = (V_{k1},\ V_{k2},\ldots),$$
$$\nu_k = V_{k1}(P_a) \qquad \text{and}$$
$$p_k = \nu_k(U_1).$$

Then $(U_k,V_k)_{k\geq 1}$ is a nested array on $(\Omega,\mathcal{F},P_a)$, see section (1.3) for the definition of a nested array.

If $P_a[\tau_a=0]=0$, then $\lim_{k\to\infty} p_k > 0$.

Indeed, since $\lim_{k\to\infty} 1_{U_1}(V_{k1}(\omega)) = 1_{[\tau_a>r_1]}(\omega)$ on $[\tau_a>0]$, we have

$$\lim_{k\to\infty} p_k = \lim_{k\to\infty} P_a[V_{k1} \in U_1] = P_a[\tau_a > r_1] > 0.$$

By corollary (1.3.10) there exists an i.i.d. sequence $(\xi_n)_{n\geq 1}$ of random variables whose U_k-subsequence is P_a-a.s. equal to $(V_{kn})_{n\geq 1}$.

If $P_a[\tau_a=0]=1$, then $\lim_{k\to\infty} p_k = 0$.

Indeed, since a is not a holding point for Y, we have

$$P_a(\{\omega : \exists K_\omega, \forall k \geq K_\omega, V_{k1} \notin U_1\}) = 1$$

which implies that $\lim_{k\to\infty} p_k = \lim_{k\to\infty} P_a[V_{k1} \in U_1] = 0$.

It follows from theorem (1.3.8) that P_a-a.s. the limits

$$t_{jn} = \lim_{k\to\infty} p_k S_{kjn} \qquad (n, j=1,2,\dots)$$

exist, where for $k>j$ S_{kjn} denotes the index of the n^{th} excursion of length greater than r_j in the sequence $(V_{kn})_{n\geq 1}$ of excursions of length greater than r_k. The P_a-a.s. defined random variable

$$N : \omega \in \Omega \to \sum_{j=1}^{\infty} \sum_{n: V_{jn}(\omega)\notin U_{j-1}} \delta_{(t_{jn}(\omega), V_{jn}(\omega))}$$

is an $\mathcal{G}$-finite Itô-Poisson point process on U_∞ with characteristic measure ν determined by

$$\nu(U_1)=1 \quad \text{and} \quad \nu|_{U_j} = \nu_j.$$

Further, the U_k-subsequence of N is the sequence $(V_{kn})_{n\geq 1}$ which implies that for every $l>0$ the $[\zeta>l]$-subsequence of N is the sequence of excursions of Y of length exceeding l.

If $(r'_k)_{k\geq 1}$ is another strictly decreasing sequence of positive real numbers with the same properties as the sequence $(r_k)_{k\geq 1}$, then the Itô-Poisson point process N' constructed as above starting from the sequence (r'_k) has a characteristic measure ν' which differs only by a multiplicative constant from ν, see corollary (1.3.9).

We continue with a proof of property (ii).

Fix $t > 0$ and choose k such that $r_k \leq t$.

Let for $0 \leq t_1 < \dots < t_l \leq t$, $0 \leq s_1 < \dots < s_n$, $A_1,\dots,A_l \in \mathcal{E}$ and $f_1,\dots,f_n \in b\mathcal{E}$

$$\Lambda = [u(t_i) \in A_i,\ i=1,\dots,l]$$

$$f(u) = \prod_{j=1}^{n} f_j(u(s_j)), \quad u \in U.$$

Then by definition of ν

$$\int_{\Lambda\cap[\zeta>t]} f(\theta_t u)\ \nu(du)$$

$$= \frac{1}{\nu_k(U_1)} \int_{\Lambda\cap[\zeta>t]} f(\theta_t u)\nu_k(du)$$

$$= \frac{1}{P_a[T_{k1}>D_{k1}+r_1]} E_a[\prod_i 1_{A_i}(V_{k1}(t_i))\cdot 1_{[\zeta(V_{k1})>t]}\cdot \prod_j f_j(V_{k1}(s_j+t))]$$

$$= \frac{1}{P_a[T_{k1}>D_{k1}+r_1]} E_a[\prod_i 1_{A_i}(Y(D_{k1}+t_i))\cdot 1_{[D_{k1}+t<T_{k1}]} \prod_j f_j(\bar{Y}(s_j))\circ\theta_{D_{k1}+t}]$$

$$= \frac{1}{P_a[T_{k1}>D_{k1}+r_1]} E_a[\prod_i 1_{A_i}(Y(D_{k1}+t_i))\cdot 1_{[D_{k1}+t<T_{k1}]}\alpha_{Y(D_{k1}+t)}(f)]$$

(by an application of the strong Markov property)

$$= \int_{\Lambda\cap[\zeta>t]} \alpha_{u(t)}(f)\ \nu(du).$$

A standard monotone class argument completes the proof of (ii).

For the proof of (i) first note that the set

$$W = \{u : u \in U_\infty \text{ and } u(s) = a \text{ for every } s \geq \zeta_u\}$$

is a measurable subset of U_∞. It is clear that $\alpha_x(1_W) = 0$ for every $x \in E$. So (ii) implies

$$\int_{[\zeta>t]} 1_W(\theta_t u)\ \nu(du) = \int_{[\zeta>t]} \alpha_{u(t)}(1_W)\ \nu(du) = 0.$$

On the other hand, if $\zeta_u > t$ then $1_W(\theta_t u) = 1_W(u)$.

Hence

$$\int_{U_\infty} 1_W(u)\ \nu(du) = \lim_{t\downarrow 0} \int_{[\zeta>t]} 1_W(u)\ \nu(du)$$

$$= \lim_{t\downarrow 0} \int_{[\zeta>t]} 1_W(\theta_t u)\ \nu(du) = 0,$$

which completes the proof of (i). Finally, $\nu([\zeta>t]) = \eta_t(E) < \infty$ for every $t > 0$. □

A family of finite measures $(\epsilon_s)_{s>0}$ on $(E,\mathcal{E})$ will be called an entrance law for the (sub-Markov) semigroup $(K_t)_{t\geq 0}$ whenever

$$\epsilon_s K_t = \epsilon_{s+t} \text{ for all } s > 0,\ t \geq 0.$$

2.2.4 Theorem. There is a one-to-one correspondence between σ-finite measures m on $(U_\infty, \mathcal{U}_\infty)$ satisfying properties (i), (ii) and (iii) of theorem (2.2.3) and entrance laws $(\epsilon_s)_{s>0}$ for the semi-group (K_t) satisfying $\epsilon_s(\{a\})=0$ for every $s > 0$.

Proof. Let us first assume that m is a σ-finite measure on $(U_\infty, \mathcal{U}_\infty)$ satisfying properties (i), (ii) and (iii) of theorem (2.2.3).

Define for $s > 0$ and $A \in \mathcal{E}$

$$\epsilon_s(A) = m(\{u : u \in U_\infty,\ u(s) \in A,\ \zeta_u > s\}).$$

$(\epsilon_s)_{s>0}$ is a family of finite measures on $(E, \mathcal{E})$ satisfying $\epsilon_s(\{a\})=0$, $s>0$. For $A \in \mathcal{E}$, denoting $A\setminus\{a\}$ by A',

$$\begin{aligned}
\epsilon_s K_t(A) &= \int m(u(s)\in dx,\ \zeta_u>s)\ E_x[1_{A'}(Y_t);\ \sigma_a>t] \\
&= \int_{[\zeta_u>s]} m(du)\ E_{u(s)}[1_{A'}(Y_t)] \\
&= \int_{[\zeta_u>s]} m(du)\ \alpha_{u(s)}[1_{A'}(u(t))] \\
&= \int_{[\zeta_u>s]} m(du)\ 1_{A'}(u(s+t)) \qquad \text{(by property (ii))} \\
&= \int_{[\zeta_u>s+t]} m(du)\ 1_{A'}(u(s+t)) \ \text{(by property (i))} \\
&= \epsilon_{s+t}(A') = \epsilon_{s+t}(A).
\end{aligned}$$

So $(\epsilon_s)_{s>0}$ is an entrance law, which proves the first half of the theorem.

To prove the second half of the theorem, let $(\epsilon_s)_{s>0}$ be an entrance law for the semigroup (K_t) satisfying $\epsilon_s(\{a\})=0$, $s > 0$.

Define for $t > 0$

$$\mathcal{G}_t = \{B : B \subset U_\infty \text{ and } B = [\zeta>t] \cap \theta_t^{-1}(A) \text{ for some } A \in \mathcal{U}_\infty\}$$

where $\mathcal{U}_\infty$ denotes the Borel σ-algebra on U_∞.

The sets

$$\{u : u \in U_\infty,\ \zeta_u > t,\ u(t+s_i) \in F_i,\ i=1,\dots,n\}$$

where $n\geq 1$, $0\leq s\leq \dots \leq s_n$ and $F_1,\dots,F_n \in \mathcal{E}$, generate the σ-algebra $\mathcal{F}_t$.

If $r<t$, then $\mathcal{U}_\infty \supset \mathcal{G}_r \supset \mathcal{G}_t$. Indeed, let $B \in \mathcal{G}_t$, say $B = [\zeta > t] \cap \theta_t^{-1}(A)$

with $A \in \mathcal{U}_\infty$. Then $B = [\zeta > r] \cap \theta_r^{-1}([\zeta>t-r] \cap \theta_{t-r}^{-1}(A)) \in \mathcal{G}_r$. Note that $\bigcup_{t>0} \mathcal{G}_t$ is a ring generating $\mathcal{U}_\infty$. The setfunction μ_t defined on $\mathcal{G}_t$ by

$$\mu_t([\zeta>t] \cap \theta_t^{-1}(A)) = \int \epsilon_t(dx)\alpha_x(A), \quad A \in \mathcal{U}_\infty,$$

is a finite measure on the ring $\mathcal{G}_t$, whilst for $r<t$

$$\begin{aligned}
\mu_r([\zeta>t]\cap\theta_t^{-1}A) &= \int \epsilon_r(dx)\ \alpha_x[\zeta>t-r,\ 1_A \circ \theta_{t-r}] \\
&= \int \epsilon_r(dx)\ E_x[1_{[\sigma>t-r]}\cdot 1_A \circ Y(\theta_{t-r})] \\
&= \int \epsilon_r(dx) \int K_{t-r}(x,dy)\alpha_y(A) \text{ (simple Markov property)} \\
&= \int \epsilon_t(dy)\ \alpha_y(A) \\
&= \mu_r([\zeta>t]\cap\theta_t^{-1}A).
\end{aligned}$$

It follows that we can define a setfunction μ on $\bigcup_{t>0} \mathcal{G}_t$ by putting

$$\mu(B) = \mu_t(B) \text{ if } B \in \mathcal{G}_t.$$

μ is a σ-finite measure on the ring $\bigcup_{t>o} \mathcal{G}_t$, and has a unique extension to a σ-finite measure on $(U_\infty,\mathcal{U}_\infty)$, see Halmos [18]. From standard monotone class arguments it follows that for $f \in b\mathcal{U}_\infty$ and $t>0$

$$\int_{[\zeta>t]} \mu(du)\ f(\theta_t u) = \int \epsilon_t(dx)\ \alpha_x(f). \tag{$*$}$$

Fix $t > 0$ and define

$$A = [u(r_i) \in A_i,\ i=1,\dots,l]$$

where $0\le r_1<\dots<r_l\le t$ and $A_1,\dots,A_l \in \mathcal{E}$.

For $0<r<r_1$ and $f \in b\mathcal{U}_\infty$

$$\begin{aligned}
&\int_{A\cap[\zeta>t]} \mu(du)\ f(\theta_t u) \\
&= \int \epsilon_r(dx) \int \alpha_x(du)[\prod_{i=1}^{l} 1_{[u(r_i-r)\in A_i]}(u)]\ 1_{[\zeta>t-r]}(u) f(\theta_{t-r}u) \\
&= \int \epsilon_r(dx) \int \alpha_x(du)[\prod_{i=1}^{l} 1_{[u(r_i-r)\in A_i]}(u)]\ 1_{[\zeta>t-r]}(u)\alpha_{u(t-r)}(f)
\end{aligned}$$

by an application of the (simple) Markov property

$$= \int_{[\zeta>r]} \mu(du)[\prod_{i=1}^{l} 1_{[u(r_i+r)\in A_i]}(\theta_r u)]\ 1_{[\zeta>t-r]}(\theta_r u)\ \alpha_{\theta_r u(t-r)}(f)$$

by $(*)$

$$= \int_{A\cap[\zeta>t]} \mu(du)\ \alpha_{u(t)}(f).$$

This proves the formula

$$\int_{\Lambda\cap[\zeta>t]} f(\theta_t u)\nu(du) = \int_{\Lambda\cap[\zeta>t]} \alpha_{u(t)}(f)\nu(du)$$

for elementary sets Λ. From a standard monotone class argument follows that this formula is true for all $\Lambda \in \sigma(u \in U_\infty \to u(r),\ r \leq t)$. As in the proof of theorem (2.2.3) it now follows that ν is concentrated on the set $\{u : u \in U_\infty \text{ and } u(s) = a \text{ for all } s \geq \zeta_u\}$. From the definition of ν it is clear that $\nu([\zeta > t]) < \infty$ for all $t > 0$, which completes the proof .□

2.2.5 <u>Remark</u>. In theorem (2.2.3) the excursions of Y are considered on the probability space $(\Omega,\mathcal{F},P_a)$. It is clear that on a probability space $(\Omega,\mathcal{F},P_x)$ we have to add to the Itô-Poisson point process (or to the i.i.d. sequence $(\xi_n)_{n\geq 1}$) a first excursion describing the process Y up to time σ_a. Let $x \in E$, $x \neq a$. The map $W : \Omega \to U$ defined by

$$(W\omega)(t) = \begin{cases} Y_t(\omega) & \text{for } t < \sigma_a(\omega) \\ a & \text{for } t \geq \sigma_a(\omega) \end{cases}$$

is a U_∞-valued random variable, describing the process Y up to time σ_a. As in the proof of theorem (2.2.3), let $(r_k)_{k\geq 1}$ be a strictly decreasing sequence of positive real numbers, $r_k \downarrow 0$ as $k \to \infty$. For $k,n = 1,2,\ldots$ $V^x_{kn}(\omega)$ denotes the n^{th} excursion from a with length exceeding r_k of the realization $Y_\bullet(\theta_{\sigma_a}\omega)$. So $V^x_{kn}(\omega) = V_{kn}(\theta_a\omega)$.

Define the vector V^x_k by $V^x_k = (V^x_{k1}, V^x_{k2},\ldots)$.

As in lemma (2.2.2) we can prove that for every $x \in E\backslash\{a\}$ the sequence $(V^x_k, U_k)_{k\geq 1}$ is a nested array on $(\Omega,\mathcal{F},P_x)$ and

$$P_x[W\in B,\ V^x_{ki} \in A_i,\ i=1,\ldots,n] = P_x[W\in B]\ P_a[V_{ki}\in A_i, i=1,\ldots,n]$$

for every $n \geq 1$, $B, A_1,\ldots, A_n \in \mathcal{U}$.

When $P_a[\tau_a = 0] = 1$, we define the point processes N^x and Q^x by

$N^x : \omega \in \Omega \to N(\theta_{\sigma_a}\omega)$, N as in theorem (2.2.3),

and

$Q^x : \omega \in \Omega \to \delta_{(0,W(\omega))}$.

It follows that the point processes N^x and Q^x defined on $(\Omega,\mathcal{F},P_x)$ are independent and as in theorem (2.2.3) the point process N^x is an Itô-Poisson point process with the same characteristic measure υ as N.

2.2.6 Remark. If state a is a holding point for Y, then there exists an i.i.d. sequence $(\xi_n)_{n\geq 1}$ on $(\Omega,\mathcal{F},P_a)$ of U_∞-valued random variables whose $[\zeta>l]$-subsequence is the sequence of excursions of Y of length greater than l. Between two consecutive excursions the process Y remains in the state a. These time intervals are exponentially distributed.

2.3 Construction of stochastic processes from Itô-Poisson point processes.

We will start this section with a list of the notations and definitions used throughout this section.

2.3.1 Notations and definitions.

As in section (2.2) (E,ρ) will denote a compact metric space with Borel σ-algebra $\mathcal{E}$ and $a \in E$ will be a given point of E. The space of càdlàg functions defined on $T=[0,\infty[$ with values in E will be denoted by U. Endowed with the Skorohod topology, U is a polish space. The Borel σ-algebra on U will be denoted by $\mathcal{U}$ and this σ-algebra is generated by the coordinate evaluations on U. The map $\zeta : U \to [0,\infty]$ is defined by

$$\zeta(u) = \zeta_u = \inf \{t : t > 0, u(t) = a \text{ or } u(t-) = a\}.$$

For $u \in U_\infty = [\zeta > 0]$ the number ζ_u is the first "time" after zero that u hits or approaches a. The map ζ is lower semi-continuous. On the space $(E,\mathcal{E})$ there will be given a Markov semigroup of kernels $(\bar{P}_t)_{t\geq 0}$ such that for every $x \in E$ there exists a probability measure α_x on $(U,\mathcal{U})$ which is concentrated on the set $\{u\in U : u(t) = a \text{ for all } t \geq \zeta_u\}$ and which has finite-dimensional distributions given by

$\alpha_x[u(t_i) \in dx_i,\ i=1,\dots,n]$

$$= \bar{P}_{t_1}(x,dx_1)\bar{P}_{t_2-t_1}(x_1,dx_2) \dots \bar{P}_{t_n-t_{n-1}}(x_{n-1},dx_n)$$

where $0 \le t_1 \le \dots \le t_n$ and $x_1,\dots,x_n \in E$.

For $t \ge 0$ the kernel K_t on $(E,\mathcal{E})$ is defined by

$$K_t(x,dy) = \alpha_x[u(t) \in dy,\ \zeta_u > t].$$

The family $(K_t)_{t\ge0}$ is a sub-Markov semigroup of kernels on $(E,\mathcal{E})$. On $(E,\mathcal{E})$ there will be given also a family of finite measures $(\eta_s)_{s>0}$ which is an entrance law for the semigroup $(K_t)_{t\ge0}$ with $\eta_s(\{a\}) = 0$ for every $s>0$. By theorem (2.2.4) there is a unique measure on $(U_\infty,\ \mathcal{U}_\infty)$, which will be denoted by ν, satisfying the three properties

(i) ν is concentrated on $\{u \in U_\infty : u(s) = a \text{ for every } s \ge \zeta_u\}$,

(ii) for each $f \in b\mathcal{U}_\infty$, $t > 0$ and $\Lambda \in \sigma(u \in U_\infty \to u(r),\ r \le t)$ we have

$$\int_{\Lambda\cap[\zeta>t]} f(\theta_t u)\ \nu(du) = \int_{\Lambda\cap[\zeta>t]} \alpha_{u(t)}(f)\ \nu(du),$$

(iii) $\nu([\zeta > t]) < \infty$ for every $t > 0$,

such that

$\eta_s(dx) = \nu([\zeta_u > s,\ u(s) \in dx]),\ s > 0.$

We will always consider ν as a measure on U by putting $\nu(U_\infty^*) = 0$. The product topological space $T \times U$, where T is equipped with the usual topology will be denoted by X and $\mathcal{M}^+$ is the space of nonnegative Borel measures on $(X,\ \mathcal{B}(X))$ which are finite on the family $\mathcal{S}$ consisting of the subsets $I \times [\zeta > t]$ where I runs through the bounded subintervals of T and $t > 0$. The $\mathcal{S}$-finite Itô-Poisson point process on U with characteristic measure ν will be denoted by P. (The existence and unicity of P follows from proposition (1.2.2).) By proposition (1.3.1) $P(\mathcal{M}_1') = 1$, where $\mathcal{M}_1'$ is the set of $\mathcal{S}$-finite point measures μ such that $\mu(\{t\} \times U) \le 1$ for every $t \ge 0$.

For $\mu \in \mathcal{M}'$, it is clear that $\text{supp}(\mu)$ is a countable set. If $\mu \in \mathcal{M}_1'$ this set can be considered as an ordered subset $(u_\sigma)_{\sigma \in J(\mu)}$ of U where $J(\mu)$ denotes the projection on T of $\text{supp}(\mu)$ and where $u_\sigma = u$ iff

$(\sigma,u) \in \mathrm{supp}(\mu)$. Let $L : U \to [0,\infty[$ be a given, measurable function on U. Define for $\sigma \in T$ and $\mu \in \mathcal{M}_1^{\cdot}$

$$B(\sigma,\mu) = \Sigma\ \{L(u_\tau) : \tau \in J(\mu) \text{ and } \tau < \sigma\}$$
$$= \int \mu(d\tau dv)\ 1_{[0,\sigma]}(\tau)L(v)$$

and

$$C(\mu) = \bigcup_{\sigma \in J(\mu)} [B(\sigma-,\mu),\ B(\sigma,\mu)[.$$

If $T = C(\mu)$ then denote by $\tilde{\mu}$ the concatenation of the functions $u_\sigma|[0,L(u_\sigma)[$, $\sigma \in J(\mu)$, that is

$$\tilde{\mu} : T \to E\ ,$$
$$\tilde{\mu}(s) = u_\sigma(s - B(\sigma-,\mu)) = \int \mu(d\tau dv)(v \cdot 1_{[0,L(v)[})(s-B(\tau-,\mu))$$

where $\sigma \in J(\mu)$ such that $s \in [B(\sigma-,\mu),\ B(\sigma,\mu)[$; the function u_σ is called the excursion straddling s.

It follows that for an $\mathcal{S}$-finite point process Q with phase space X such that

$$Q(\{\mu \in \mathcal{M}_1^{\cdot} : T = C(\mu)\}) = 1$$

the maps Y_s, $s \geq 0$, defined by

$$Y_s : \mu \in \{\mu \in \mathcal{M}_1^{\cdot} : T = C(\mu)\} \to \tilde{\mu}(s)$$

are Q-a.e. defined random variables on the probability space $(\mathcal{M}^+,\mathcal{B}(\mathcal{M}^+),Q)$.

In what follows we want to consider a construction of this kind for the Itô-Poisson point process P. With an extra assumption about the characteristic measure ν of P it will turn out that the process $Y = (Y_t)_{t\geq 0}$ is a Markov process. So it is natural to consider a family of probability measures $(P_x)_{x\in E}$ on $(\mathcal{M}^+,\mathcal{B}(\mathcal{M}^+))$, where P_x is the probability distribution of the process Y starting in x. So we have to add a first point to P corresponding to a start from x taking in account the given transition probabilities $(\bar{P}_t)$.

Consider the measurable map

$$u \in U \to \delta_{(0,u)} \in \mathcal{M}^+.$$

For $x \neq a$, Q_x denotes the image of the probability measure α_x under this map. Q_x is a point process with phase space X. The intensity measure i_{Q_x}, the Palm measures $(Q_x)_{(\tau,v)}$ and the Laplace transform $\hat{Q}_x$ are given by

$$i_{Q_x} = \delta_0 \otimes \alpha_x,$$

$$(Q_x)_{(\tau,v)} = \delta_{\delta_{(\tau,v)}}, \qquad (\tau,v) \in X,$$

$$\hat{Q}_x(f) = \int \alpha_x(du)\, e^{-f(0,u)}, \qquad f \in \mathcal{B}(X)_+.$$

Define

$$P_x = \begin{cases} Q_x * P & \text{if } x \in E\setminus\{a\} \\ P & \text{if } x = a \end{cases},$$

$(P_x)_{x \in E}$ is a family of point processes on X. The most important properties of the point processes P_x are collected in the following lemma, whose straightforward proof is deleted.

2.3.2 Lemma. For $x \neq a$, the intensity measure i_{P_x}, the Palm measures $(P_x)_{(\tau,v)}$ and the Laplace transform $\hat{P}_x$ of the point process P_x are given by

$$i_{P_x} = i_{Q_x} + i_P,$$

$$(P_x)_{(\tau,v)} = \begin{cases} P_x * \delta_{\delta_{(\tau,v)}} & \text{for } v \in U \text{ and } \tau > 0 \\ \delta_{\delta_{(0,v)}} * P & \text{for } v \in U \text{ and } \tau = 0 \end{cases},$$

$$(\hat{P}_x)(f) = \int \alpha_x(du)\, e^{-f(0,u)} \exp\left[-\int (1-e^{-f})\, d\lambda \otimes \nu\right], \quad f \in \mathcal{B}(X)_+.$$

Further

$$P_x(\mathcal{M}_1') = 1.$$

Let $\Omega = \mathcal{M}_1'$ and $\mathcal{F}$ the trace of $\mathcal{B}(\mathcal{M}^+)$ on Ω. Our basic family of probability spaces will be $(\Omega, \mathcal{F}, P_x)$, $x \in E$.

Define for $\omega \in \Omega$ and $\tau \geq 0$

$$A(\tau,\omega) = \int_X \omega(d\sigma du)\, 1_{]0,\tau]}(\sigma) \zeta_u.$$

The random variable $A(\tau)$ is the sum of the lengths of the excursions up to and including time τ, leaving out the excursion at time 0. As a function of τ, $A(\tau,\omega)$ is a non-decreasing càdlàg function on $[0,\infty[$ for every $\omega \in \Omega$. The Laplace transform of the random variable $A(\tau)$ is given by:

$$\int_\Omega e^{-\lambda A(\tau)}\, dP_x = \exp\left[-\tau \int_U (1-e^{-\lambda\zeta_u})\nu(du)\right],\ \lambda > 0,\ x \in E.$$

From now on we will assume that

$$\int_U (1-e^{-\zeta_u})\nu(du) < \infty.$$

Then $A(\tau)$ is P_x-a.s. finite for every $x \in E$. Note that the family of random variables $(A(\tau))_{\tau\geq 0}$ is a subordinator whose Levy measure is the image $\zeta(\nu)$ of the measure ν under the map ζ. See for subordinators Itô [24]. Addition of a linear term $\gamma\tau$, $\gamma \geq 0$, to $A(\tau)$ gives us the general form of a subordinator with Levy measure $\zeta(\nu)$. Define for $\tau \geq 0$ and $\omega \in \Omega$

$$\sigma_a(\omega) = \int \omega(d\sigma du)\ 1_{\{0\}}(\sigma)\zeta_u$$

and

$$B(\tau,\omega) = \sigma_a(\omega) + A(\tau,\omega) + \gamma\tau.$$

It follows from a straightforward calculation that the Laplace transform of the random variable $B(\tau)$ is given by

$$\int e^{-\lambda B(\tau)} dP_x = \int e^{-\lambda\sigma_a} dP_x \cdot \exp[-\tau(\lambda\gamma + \int (1-e^{-\lambda\zeta_u})\nu(du))].$$

For $\omega \in \Omega$, denote by $R(\omega)$ the range of $B(\cdot,\omega)$:

$$R(\omega) = \{s \in [0,\infty[\ :\ \exists\tau\ :\ s = B(\tau,\omega)\}$$

and let $\varphi(\cdot,\omega)$ be the right continuous inverse of $B(\cdot,\omega)$:

$$\varphi(s,\omega) = \inf\ \{\tau\ :\ B(\tau,\omega) > s\},\quad s \geq 0.$$

It follows from the definition of φ that

$$B(\varphi(s,\omega)-,\omega) \leq s \leq B(\varphi(s,\omega),\omega)$$

for every $s \geq 0$, where $B(0-,\omega) = 0$. Let $J(\omega)$ be the projection of the support of the measure ω on T:

$$J(\omega) = \{\sigma \in T\ :\ \omega(\{\sigma\} \times U) = 1\},\quad \omega \in \Omega.$$

Note that $J(\omega)$ is P_x-a.s. a discrete subset of T if $\nu(U) < \infty$, and a countable, dense subset of T if $\nu(U) = +\infty$.

Define for $\omega \in \Omega$

$$C(\omega) = \bigcup_{\sigma \in J(\omega)} [B(\sigma-,\omega),\ B(\sigma,\omega)[.$$

With the above introduced definitions and notations we have the following lemma.

2.3.3 Lemma. Let $x \in E$. Then P_x-a.s.

$$T = \begin{cases} R(\omega) + C(\omega) & \text{if } \nu(U) = +\infty \text{ or } \gamma > 0 \\ \\ C(\omega) & \text{if } 0 < \nu(U) < \infty \text{ and } \gamma = 0 \end{cases} .$$

where the union is a union of disjoint sets.

Proof: Assume that $\nu(U) = +\infty$ or $\gamma > 0$. It is clear that the function $B(\cdot,\omega)$, is P_x-a.s. a strictly increasing function, $B(\tau,\omega) \uparrow \infty$ as $\tau \to \infty$. The assertion in the lemma follows from appendix (A3.3).

We continue with the case $0 < \nu(U) < \infty$ and $\gamma = 0$. In this case $J(\omega)$ is P_x-a.s. a discrete set, which can be written as $J(\omega) = (\sigma_n(\omega))_{n\geq 1}$ with $\sigma_1(\omega) < \sigma_2(\omega) < \ldots$ Further $\sigma_n(\omega) \to \infty$ as $n \to \infty$. Since $\gamma = 0$, $B(\sigma_n-,\ \omega) = B(\sigma_{n-1},\omega)$ and the assertion of the lemma follows. □

Let $\omega \in \Omega$ and $t \geq 0$. If $t \in C(\omega)$, u is the excursion (in ω) straddling t, if there exists a $\tau \geq 0$ such that $t \in [B(\tau-,\omega),\ B(\tau,\omega)[$ and $\omega_{(\tau,u)}=1$; note that $\tau = \varphi(t,\omega)$ and that $\zeta_u = B(\varphi(t,\omega),\omega) - B(\varphi(t,\omega)-,\omega)$.

With $\omega \in \Omega$ we associate a function $\tilde{\omega} : T \to E$ defined by

$$\tilde{\omega}(t) = \begin{cases} u(t - B(\varphi(t,\omega)-,\omega)) & \text{if } t \in C(\omega) \\ \\ a & \text{if } t \notin C(\omega) \end{cases}, \quad t \in T,$$

where u is the excursion straddling t. Note that

$$\tilde{\omega}(t) = \int \omega(d\sigma du)\ (u \cdot 1_{[0,\zeta_u[})(t-B(\sigma-,\omega)) \quad \text{if } t \in C(\omega),$$

and

$$1_{C(\omega)}(t) = \int \omega(d\sigma du)\ 1_{[0,\zeta_u[}(t-B(\sigma-,\omega)).$$

It follows that the map $\omega \in (\Omega,\mathcal{F}) \to \tilde{\omega} \in (E^T,\ \mathcal{E}^T)$ is measurable. Denote the coordinate evaluations on E^T by Y_t, $t \geq 0$, i.e.

$$Y_t : E^T \to E,\ Y_t(f) = f(t),$$

and the image of the probability measure P_x under the map $\omega \to \tilde{\omega}$ by $\mathbf{P}_x$. Then, $Y = \{Y_t : t \geq 0\}$ is an E-valued stochastic process on the probability space $(E^T, \mathcal{E}^T, \mathbf{P}_x)$. We continue with the calculation of the finite-dimensional distributions of the process Y. From the definition of the measure ν it follows that

$$\nu([u(s) \in dx,\ \zeta_u > s+l]) = \eta_s(dx)\ \bar{P}_l(x,\ E\backslash\{a\}),\ s,l > 0.$$

Writing β_x for the α_x-distribution of ζ,

$$\beta_x(dl) = \alpha_x[\zeta \in dl] = d\bar{P}_l(x,\ \{a\}),$$

we get

$$\nu[u(s) \in dx,\ \zeta_u - s \in dl] = \eta_s(dx)\beta_x(dl), \quad s,l > 0.$$

2.3.4 Proposition. Let $f \in b\mathcal{E}$ such that $f(a) = 0$.

If $\nu(U) = +\infty$ or $\gamma > 0$ then

$$E_a[f(Y_t)] = \begin{cases} \int P(d\omega) \int_0^t d\varphi(q,\omega) \int \eta_{t-q}(dy)\ f(y) & \text{if } t > 0 \\ 0 & \text{if } t = 0 \end{cases}$$

and for $x \neq a$

$$E_x[f(Y_t)] = \begin{cases} \int K_t(x,dy)f(y) + \int_0^t \beta_x(dl)\ {}_a[f(Y_{t-l})], & \text{if } t > 0 \\ \int K_0(x,dy)f(y) + \beta_x(\{0\})\ {}_a[f(Y_0)] & \text{if } t = 0 \end{cases}$$

If $0 < \nu(U) < \infty$ and $\gamma=0$ the same formulas hold except for the case $x=a$ and $t=0$:

$$E_a[f(Y_0)] = \frac{1}{\nu(U)} \int \nu(du)\ f[u(0)].$$

Proof. The proof is based on an application of the Palm formula, see section (1.2). Let $t > 0$.

$$E_a[f(Y_t)] = \int P(d\omega)\ 1_{C(\omega)}(t) \int \omega(d\sigma du)\ (f\circ u\cdot 1_{[0,\zeta_u[})(t-B(\sigma-,\omega))$$
$$= \int P(d\omega) \int \omega(d\sigma du)\ (f\circ u\cdot 1_{[0,\zeta_u[})(t-B(\sigma-,\omega))$$

$$= \int_0^\infty d\sigma \int \nu(du) \int P(d\omega)\ (f\circ u\cdot 1_{[0,\zeta_u[})(t-B(\sigma-,\omega + \delta_{(\sigma,u)}))$$

$$= \int P(d\omega) \int_0^\infty d\sigma \int \eta_{t-B(\sigma-,\omega)}(dy)\ f(y)$$

$$= \int P(d\omega) \int_0^t d\varphi(q,\omega) \int \eta_{t-q}(dy) f(y)$$

where in the last step we have used an integration formula for right continuous inverses for which we refer to appendix A3.

For $t=0$ and $\nu(U) = \infty$ or $\gamma > 0$

$$P[0 \in C(\omega)] = P[B(0,\omega) > 0]$$

$$= P[\omega(\{0\} \times U) = 1] = 0 \text{ since } \lambda\otimes\nu(\{0\} \times U) = 0.$$

It follows that $P_a[Y_0 = a] = 1$, so $E_a[f(Y_0)] = 0$.

If $0 < \nu(U) < \infty$ and $\gamma=0$, then the formula for $E_a[f(Y_0)]$ follows from theorem (1.3.4). Let $x \neq a$ and $t > 0$. For both cases we have

$$E_x[f(Y_t)]$$

$$= \int Q_x * P(d\omega) \int \omega(d\sigma du)\ (f\circ u\cdot 1_{[0,\zeta_u[})(t-B(\sigma-,\omega))$$

$$= \int Q_x(d\omega') \int \omega'(d\sigma du)\ 1_{[0]}(\sigma)(f\circ u\cdot 1_{[0,\zeta_u[})(t)$$

$$+ \int Q_x(d\omega') \int P(d\omega) \int \omega(d\sigma du)\ 1_{]0,\infty[}(\sigma)(f\circ u\cdot 1_{[0,\zeta_u[})(t-B(\sigma-,\omega+\omega'))$$

$$= \int \alpha_x(du) 1_{[\zeta_u > t]} f(u(t))$$

$$+ \int \alpha_x(dv) \int P(d\omega) \int \omega(d\sigma du)\ (f\circ u\cdot 1_{[0,\zeta_u[})(t-B(\sigma-,\omega)-\zeta_v)$$

$$= \int K_t(x,dy) f(y) + \int_0^t \beta_x(dl) E_a[f(Y_{t-l})].$$

For $t=0$ we only have to note that

$$\int \alpha_x(dv) \int P(d\omega) \int \omega(d\sigma du)\ (f\circ u\cdot 1_{[0,\zeta_u[})(0-B(\sigma-,\omega)-\zeta_v)$$

$$= \alpha_x[\zeta_v = 0]\cdot E_a[f(Y_0)]$$

$$= \beta(\{0\})\cdot E_a[f(Y_0)].$$

Which completes the proof of the proposition. □

Define the measure Φ on $[0,\infty[$ and the kernels $(S_t)_{t>0}$ and S_o on $(E,\mathcal{E})$ by:

$$\Phi(dq) = \int_\Omega P(d\omega)d\varphi(q,\omega),$$

$$S_t(x,dy) = \begin{cases} (\Phi * \eta)_t(dy) & \text{for } x=a,\ y\neq a \\ K_t(x,dy) + \int_0^t \beta_x(dl)(\Phi * \eta)_{t-l}(dy) & \text{for } x,y \neq a, \end{cases}$$

$$S_t(x,\{a\}) = 1 - S_t(x, E\setminus\{a\}).$$

If $\nu(U) = +\infty$ or $\gamma > 0$ then

$$S_o(x,dy) = \begin{cases} 0 & \text{for } x=a,\ y\neq a \\ K_o(x,dy) & \text{for } x,y\neq a \end{cases}$$

If $0 < \nu(U) < \infty$ and $\gamma = 0$ then

$$S_o(x,dy) = \begin{cases} \dfrac{1}{\nu(U)}\ \nu(u(0) \in dy,\ \zeta_u > 0) & \text{for } x=a,\ y\neq a \\ K_o(x,dy) + \dfrac{\beta_x(\{0\})}{\nu(U)}\ \nu(u(0) \in dy,\ \zeta_u > 0) & \text{for } x,y\neq a \end{cases}$$

In both cases

$$S_o(x,\{a\}) = 1 - S_o(x,E\setminus\{a\}).$$

The kernels $(S_t)_{t\geq 0}$ are a family of Markov kernels on $(E,\mathcal{E})$.

Then the statement of proposition (2.3.4) can be written as

$$E_x[f(Y_t)] = S_t f(x), \quad t \geq 0,\ f \in b\mathcal{E}.$$

Define for $t \geq 0$ the map $\psi_t : \Omega \to \Omega$ by

$$\psi_t(\omega) = \begin{cases} T_{\varphi(t)}(\omega) & \text{if } t \in R(\omega) \\ \delta_{(0,\theta_{t-B(\varphi(t,\omega)-,\omega)}u)} + T_{\varphi(t)}(\omega) & \text{if } t \notin R(\omega) \end{cases}$$

where u is the excursion straddling t and where T_φ is defined as in section (1.3) p.37.

The meaning of the map ψ_t is explained in the following lemma.

2.3.5 Lemma. For $s,t \geq 0$ and $\omega \in \Omega$ we have

$$\varphi(s+t,\omega) = \varphi(t,\omega) + \varphi(s,\ \psi_t\omega)$$

and

$$Y_s[(\psi_t\omega)^{\sim}] = Y_{s+t}(\tilde{\omega}).$$

Proof. First note that $\sigma_a(\psi_t\omega) = B(\varphi(t,\omega),\omega)-t$.

Indeed, if $t \in R(\omega)$, then

$$\begin{aligned}\sigma_a(\psi_t\omega) &= \sigma_a(T_{\varphi(t)}\omega)\\ &= \int (T_{\varphi(t)}\omega)(d\sigma du)1_{\{0\}}(\sigma)\zeta_u\\ &= \int \omega(d\sigma du)1_{]\varphi(t,\omega),\infty[}(\sigma)1_{\{0\}}(\sigma-\varphi(t,\omega))\zeta_u\\ &= 0\end{aligned}$$

and the result follows, as $B(\varphi(t,\omega),\omega) = t$ for $t \in R(\omega)$.

If $t \notin R(\omega)$ then

$$\begin{aligned}\sigma_a(\psi_t\omega) &= \sigma_a(\delta_{(0,\theta_{t-B(\varphi(t,\omega)-,\omega)}u)}) + \sigma_a(T_{\varphi(t)}\omega)\\ &= \zeta_u - (t - B(\varphi(t,\omega)-,\omega))\\ &= B(\varphi(t,\omega),\omega) - t\end{aligned}$$

since u is the excursion straddling t. We continue with the calculation of $B(\tau,\psi_t\omega)$.

$$\begin{aligned}B(\tau,\psi_t\omega) &= \sigma_a(\psi_t\omega) + A(\tau,\psi_t\omega) + \gamma\tau\\ &= B(\varphi(t,\omega),\omega) - t + \int \omega(d\sigma du)\ 1_{]\varphi(t,\omega),\varphi(t,\omega)+\tau]}(\sigma)\zeta_u + \gamma\tau\\ &= B(\varphi(t,\omega),\omega) - t + B(\varphi(t,\omega) + \tau,\omega) - B(\varphi(t,\omega),\omega)\\ &= B(\varphi(t,\omega) + \tau,\omega) - t. \qquad (*)\end{aligned}$$

Hence

$$\begin{aligned}\varphi(s,\psi_t\omega) &= \inf\ \{\tau : B(\tau,\ \psi_t\omega) > s\}\\ &= \inf\ \{\tau : B(\varphi(t,\omega) + \tau,\omega) > s + t\} \text{ by formula } (*)\\ &= -\varphi(t,\omega) + \varphi(s+t,\omega),\end{aligned}$$

which proves the first part of the lemma.

For the second part, suppose first that $s \in R(\psi_t\omega)$.

Then, for some $\tau \geq 0$

$$B(\tau,\ \psi_t\omega) = s,$$

hence by formula (*)

$$B(\varphi(t,\omega) + \tau,\omega) = t + B(\tau,\ \psi_t\omega) = t + s,$$

and it follows that $s+t \in R(\omega)$.

So $Y_s((\psi_t\omega)^\sim) = Y_{s+t}(\tilde{\omega}) = a$ by definition of Y, see p.72. Suppose now that $s \notin R(\psi_t\omega)$. Let $s < \sigma_a(\psi_t\omega) = B(\varphi(t,\omega),\omega) - t$. Then $\varphi(s+t,\omega) = \varphi(t,\omega)$ and there is one excursion (in ω) straddling both t and s+t, so

$$\begin{aligned} Y_s((\psi_t\omega)^\sim) &= u(s+t - B(\varphi(t,\omega)-,\omega)) \\ &= u(s+t - B(\varphi(t+s,\omega)-,\ \omega)) = Y_{s+t}(\tilde{\omega}) \end{aligned}$$

where u is the excursion straddling both t and s+t. For $s > \sigma_a(\psi_t\omega)$ we have $B(0,\psi_t\omega) < s$ so $\varphi(s,\psi_t\omega) > 0$. It follows that

$$\begin{aligned} (\psi_t\omega)(\varphi(s,\psi_t\omega),u) &= (T_{\varphi(t)}\omega)(\varphi(s+t,\omega)-\varphi(t,\omega),u) \\ &= \omega(\varphi(s+t,\omega),u) \end{aligned}$$

and the excursion straddling s in $\psi_t\omega$ is the same as the excursion straddling s+t in ω. Note that $B(\varphi(s,\psi_t\omega)-,\ \psi_t\omega) = B(\varphi(s+t,\omega)-,\omega) - t$.

Indeed
$$\begin{aligned} B(\varphi(s,\psi_t\omega)-,\ \psi_t\omega) &= \lim_{\epsilon\downarrow 0} B(\varphi(s,\psi_t\omega)-\epsilon,\ \psi_t\omega) \\ &= \lim_{\epsilon\downarrow 0} B(\varphi(s+t,\omega) - \varphi(t,\omega) - \epsilon,\ \psi_t\omega) \\ &= \lim_{\epsilon\downarrow 0} B(\varphi(s+t,\omega)-\epsilon,\omega) - t \text{ by formula } (*) \\ &= B(\varphi(s+t,\omega)-,\omega) - t. \end{aligned}$$

It follows that

$$\begin{aligned} Y_s((\psi_t\omega)^\sim) &= u(s - B(\varphi(s,\psi_t\omega)-,\ \psi_t\omega)) \\ &= u(s+t - B(\varphi(s+t,\omega)-,\omega)) = Y_{s+t}(\tilde{\omega}), \end{aligned}$$

where u is the excursion straddling s in $\psi_t\omega$.□

2.3.6 <u>Theorem</u>. Let $n \geq 2$, $f_1,\dots,f_n \in b\mathcal{E}$, $0 \leq t_1 \leq \dots \leq t_n$ and $x \in E$ then

$$\begin{aligned} &E_x\left[\prod_{i=1}^{n} f_i(Y_{t_i})\right] \\ &= \int S_{t_1}(x,dy_1) \int S_{t_2-t_1}(y_1,dy_2)\dots\int S_{t_n-t_{n-1}}(y_{n-1},dy_n)f_1(y_1)\dots f_n(y_n). \end{aligned}$$

<u>Proof</u>. We will only consider the case $x \neq a$ and $t_1 > 0$. The proof for the other cases is analogous and is therefore deleted.

$$E_x\left[\prod_{i=1}^{n} f_i(Y_{t_i})\right] = E_x\left[\bar{f}_1(Y_{t_1})\prod_{i=2}^{n} f_i(Y_{t_i})\right] + f_1(a)\, E_x\left[\prod_{i=2}^{n} f_i(Y_{t_i})\right]$$

where $\bar{f}_1$ is the function on E defined by $\bar{f}_1(x) = f_1(x)-f(a)$.

Since $\bar{f}_1(a) = 0$,

$$E_x[\bar{f}_1(Y_{t_1}) \prod_{i=2}^{n} f_i(Y_{t_i})]$$

$$= E_x[1_C(t_1)\cdot\bar{f}_1(Y_{t_1}) \prod_{i=2}^{n} f_i(Y_{t_i})]$$

$$= \int P_x(d\omega) \int \omega(d\sigma du)(\bar{f}_1 o u\cdot 1_{[0,\zeta_u[})(t_1-B(\sigma-,\omega)) \prod_{i=2}^{n} f_i(Y_{t_i-t_1})((\psi_{t_1}\omega)^{\sim})$$

$$= \int Q_x(d\omega') \int P(d\omega) \int (\omega+\omega')(d\sigma du)(\bar{f}_1 o u\cdot 1_{[0,\zeta_u[})(t_1-B(\sigma-,\omega+\omega'))\cdot$$

$$\prod_{i=2}^{n} f_i(Y_{t_i-t_1})(\psi_{t_1}(\omega+\omega')^{\sim})$$

$$= \int Q_x(d\omega') \int P(d\omega) \int \omega'(d\sigma du) \text{ --- } + \int Q_x(d\omega') \int P(d\omega) \int \omega(d\sigma du) \text{ --- }$$

$$= I + II.$$

$$I = \int Q_x(d\omega') \int \omega'(d\sigma du)(\bar{f}_1 o u\cdot 1_{[0,\zeta_u[})(t_1)\int P(d\omega) \prod_{i=2}^{n} f_i(Y_{t_i-t_1})(\psi_{t_1}(\omega+\omega')^{\sim})$$

$$= \int \alpha_x(du)(\bar{f}_1 o u\cdot 1_{[0,\zeta_u[})(t_1)\int P(d\omega) \prod_{i=2}^{n} f_i(Y_{t_i-t_1})(\psi_{t_1}(\omega+\delta_{(0,u)})^{\sim})$$

It is clear that $\zeta_u > t_1$ implies $B(\tau,\omega+\delta_{(0,u)}) = B(\tau,\omega) + \zeta_u > t_1$ for every $\tau \geq 0$. So $\varphi(t_1, \omega+\delta_{(0,u)}) = 0$ and $t_1 \notin R(\omega+\delta_{(0,u)})$. It follows that $\psi_{t_1}(\omega+\delta_{(0,u)}) = \delta_{(0,\theta_{t_1}u)} + T_o(\omega+\delta_{(0,u)})$

$$= \delta_{(0,\theta_{t_1}u)} + \omega \quad \text{P-a.s.}$$

$$= \int \alpha_x(du)(\bar{f}_1 o u\cdot 1_{[0,\zeta_u[})(t_1) \int P(d\omega) \prod_{i=2}^{n} f_i(Y_{t_i-t_1})((\omega+\delta_{(0,\theta_{t_1}u)})^{\sim})$$

$$= \int K_{t_1}(x,dy)\ \bar{f}_1(y) \int \alpha_y(du) \int P(d\omega) \prod_{i=2}^{n} f_i(Y_{t_i-t_1})((\omega+\delta_{(0,u)})^{\sim})$$

$$= \int K_{t_1}(x,dy)\ \bar{f}_1(y)\ E_y[\prod_{i=2}^{n} f_i(Y_{t_i-t_1})];$$

$$II = \int \alpha_x(dv) \int P(d\omega) \int \omega(d\sigma du)(\bar{f}_1 o u\cdot 1_{[0,\zeta_u[})(t_1-\zeta_v-B(\sigma-,\omega))\cdot$$

$$\prod_{i=2}^{n} f_i(Y_{t_i-t_1})((\psi_{t_1}(\omega+\delta_{(0,v)}))^{\sim})\ .$$

$$= \int \alpha_x(dv) \int_0^\infty d\sigma \int \nu(du) \int P(d\omega)(\bar{f}_1 \circ u \cdot 1_{[0,\zeta_u[})(t_1-\zeta_v-B(\sigma-,\omega+\delta_{(\sigma,u)}))\cdot$$

$$\prod_{i=2}^{n} f_i(Y_{t_i-t_1})(\psi_{t_1}(\omega+\delta_{(0,v)}+ \delta_{(\sigma,u)})^{\sim})$$

by an application of the Palm formula, see section (1.2).

If $t_1-\zeta_v-B(\sigma-,\omega) < \zeta_u$, then $B(\sigma-, \omega + \delta_{(0,v)} + \delta_{(\sigma,u)}) > t_1$.

Hence $\varphi(t_1,\omega+\delta_{(0,v)}+\delta_{(\sigma,u)}) \leq \sigma$.

If $0 \leq t_1-\zeta_v-B(\sigma-,\omega)$, then $B(\sigma-,\omega+\delta_{(0,v)}+\delta_{(\sigma,u)}) \leq t_1$.

Hence $\varphi(t_1,\omega+\delta_{(0,v)}+\delta_{(\sigma,u)}) \geq \sigma$. It follows that

$\varphi(t_1,\omega+\delta_{(0,v)}+\delta_{(\sigma,u)}) = \sigma$ and $t_1 \notin R(\omega+\delta_{(0,v)}+\delta_{(\sigma,u)})$

So $\psi_{t_1}(\omega+\delta_{(0,v)}+\delta_{(\sigma,u)}) = \delta_{(0,\bar{u})} + T_\sigma(\omega+\delta_{(0,v)} + \delta_{(\sigma,u)})$

$= \delta_{(0,\bar{u})} + T_\sigma(\omega)$ P-a.s.

where $\bar{u} = \theta_{t_1-B(\sigma-,\omega)-\zeta_v}u$.

$$= \int \alpha_x(dv) \int_0^\infty d\sigma \int \nu(du) \int P(d\omega)(\bar{f}_1 \circ u \cdot 1_{[0,\zeta_u[})(t_1-\zeta_v-B(\sigma-,\omega))\cdot$$

$$\prod_{i=2}^{n} f_i(Y_{t_i-t_1})(\delta_{(0,\bar{u})} + T_\sigma\omega)$$

$$= \int \alpha_x(dv) \int_0^\infty d\sigma \int \nu(du) \int P(d\omega)(\bar{f}_1 \circ u \cdot 1_{[0,\zeta_u[})(t_1-\zeta_v-B(\sigma-,\omega))\cdot$$

$$\int P(d\omega') \prod_{i=2}^{n} f_i(Y_{t_i-t_1})(\omega'+\delta_{(0,\bar{u})})$$

by an application of the renewal property (see theorem (1.3.3))

$$= \int \alpha_x(dv) \int (\phi * \eta)_{t_1-\zeta_v}(dy)\bar{f}_1(y) \int \alpha_y(du)$$

$$\int P(d\omega') \prod_{i=2}^{n} f_i(Y_{t_i-t_1})(\omega'+\delta_{(0,u)})$$

$$= \int_0^{t_1} \beta_x(dl) \int (\phi * \eta)_{t_1-l}(dy)\bar{f}_1(y)\, E_y[\prod_{i=2}^{n} f_i(Y_{t_i-t_1})].$$

It follows that

$$E_x[\prod_{i=1}^{n} f_i(Y_{t_i})] = \int_{E\setminus\{a\}} S_{t_1}(x,dy)\bar{f}_1(y)E_y[\prod_{i=2}^{n} f_i(Y_{t_i-t_1})]$$

$$+ f_1(a)\, E_a[\prod_{i=2}^{n} f_i(Y_{t_i-t_1})]$$

$$= \int_E S_{t_1}(x,dy) f_1(y)\, E_y[\prod_{i=2}^{n} f_i(Y_{t_i-t_1})].$$

An induction argument completes the proof of the theorem.□

As a consequence of theorem (2.3.6), $(S_t)_{t\geq 0}$ is a Markov semigroup on $(E,\mathcal{E})$ and Y is the canonical representation of the Markov process with transition semigroup $(S_t)_{t\geq 0}$.

Let $(V_\lambda)_{\lambda>0}$ be the resolvent of the semigroup $(S_t)_{t\geq 0}$

$$V_\lambda : b\mathcal{E} \to b\mathcal{E} ,$$

$$V_\lambda f(x) = \int_0^\infty e^{-\lambda t} S_t f(x)\, dt = E_x \int_0^\infty e^{-\lambda t} f(Y_t) dt, \quad x \in E.$$

Denote the resolvent of the semigroup $(K_t)_{t\geq 0}$ by $(G_\lambda)_{\lambda>0}$.

Define for $\lambda > 0$ and $f \in b\mathcal{E}$

$$\hat{\eta}_\lambda(f) = \int_0^\infty dt\ e^{-\lambda t} \int_E \eta_t(dx)\ f(x).$$

This integral is finite because of the assumption

$$\int (1-e^{-\zeta_u})\ \nu(du) = \hat{\eta}_1(1) < \infty.$$

Define for $\lambda > 0$ and $x \in E$

$$z_\lambda(x) = \int \alpha_x(du)\ e^{-\lambda\zeta_u}.$$

In the next lemma we prove some relations between (G_λ), $(\hat{\eta}_\lambda)$ and z_λ.

2.3.7 Lemma. Let $\lambda, \mu > 0$ and $f \in b\mathcal{E}$, then

(i) $(\mu-\lambda)\hat{\eta}_\lambda(G_\mu f) = \hat{\eta}_\lambda(f) - \hat{\eta}_\mu(f)$,

(ii) $z_\lambda = 1-\lambda G_\lambda 1$,

(iii) $(\mu-\lambda)\hat{\eta}_\mu(z_\lambda) = \mu\hat{\eta}_\mu(1) - \lambda\hat{\eta}_\lambda(1)$.

Proof.

(i) $$(\mu-\lambda)\hat{\eta}_\lambda(G_\mu f) = (\mu-\lambda) \int_0^\infty dt\ e^{-\lambda t} \int \eta_t(dx) \int_0^\infty e^{-\mu s}\ K_s f(x) ds$$

$$= (\mu-\lambda) \int_0^\infty dt\ e^{-\lambda t} \int_0^\infty e^{-\mu s}\ \eta_{s+t}(f)ds$$

$$= \int_0^\infty ds\ e^{-\mu s}\ \eta_s(f) \int_0^s (\mu-\lambda)e^{(\mu-\lambda)t}dt$$

$$= \hat{\eta}_\lambda(f) - \hat{\eta}_\mu(f).$$

(ii) $$z_\lambda(x) = \int \alpha_x(du)e^{-\lambda\zeta_u}$$

$$= \int \alpha_x(du) \int_0^\infty ds\ 1_{]\zeta_u,\infty[}(s)\lambda e^{-\lambda s}ds$$

$$= \int_0^\infty \lambda e^{-\lambda s}(1-K_s1(x))ds$$

$$= 1-\lambda G_\lambda 1(x).$$

(iii) $$(\mu-\lambda)\ \hat{\eta}_\mu(z_\lambda) = (\mu-\lambda)\hat{\eta}_\mu(1-\lambda G_\lambda 1) \qquad \text{(by (ii))}$$

$$= (\mu-\lambda)\hat{\eta}_\mu(1)+\lambda(\hat{\eta}_\mu(1) - \hat{\eta}_\lambda(1)) \qquad \text{(by (i))}$$

$$= \mu\hat{\eta}_\mu(1) - \lambda\hat{\eta}_\lambda(1).\ \square$$

We continue with a theorem which gives an expression for the resolvent $(V_\lambda)_{\lambda>0}$.

2.3.8 Theorem. Let $\lambda > 0$, $f \in b\mathcal{E}$. Then

$$V_\lambda f(x) = G_\lambda f(x) + z_\lambda(x)\ V_\lambda f(a) \text{ where}$$

$$V_\lambda f(a) = \frac{\hat{\eta}_\lambda(f) + \gamma f(a)}{\lambda\gamma + \lambda\hat{\eta}_\lambda(1)}.$$

Proof. Let $x \in E\setminus\{a\}$ then

$$V_\lambda f(x) = \int P_x(d\omega) \int \omega(d\sigma du) \int_0^\infty e^{-\lambda t}\ (f\circ u\cdot 1_{[0,\zeta_u[})(t-B(\sigma-,\omega))dt$$

$$+ \int P_x\ (d\omega) \int_0^\infty 1_{C^*(\omega)}(t)\ e^{-\lambda t}f(a)dt$$

$$= \int P_x(d\omega) \int \omega(d\sigma du) \int_0^\infty e^{-\lambda t} (\bar{f}\circ u \cdot 1_{[0,\zeta_u[})(t-B(\sigma-,\omega))dt + \frac{1}{\lambda} f(a)$$

where $\bar{f}(\cdot) = f(\cdot) - f(a)$.

$$= \int \alpha_x(dv) \int_0^{\zeta_v} e^{-\lambda t}\bar{f}(v(t))dt + \int \alpha_x(dv) \int_{\zeta_v}^\infty e^{-\lambda t} E_a[\bar{f}(Y_{t-\zeta_v})]dt + \frac{1}{\lambda} f(a)$$

$$= \int_0^\infty e^{-\lambda t} K_t \bar{f}(x)dt + \int \alpha_x(dv) e^{-\lambda\zeta_v} . V_\lambda \bar{f}(a) + \frac{1}{\lambda} f(a)$$

$$= \int_0^\infty e^{-\lambda t} K_t f(x)dt + (1 - \lambda \int_0^\infty e^{-\lambda t} K_t 1(x)dt) \cdot V_\lambda f(a) \qquad (*)$$

$$V_\lambda f(a) = \int P(d\omega) \int \omega(d\sigma du) \int_0^\infty e^{-\lambda t} (\bar{f}\circ u \cdot 1_{[0,\zeta_u[})(t-B(\sigma-,\omega))dt + \frac{1}{\lambda} f(a)$$

$$= \int_0^\infty d\sigma \int \nu(du) \int P(d\omega) e^{-\lambda B(\sigma-,\omega)} \int_0^{\zeta_u} e^{-\lambda t} \bar{f}(u(t))dt + \frac{1}{\lambda} f(a)$$

$$= \int_0^\infty d\sigma \int \nu(du) \exp[-\sigma(\lambda\gamma + \int (1-e^{-\lambda\zeta_w})\nu(dw))] \cdot$$

$$\int_0^\infty e^{-\lambda t} \bar{f}(u(t)) 1_{[\zeta_u > t]} dt + \frac{1}{\lambda} f(a)$$

$$= \frac{1}{\lambda\gamma + \int (1-e^{-\lambda\zeta_w})\nu(dw)} \int_0^\infty e^{-\lambda t} \int \nu(du) \bar{f}(u(t)) 1_{[\zeta_u > t]} dt + \frac{1}{\lambda} f(a).$$

Note that $\int (1-e^{-\lambda\zeta_w}) \nu(dw) = \lambda\hat{\eta}_\lambda(1)$,

and $\int_0^\infty e^{-\lambda t} \int \nu(du) \bar{f}(u(t)) 1_{[\zeta_u > t]} dt = \hat{\eta}_\lambda(\bar{f})$.

Hence

$$V_\lambda f(a) = \frac{\hat{\eta}_\lambda(f) + \gamma f(a)}{\lambda\gamma + \lambda\hat{\eta}_\lambda(1)} . \qquad (1)$$

Further

$$1 - \lambda \int_0^\infty e^{-\lambda t} K_t 1(x)dt = \int_0^\infty \lambda e^{-\lambda t} (1 - K_t 1(x))dt$$

$$= \int \alpha_x(du) \int_0^\infty \lambda e^{-\lambda t}\, 1_{[\zeta_u \leq t]} dt$$

$$= \int \alpha_x(du)\, e^{-\lambda \zeta_u}. \qquad (2)$$

Substitution of (1) and (2) in formula (*) completes the proof. □

The next theorem states that the resolvent (V_λ) inherits the Ray property of the resolvent (G_λ) if the following extra condition is satisfied: $z_\lambda(x) < 1$ for every $x \neq a$.

2.3.9 Theorem. If $z_\lambda(x) < 1$ for every $x \neq a$ and if (G_λ) is a Ray resolvent, then the same is true for (V_λ). In this case the process Y has càdlàg paths P_x-a.s. and is therefore a strong Markov process.

Proof. If (G_λ) is a Ray resolvent, then the same proof as in Rogers [45] can be used to prove the Ray property for the resolvent (V_λ). By construction, Y is the canonical representation of the Markov process corresponding to (V_λ). So P_x-a.s. the limits $X_t = \lim_{\mathbb{Q}\ni q \downarrow\downarrow t} Y_q$ exist for all $t \geq 0$ and the process $X=(X_t)_{t\geq 0}$ is a càdlàg version of the process Y which has the strong Markov property, see Williams [59], ch.III, p.194 ff. So it is sufficient to prove that the processes X and Y are P_x-equal. It is clear that $X_t(\omega)=Y_t(\omega)$ for every $t \in C(\omega)$. So there is nothing to prove in the case $\gamma=0$ and $0 < \nu(U) < \infty$, since in this case $T=C(\omega)$ (see lemma (2.3.3)). Suppose now that $\gamma > 0$ or $\nu(U) = +\infty$ and that $t \in R(\omega)$, say $t=B(r,\omega)$.

If $\nu(U)<\infty$, then there is a first interval $[B(\sigma-,\omega), B(\sigma,\omega)[$ following t, i.e.

$t < B(\sigma-,\omega)$ and $[t,B(\sigma-,\omega)[\subset R(\omega)$,

and it is clear that $X_t(\omega) = a = Y_t(\omega)$.

If $\nu(U)=+\infty$, there is no first interval $[B(\sigma-,\omega),B(\sigma,\omega)[$ following t, since this would imply that $\omega([r,\sigma[\times U) = 0$ for some $\sigma > r$ which is impossible. So for every $\varepsilon > 0$, the interval $[t, t+\varepsilon]$ contains an excursion interval. Since we can choose in each excursion interval a rational number q so that $Y_q(\omega)$ is in an arbitrary small neighbourhood of a, it follows that $X_t(\omega) = a = Y_t(\omega)$. □

The following theorem gives an explicit formula for the Blumenthal-Getoor local time for Y at a, see Blumenthal & Getoor [3], ch. V. In this theorem the map $\varphi(t)$ defined on $\mathcal{M}_1'$ is considered as a (P_x-a.s.) defined map on the sample space E^T of the process Y, which is possible since the map $\omega \in \mathcal{M}_1' \to \tilde{\omega} \in E^T$ is an injection.

2.3.10 Theorem. Under the assumptions of theorem (2.3.8), the Blumenthal-Getoor local time $L=(L_t)_{t\geq 0}$ at the state a of the process Y is given by

$$L_t = [\gamma + \int (1-e^{-\zeta_u})\nu(du)]\cdot\varphi(t).$$

Proof. Let σ_α be the first time that Y hits or approaches the state a.

$$\begin{aligned} E_x(e^{-\sigma_a}) &= \int P_x(d\omega)e^{-B(0,\omega)} \\ &= \int Q_x(d\omega')\int P(d\omega)\, e^{-B(0,\omega+\omega')} \\ &= \int Q_x(d\omega')e^{-B(0,\omega')}. \end{aligned} \tag{1}$$

$$\begin{aligned} &\int P_x(d\omega)\int_0^\infty e^{-t}d\varphi(t,\omega) \\ &= \int P_x(d\omega)\int_0^\infty e^{-t}\, 1_{[B(0,\omega),\infty[}(t)d\varphi(t,\omega) \\ &= \int Q_x(d\omega')\int P(d\omega)\int_0^\infty e^{-t}\, 1_{[B(0,\omega+\omega'),\infty[}(t)\, d\varphi(t,\omega+\omega') = (*). \end{aligned}$$

Note that $B(0,\omega+\omega') = B(0,\omega')$ for P almost every ω, and

$B(\tau,\omega') = B(0,\omega')$ for Q_x almost every ω'.

So $\varphi(t,\omega+\omega') = \inf\{\tau\colon B(\tau,\omega+\omega') > t\}$

$$= \inf \{\tau: B(\tau,\omega) + B(0,\omega') > t\}$$

$$= \varphi(t-B(0,\omega'),\omega) \text{ for } t > B(0,\omega').$$

Substitution in (*) yields:

$$(*) = \int Q_x(d\omega') \int P(d\omega) \int_0^\infty e^{-t}\, 1_{[B(0,\omega'),\infty[}(t)\, d\varphi(t-B(0,\omega'),\omega)$$

$$= \int Q_x(d\omega')\, e^{-B(0,\omega')} \cdot \int P(d\omega) \int_0^\infty e^{-t} d\varphi(t,\omega). \tag{2}$$

$$\int P(d\omega) \int_0^\infty e^{-t} d\varphi(t,\omega) = \int_0^\infty dt \int P(d\omega)\, e^{-B(t,\omega)}$$

$$= \frac{1}{\gamma + \int (1-e^{-\zeta_u})\nu(du)}. \tag{3}$$

Substitution of (2) and (3) in (1) yields

$$E_x(e^{-\sigma_a}) = [\gamma + \int (1-e^{-\zeta_u})\nu(du)] \int P_x(d\omega) \int_0^\infty e^{-t} d\varphi(t,\omega).$$

Since the Itô-Poisson point process of excursions from a can be reconstructed from Y, it follows that $\varphi(t)$ is measurable with respect to the σ-algebra $\sigma(Y_t : t \geq 0)$ generated by the process Y. An application of Galmarino's test, see Dellacherie & Meyer [7] p. 149, yields that the process $(\varphi(t))_{t\geq 0}$ is adapted to the filtration of the process Y. Finally, it follows from lemma (2.3.5) that the functional $(\varphi(t))_{t\geq 0}$ is additive, which completes the proof of the theorem. □

We end this section by a short description of the process Y^δ, which is the process Y, constructed as above from the family of point processes (P_x), with killing on state a at a rate δ proportional to the local time. This is also an example of the construction of a stochastic process from a more general point process than an Itô-Poisson point process. Let for $s \geq 0$ the point process P^s be defined as the image of P under the mapping $\omega \in \mathcal{M}^+ \to 1_{[0,s]\times U}\omega \in \mathcal{M}^+$. It is clear that P^s is a Poisson point process on X with intensity measure $1_{[0,s]}\lambda \otimes \nu$ where ν is the characteristic measure of P. Let for $\delta>0$ the point process S^δ be the Cox process on X defined by

$$S^\delta = \int_0^\infty \delta e^{-\delta s} P^s ds,$$

and let for $x \in E$ the family of point processes S_x^δ be defined by

$$S_x^\delta = \begin{cases} Q_x * S^\delta & \text{for } x \neq a \\ S^\delta & \text{for } x = a \end{cases},$$

where the point process Q_x is defined as in the beginning of this section. Let the map $\kappa : \mathcal{M}^+ \to [0,\infty]$ be defined by

$$\kappa(\omega) = \inf \{\tau : \omega([\tau,\infty[\times U) = 0\}$$

and define

$$B(\tau,\omega) = \begin{cases} \int \omega(d\sigma du) 1_{[0,\tau]}(\sigma)\zeta_u + \gamma\tau & \text{for } \tau \leq \varphi(\omega) \\ B(\varphi(\omega),\omega) & \text{for } \tau > \varphi(\omega) \end{cases}$$

where γ is a positive constant.

The process Y^δ is now constructed from the family of point processes (S_x^δ) in the following way. Until time $B(\infty,\omega)$ the construction is the same as for the process Y associated with the family of point processes (P_x). At time $B(\infty,\omega)$ the process Y^δ is killed. It can be shown that Y^δ has the simple Markov property.

We will only give an expression for the resolvent of the process Y^δ. Let $(V_\lambda^\delta)_{\lambda>0}$ be the resolvent of the process Y^δ.

$$V_\lambda^\delta f(x) = E_x \int_0^\infty e^{-\lambda t} f(Y_t^\delta) dt, \quad x \in E, \ f \in b\mathcal{E}.$$

2.3.11 Theorem. Let $\lambda > 0$ and $f \in b\mathcal{E}$. Then

$$V_\lambda^\delta f(x) = G_\lambda f(x) + z_\lambda(x)\, V_\lambda^\delta f(a) \text{ where}$$

$$V_\lambda^\delta f(a) = \frac{\hat{\eta}_\lambda(f) + \gamma f(a)}{\delta + \lambda\gamma + \lambda\hat{\eta}_\lambda(1)}.$$

Proof. We only calculate $V_\lambda^\delta f(a)$. The rest of the proof is analogous to the proof of theorem (2.3.8). Suppose first that f(a)=0.

$$V_\lambda^\delta f(a) = \int S^\delta(d\omega)\int \omega(d\sigma du)\int_0^\infty e^{-\lambda t}(f\circ u\cdot 1_{[0,\zeta_u[})(t-B(\sigma-,\omega))dt$$

$$= \int_0^\infty ds\ \delta e^{-\delta s}\int P(d\omega)\int \omega(d\sigma du)1_{[0,s]}(\sigma)\int_0^\infty dt\ e^{-\lambda t}(f\circ u\cdot 1_{[0,\zeta_u[})(t-B(\sigma-,\omega))$$

$$= \int_0^\infty ds\ \delta e^{-\delta s}\int_0^s d\sigma\int \nu(du)\int P(d\omega)\int_0^\infty dt\ e^{-\lambda t}(f\circ u\cdot 1_{[0,\zeta_u[})(t-B(\sigma-,\omega))$$

$$= \int_0^\infty d\sigma\int \nu(du)\int P(d\omega)e^{-\delta\sigma-\lambda B(\sigma-,\omega)}\int_0^{\zeta_u} dt\ e^{-\lambda t}f(u(t))$$

$$= \frac{1}{\delta + \lambda\gamma + \int(1-e^{-\lambda\zeta_w})\nu(dw)}\int_0^\infty dt\ e^{-\lambda t}\int \nu(du)f(u(t))1_{[\zeta_u>t]}$$

$$= \frac{\hat{\eta}_\lambda(f)}{\delta + \lambda\gamma + \lambda\hat{\eta}_\lambda(1)}\ .$$

Further

$$V_\lambda^\delta 1(a) = \int S^\delta(d\omega)\int_0^{B(\infty,\omega)} dt\ e^{-\lambda t}$$

$$= \int_0^\infty ds\ \delta\ e^{-\delta s}\int P(d\omega)\int_0^{B(s,\omega)} dt\ e^{-\lambda t}$$

$$= \frac{1}{\lambda}\int_0^\infty ds\ \delta\ e^{-\delta s}\ [1 - e^{-s(\lambda\gamma+\lambda\hat{\eta}_\lambda(1))}]$$

$$= \frac{\gamma+\hat{\eta}_\lambda(1)}{\delta + \lambda\gamma + \lambda\hat{\eta}_\lambda(1)}\ .$$

So for $f \in b\mathcal{E}$

$$V_\lambda^\delta f(a) = [V_\lambda^\delta(f-f(a)\cdot 1)](a) + f(a)V_\lambda^\delta 1(a)$$

$$= \frac{\hat{\eta}_\lambda(f-f(a)\cdot 1)+(\gamma+\hat{\eta}_\lambda(1))f(a)}{\delta + \lambda\gamma + \lambda\hat{\eta}_\lambda(1)}.$$

$$= \frac{\hat{\eta}_\lambda(f) + \gamma f(a)}{\delta + \lambda\gamma + \lambda\hat{\eta}_\lambda(1)}\ .\ \Box$$

CHAPTER 3

APPLICATIONS

In chapter 2 we saw how to construct for a Ray process Y the Itô-Poisson point process of excursions from a recurrent state a, which is not a holding point and for which $P_a[\tau_a=0]=1$ where τ_a is the infimum of the times $t > 0$ at which Y hits or approaches the state a. In the first section of this chapter we will show how one can get an explicit formula for the characteristic measure of the Itô-Poisson point process of excursions from zero for standard Brownian motion using the elementary calculations of the distribution of Brownian excursions in Chung [6]. By means of adjunction of a Radon-Nikodym factor we get from this result an explicit formula for the characteristic measure of the Itô-Poisson point process of excursions from zero for Brownian motion with constant drift. This will be done in section (3.2).
A well-known problem which can be treated with excursion theory is to describe all strong Markov processes which behave like a given strong Markov process outside a given state a (or more generally outside some subset of states D). We will consider the problem to describe all Ray processes on $[0,\infty[$ which behave like Brownian motion outside zero. This problem was first treated by Feller [11] using theory of differential equations. Feller's solution was that the infinitesimal generator of such a process is the differential operator $\mathscr{G} = \frac{1}{2}\frac{d^2}{dx^2}$ on $C_2([0,\infty[)$ with domain D defined by

$$D=C_2([0,\infty[) \cap \{u : p_1u(0) - p_2u'(0) + p_3u''(0) = \int p_4(dx)[u(x)-u(0)]\}$$

where p_1, p_2 and p_3 are nonnegative real numbers and p_4 a σ-finite measure on $]0,\infty[$ such that

$$p_1 + p_2 + p_3 + \int p_4(dx)(1-e^{-x}) = 1.$$

Itô and McKean constructed in [26] the sample paths of these processes, which they called Feller's Brownian motions, from the reflecting Brownian motion and its local time and (independent) exponential holding times and differential processes. Rogers derived in [45] Feller's result using resolvent identities. We will give in section (3.3) an interpretation of the parameters p_1, p_2, p_3 and the measure p_4 by means of excursion theory. In the following section we will use these results to construct a model for random motion on an n-pod E_n, that is a tree with one single vertex 0 and with n legs having infinite length. This problem comes from a problem which arises in considering the movement of nutrients in the root system of a plant and also has possible application to the spread of pollutants in a stream system and to the analysis and design of circulatory systems, see Frank and Durham [12]. We will construct all strong Markov processes which behave like standard Brownian motion restricted to a half line, when restricted to a single leg.

In the last section we will construct a Markov process on $[0,\infty[$ of the following description: starting at a point $x \in]0,\infty[$ it evolves like a given strong Markov process until reaching 0 where it waits a length of time having an exponential distribution with parameter α after which time it jumps independently to a new position in $]0,\infty[$ according to a given probability measure η and then proceeds as before.

3.1 The Itô-Poisson point process attached to Brownian motion.

In this section we will apply the results of section (2.2) to Brownian motion. In particular, we will derive an explicit formula for the characteristic measure ν of the Itô-Poisson point process of excursions from zero. The derivation is based on theorem (2.2.3) and on the elementary calculations of the distribution of Brownian excursions in Chung [6].

Let $B = (B_t)_{t\geq 0}$ be a standard one-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, P_o)$. Let $r>0$ and let V_1 be the first excursion of B from zero with length greater than r. Following Chung [6], we introduce the notations:

$$\beta(r) = \inf \{t>r : B_t = 0\},$$

$$\gamma(r) = \sup \{t<r : B_t = 0\},$$

$$L(r) = \beta(r) - \gamma(r).$$

So $]\gamma(r), \beta(r)[$ is the excursion interval containing r. As $P_o[B_r=0] = 0$ and B has continuous realizations, $L(r) > 0$ (P_o-a.s.). Denote as in section (2.2), the first excursion interval of length $> r$ by $]D_1, T_1[$.

For $n \geq 1$, $0 < t_1 < \dots < t_n$ and $x_1, \dots, x_n \in \mathbb{R}_+$ (or $x_1, \dots, x_n \in \mathbb{R}_-$)

$$\begin{aligned} P_o[V_1(t_i) \in dx_i,\ i=1,\dots,n] &= P_o[V_1(t_i) \in dx_i,\ i=1,\dots,n,\ L(r) \leq r] \\ &\quad + P_o[V_1(t_i) \in dx_i,\ i=1,\dots,n,\ L(r) > r] \\ &= I + II. \end{aligned} \tag{1}$$

It is clear that

$$[L(r) \leq r] = [D_1 \geq \beta(r)] \text{ and that}$$

$$D_1 = \beta(r) + D_1 \circ \theta_{\beta(r)} \text{ on } [D_1 \geq \beta(r)].$$

It follows that

$$\begin{aligned} I &= P_o[V_1(t_i) \circ \theta_{\beta(r)} \in dx_i,\ i=1,\dots,n,\ \beta(r) - \gamma(r) \leq r] \\ &= P_o[\beta(r) - \gamma(r) \leq r] \cdot P_o[V_1(t_i) \in dx_i,\ i=1,\dots,n] \end{aligned} \tag{2}$$

by an application of the strong Markov property on stopping time $\beta(r)$. It is also clear that $[L(r) > r] = [T_1 = \beta(r),\ D_1 = \gamma(r)]$. It follows that

$$II = P_o[B_{\gamma(r)+t_i} \in dx_i,\ i=1,\dots,n,\ L(r) > \max(t_n, r)]. \tag{3}$$

Substitution of (2) and (3) in (1) yields

$$P_o[V_1(t_i) \in dx_i,\ i=1,\dots,n]$$

$$= \frac{\mathbb{P}_o[B(\gamma(r)+t_i) \in dx_i, i=1,\dots,n,\ L(r) > \max(t_n,r)]}{\mathbb{P}_o[L(r) > r]} . \qquad (4)$$

A simple calculation using Chung [6], formula (2.20) results in

$$P_o[L(r) \geq r] = \frac{2}{\pi} \qquad (5)$$

and the same reasoning as in Chung [6], theorem 6 yields for $l > \max(t_n, r)$

$$P_o[\gamma(r) \in ds,\ B(\gamma(r) + t_i) \in dx_i,\ i=1,\dots,n,\ L(r) \in dl]$$

$$= p(s;0,0)g(t_1;0,x_1)q(t_2-t_1;x_1,x_2)\dots$$

$$q(t_n-t_{n-1};x_{n-1},x_n)g(l-t_n;0,x_n)ds\,dx_1\dots dx_n dl$$

where

$$p(t;x,y) = \exp\left[-\frac{1}{2t}(x-y)^2\right],$$

$$g(t;0,y) = \left[\frac{1}{2\pi t}\right]^{\frac{1}{2}} \frac{|y|}{t} \exp\left[-\frac{1}{2t} y^2\right],$$

$$q(t;x,y) = p(t;x,y) - p(t;x,-y).$$

The probabilistic interpretations of p,g and q are given by

$$P_x[B(t) \in dy] = p(t;x,y)dy ,$$

$$P_o[\sigma_y \in dt] = g(t;0,y)dt,$$

$$P_x[B(t) \in dy,\ \sigma_o > t] = q(t;x,y)dy,$$

for $t > 0$ and $x \cdot y > 0$.

Hence for $t_n > r$

$$P_o[B(\gamma(r) + t_i) \in dx_i,\ i=1,\dots,n,\ L(r) > t_n]$$

$$= \left[\frac{2r}{\pi}\right]^{\frac{1}{2}} g(t_1;0,x_1)q(t_2-t_1;x_1,x_2)\dots q(t_n-t_{n-1};x_{n-1},x_n)dx_1\dots dx_n . \qquad (6)$$

Substitution of (5) and (6) in (4) gives for $t_n > r$

$$P_o[V_1(t_i) \in dx_i,\ i=1,\dots,n]$$

$$= \left[\frac{\pi r}{2}\right]^{\frac{1}{2}} g(t_1;0,x_1)q(t_2-t_1;x_1,x_2)\dots q(t_n-t_{n-1};x_{n-1},x_n)dx_1\dots dx_n .$$

This formula enables us to calculate some important quantities.

$$P_o[T_1 < \infty] = \int P_o[V_1(r) \in dx]$$

$$= 2\ (2\pi r)^{\frac{1}{2}} \int_0^\infty g(r;0,x)dx = 1$$

and

$$P_o[T_1 - D_1 > r + s] = \left[\frac{r}{r+s} \right]^{½}.$$

Let $(r_k)_{k\geq1}$ be a strictly decreasing sequence of positive real numbers, such that $\lim_{k\to\infty} r_k = 0$. Let $(U_k,V_k)_{k\geq1}$ be defined as in the proof of theorem (2.2.3), i.e. V_k is the sequence of excursions from 0 of length greater than r_k and U_k is the set of functions $u \in U = D_{[0,\infty[}(\mathbb{R})$ for which $\zeta_u > r_k$. Then

$$\lim_{k\to\infty} p_k = \lim_{k\to\infty} P_o(V_{k1} \in U_1) = \lim_{k\to\infty} \left[\frac{r_k}{r_k+r_1} \right]^{½} = 0,$$

so there exists an Itô-Poisson point process N on U whose $[\zeta > l]$-subsequence is the sequence of excursions of B of length greater than l. The characteristic measure ν of N is given by

$$\nu[u(t_i) \in dx_i,\ i=1,\dots,n]$$
$$=\left[\frac{\pi r_1}{2} \right]^{½} g(t_1;0,x_1)q(t_2-t_1;x_1,x_2)\dots q(t_n-t_{n-1};x_{n-1},x_n)dx_1\dots dx_n$$

where $0 < t_1 < \dots < t_n$ and $x_1,\dots,x_n > 0$ (or $x_1,\dots,x_n < 0$). Taking $r_1 = \frac{8}{\pi}$ we get the same normalization of ν as in Ikeda & Watanabe [22], p. 124. With our notations it is more convenient to take $r_1 = \frac{2}{\pi}$.

3.2 The Itô-Poisson point process attached to Brownian motion with constant drift.

With the results of section (3.1) for standard Brownian motion, it is not very difficult to write down a formula for the characteristic measure of the Itô-Poisson point process of excursions from zero of Brownian motion with constant drift. The passage from Brownian motion densities to densities of Brownian motion with drift is done by adjunction of a Radon-Nikodym factor, see for instance Imhof & Kummerling [23], section 5. Let $Y = (Y_t)_{t\geq0}$ be Brownian motion with constant drift δ, i.e.

$$Y(t) = \delta t + B(t),\ t \geq 0.$$

A straightforward calculation yields

$$P_x[Y(t_i) \in dy_i, i=1,\dots,n]$$
$$= \exp[\delta(y_n-x) - \frac{1}{2}\delta^2 t_n]\ P_x[B(t_i) \in dy_i, i=1,\dots,n]$$

for $0 \leq t_1 \leq \dots \leq t_n$ and $y_1, \dots, y_n \in \mathbb{R}$.

It follows that the Radon-Nikodym derivatives $\bar{p}(t;x,y)$, $\bar{g}(t;0,y)$ and $\bar{q}(t;x,y)$ of the measures $P_x[Y(t) \in dy]$, $P_o[\sigma_y \in dt]$ and $P_x[Y(t) \in dy,\ \sigma_o > 0]$ with respect to the Lebesgue measure are given by

$$\bar{p}(t;x,y) = \exp\ [\delta(y-x) - \frac{1}{2}\delta^2 t]\ p(t;x,y),$$
$$\bar{g}(t;0,y) = \exp\ [\delta y - \frac{1}{2}\delta^2 t]\ g(t;0,y),$$
$$\bar{q}(t;x,y) = \exp\ [\delta(y-x) - \frac{1}{2}\delta^2 t]\ q(t;x,y),$$

for $t > 0$ and $x \cdot y > 0$.

Let $]\bar{\gamma}(r), \bar{\beta}(r)[$ be the excursion interval of Y containing r with length $\bar{L}(r) = \bar{\beta}(r) - \bar{\gamma}(r)$ and let $]\bar{D}_1, \bar{T}_1[$ resp. $\bar{V}_1$ be the first excursion interval resp. the first excursion of the process Y with length greater than r, then a simple calculation yields

$$P_o[Y(\bar{\gamma}(r) + t_i) \in dy_i,\ i=1,\dots,n,\ \bar{L}(r) > t_n]$$
$$= \left(\int_0^r \left[\frac{1}{2\pi s}\right]^{1/2} \exp\left(-\frac{1}{2}\delta^2 s\right) ds\right) \bar{g}(t_1;0,y_1) \cdot \left\{\prod_{i=1}^{n-1} \bar{q}(t_{i+1}-t_i; y_i, y_{i+1})\right\} dy_1 \dots dy_n$$

for $0 < t_1 < \dots < t_n$; $y_1, \dots, y_n > 0$ (or $y_1, \dots, y_n < 0$) and $t_n > r$.

It follows that

$$P_o[\bar{L}(r) > r] = \left(\int_0^r \left[\frac{1}{2\pi s}\right]^{1/2} \exp\left(-\frac{1}{2}\delta^2 s\right) ds\right) \int_{-\infty}^{\infty} \bar{g}(r;0,x)\,dx$$

and

$$P_o[\bar{V}_1(t_i) \in dy_i,\ i=1,\dots,n]$$

$$= \frac{1}{\int \bar{g}(r;0,x)dx} \bar{g}(t_1;0,y_1)\bar{q}(t_2-t_1;y_1,y_2)\cdots$$

$$\bar{q}(t_n-t_{n-1};y_{n-1},y_n)\, dy_1 \,\ldots\, dy_n .$$

So

$$P_o[\bar{T}_1 < \infty] = 1$$

and

$$P_o[\bar{T}_1-\bar{D}_1 > r + s] = \frac{\int \bar{g}(r+s;0,x)dx}{\int \bar{g}(r;0,x)dx} .$$

Since

$$\int_{-\infty}^{+\infty} \bar{g}(r;0,x)dx = \left[\frac{1}{r} \right]^{½} \exp(- \frac{1}{2} \delta^2 r)\ E_o(|B(1)|\ \exp [\delta B(1)\sqrt{r}])$$

we get

$$\lim_{r\downarrow 0} \frac{\int \bar{g}(r+s;0,x)dx}{\int \bar{g}(r;0,x)dx}$$

$$= \lim_{r\downarrow 0} \left[\frac{r}{r+s} \right]^{½} \exp(- \frac{1}{2} \delta^2 r)\ \frac{E_o\ |B(1)|\ \exp[\delta B(1)(r+s)^{½}]}{E_o\ |B(1)\ \exp[\delta B(1)r^{½}]} = 0.$$

So in the same way as for standard Brownian motion it follows that there exists an Itô-Poisson point process $\bar{N}$ on U whose $[\zeta > l]$ subsequence is the sequence of excursions of Y of length greater than l. An appropriate choice for r_1 yields the following formula for the characteristic measure $\bar{\nu}$ of $\bar{N}$:

$$\bar{\nu}[u(t_i) \in dy_i,\ i=1,\ldots,n]$$

$$= \bar{g}(t_1;0,y_1)\bar{q}(t_2-t_1;y_1,y_2)\ldots\bar{q}(t_n-t_{n-1};y_{n-1},y_n)dy_1\ldots dy_n$$

where $0 < t_1 < \ldots < t_n$ and $y_1,\ldots,y_n > 0$ (or $y_1,\ldots,y_n < 0$).

3.3 Feller's Brownian motions.

Let $B = (B_t)_{t\geq 0}$ be a standard one-dimensional Brownian motion on a probability space $(\Omega,\mathcal{F},P_o)$. The process $Y = (Y_t)_{t\geq 0}$ defined by $Y_t=|B_t|$ is called reflecting Brownian motion. Let $r > 0$ be given and let

V^Y(resp. V^B) be the first excursion from zero of the process Y (resp. B) with length greater than r. Then, for $n \geq 1$, $0 < t_1 < \ldots < t_n$ and $x_1, \ldots, x_n \in \mathbb{R}_+$, we have

$$P_o[V^Y(t_i) \in dx_i,\ i=1,\ldots,n]$$

$$= P_o[V^B(t_i) \in dx_i,\ i=1,\ldots,n] + P_o[-V^B(t_i) \in dx_i,\ i=1,\ldots,n]$$

$$= 2\left[\frac{\pi r}{2}\right]^{½} g(t_1;0,x_1)q(t_2-t_1;x_1,x_2)\ldots q(t_n-t_{n-1};x_{n-1},x_n)dx_1\ldots dx_n$$

since $g(t;0,x)=g(t;0,-x)$ and $q(t;x,y)=q(t;-x,-y)$. (See (3.1) for the definitions of $g(t;0,x)$ and $q(t;x,y)$.) It is now clear that the characteristic measure ν of the Itô-Poisson point process of excursions from zero of reflecting Brownian motion is given by

$$\nu[u(t_i) \in dx_i,\ i=1,\ldots,n]$$

$$= g(t_1;0,x_1)q(t_2-t_1;x_1,x_2)\ldots q(t_n-t_{n-1};x_{n-1},x_n)dx_1\ldots dx_n$$

where $0 < t_1 < \ldots < t_n$ and $x_1,\ldots,x_n \in \mathbb{R}_+$. The characteristic measure of the Itô-Poisson point process of excursions from zero of Brownian and reflecting Brownian motion correspond to the same semigroup $(K_t)_{t\geq 0}$ which is defined by

$$K_t(x,dy) = q(t;x,y)dy.$$

The entrance laws $(\eta_s)_{s>0}$ are given by

$$\eta_s(dy) = g(s;0,y)dy \text{ for Brownian motion}$$

and

$$\eta_s(dy) = 1_{]0,\infty[}(y)g(s;0,y)dy \text{ for reflecting Brownian motion.}$$

It is easy to see that all strong Markov processes Y, which behave like Brownian motion until the first hitting τ_o^Y of 0, i.e.

$$P_x[Y_{t_i} \in dy_i,\ i=1,\ldots,n,\ \tau_o^Y > t] = P_x[B_{t_i} \in dy_i,\ i=1,\ldots,n,\ \tau_o > t]$$

$$(0 \leq t_1 < \ldots < t_n \leq t),$$

have characteristic measures (for the Itô-Poisson point process of excursions from zero), which correspond to the semigroup (K_t). It is clear from the construction of these processes from Itô-Poisson point processes that the converse also holds. A problem extensively studied in the literature is to describe all the Ray processes on $[0,\infty[$, which behave like Brownian motion until the first hitting or approach of 0, see among others Feller [11], Itô-McKean [26] and Rogers [45]. It follows from an application of the strong Markov property that the resolvent $(V_\lambda)_{\lambda>0}$ of such a process satisfies the following formula

$$V_\lambda f(x) = G_\lambda f(x) + e^{-x\sqrt{2\lambda}}\, V_\lambda f(0)\ , \quad f \in C_0([0,\infty[),\ x \geq 0,$$

where

$$G_\lambda f(x) = \int_0^\infty e^{-\lambda t}\, K_t f(x)dt$$

$$= E_x \int_0^{\tau_0} e^{-\lambda t} f(B_t)dt.$$

Rogers [45] gives the following characterization of $V_\lambda f(0)$: there exist non-negative constants p_1, p_2, p_3 and a non-negative measure p_4 on $]0,\infty[$ such that

$$\int_{]0,\infty[} p_4(dx)(1-e^{-x}) < \infty$$

and such that

$$V_\lambda\ f(0) = \frac{2p_2 \int_0^\infty e^{-x\sqrt{2\lambda}} f(x)dx + p_3 f(0) + \int_{]0,\infty[} p_4(dx) G_\lambda f(x)}{p_1 + p_2\sqrt{2\lambda} + \lambda p_3 + \int_{]0,\infty[} p_4(dx)(1-e^{-x\sqrt{2\lambda}})}.$$

Actually we should have considered these processes on the one-point compactification $[0,\infty]$ of $[0,\infty[$, the point ∞ playing the role of a cemetery, where the process is sent to when killed. We have left this out to avoid annoying technicalities. Rogers derived this formula from the resolvent equation for the process Y and he remarks that the constants p_1, p_2 and p_3 and the measure p_4 have natural interpretations in excursion theory. Indeed, let for $s > 0$ and $n \geq 0$ the measure ϵ_{xs} on $[0,\infty[$ be defined by

$$\epsilon_{xs}(dy) = q(s;x,y)dy \qquad \text{for } x > 0$$

and

$$\epsilon_{0s}(dy) = g(s;0,y)dy.$$

The families $(\epsilon_{xs})_{s>0}$, $x \geq 0$, of finite measures on $[0,\infty[$ are entrance laws for the semigroup (K_t). According to theorem (2.2.4), the semigroup (K_t) and the entrance law $(\epsilon_{xs})_{s>0}$ determine a unique σ-finite measure ν_x on the excursion space $(U_\infty, \mathcal{U}_\infty)$ satisfying property (i), (ii) and (iii) of theorem (2.2.3) such that

$$\epsilon_{xs}(dy) = \nu_x([\zeta_u > s,\ u(s) \in dy]).$$

It is clear that ν_o is the characteristic measure of the Itô-Poisson point process of excursions from zero of standard Brownian motion, see section (3.1) and for $x > 0$ the measure ν_x is identical to the distribution α_x on $(U_\infty, \mathcal{U}_\infty)$ of standard Brownian motion started from x, which is absorbed in state 0. Let p be a nonnegative measure on $[0,\infty[$ such that $p([x,\infty[) < \infty$ for every $x > 0$. Define the measure ν on $(U_\infty, \mathcal{U}_\infty)$ by

$$\nu = \int_{[0,\infty[} p(dx)\nu_x.$$

Then the family $(\eta_s)_{s>0}$ defined by

$$\begin{aligned}\eta_s(dy) &= \nu([\zeta_u > s,\ u(s) \in dy]) \\ &= \int_{[0,\infty[} p(dx)\epsilon_{xs}(dy)\end{aligned}$$

is an entrance law for the semigroup (K_t). For $\lambda>0$ and bounded, measurable functions f on $[0,\infty[$ we have

$$\begin{aligned}\hat{\eta}_\lambda(f) &= \int_0^\infty ds e^{-\lambda s} \int \eta_s(dy)f(y) \\ &= p(\{0\}) \int_0^\infty e^{-y\sqrt{2\lambda}}\, f(y)dy + \int_{]0,\infty[} p(dx)G_\lambda f(x).\end{aligned}$$

It follows that

$$\begin{aligned}\int_U (1-e^{-\zeta_u})\nu(du) &= \hat{\eta}_1(1) \\ &= \frac{1}{\sqrt{2}}\, p(\{0\}) + \int_{]0,\infty[} p(dx)(1-e^{-x\sqrt{2}}).\end{aligned}$$

Let P be the Itô-Poisson process on $[0,\infty[$ with characteristic measure

ν. As in section (2.3) we assume the finiteness of $\int_U (1-e^{-\zeta_u})\nu(du)$ to guarantee that the sum $A(\tau)$ of the lengths of the excursions up to time τ is finite. This condition is equivalent to the following condition on the measure p

$$\int_{]0,\infty[} p(dx)(1-e^{-x\sqrt{2}}) < \infty.$$

Let $\gamma \geq 0$ and define as in section (2.3)

$$B(\tau,\omega) = \sigma_0(\omega) + A(\tau,\omega) + \gamma\tau.$$

Finally, let $\delta \geq 0$ be the killing-rate in local time at state 0. If $(V_\lambda)_{\lambda>0}$ is the resolvent of the strong Markov process attached to P then by theorem (2.3.11) we have

$$V_\lambda f(x) = \begin{cases} G_\lambda f(x) + e^{-x\sqrt{2\lambda}}\, V_\lambda f(0) & \text{for } x \neq 0 \\[2ex] \dfrac{p(\{0\}) \int_0^\infty e^{-y\sqrt{2\lambda}} f(y)dy + \int_{]0,\infty[} p(dx) G_\lambda f(x) + \gamma f(0)}{\delta + \lambda\gamma + \frac{1}{2} p(\{0\})\sqrt{2\lambda} + \int_{]0,\infty[} p(dx)(1-e^{-x\sqrt{2\lambda}})} & \text{for } x = 0. \end{cases}$$

It follows that p_1 is the killing-rate in local time at state 0. The constant p_3 corresponds to γ, which is a measure for the stickiness at state 0 as will be explained in section (4.2). Further, it is easy to see that ν_x is concentrated on the set of excursions $\{u \in U_\infty : u(0)=x\}$, so $p_4(dx)$ is the rate in local time at zero by which there appear excursions from zero starting at x. The constant $2p_2$ is the rate in local time at zero by which the process exits zero continuously.

3.4 Brownian motion on an n-pod.

In this example we will construct Markov processes on an n-pod E_n. As a set E_n is defined by

$$E_n =]0,\infty[\times \{1,\dots,n\} \cup \{0\}.$$

Let $d_n : E_n \times E_n \to \mathbb{R}$ be defined by

$$d_n[(x,i),\ (y,j)] = \begin{cases} x+y & \text{for } i \neq j \\ |x-y| & \text{for } i = j \end{cases},$$

$$d_n[0,(x,i)] = x,$$

$$d_n[0,0] = 0.$$

The function d_n is a metric on E_n and the topological space (E_n, d_n) is a locally compact, second countable Hausdorff space. The topological space E_n is called an n-pod. Denote by $\mathcal{E}_n$ the Borel σ-algebra on E_n and by A_k, $k=1,\dots,n$, the subset

$A_k = \{e \in E_n : e = 0 \text{ or } e = (x,k) \text{ for some } x > 0\}$

of E_n. The subset A_k is called the k^{th} axis of E_n. It is clear that E_2 is homeomorphic to the real line $\mathbb{R}$ and that A_i is homeomorphic to $[0,\infty[$. We want to consider strong Markov processes Y on E_n, which behave like Brownian motion until the first hitting or approach σ_o of state 0, i.e.

$$P_{(x,i)}([Y_t \in (y,y+dy)\times\{j\},\ \sigma_o > t]) = \begin{cases} q(t;x,y)\lambda_i(dy) & \text{for } j=i \\ 0 & \text{for } j\neq i \end{cases}$$

where λ_i is the image of the Lebesgue measure on $[0,\infty[$ under the map $\varphi_i : [0,\infty[\to A_i$ defined by

$$\varphi_i(x) = \begin{cases} (x,i) & \text{for } x > 0 \\ 0 & \text{for } x = 0 \end{cases}.$$

and where $q(t;x,y)$ is defined as in section (3.1).

Define for $t > 0$ the kernel P_t^n on $(E_n, \mathcal{E}_n)$ by

$$P_t^n(e,F) = \int_0^\infty 1_F((y,i))q(t;x,y)\lambda_i(dy) + 1_F(0)\{1-\int_0^\infty q(t;x,y)\lambda_i(dy)\} \text{ if } e=(x,i)$$

and $P_t^n(0,F) = 1_F(0)$.

The family of kernels $(P_t^n)_{t\geq 0}$, where P_o^n is the identity kernel on $(E_n, \mathcal{E}_n)$, is a Markov semigroup of kernels on $(E_n, \mathcal{E}_n)$ which can be shown to correspond to a Feller-Dynkin semigroup on $C_o(E_n)$ (see Williams [59], p.115 for the definition of a Feller-Dynkin semigroup). Let α_e, $e \in E_n$, be the measure on the function space $U^{(n)} = D_{E_n}([0,\infty[)$ (see appendix A2) whose finite dimensional distributions are given by

$$\alpha_e[u(t_i) \in de_i,\ i=1,\dots,m]$$
$$= P^n_{t_1}(e,de_1)P^n_{t_2-t_1}(e_1,de_2)\dots p^n_{t_m-t_{m-1}}(e_{m-1},de_m)$$

where $m\geq 1$, $0\leq t_1\leq\dots\leq t_m$ and $e_1,\dots,e_m \in E_n$. The measure α_e is concentrated on the set $\{u \in U^{(n)}:\ u(t)=0 \text{ for all } t\geq\zeta_u\}$ where ζ_u is defined by $\zeta_u=\inf\{t>0 : u(t)=0 \text{ or } u(t-)=0\}$. Denote by $(K^n_t)_{t\geq 0}$ the sub-Markov semigroup of kernels on $(E_n,\mathcal{E}_n)$ given by

$$K^n_t(e,dy) = \alpha_e[u(t)\in dy,\ \zeta_u>t]$$
$$= \begin{cases} q(t;x,y)\lambda_i(dy) & \text{for } e=(x,i) \\ 0 & \text{for } e=0 \end{cases}.$$

Let Y be a strong Markov process on E_n, which behaves like Brownian motion until the first hitting or approach of 0, and let υ be the characteristic measure of the Itô-Poisson point process of excursions from zero of the process Y. It follows then that the measure υ is determined by an entrance law $(\eta_s)_{s>0}$ for the semigroup (K^n_t) with $\eta_s(\{0\})=0$ for every $s>0$ (same reasoning as for reflecting Brownian motion, see section (3.3)). We have

$$\int\eta_s(de)K^n_t(e,dy) = \sum_{i=1}^{n}\int_0^\infty\eta^i_s(dx)q(t;x,y)\lambda_i(dy) = \sum_{i=1}^{n} 1_{A_i}(y)\eta_{s+t}(dy)$$

where $\eta^i_s = \varphi_i^{-1}[1_{A_i}\cdot\eta_s]$ is the φ_i^{-1}-image of the restriction of η_s to the axis A_i. It follows that for $i=1,\dots,n$

$$\left(\int_0^\infty\eta^i_s(dx)q(t;x,y)\right)dy = \eta^i_s(dy).$$

As in section (3.3) there exist measures $p^{(i)}$, $i=1,\dots,n$ on $[0,\infty[$ such that

$$\int_0^\infty p^{(i)}(dx)(1-e^{-x}) < \infty$$

and

$$\eta^i_s(dy) = \int_{[0,\infty[} p^{(i)}(dx)\epsilon_{xs}(dy),$$

where ϵ_{xs} is defined as in section (3.3). It follows that the resolvent

$(V_\lambda)_{\lambda>0}$ of the strong Markov process Y is given by

$$V_\lambda f(x,i) = \int_0^\infty G_\lambda(x,dy)f(y,i) + e^{-x\sqrt{2\lambda}}V_\lambda f(0)$$

with

$$V_\lambda f(0) = \frac{\sum_{i=1}^n p^{(i)}(\{0\}) \int_0^\infty e^{-y\sqrt{2\lambda}} f(y,i)dy + \sum_{i=1}^n \int_{]0,\infty[} p^{(i)}(dx)\int_0^\infty G_\lambda(x,dy)f(y,i) + \gamma f(0)}{\delta + \lambda\gamma + \frac{1}{2}\sqrt{2\lambda}\sum_{i=1}^n p^{(i)}(\{0\}) + \sum_{i=1}^n \int_{]0,\infty[} p^{(i)}(dx)(1-e^{-x\sqrt{2\lambda}})}$$

where γ and δ are nonnegative constants and $f \in b\mathcal{E}_n$.

As a special case take $n=2$, $\delta=\gamma=p^{(1)}(]0,\infty[)=p^{(2)}(]0,\infty[)=0$ and $\alpha + \beta > 0$ where $\alpha=p^{(1)}(\{0\})$ and $\beta=p^{(2)}(\{0\})$. Then we get

$$V_\lambda f(0) = \left[\frac{2}{\lambda}\right]^{1/2} \frac{1}{\alpha+\beta} \{\alpha \int_0^\infty e^{-y\sqrt{2\lambda}} f(y,1)dy + \beta \int_0^\infty e^{-y\sqrt{2\lambda}} f(y,2)dy\}.$$

If we map E_2 on $\mathbb{R}$ via the map ψ defined by

$$\psi(y,1) = y$$

$$\psi(y,2) = -y$$

$$\psi(0) = 0$$

and if we use again f for the map $f\circ\psi$ on $\mathbb{R}$, we get after some straightforward calculation that

$$V_\lambda f(0) = \left[\frac{2}{\lambda}\right]^{1/2} \frac{1}{\alpha+\beta} \{\alpha \int_0^\infty e^{-y\sqrt{2\lambda}} f(y)dy + \beta \int_0^\infty e^{-y\sqrt{2\lambda}} f(-y)dy\}$$

and

$$V_\lambda f(x) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\lambda}} [e^{-\sqrt{2\lambda}|x-y|} + \text{sign}(y) \frac{\alpha-\beta}{\alpha+\beta} e^{-\sqrt{2\lambda}(|x|+|y|)}] f(y)dy$$

for all $x \in \mathbb{R}\backslash\{0\}$, in which

$$\text{sgn}(y) = \begin{cases} -1 & \text{for } y < 0 \\ 0 & \text{for } y = 0 \\ 1 & \text{for } y > 0 \end{cases}.$$

This is the resolvent of so-called skew Brownian motion, see Itô and McKean [27], p. 115. The numbers $\frac{\alpha}{\alpha+\beta}$ and $\frac{\beta}{\alpha+\beta}$ may be interpreted as the probabilities for an excursion on the right- resp. left hand side of 0, see Harrison and Shepp [21] and section (4.2).

In [12] Frank and Durham give an intuitive description of symmetric Brownian motion on a 3-pod, which corresponds to the case $\delta=\gamma=0$ and $p^{(1)}=p^{(2)}=p^{(3)}=\frac{1}{3}\delta_0$. We end this section by remarking that a similar construction is possible for processes which behave outside zero like Brownian motion with constant drift.

3.5 A remark by Blumenthal's construction of the Markov process attached to an Itô-Poisson point process.

In [2] Blumenthal constructs for a given characteristic measure and entrance law the extension of the original process whose entrance law is the given one, claiming that this construction is the one that Itô was referring to in [25]. Let E be a compact metric space and let $a \in E$ be a fixed point. Let $X = (X_t)_{t\geq 0}$ be a Ray process with space E. Blumenthal considers the case $E = [0,\infty[$, $a=0$ and X is a standard Markov process. By a standard construction for Markov processes, X can be considered as a Ray process on the one-point compactification $[0,\infty]$ of $[0,\infty[$, see Getoor [15]. Denote as in section (2.2) by α_x the distribution on $U = D_E([0,\infty[)$ of the process X absorbed in a after the first hitting or approach σ_a of the point a and by $(K_t)_{t\geq 0}$ the sub-Markov semigroup of kernels on $(E,\mathcal{E})$ defined by

$$\alpha_t(x,dy)=\alpha_x[u(t)\in dy,\ \zeta_u > t].$$

Blumenthal's construction is based on an approximation with Markov processes of the following type. Starting at a point $x \in E\backslash\{a\}$ the process evolves according to the transition probabilities of the process X until reaching the state a where it waits a length of time having an exponential distribution with mean $\alpha>0$ after which time it jumps independently to a new position in $E\backslash\{a\}$ according to a given probability distribution η and then proceeds as before. The measure η is called the jumping in measure and α is called the holding parameter. For the existence of such a Markov process Blumenthal refers to Meyer [41]. A standard calculation using the strong Markov property yields for the resolvent $(U_\lambda)_{\lambda>0}$ of this process the formula

$$U_\lambda f(x) = G_\lambda f(x) + E_x(e^{-\lambda\sigma_a}) \frac{\frac{1}{\alpha} f(a) + \int\eta(dy)G_\lambda f(y)}{\frac{1}{\alpha}\lambda + \int\eta(dy)E_y(1-e^{-\lambda\sigma_a})}$$

where

$$G_\lambda f(x) = \int_0^\infty e^{-\lambda t}K_t f(x)dt \text{ and } f \in b\mathcal{E}.$$

It is not difficult to construct this process with the methods of section (2.3). Define the family of finite measures $(\eta_s)_{s>0}$ on $(E,\mathcal{E})$ by

$$\eta_s(dy) = \int\eta(dx)K_s(x,dy).$$

It is clear that the family $(\eta_s)_{s>0}$ is an entrance law for the semigroup (K_t) satisfying the property $\eta_s(\{a\})=0$ for every $s>0$. Let ν be the σ-finite measure on $(U,\mathcal{U})$ corresponding to the entrance law (η_s) and the semigroup (K_t), see theorem (2.2.4), and let P be the Itô-Poisson point process on U with characteristic measure ν. Let Y be the Markov process attached to P as in section (2.3), then by theorem (2.3.8) the resolvent $(V_\lambda)_{\lambda>0}$ of Y is given by

$$V_\lambda f(x) = G_\lambda f(x) + E_x(e^{-\lambda\sigma_a}) \frac{\hat{\eta}_\lambda(f) + \gamma f(a)}{\lambda\gamma + \lambda\hat{\eta}_\lambda(1)}, \quad f \in b\mathcal{E}.$$

Since

$$\hat{\eta}_\lambda(f) = \int_0^\infty dt\, e^{-\lambda t} \int\eta(dx)K_t f(x) = \int\eta(dx)G_\lambda f(x)$$

and, for $x\neq a$,

$$\lambda G_\lambda 1(x) = \int_0^\infty \lambda e^{-\lambda t}K_t 1(x)dt$$

$$= \int_0^\infty \lambda e^{-\lambda t} E_x(1_{[\sigma_a > t]})dt$$

$$= E_x \int_0^{\sigma_a} \lambda e^{-\lambda t} dt$$

$$= E_x(1-e^{-\sigma_a}),$$

the process Y with $\gamma = \frac{1}{\alpha}$ is the above described Markov process. The strong Markov property for Y follows from the assumptions in Blumenthal [2] about the resolvent $(G_\lambda)_{\lambda>0}$, see theorem (2.3.9).

CHAPTER 4

RANDOM WALK APPROXIMATIONS

In chapter 3 we derived an expression for the characteristic measure ν of the Itô-Poisson point process P of excursions from zero of reflecting Brownian motion. Let e_u be the sign of excursion u:

$e_u = -1$ or $+1$ according as $u(t) < 0$ or > 0 for $t \in]0,\zeta_u[$. .

The measure ν is concentrated on the set of excursions with sign +1. Consider the product $U \times \{-1,1\}$ of the excursion space U and the set $\{-1,1\}$. Let p be a probability distribution on $\{-1,1\}$ and let ν' be the image of the product measure $\nu \otimes p$ under the map $(u,e) \in U \times \{-1,1\} \to e \cdot u \in U$. Denote by P' the Itô-Poisson point process with characteristic measure ν'; one should think about P' as the point process derived from P by changing independently the signs of the excursions with probability $1-\alpha = p(\{-1\})$. It turns out that the Markov process attached to P' is skew Brownian motion X^α, see section (3.4). In [21] Harrison and Shepp gave a random walk approximation of the process X^α. They considered a Markov chain $(S_n)_{n\geq 0}$ on $\mathbb{Z}$ with $S_o = 0$ and with transition probabilities given by

$$P[S_{n+1} = S_n+1 \,|\, S_n] = P[S_{n+1} = S_n-1 \,|\, S_n] = \frac{1}{2} \qquad \text{if } S_n \neq 0,$$
$$P[S_{n+1} = 1 \,|\, S_n = 0] = \alpha,$$
$$P[S_{n+1} = -1 \,|\, S_n = 0] = 1-\alpha.$$

Let now $X_n = (X_n(t))_{t\geq 0}$ be the process defined by

$$X_n(t) = n^{-\frac{1}{2}} \cdot S_{[nt]}.$$

Harrison and Shepp proved in [21] among other things the weak convergence of the sequence $\{X_n,\ n \geq 1\}$ to the process X^α. They conclude their article with a remark that this result can be extended to a Markov chain with a more general type of behaviour at the origin. We will consider the following behaviour at the origin:

$$P[S_{n+1} = k | S_n = 0] = p_k$$

with $\{p_k,\ k \in \mathbb{Z}\}$ a probability distribution on $\mathbb{Z}$.

In the construction of a Markov process from an Itô-Poisson point process P we added a linear term $\gamma\tau$ to the sum of the lengths of the excursions up to and including time τ. Section (4.3) is devoted to a random walk approximation of the process Y_γ constructed in this way from the Itô-Poisson point process of excursions from 0 of standard Brownian motion, see section (3.1). It will turn out that such an approximation has to be based on a sequence $S_n=(S_{nk})_{k\geq 0}$, n=1,2,... of random walks on $\mathbb{Z}$, with transition probabilities given by:

$$P[S_{n+1} = S_n+1 | S_n] = P[S_{n+1} = S_n-1 | S_n] = \frac{1}{2} \qquad \text{if } S_n \neq 0,$$

$$P[S_{n,k+1} = 0 | S_{n,k} = 0] = \gamma_n \quad ,$$

$$P[S_{n,k+1} = +\ 1 | S_{n,k} = 0] = \frac{1}{2}\,(1-\gamma_n) \ .$$

It will turn out that we have to take $\gamma_n = 1 - \dfrac{1}{1+\gamma\cdot n^{1/2}}$.

4.1 Approximation by discrete semigroups.

Let $S_n = (S_{nk})_{k\in\mathbb{N}}$, n=1,2,..., be a sequence of Markov chains on $\mathbb{Z}$ with transition matrices $P_n : \mathbb{Z} \times \mathbb{Z} \to [0,1]$. Define for $n \geq 1$ the process $X_n = (X_n(t))_{t\geq 0}$ with continuous time parameter by

$$X_n(t) = n^{-1/2}\cdot S_{n,[nt]}.$$

The probability distribution of the process X_n is completely determined by the transition matrix P_n and the initial distribution υ of the Markov chain S_n. We will denote the distribution of the process X_n by $P_\mu^{(n)}$, where μ is the distribution of $X_n(0)$: $\mu(\{k\cdot n^{-1/2}\}) = \upsilon(\{k\})$, $k \in \mathbb{Z}$. If $\mu=\delta_x$ we will write simply $P_x^{(n)}$. The measure $P_\mu^{(n)}$ is a probability measure on the space of càdlàg functions on $[0,\infty[$, which will be denoted by D in this chapter. See appendix A2. Let for a finite, strictly increasing sequence $(t_i)_{1\le i\le k}$ of nonnegative real numbers $\pi_{t_1\dots t_k}$ be the projection of D on $\mathbb{R}^k$:

$$\pi_{t_1\dots t_k}(u) = (u(t_1),\dots,u(t_k)),\quad u \in D.$$

Then the finite-dimensional distributions of $P_\mu^{(n)}$ are given by

$$\pi_{t_1\dots t_k} P_\mu^{(n)}(A)$$

$$= \sum_{m_o\in\mathbb{Z}} \mu(\{m_o\cdot n^{-1/2}\})\ P_n^{[nt_1]}(m_o,m_1) \prod_{i=2}^{k} P_n^{[nt_i]-[nt_{i-1}]}(m_{i-1},m_i)$$

where $A \in \mathcal{B}(\mathbb{R}^k)$, and where the summation is over all vectors $(m_o,\dots,m_k) \in \mathbb{Z}^{k+1}$ such that $n^{-1/2}\cdot(m_1,\dots,m_k) \in A$. Let $C_o(\mathbb{R})$ be the Banach space of (bounded) continuous functions on $\mathbb{R}$ which vanish at infinity, normed by the supnorm and let for $n \ge 1$

Σ_n be the space of functions f on the discrete space $n^{-1/2}\cdot\mathbb{Z}$ which vanish at infinity, normed by the supnorm $\|\cdot\|_n$:

$$\|f\|_n = \sup\{|f(x)| : x \in n^{-1/2}\cdot\mathbb{Z}\},\text{ and}$$

$\mathcal{P}_n$ be the operator from $C_o(\mathbb{R})$ to Σ_n which assigns to $f \in C_o(\mathbb{R})$ its restriction to $n^{-1/2}\cdot\mathbb{Z}$.

The sequence $(\Sigma_n, \mathcal{P}_n)_{n\ge1}$ is an approximation of the Banach space $C_o(\mathbb{R})$, see Kato [32], p. 512. Define for $n\ge1$ the operator U_n on Σ_n by

$$U_nf(x) = \int f[u(\tfrac{1}{n})]\ P_x^{(n)}(du)$$

$$= \sum_{j\in\mathbb{Z}} P_n(x\cdot n^{1/2},\ j)\ f(j\cdot n^{-1/2}),\qquad x \in n^{-1/2}\cdot\mathbb{Z},\ f \in \Sigma_n.$$

U_n is a bounded linear operator on Σ_n. Denote by $\mathcal{U}_n = (U_n(t))_{t\ge0}$ the extension to $t \in [0,\infty[$ of the discrete semigroup on Σ_n with time unit $\tau_n = \frac{1}{n}$ and with generator $n(U_n-I)$:

$$[U_n(t)f](x) = \int f\ d(\pi_tP_x^{(n)})$$

$$= \sum_{j\in\mathbb{Z}} (P_n)^{[nt]}(x\cdot n^{1/2}, j)f(j\cdot n^{-1/2}),\quad x \in n^{-1/2}\cdot\mathbb{Z},\ f \in \Sigma_n.$$

So $U_n = U_n(\tau_n)$. Let $T = (T(t))_{t\ge0}$ be a semigroup on $C_o(\mathbb{R})$. The sequence $(\mathcal{U}_n)_{n\ge1}$ is an approximation of the semigroup T if for every $t \in [0,\infty[$

and for every $f \in C_o(\mathbb{R})$ it is true that

$$\lim_{n\to\infty} \|U_n(k_n\cdot\tau_n)(\mathscr{P}_n f) - \mathscr{P}_n[T(t_o)f]\|_n = 0$$

for any sequence $\{k_n\}$ of nonnegative integers such that $k_n\cdot\tau_n \to t_o$. A necessary and sufficient condition for the sequence $(\mathscr{U}_n)_{n\geq 1}$ to be an approximation of the semigroup T is given by the theorem of Trotter-Kato, see Kato [32], p.511. With our notations this condition is that for some $\lambda > 0$

$$\lim_{n\to\infty} \| \int_0^\infty (1 + \frac{\lambda}{n})^{-[nt]-1} U_n(t)(\mathscr{P}_n f)dt - \mathscr{P}_n[\int_0^\infty e^{-\lambda t}T(t)f\ dt]\|_n = 0$$

for every $f \in C_o(\mathbb{R})$. Assume that the semigroup T is a Markov semigroup and that for every probability measure m on $\mathbb{R}$ there exists a (probability) measure P_m on D with finite-dimensional distributions given by

$$\pi_{t_1\ldots t_k} P_m(A)$$
$$= \int m(dx_o)\int T_{t_1}(x_o,dx_1) \int T_{t_2-t_1}(x_1,dx_2)\ldots$$
$$\int T_{t_k-t_{k-1}}(x_{k-1},dx_k)\ 1_A(x_1,\ldots,x_k),\quad 0\leq t_1< \ldots <t_k,\ A \in \mathscr{B}(\mathbb{R}^k),$$

where $T_t(x,dy)$ denotes the unique Markov kernel on $(\mathbb{R},\mathscr{B}(\mathbb{R}))$ such that

$$T(t)f(x) = \int T_t(x,dy)f(y),\ t \geq 0,\ x \in \mathbb{R} \text{ and } f \in C_o(\mathbb{R}).$$

With these notations and assumptions we can prove the following theorem.

4.1.1 Theorem. Let $(\upsilon_n)_{n\geq 1}$ be a sequence of probability measures with $\mathrm{supp}(\upsilon_n) \subset n^{-1/2}\cdot\mathbb{Z}$ and let m be a probability measure on $\mathbb{R}$. If $(\mathscr{U}_n)_{n\geq 1}$ is an approximation of T and if $\upsilon_n \Rightarrow m$ as $n\to\infty$, then

$$\pi_{t_1\ldots t_k} P^{(n)}_{\upsilon_n} \Rightarrow \pi_{t_1\ldots t_k} P_m \quad \text{as } n \to \infty$$

for every finite, strictly increasing sequence $(t_i)_{1\leq i\leq k}$ of nonnegative real numbers.

<u>Proof</u>. The proof is by induction on the number of elements of the sequence $(t_i)_{1\le i\le k}$. Let $t \ge 0$ and $f \in C_o(\mathbb{R})$. Then

$$\left|\int f\, d\pi_t P^{(n)}_{\upsilon_n} - \int f\, d\pi_t P_m\right|$$
$$= \left|\int U_n(t)(\mathscr{P}_n f)d\upsilon_n - \int T(t)f dm\right|$$
$$\le \left|\int U_n(t)(\mathscr{P}_n f)d\upsilon_n - \int U_n([nt]\cdot\tfrac{1}{n})(\mathscr{P}_n f)d\upsilon_n\right|$$
$$+ \left|\int U_n([nt]\cdot\tfrac{1}{n})(\mathscr{P}_n f)d\upsilon_n - \int \mathscr{P}_n[T(t)f]d\upsilon_n\right|$$
$$+ \left|\int \mathscr{P}_n[T(t)f]d\upsilon_n - \int T(t)f dm\right|$$
$$\le 0 + \|U_n([nt]\cdot\tfrac{1}{n})(\mathscr{P}_n f) - \mathscr{P}_n[T(t)f]\|_n$$
$$+ \left|\int T(t)f\, d\upsilon_n - \int T(t)f\, dm\right| \to 0 \quad \text{as } n \to \infty,$$

the second term since (U_n) approximates T, the third term since (υ_n) converges weakly to m.

Hence

$$\pi_t P^{(n)}_{\upsilon_n} \Rightarrow \pi_t P_m \qquad \text{as } n\to\infty,$$

which proves the theorem for sequences of length one.

Assume that the theorem is correct for sequences of length k. Let $0 \le t_1 < \ldots < t_k < t_{k+1}$ and let $f_1,\ldots,f_{k+1} \in C_o(\mathbb{R})$. Denoting the function $(x_1,\ldots,x_{k+1}) \in \mathbb{R}^{k+1} \to \prod_{i=1}^{k+1} f_i(x_i)$ by $\bigotimes_{i=1}^{k+1} f_i$ we get:

$$\left|\int \bigotimes_{i=1}^{k+1} f_i\, d\pi_{t_1\ldots t_k t_{k+1}} P^{(n)}_{\upsilon_n} - \int \bigotimes_{i=1}^{k+1} f_i\, d\pi_{t_1\ldots t_k t_{k+1}} P_m\right|$$
$$= \left|\int f_1\otimes\ldots\otimes f_{k-1} \otimes (f_k\cdot U_n(\tfrac{[nt_{k+1}]-[nt_k]}{n})\mathscr{P}_n f_{k+1})\, d\pi_{t_1\ldots t_k} P^{(n)}_{\upsilon_n}\right.$$
$$\left. - \int f_1\otimes\ldots\otimes f_{k-1} \otimes (f_k\cdot T(t_{k+1}-t_k)f_{k+1})\, d\pi_{t_1\ldots t_k} P_m\right|$$

(by the simple Markov property)

$$\le \|f_1\|_\infty\cdot\ldots\cdot\|f_k\|_\infty\cdot\|U_n(\tfrac{[nt_{k+1}]-[nt_k]}{n})\, \mathscr{P}_n f_{k+1} - \mathscr{P}_n[T(t_{k+1}-t_k)f_{k+1}]\|_n$$
$$+ \left|\int f_1\otimes\ldots\otimes f_{k-1}\otimes(f_k\cdot T(t_{k+1}-t_k)f_{k+1})\, d(\pi_{t_1\ldots t_k} P^{(n)}_{\upsilon_n} - \pi_{t_1\ldots t_k} P_m)\right|$$
$$\to 0 \text{ as } n \to \infty,$$

the first term since (U_n) approximates T and the second by the induction hypothesis. □

4.2 Skew Brownian Motion.

In [27] Itô and McKean introduced skew Brownian motion as an example of a diffusion process. They defined skew Brownian motion X^α as follows: consider standard Brownian motion and alter independently of the process the excursions away from zero, each excursion being positive with probability α and negative with probability $\beta = 1-\alpha$; α is a parameter, $\alpha \in [0,1]$. In [52] Walsh gives an expression for the transition density $q^\alpha(t;x,y)$ of X^α:

$$q^\alpha(t;x,y) = p(t;x,y) + (\alpha-\beta)\ \mathrm{sgn}(y)\ p(t;0,|x|+|y|), t>0,\ x,y \in \mathbb{R},$$

where

$$p(t;x,y) = \left[\frac{1}{2\pi t}\right]^{1/2} \exp\left[-\frac{1}{2t}(x-y)^2\right],$$

$$\mathrm{sgn}(y) = \begin{cases} -1 & \text{for} \quad y < 0 \\ 0 & \text{for} \quad y = 0 \\ 1 & \text{for} \quad y > 0. \end{cases}$$

It follows by direct calculation that $(q^\alpha(t;.,.))_{t>0}$ is a Markov transition density on $\mathbb{R}$, i.e.

(i) $q^\alpha(t;x,y) \geq 0$;

(ii) $q^\alpha(t;.,.)$ is $\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R})$ - measurable;

(iii) $\int_{-\infty}^{+\infty} q^\alpha(t;x,y)\ dy = 1$;

(iv) $\int_{-\infty}^{+\infty} q^\alpha(s;x,y)\ q^\alpha(t;y,z)\ dy = q^\alpha(s+t;x,z)$.

Let $T_\alpha(t)$, $t > 0$, be the integral operator with kernel $q^\alpha(t;.,.)$ on the Banach space B of bounded, Borel measurable functions f on $\mathbb{R}$, normed by the supnorm:

$$(T_\alpha(t)f)(x) = \int_{-\infty}^{+\infty} f(y)q^\alpha(t;x,y)dy,\ x \in \mathbb{R}.$$

With $T_\alpha(0)$ the identity operator on B, the family $T_\alpha = (T_\alpha(t))_{t\geq 0}$ is a Markov semigroup on B. Remember that a one-parameter family $(P_t)_{t\geq 0}$ of bounded linear operators on B is called a Markov semigroup on B if

(i) $\forall t \geq 0,\ \forall f \in B,\ 0 \leq f \leq 1 \Rightarrow 0 \leq P_t f \leq 1$;

(ii) $\forall t \geq 0,\ P_t 1 = 1$;

(iii) $\forall s \geq 0,\ \forall t \geq 0,\ P_{s+t} = P_s P_t$;

(iv) $\forall t \geq 0,\ \forall (f_n)_{n\geq 1} \subset B$: $f_n \downarrow 0 \Rightarrow P_t f_n \downarrow 0$ (pointwise).

A Markov semigroup $(P_t)_{t\geq 0}$ on B is called a Feller-Dynkin semigroup on $C_o(\mathbb{R})$ if we have in addition that

(v) $\forall t \geq 0,\ P_t : C_o(\mathbb{R}) \to C_o(\mathbb{R})$;

(vi) $\forall f \in C_o(\mathbb{R}),\ \lim_{t\downarrow 0} \|P_t f - f\|_\infty = 0$.

So a Feller-Dynkin semigroup is a strongly continuous Markov semigroup of linear operators on $C_o(\mathbb{R})$. See Williams [59] for further information.

4.2.1 Proposition. The semigroup T_α is a Feller-Dynkin semigroup on $C_o(\mathbb{R})$.

Proof. We only need to prove that $T_\alpha(t)$ is an operator on $C_o(\mathbb{R})$ and that T_α is strongly continuous. For $f \in C_o(\mathbb{R})$ and $x \in \mathbb{R}$ we have

$$T_\alpha(t)f(x)$$

$$= \int_{-\infty}^{+\infty} p(t;x,y)f(y)dy + (\alpha-\beta)\{\int_0^\infty p(t;0,|x|+y)f(y)dy - \int_{-\infty}^{+\infty} p(t;0,|x|-y)f(y)dy\}$$

$$= \int_{-\infty}^{+\infty} p(t;0,y)f(x+y)dy + (\alpha-\beta)\{\int_{|x|}^{\infty} p(t;0,y)f(y-|x|)dy - \int_{-\infty}^{-|x|} p(t;0,y)f(y+|x|)dy\}.$$

Since the integrands are all bounded by the integrable function $\|f\|_\infty p(t;0,.)$, we may take limits for the variable x under the integral sign and from this it can easily be seen that $T_\alpha(t)f \in C_o(\mathbb{R})$.

To prove the strong continuity of the semigroup, we first note that for $f \in C_o(\mathbb{R})$ and $x \in \mathbb{R}$

$$T_\alpha(t)f(x) - f(x)$$

$$= \int_{-\infty}^{+\infty} p(t;x,y)[f(y)-f(x)]dy + (\alpha-\beta)\int_{-\infty}^{+\infty} \mathrm{sgn}(y)p(t;0,|x|+|y|)[f(y)-f(x)]dy$$

$$= \int_{-\infty}^{+\infty} p(t;0,y)[f(x+y)-f(x)]dy + (\alpha-\beta) \int_0^{\infty} p(t;0,|x|+y)[f(y)-f(-y)]dy.$$

As $f \in C_0(\mathbb{R})$, f is uniformly continuous. Choose $\epsilon>0$. There exists a $\delta>0$ such that $|y-y'| < 2\delta \Rightarrow |f(y)-f(y')|< \epsilon$. Then:

$$|T_\alpha(t)f(x) - f(x)|$$
$$\leq \epsilon \int_{-\delta}^{\delta} p(t;0,y)dy + 4\|f\|_\infty \int_{\delta}^{\infty} p(t;0,y)dy + |\alpha-\beta|\epsilon \int_0^{\delta} p(t;0,|x|+y)dy$$
$$+ 2|\alpha-\beta| \, \|f\|_\infty \int_{\delta}^{\infty} p(t;0,|x|+y)dy$$
$$\leq (\epsilon+|\alpha-\beta|\epsilon) + (4\|f\|_\infty + 2|\alpha-\beta|\|f\|_\infty) \int_{\delta\cdot t^{-1/2}}^{\infty} p(1;0,y)dy.$$

It follows that $\lim_{t\downarrow 0} \|T_\alpha(t)f-f\|_\infty \leq 3\epsilon$ for every $\epsilon>0$ which implies the strong continuity of T_α.□

From proposition (4.2.1) follows the existence of a strong Markov process with transition density $q^\alpha(t;x,y)$ and with right continuous paths. In fact with probability 1 this Markov process has continuous paths as follows from:

$$\left| \int_{\mathbb{R}-]x-\epsilon,x+\epsilon[} q^\alpha(t;x,y)dy \right|$$
$$\leq 2\left(\int_{-\infty}^{x-\epsilon} + \int_{x+\epsilon}^{\infty} \right) p(t;x,y)dy = 4 \int_{\epsilon\cdot t^{-1/2}}^{\infty} p(1;0,y)dy = o(t) \text{ as } t \downarrow 0.$$

See Dynkin [8], chapter three, theorem (3.5). So $q^\alpha(t;x,y)$ is the transition density of a diffusion process. In the next two propositions we give expressions for the resolvent R^α_λ and the infinitesimal generator G_α of the semi-group T_α.

4.2.2 Proposition. Let $\lambda > 0$, $f \in C_0(\mathbb{R})$ and $x \in \mathbb{R}$. Then

$$R^\alpha_\lambda f(x) = \int_0^{\infty} e^{-\lambda t} T_\alpha(t)f(x)dt$$

$$= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\lambda}} [e^{-\sqrt{2\lambda}|x-y|} + \mathrm{sgn}(y)(\alpha-\beta)\, e^{-\sqrt{2\lambda}(|x|+|y|)}]f(y)dy.$$

Proof. The result follows immediately from the formula:

$$\int_0^\infty e^{-\lambda t} p(t;x,y)dt = \frac{1}{\sqrt{2\lambda}} e^{-\sqrt{2\lambda}|x-y|}. \square$$

4.2.3 Proposition. Let G_α be the infinitesimal generator of the semigroup T_α. Then

$$\mathcal{D}(G_\alpha) = \{h \in C_o(\mathbb{R}) : h^{(2)} \text{ exists on } \mathbb{R}\setminus\{0\};\ \alpha h'(0+) = \beta h'(0-);\ h^{(2)}(0+) = h^{(2)}(0-);\ h^{(2)} \text{ continuous on } \mathbb{R}\setminus\{0\};\ \lim_{|x|\to\infty} h^{(2)}(x)=0\}$$

$$(G_\alpha h)(x) = \begin{cases} \frac{1}{2} h^{(2)}(x) & \text{for } x \neq 0, \\ \frac{1}{2} h^{(2)}(0+) & \text{for } x = 0 \end{cases} \qquad h \in \mathcal{D}(G_\alpha).$$

Proof. Let $h \in \mathcal{D}(G_\alpha)$. Since $\mathcal{D}(G_\alpha) = R^\alpha_\lambda(C_o(\mathbb{R}))$, there exists an $f \in C_o(\mathbb{R})$ such that $h = R^\alpha_\lambda f \in C_o(\mathbb{R})$.

It follows that

$$h(x) = \frac{1}{\sqrt{2\lambda}} e^{-x\sqrt{2\lambda}} \int_{-\infty}^{x} e^{y\sqrt{2\lambda}} f(y)dy + \frac{1}{\sqrt{2\lambda}} e^{x\sqrt{2\lambda}} \int_{x}^{\infty} e^{-y\sqrt{2\lambda}} f(y)dy$$
$$+ \frac{1}{\sqrt{2\lambda}} e^{-|x|\sqrt{2\lambda}} \int_{-\infty}^{+\infty} (\alpha-\beta)\mathrm{sgn}(y)\, e^{-|y|\sqrt{2\lambda}} f(y)dy.$$

By direct calculation we find:

$h^{(2)}$ exists on $R\setminus\{0\}$;

$\alpha h'(0+) = \beta h'(0-)$;

$h^{(2)}(0+) = h^{(2)}(0-)$;

$h^{(2)}(x) = 2\lambda h(x) - 2f(x)$ for $x \neq 0$;

$h^{(2)}$ is continuous on $\mathbb{R}\setminus\{0\}$ and $\lim_{|x|\to\infty} h^{(2)}(x) = 0$.

From $(\lambda-G_\alpha)R_\lambda^\alpha f = f$ we conclude that

$$G_\alpha h(x) = \frac{1}{2} h^{(2)}(x) \text{ for } x \neq 0.$$

Further:

$$G_\alpha h(0) = \lim_{t\downarrow 0} \frac{1}{t} [T_\alpha(t)h(0)-h(0)]$$

$$= \lim_{t\downarrow 0} \{\int_{-\infty}^{0} \frac{2\beta}{t} p(t;0,y)[h(y)-h(0)]dy + \int_{0}^{\infty} \frac{2\alpha}{t} p(t;0,y)[h(y)-h(0)]dy\}.$$

Now we use a Taylor expansion for h:

$$h(y) = h(0) + yh'(0+) + \frac{1}{2} y^2 h^{(2)}(0+) + \epsilon(y)y^2 \text{ for } y > 0$$

$$h(y) = h(0) + yh'(0-) + \frac{1}{2} y^2 h^{(2)}(0-) + \epsilon(y)y^2 \text{ for } y < 0$$

where $\epsilon(y) \to 0$ as $y \to 0$.

Using $\alpha h'(0+) = \beta h'(0-)$ and $h^{(2)}(0-) = h^{(2)}(0+)$ we get:

$$G_\alpha h(0) = \lim_{t\downarrow 0} \{2\beta h'(0-)\int_{-\infty}^{+\infty} \frac{1}{t} p(t;0,y)y\, dy + (\alpha+\beta)h^{(2)}(0+) \int_{0}^{\infty} y^2\frac{1}{t} p(t;0,y)dy$$

$$+ \int_{-\infty}^{0} \frac{2\beta}{t} p(t;0,y)\epsilon(y)y^2 dy + \int_{0}^{\infty} \frac{2\alpha}{t} p(t;0,y)\epsilon(y)y^2 dy\}$$

$$= \frac{1}{2} h^{(2)}(0+)$$

since $\epsilon(y)$ is bounded by $\|h^{(2)}\|_\infty$ and tends to 0 for $y \to 0$.

It follows that the operator C defined by

$$\mathfrak{D}(C) = \{h \in C_0(\mathbb{R}) : h^{(2)} \text{ exists on } \mathbb{R}\backslash\{0\};\ \alpha h'(0+) = \beta h'(0-);$$
$$h^{(2)}(0+) = h^{(2)}(0-);\ h^{(2)} \text{ continuous on } \mathbb{R}\backslash\{0\};$$
$$\lim_{|x|\to\infty} h^{(2)}(x)=0\}$$

$$Cf(x) = \begin{cases} \frac{1}{2} h^{(2)}(x) & \text{for } x \neq 0 \\ \frac{1}{2} h^{(2)}(0+) & \text{for } x = 0 \end{cases} \qquad h \in \mathfrak{D}(C),$$

is an extension of G_α. Let $h \in \mathfrak{D}(C)$ and let $Ch = h$. By direct calculation it follows that $h(x) \equiv 0$. From the corollary to theorem 1.1 in Dynkin [8] chapter one we conclude that $C = G_\alpha$.□

In what follows, we will use without further mentioning the notations and definitions of section (4.1). In [21] Harrison & Shepp gave a random walk approximation of the process X^{α}. Let S_n be the Markov chain on $\mathbb{Z}$ starting in 0 with transition matrix $P_n=P$ not depending on n defined by

$$P(m,m-1) = P(m,m+1) = \frac{1}{2} \qquad \text{for } m \neq 0 ,$$
$$P(0,k) = p_k \qquad \text{for } k \in \mathbb{Z}$$

where $\{p_k : k \in \mathbb{Z}\}$ is a probability distribution on $\mathbb{Z}$. Harrison & Shepp proved in [21] that for the case

$p_1 = \alpha,\ p_{-1} = \beta\ ,\ \alpha+\beta = 1,$

the sequence of probability measures $(P_0^{(n)})$ on D converges weakly to the distribution P_0^{α} of the Feller-Dynkin process starting in 0 with transition semigroup T_{α}. They conclude their article by remarking that this result can be proven for more general probability distributions $\{p_k : k \in \mathbb{Z}\}$. With these notations and definitions we will prove the following theorem.

4.2.4 <u>Theorem</u>. Let $(\upsilon_n)_{n\geq 1}$ be a sequence of probability measures on $\mathbb{R}$ with $\text{supp}(\upsilon_n) \subset n^{-1/2}\cdot\mathbb{Z}$ converging weakly to a probability measure m on $\mathbb{R}$. If $0 < \Sigma|k|p_k < \infty$, then for every $k \geq 1$ and $0 \leq t_1 < \ldots < t_k$ the finite dimensional distributions $\pi_{t_1\ldots t_k} P_{\upsilon_n}^{(n)}$ converge weakly to the finite dimensional distribution $\pi_{t_1\ldots t_k} P_m^{\alpha}$ as $n \to \infty$,

where in $\alpha = \dfrac{\Sigma k^+ p_k}{\Sigma k p_k}$ and P_m^{α} is the distribution of the Feller-Dynkin process with initial distribution m and with transition semigroup T_{α}.

<u>Proof</u>. Once we have proved that the sequence $(\mathcal{U}_n)_{n\geq 1}$ is an approximation of the semigroup T_{α} for a probability distribution $\{p_k : k \in \mathbb{Z}\}$ satisfying the inequality $0 < \Sigma|k|p_k < \infty$, statement (i) follows directly from theorem (4.1.1). □

So the proof of theorem (4.2.4) is complete once we have proved that the sequence $(\mathcal{U}_n)_{n\geq 1}$ approximates T_α. The rest of this section is devoted to this proof which is split up in a series of lemmas.

4.2.5 Approximation lemma. If $0 < \Sigma|k|p_k < \infty$, then the sequence $(\mathcal{U}_n)_{n\geq 1}$ is an approximation of the semigroup T_α, i.e. for each $t_o \in [0,\infty)$ and for each $f \in C_o(\mathbb{R})$ it is true that

$$\lim_{n\to\infty} \|U_n(k_n\cdot\frac{1}{n})(\mathcal{P}_n f) - \mathcal{P}_n[T_\alpha(t_o)f]\|_n = 0$$

for any sequence $\{k_n\}$ of nonnegative integers such that $k_n\cdot\frac{1}{n} \to t_o$ as $n \to \infty$.

To prove the approximation lemma we use the theorem of Trotter-Kato which says that the sequence $(\mathcal{U}_n)$ is an approximation of T_α if for some $\lambda > 0$ the next statement holds:

$$\lim_{n\to\infty} \| \int_0^\infty (1 + \frac{\lambda}{n})^{-[nt]-1} U_n(t)(\mathcal{P}_n f)dt - \mathcal{P}_n(\int_0^\infty e^{-\lambda t}T_\alpha(t)f\ dt)\|_n = 0$$

for every $f \in C_o(\mathbb{R})$.

From now on $\lambda > 0$ is fixed. We introduce the following notation

$$I_\alpha f = \int_0^\infty e^{-\lambda t}T_\alpha(t)f\ dt \qquad \text{for } f \in C_o(\mathbb{R})$$

$$I_n f = \int_0^\infty (1+ \frac{\lambda}{n})^{-[nt]-1} U_n(t)f\ dt \qquad \text{for } f \in \Sigma_n.$$

Our first step consists of calculating $I_n f$.

4.2.6 Lemma. Let $\lambda > 0$, $f \in \Sigma_n$ and $x \in n^{-1/2}\cdot \mathbb{Z}$ then

$$(I_n f)(x) = \frac{1}{\lambda+n} \sum_{j\in\mathbb{Z}} G_{(1+\frac{\lambda}{n})^{-1}} (x\cdot n^{1/2}, j)f(j\cdot n^{-1/2})$$

where

$$G_u(i,j) = \sum_{k\geq 0} (uP)^k(i,j), \quad i,j \in \mathbb{Z}.$$

<u>Proof</u>. $\int_0^\infty (1+\frac{\lambda}{n})^{-[nt]-1} U_n(t)f(x)dt$

$$= \int_0^\infty (1+\frac{\lambda}{n})^{-[nt]-1} (\sum_{j\in\mathbb{Z}} P^{[nt]}(x\cdot n^{1/2}, j)f(j\cdot n^{-1/2}))dt$$

$$= \sum_{k=0}^\infty \int_{[\frac{k}{n},\frac{k+1}{n}[} (1+\frac{\lambda}{n})^{-k-1} (\sum_{j\in\mathbb{Z}} P^k(x\cdot n^{1/2}, j)f(j\cdot n^{-1/2}))dt$$

$$= \sum_{k=0}^\infty \sum_{j\in\mathbb{Z}} \frac{1}{n} (1+\frac{\lambda}{n})^{-k-1} P^k(x\cdot n^{1/2}, j)f(j\cdot n^{-1/2})$$

$$= \frac{1}{\lambda+n} \sum_{j\in\mathbb{Z}} G_{(1+\frac{\lambda}{n})^{-1}} (x\cdot n^{1/2}, j)f(j\cdot n^{-1/2}). \square$$

For the calculation of the potential matrix G_u we proceed in the following way. Let $Z = (Z_n, n\in\mathbb{N})$ be the Markov chain starting in $x \in \mathbb{Z}$ with transition matrix uP, $0 < u < 1$. (see page 117 for the definition of P). For $y \in \mathbb{Z}$ we define N_y as the total number of visits of the process Z to the state y:

$$N_y = \sum_{k=0}^\infty 1_{\{y\}} \circ Z_k.$$

Then we have

$$E_x(N_y) = \sum_{k=0}^\infty P_x[Z_k = y]$$

$$= \sum_{k=0}^\infty (uP)^k (x,y) = G_u(x,y).$$

An application of the strong Markov property gives:

$$P_x[N_y \geq k] = P_x[\sigma_y < \infty] (P_y[\sigma_y^1 < \infty])^{k-1}$$

where $\sigma_y = \inf \{n \geq 0 : Z_n = y\}$

$\sigma_y^1 = \inf \{n \geq 1 : Z_n = y\}$

$\inf \phi = +\infty$.

We conclude that

$$G_u(x,y) = E_x[N_y]$$

$$= \sum_{k\geq 1} P_x[N_y \geq k]$$

$$= \frac{P_x[\sigma_y < \infty]}{1-P_y[\sigma_y^1 < \infty]}.$$

Put $h(x,y) = P_x[\sigma_y < \infty]$, $x,y \in \mathbb{Z}$.

The function h has the following properties:

(i) $h(x,y) = \frac{1}{2} u\, h(x-1,y) + \frac{1}{2} u\, h(x+1,y)$, $x,y \in \mathbb{Z}$, $x \neq 0,y$ ("Markov property");

(ii) $h(y,y) = 1$;

(iii) $h(0,y) = \sum_{k\in\mathbb{Z}} u p_k h(k,y)$, $y \neq 0$;

(iv) $0 \leq h(x,y) \leq 1$.

Fix y. *Consider first the case* $y < 0$.

Equation (i) is a difference equation with characteristic equation

$$\frac{1}{2} u\, z^2 - z + \frac{1}{2} u = 0.$$

The solutions of this equation are ξ and ξ^{-1} where $\xi = \frac{1}{u}(1- (1-u^2)^{½})$.

It follows that $0 < \xi < 1$ and that h must be of the following form:

$$h(x,y) = \begin{cases} A_y\xi^x + B_y\xi^{-x} & \text{for } x \leq y \\ D_y\xi^x + C_y\xi^{-x} & \text{for } y \leq x \leq 0, \\ E_y\xi^x + F_y\xi^{-x} & \text{for } x \geq 0 \end{cases}$$

where $A_y,\ldots,F_y$ are constants, only depending on y and satisfying the relations

$$A_y\xi^y + B_y\xi^{-y} = C_y\xi^{-y} + D_y\xi^y$$

$$C_y + D_y = E_y + F_y.$$

Property (iv) implies that

$$A_y = F_y = 0$$

and it follows from property (ii) that

$$C_y\ \xi^{-y} + D_y\ \xi^y = 1.$$

It follows that for $y < 0$

$$h(x,y) = \begin{cases} \xi^{y-x} & \text{for } x \leq y \\ C_y\xi^{|x|} + D_y\xi^x & \text{for } x \geq y \end{cases}$$

and : $C_y\xi^{-y} + D_y\xi^{y} = 1$.

Analogous calculations yield for y=0

$$h(x,0) = \xi^{|x|}$$

and for y > 0

$$h(x,y) = \begin{cases} C_y\xi^{|x|} + D_y\xi^{-x} & \text{for } x \leq y \\ \xi^{x-y} & \text{for } x \geq y \end{cases}$$

and : $C_y\ \xi^{y} + D_y\ \xi^{-y} = 1$.

4.2.7 <u>Lemma</u>

$$G_u(x,y) = \frac{\xi^{|x-y|}}{\Delta} + \gamma_y \frac{\xi^{|x|+|y|}}{\Delta}.$$

where: $\gamma_y = \dfrac{\xi^{-|y|}\ \Sigma up_k\xi^{|k-y|} - 1 + \Delta 1_{\{y=0\}}}{1 - \Sigma up_k\xi^{|k|}}$,

$$\Delta = (1-u^2)^{½};$$

$$\xi = \frac{1}{u}(1-\Delta).$$

<u>Proof</u>. An application of the Markov property gives for y < 0:

$$\begin{aligned} P_y[\sigma_y^1<\infty] &= \frac{1}{2}uP_{y+1}[\sigma_y<\infty] + \frac{1}{2}uP_{y-1}[\sigma_y < \infty] \\ &= \frac{1}{2}uh(y+1,y) + \frac{1}{2}uh(y-1,y) \\ &= \frac{1}{2}u(C_y\xi^{-y-1} + D_y\xi^{y+1}) + \frac{1}{2}u\xi \\ &= \frac{1}{2}(1+\Delta)C_y\xi^{-y} + \frac{1}{2}(1-\Delta)D_y\xi^{y} + \frac{1}{2}(1-\Delta) \\ &= 1 - \Delta D_y\xi^{y}. \end{aligned}$$

In the same way we get for y = 0:

$$P_o[\sigma_o^1 < \infty] = \Sigma\ up_k\xi^{|k|}$$

and for y > 0

$$P_y[\sigma_y^1 < \infty] = 1 - \Delta D_y\xi^{-y}.$$

Substitution of the expressions we have found for $P_x[\sigma_y < \infty]$ (=h(x,y)) and $P_y[\sigma_y^1 < \infty]$ in the formula

$$G_u(x,y) = \frac{P_x[\sigma_y<\infty]}{1-P_y[\sigma_y^1<\infty]}$$

yields

$$G_u(x,y) = \frac{\xi^{|x-y|}}{\Delta} + \gamma_y \frac{\xi^{|x|+|y|}}{\Delta}, \quad \text{where } \gamma_y = \frac{C_y}{D_y}.$$

Substitution in property (iii) of the function h gives us an expression for γ_y. We take again the case $y < 0$:

$$h(0,y) = C_y + D_y$$

$$h(0,y) = \Sigma_k up_k h(k,y) \quad \text{(condition (iii))}$$

$$= \Sigma_{k\leq y} up_k \xi^{y-k} + \Sigma_{k>y} up_k[C_y\xi^{|k|} + D_y\xi^k]$$

$$C_y\xi^{-y} + D_y\xi^y = 1.$$

Hence

$$\gamma_y + 1 = \frac{1}{D_y} \Sigma_{k\leq y} up_k \xi^{y-k} + \Sigma_{k>y} up_k\xi^k + \gamma_k \Sigma_{k>y} up_k \xi^{|k|}$$

and

$$\gamma_y\xi^{-y} + \xi^y = \frac{1}{D_y}.$$

So

$$\gamma_y = \frac{\Sigma_{k\leq y}up_k\xi^{2y-k} + \Sigma_{k>y}up_k\xi^k - 1}{1 - \Sigma up_k \xi^{|k|}}$$

$$= \frac{\xi^{-|y|}\Sigma\, up_k\xi^{|k-y|} - 1}{1 - \Sigma up_k\xi^k}.$$

The same kind of calculation for y=0 and y>0 gives the result.□

Combination of lemma (4.2.6) and lemma (4.2.7) gives us the following formula for $I_n(f)$:

For $f \in \Sigma_n$ and $x \in n^{-1/2}\cdot\mathbb{Z}$

$$(I_nf)(x) = \int_{-\infty}^{+\infty} k_n(x,y)f(y_n)dy$$

where

$$y_n = \frac{[y\cdot n^{1/2}]}{n^{1/2}},$$

$$k_n(x,y) = \left[2\lambda + \frac{\lambda^2}{n}\right]^{-1/2} \left(\xi_n^{|x-y_n|n^{1/2}} + \gamma_{[y\cdot n^{1/2}]}\, \xi_n^{(|x|+|y_n|)n^{1/2}}\right).$$

$$\xi_n = 1 + \frac{\lambda}{n} - [(\frac{\lambda}{n})^2 + \frac{2\lambda}{n}]^{½},$$

$$\gamma_{[y\cdot n^{½}]} = \frac{\xi_n^{-|[y\cdot n^{½}]|}\ \Sigma u p_k \xi_n^{|k-[y\cdot n^{½}]|} - 1 + \Delta\, 1_{\{y=0\}}}{1 - \Sigma u p_k \xi_n^{|k|}}$$

with $u = (1 + \frac{\lambda}{n})^{-1}$.

Denote the kernel of the resolvent of the semigroup T_α by $k(x,y)$,

$$k(x,y) = \frac{1}{\sqrt{2\lambda}} (e^{-\sqrt{2\lambda}|x-y|} + (\alpha-\beta)\operatorname{sgn}(y)e^{-\sqrt{2\lambda}(|x|+|y|)}).$$

The following lemma contains estimates for ξ_n^x and $\gamma_{[y\cdot n^{½}]}$.

4.2.8 Lemma.

(i) $\lim_{n\to\infty} \{\sup_{x\geq 0} n^{1-\delta}|\xi_n^x - \exp(-x\left[\frac{2\lambda}{n}\right]^{½})|\} = 0$ for every $\delta > 0$.

(ii) For every $\delta > 0$, $\eta > 0$ and $M > 0$, there exists a number N such that for all $n \geq N$ we have:

$$\sup_{y\in[-M,M]\setminus]-\eta,\eta[} |\gamma_{[y\cdot n^{½}]} - (\alpha-\beta)\operatorname{sgn}(y)| < \delta.$$

Moreover there exists a constant V such that

$$|\gamma_{[y\cdot n^{½}]} - (\alpha-\beta)\operatorname{sgn}(y)| \leq V \text{ for all } y \in [-M,M].$$

Proof. See Appendix A4. □

Proof of the approximation lemma (4.2.5).

Note that it is sufficient to prove the lemma for $f \in C_o(\mathbb{R})$ with compact support. Indeed let $f,g \in C_o(\mathbb{R})$. If $\|f-g\|_\infty < \epsilon$ then

$$\begin{aligned}
&\|I_n(\mathcal{P}_n f) - \mathcal{P}_n(I_\alpha f)\|_n \\
&\leq \|I_n(\mathcal{P}_n f) - I_n(\mathcal{P}_n g)\|_n + \|I_n(\mathcal{P}_n g) - \mathcal{P}_n(I_\alpha g)\|_n + \|\mathcal{P}_n(I_\alpha g) - \mathcal{P}_n(I_\alpha f)\|_n \\
&\leq \|I_n(\mathcal{P}_n g) - \mathcal{P}_n(I_\alpha g)\|_n + \epsilon \int_0^\infty [(1+\frac{\lambda}{n})^{-[nt]-1} + e^{-\lambda t}]dt \\
&\leq \|I_n(\mathcal{P}_n g) - \mathcal{P}_n(I_\alpha g)\|_n + \frac{2}{\lambda}\epsilon.
\end{aligned}$$

As the functions with compact support are dense in $C_o(\mathbb{R})$, we can restrict ourselves to $f \in C_0(\mathbb{R})$ with compact support. From now on we fix a function $f \in C_0(\mathbb{R})$ with compact support, say

$$f(y) = 0 \text{ for } |y| > M.$$

Let $\epsilon > 0$ and $\eta > 0$. From lemma (4.2.8) it follows that for n sufficiently large:

1. $|\gamma_{[y\cdot n^{1/2}]} - (\alpha-\beta)\text{sgn}(y)| \le \epsilon$, for all $y \in [-M,M]\backslash]-\eta,\eta[$;
2. $\sup_{x,y} |\xi_n^{|x-y_n|n^{1/2}} - e^{-|x-y_n|(2\lambda)^{1/2}}| \le \epsilon$,

   $\sup_{x,y} |\xi_n^{(|x|+|y_n|)n^{1/2}} - e^{(|x|+|y_n|)(2\lambda)^{1/2}}| \le \epsilon$;
3. $| (2\lambda)^{-1/2} - \left[2\lambda + \frac{\lambda^2}{n}\right]^{1/2} | \le \epsilon$;
4. $0 < \xi_n < 1$;
5. $|\gamma_{[y\cdot n^{1/2}]} - (\alpha-\beta) \text{ sgn}(y)| \le V$ for all $y \in [-M,M]$.

Hence

$$|k_n(x,y) - k(x,y_n)| \le \begin{cases} 5\epsilon(2\lambda)^{-1/2} & \text{for } x \in \mathbb{R},\ y \in [-M,M]\backslash-]\eta,\eta[\\ (4\epsilon+V)(2\lambda)^{-1/2} & \text{for } x \in \mathbb{R},\ y \in -]\eta,\eta[. \end{cases}$$

It follows that for all n sufficiently large

$$\begin{aligned} &\|I_n(\mathcal{P}_n f) - \mathcal{P}_n(I_\alpha f)\|_n \\ &= \sup_{x \in n^{-1/2}\cdot\mathbb{Z}} \left| \int_{-\infty}^{+\infty} k_n(x,y)f(y_n)dy - \int_{-\infty}^{+\infty} k(x,y)f(y)dy \right| \\ &\le \sup_{x \in n^{-1/2}\cdot\mathbb{Z}} \left| \int_{-\infty}^{\infty} (k_n(x,y)-k(x,y_n))f(y_n)dy \right| \\ &\qquad + \left\| \int_{-\infty}^{+\infty} k(\cdot,y_n)f(y_n)dy - \int_{-\infty}^{+\infty} k(\cdot,y)f(y)dy \right\|_\infty \\ &\le 10(2\lambda)^{-1/2} M\|f\|_\infty\epsilon + 2(2\lambda)^{-1/2} (4\epsilon + V)\eta\|f\|_\infty + \epsilon. \end{aligned}$$

Since ϵ and η are arbitrarily chosen,

$$\lim_{n\to\infty} \|I_n(\mathcal{P}_n f) - \mathcal{P}_n(I_\alpha f)\|_n = 0$$

which completes the proof of the lemma. □.

4.3 Stickiness.

In section (3.1) we derived for standard Brownian motion $(B_t)_{t\geq 0}$ an explicit formula for the finite-dimensional distributions of the characteristic measure ν of the Itô-Poisson point process of excursions from zero:

$$\nu[u(t_i) \in dx_i,\ i=1,\dots,n]$$
$$= g(t_1;0,x_1)q(t_2-t_1;x_1,x_2)\dots q(t_n-t_{n-1};x_{n-1},x_n)dx_1\dots dx_n$$

for $0 < t_1 < \dots < t_n$ and $x_1,\dots,x_n > 0$ (or $x_1,\dots,x_n < 0$). See section (3.1) for the definitions of $g(t;0,x)$ and $q(t;x,y)$. Let for $s > 0$ the measure η_s on $\mathbb{R}$ be defined by

$$\eta_s(dy) = \nu[u(s) \in dy,\ \zeta_u > s]$$
$$= g(s;0,y)dy.$$

The measure η_s is a finite measure: $\eta_s(\mathbb{R}) = 2(2\pi s)^{-1/2}$.
Let, as in section (2.2), for $t \geq 0$ the kernel K_t on $\mathbb{R}$ be defined by

$$K_t(x,dy) = P_x[B_t \in dy,\ \sigma_0 > t].$$

The family of kernels $(K_t)_{t\geq 0}$ is a sub-Markov semigroup on $\mathbb{R}$ and the family of finite measures $(\eta_s)_{s>0}$ on $\mathbb{R}$ is an entrance law for the semigroup (K_t) satisfying $\eta_s(\{0\}) = 0$ for every $s > 0$. The resolvent $(G_\lambda)_{\lambda>0}$ of the semigroup (K_t) is given by

$$G_\lambda f(x) = \frac{1}{\sqrt{2\lambda}} \int_0^{+\infty} (e^{-\sqrt{2\lambda}|x-y|} - e^{-\sqrt{2\lambda}|x+y|})f(y)dy \qquad \text{for } x > 0,$$

and

$$G_\lambda f(x) = \frac{1}{\sqrt{2\lambda}} \int_{-\infty}^{0} (e^{-\sqrt{2\lambda}|x-y|} - e^{-\sqrt{2\lambda}|x+y|})f(y)dy \qquad \text{for } x < 0.$$

Note that we can write:

$$G_\lambda f(x) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\lambda}} (e^{-\sqrt{2\lambda}|x-y|} + e^{-\sqrt{2\lambda}(|x|+|y|)})f(y)dy, \quad x \in \mathbb{R}. \qquad (1)$$

The measure α_x (see section (2.2),p.60) is the distribution of Brownian motion absorbed in 0. Let P be the Itô-Poisson point process with characteristic measure ν and let Y_γ be the Markov process constructed starting from the family of point processes $(P_x)_{x\in\mathbb{R}}$ where the term $\gamma\tau$ has been added to the sum $(\sigma_a(\omega)+A(\tau,\omega))$ of the lengths of the excursions up to time τ, see section (2.3). In this case we have

$$\int_U (1-e^{-\zeta_u})\nu(du) = \hat{\eta}(1)$$

$$= \int_0^\infty e^{-s}\, 2(2\pi s)^{-1/2}\, ds = 2^{1/2}.$$

Let $(V_\lambda^\gamma)_{\lambda>0}$ be the resolvent of the process Y_γ.

4.3.1 Proposition. For $\lambda > 0$ and f a bounded, measurable function on $\mathbb{R}$ we have:

$$V_\lambda^\gamma f(x) = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\lambda}} e^{-\sqrt{2\lambda}|x-y|} f(y)dy - \frac{\lambda\gamma}{\lambda\gamma+\sqrt{2\lambda}} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\lambda}} e^{-\sqrt{2\lambda}(|x|+|y|)} f(y)dy + \frac{\gamma}{\lambda\gamma+\sqrt{2\lambda}} e^{-\sqrt{2\lambda}|x|}\, f(0).$$

Proof. By theorem (2.3.6) we have

$$V_\lambda^\gamma f(x) = \begin{cases} G_\lambda f(x) + z_\lambda(x) V_\lambda f(0) & \text{for } x \neq 0 \\ \dfrac{\hat{\eta}_\lambda(f)+\gamma f(0)}{\lambda\gamma + \lambda\hat{\eta}_\lambda(1)} & \text{for } x = 0 \end{cases} \qquad (2)$$

with

$$z_\lambda(x) = \int \alpha_x(du)\, e^{-\lambda\zeta_u}.$$

It is well known that for Brownian motion we have

$$z_\lambda(x) = e^{-\sqrt{2\lambda}|x|}, \qquad (3)$$

see Williams [59], p. 85. For $y \neq 0$ and $\lambda > 0$

$$\int_0^\infty e^{-\lambda t} g(t;0,y)dt = e^{-\sqrt{2\lambda}|y|}.$$

Hence $\hat{\eta}_\lambda(f) = \int_0^\infty e^{-\lambda t} \int_{-\infty}^{+\infty} \eta_t(dy) f(y) dt$

$$= \int_{-\infty}^{+\infty} e^{-\sqrt{2\lambda}|y|} f(y)dy. \tag{4}$$

Substitution of (3) and (4) in (2) yields:

$$V_\lambda^\gamma f(x) = G_\lambda f(x) + e^{-\sqrt{2\lambda}|x|} \frac{\int_{-\infty}^{+\infty} e^{-\sqrt{2\lambda}|y|} f(y)dy + \gamma f(0)}{\lambda\gamma + \sqrt{2\lambda}}.$$

The result follows from this expression after substitution of (1) for $G_\lambda f(x)$.□

Note that for $\gamma=0$ we get the resolvent of standard Brownian motion. In this case the construction given in section (2.3) is a pathwise reconstruction of the original process B.

For $\gamma=+\infty$, the family (V_λ^∞) is the resolvent of Brownian motion absorbed in 0. Finally, we note that one can also verify directly from the formula for V_λ^γ in proposition (4.3.1) that the family $(V_\lambda^\gamma)_{\lambda>0}$ satisfies the resolvent equation. Since $(V_\lambda^0)_{\lambda>0}$ is the resolvent of standard Brownian motion, it is a strongly continuous Markov resolvent on $C_0(\mathbb{R})$, and it follows from the formula

$$V_\lambda^\gamma f(x) = V_\lambda^0 f(x) - \frac{\lambda\gamma}{\lambda\gamma+\sqrt{2\lambda}} \frac{1}{\sqrt{2\lambda}} e^{-\sqrt{2\lambda}|x|} \int_{-\infty}^{+\infty} e^{-|y|\sqrt{2\lambda}}[f(y)-f(0)]dy,$$

that $(V_\lambda^\gamma)_{\lambda>0}$ is also a strongly continuous Markov resolvent on $C_0(\mathbb{R})$. The theorem of Hille-Yosida implies the existence of a unique strongly continuous Markov semigroup $\{Q_t^\gamma,\ t \geq 0\}$ on $C_0(\mathbb{R})$ such that

$$\int_0^\infty e^{-\lambda t}\, Q_t^\gamma f dt = V_\lambda^\gamma f \quad , \qquad \lambda > 0,\ f \in C_0(\mathbb{R}).$$

See Williams [59], p.110. Let A_γ be the infinitesimal generator of the semigroup $(Q_t^\gamma)_{t\geq0}$.

4.3.2 Proposition. If $0 < \gamma < \infty$ and

$$D = \{h \in C_0(\mathbb{R}) : h^{(2)} \text{ exists on } \mathbb{R}\backslash\{0\};\ h'(0+) = h'(0-) + \gamma h^{(2)}(0+);$$
$$h^{(2)}(0+) = h^{(2)}(0-);\ h^{(2)} \text{ continuous on } \mathbb{R}\backslash\{0\};$$
$$\lim_{|x|\to\infty} h^{(2)}(x) = 0\},$$

then $\mathcal{D}(A_\gamma) = D$ and for $h \in D$

$$(A_\gamma h)(x) = \begin{cases} \frac{1}{2} h^{(2)}(x) & \text{for } x \neq 0 \\ \frac{1}{2} h^{(2)}(0+) & \text{for } x = 0 \end{cases}.$$

Proof. Fix $\lambda > 0$.

$$\mathcal{D}(A_\gamma) = V_\lambda^\gamma(C_0(\mathbb{R})).$$

Let $h \in \mathcal{D}(A_\gamma)$, say $h = V_\lambda^\gamma f$, $f \in C_0(\mathbb{R})$. By direct calculation we find that $h \in D$ and

$$h^{(2)}(x) = 2\lambda h(x) - 2f(x) \text{ for } x \neq 0.$$

From $(\lambda - A_\gamma)V_\lambda^\gamma f = f$ we conclude that

$$(A_\gamma h)(x) = \begin{cases} \frac{1}{2} h^{(2)}(x) & \text{for } x \neq 0 \\ \frac{1}{2} h^{(2)}(0+) & \text{for } x = 0 \end{cases}.$$

It follows that the operator C defined by

$$\mathcal{D}(C) = D$$

$$Ch(x) = \begin{cases} \frac{1}{2} h^{(2)}(x) & \text{for } x \neq 0 \\ \frac{1}{2} h^{(2)}(0+) & \text{for } x = 0 \end{cases}.$$

is an extension of A_γ.

Let $h \in \mathcal{D}(C)$ and $Ch = h$. By direct calculation it follows that $h = 0$. From the corollary to theorem 1.1 in Dynkin [8] chapter one we conclude that $C = A_\gamma$. □

Remember the definition of the complementary error function

$$\mathrm{erfc}\ (x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-v^2} dv.$$

Define the function $\epsilon : \mathbb{R} \to \mathbb{R}$ by

$$\epsilon(x) = e^{x^2} \mathrm{erfc}\ (x)\ , \quad x \in \mathbb{R}.$$

4.3.3 Proposition. Let $f \in C_0(\mathbb{R})$ and $0 < \gamma < \infty$. Then

$$Q_t^\gamma f(x) = \int_{-\infty}^{+\infty} \{p(t;x,y)-p(t;0,|x|+|y|)[1 - \frac{\sqrt{2\pi t}}{\gamma}\, \epsilon\left[\frac{|x|+|y|}{\sqrt{2t}} + \frac{\sqrt{2}t}{\gamma}\right]]\} f(y)dy$$
$$+ \sqrt{2\pi t}\ p(t;0,x)\epsilon\left[\frac{|x|}{\sqrt{2t}} + \frac{\sqrt{2}t}{\gamma}\right] f(0).$$

Proof. By inversion of the Laplace transform $V_\lambda^\gamma f$ of $Q_t^\gamma f$. □

Before giving a random walk approximation of the process Y_γ, we will discuss some properties of the family of processes $(Y_\gamma)_{0\leq\gamma\leq\infty}$. In the first place consider the behaviour at state 0. For $\gamma > 0$ the probability of the event $[Y_\gamma(t)=0]$ and the expected sojourn time in 0 up to time t are positive. If we start the process Y_γ in 0 we get:

$$P_0[Y_\gamma(t)=0] = \epsilon(\frac{\sqrt{2t}}{\gamma})$$

and

$$E_0 \int_0^s 1_{\{0\}}(Y_\gamma(t))dt = \frac{\gamma^2}{2}\, [\epsilon(\frac{\sqrt{2s}}{\gamma}) - 1] + \gamma\left[\frac{2s}{\pi}\right]^{1/2}.$$

On the other hand, the set $\{t \geq 0 : Y_\gamma(t) = 0\}$ does not contain intervals of positive length P_0-a.s.. Indeed, since the characteristic measure ν of the underlying Itô-Poisson point process has infinite mass, the range R of the function B (see section (2.3), p. 71) does not contain intervals of positive length. The processes Y_γ have continuous realizations. This can be seen from the construction in section (2.3), but it can also be verified directly from the transition semigroup (Q_t^γ). Indeed, since $0 < \epsilon(x) < 1$ for every $x \in\ ]0,\infty[$, we have for

$t < \frac{\gamma^2}{2\pi}$, $\delta>0$ and $x\neq 0$

$$(Q_t^\gamma 1_{\mathbb{R}\setminus]x-\delta,x+\delta[})(x) \le \left(\int_{-\infty}^{x-\delta} + \int_{x+\delta}^{\infty}\right)p(t;x,y)dy + \exp\left[-\frac{x^2}{2t}\right]$$
$$= 2\int_{\delta\cdot t^{-1/2}}^{\infty} p(1;0,y)dy + \exp\left[-\frac{x^2}{2t}\right]$$

and for x=0 we get the same estimation but without the term $\exp\left[-\frac{x^2}{2t}\right]$.
It follows that

$$(Q_t^\gamma 1_{\mathbb{R}\setminus]x-\delta,x+\delta[})(x) = o(t) \quad \text{as } t \downarrow 0$$

and it follows by Dynkin [8], chapter 3, theorem (3.5) that Y_γ has continuous realizations. A straightforward, but tedious calculation yields that all moments of $Y_\gamma(t)$ exist, especially

$$E_x(Y_\gamma(t)) = x$$

and

$$E_x[(Y_\gamma(t)-x)^2] = t - \int_0^t (2\pi s)^{1/2} p(s;0,x)\ \epsilon\left[\frac{\sqrt{2s}}{\gamma} + \frac{|x|}{2s}\right] ds$$
$$- x^2\ (2\pi t)^{1/2}\ p(t;0,x)\epsilon\left[\frac{\sqrt{2t}}{\gamma} + \frac{|x|}{2t}\right].$$

It follows by the inequality of Cauchy-Schwarz that

$$E_x[(Y_\gamma(t)-x)^4] \le t^2$$

and

$$E_x[(Y_\gamma(t_2)-Y_\gamma(t_1))^4] \le (t_2-t_1)^2.$$

As a consequence the family $(Y_\gamma)_{0\le\gamma\le\infty}$ is tight, see Billingsley [1], p.95 and appendix A2. Let $(\gamma_n)_{n\ge1}$ be a convergent sequence in $[0,\infty]$, say $\gamma_n \to \gamma_0$. Denote the distribution of the process Y_γ with initial distribution μ by $P_\mu^{(\gamma)}$.

As $\|V_\lambda^\gamma f - V_\lambda^{\gamma'} f\|_\infty \le \frac{1}{\sqrt{2\lambda}} \left| \frac{\lambda\gamma}{\lambda\gamma+\sqrt{2\lambda}} - \frac{\lambda\gamma'}{\lambda\gamma'+\sqrt{2\lambda}} \right| \frac{4}{\sqrt{2\lambda}} \|f\|_\infty$, the finite dimensional distributions of the sequence $(P_\mu^{(\gamma_n)})_{n\ge1}$ converge to those of $P_\mu^{(\gamma_o)}$ and we may conclude that $\gamma_n \to \gamma_o$ as $n \to \infty$ implies that

$$P_\mu^{(\gamma_n)} \to P_\mu^{(\gamma_o)} \qquad \text{as } n \to \infty.$$

We continue with a random walk approximation of the process Y_γ. Let $S_n=(S_{nk})_{k\geq 0}$ be a Markov chain on $\mathbb{Z}$ with transition matrix $P_n:\mathbb{Z} \times \mathbb{Z} \to [0,1]$ given by

$$P_n(i,i+1) = P_n(i,i-1) = \frac{1}{2} \text{ for } i \neq 0,$$

$$P_n(0,0) = \alpha_n,\ 0 < \alpha_n < 1,$$

$$P_n(0,1) = P_n(0,-1) = \frac{1}{2}(1-\alpha_n).$$

Define the process $X_n=(X_n(t))_{t\geq 0}$ by

$$X_n(t) = n^{-1/2} \cdot S_{n,[nt]},\ t \geq 0.$$

Denote as in section (4.1) the distribution of the process X_n by $P_\mu^{(x)}$, where μ is the distribution of $X_n(0)$; if υ_n is the initial distribution of the Markov chain S_n, then $\mu(\{k\cdot n^{-1/2}\}) = \upsilon_n(\{k\})$, $k \in \mathbb{Z}$. If $P_\upsilon^{(\gamma)}$ is the distribution of Y_γ and if $\upsilon_n \Rightarrow \upsilon$, we want to choose the parameters α_n in such a way that $P_{\upsilon_n}^{(n)} \Rightarrow P_\upsilon^{(\gamma)}$ as $n \to \infty$. We will use the same methods as in section (4.1) and (4.2). Let us start out with a summary of the notations used in these sections.

For $n \geq 1$

Σ_n is the space of functions f on the discrete space $n^{-1/2}\cdot\mathbb{Z}$, which vanish at infinity, normed by

$$\|f\|_n = \sup \{|f(x)| : x \in n^{-1/2}\cdot\mathbb{Z}\}$$

and

$\mathcal{P}_n$ is the operator from $C_o(\mathbb{R})$ to Σ_n which assigns to $f \in C_o(\mathbb{R})$ its restriction to $n^{-1/2}\cdot\mathbb{Z}$.

The sequence $(\Sigma_n, \mathcal{P}_n)_{n\geq 1}$ is an approximation of the Banach space $C_o(\mathbb{R})$. Further for $n\geq 1$

U_n is the operator on Σ_n defined by

$$U_n f(x) = \sum_{j\in\mathbb{Z}} P_n(x\cdot n^{1/2}, j)\, f(j\cdot n^{-1/2}),\ x \in n^{-1/2}\cdot\mathbb{Z},\ f \in \Sigma_n.$$

The operator U_n is a bounded, linear operator on Σ_n. The extension to $t \in [0,\infty[$ of the discrete semigroup on Σ_n with time unit $\tau_n = \frac{1}{n}$ and with generator $n(U_n-I)$ is denoted by $\mathcal{U}_n = (U_n(t))_{t\geq 0}$:

$$[U_n(t)f](x) = \sum_{j\in\mathbb{Z}} (P_n)^{[nt]}(x\cdot n^{1/2}, j)f(j\cdot n^{-1/2}),\ x \in n^{-1/2}\cdot \mathbb{Z},\ f \in \Sigma_n.$$

In the next proposition we will prove that the sequence $(\mathcal{U}_n)_{n\geq 1}$ is an approximation of the semigroup $(Q_t^\gamma)_{t\geq 0}$ if we choose $\alpha_n = \dfrac{\gamma\cdot n^{1/2}}{1+\gamma\cdot n^{1/2}}$

4.3.4 <u>Proposition</u>. Let $f \in C_o(\mathbb{R})$ and $\alpha_n = \dfrac{\gamma\cdot n^{1/2}}{1+\gamma\cdot n^{1/2}}$. Then for every $t_o \in [0,\infty[$ it is true that

$$\lim_{n\to\infty} \|U_n(\tfrac{1}{n} k_n)(\mathcal{P}_n f) - \mathcal{P}_n(Q^\gamma_{t_o} f)\|_n = 0$$

for any sequence $(k_n)_{n\geq 1}$ of nonnegative integers such that $\lim_{n\to\infty} \frac{1}{n} k_n = t_o$.

<u>Proof</u>. To prove the proposition we use the theorem of Trotter-Kato which states that the sequence $(\mathcal{U}_n)$ is an approximation of the semigroup (Q_t^γ) if for some $\lambda > 0$

$$\lim_{n\to\infty} \|I_n(\mathcal{P}_n f) - \mathcal{P}_n(V_\lambda^\gamma f)\|_n = 0$$

for every $f \in C_o(\mathbb{R})$, where for $g \in \Sigma_n$

$$I_n g = \int_0^\infty (1+ \frac{\lambda}{n})^{-[nt]-1}\, U_n(t)g\, dt.$$

It follows from lemma (4.2.6) that for $\lambda > 0$, $g \in \Sigma_n$ and $x \in n^{-1/2}\cdot\mathbb{Z}$

$$(I_n g)(x) = \frac{1}{\lambda+n} \sum_{j\in\mathbb{Z}} G^{(n)}_{(1+\frac{\lambda}{n})^{-1}} (x\cdot n^{1/2}, j)g(j\cdot n^{-1/2})$$

where for $0 < u < 1$

$$G_u^{(n)} = \sum_{k=0}^\infty (u\cdot P_n)^k.$$

An application of lemma (4.2.7) yields for $x,y \in \mathbb{Z}$

$$G_u^{(n)}(x,y) = \frac{1}{\Lambda}\, \xi^{|x-y|} + \gamma_y\, \frac{1}{\Lambda}\, \xi^{|x|+|y|}$$

where

$$\Delta = (1-u^2)^{1/2},$$

$$\xi = \frac{1}{u}(1-\Delta),$$

and

$$\gamma_y = \begin{cases} \dfrac{\alpha_n(u-1)+\alpha_n\Delta}{\Delta(1-\alpha_n)+\alpha_n(1-u)} & \text{for } y = 0 \\ \dfrac{\alpha_n(u-1)}{\Delta(1-\alpha_n)+\alpha_n(1-u)} & \text{for } y \neq 0. \end{cases}$$

It follows that for $\lambda > 0$, $f \in C_0(\mathbb{R})$ and $x \in n^{-1/2}\cdot\mathbb{Z}$

$$[I_n(\mathcal{P}_n f)](x)$$

$$= \frac{1}{\lambda+n} \sum_{j\in\mathbb{Z}} G^{(n)}_{(1+\frac{\lambda}{n})^{-1}}(x\cdot n^{1/2}, j)\ f(j\cdot n^{-1/2})$$

$$= \left[2\lambda + \frac{\lambda^2}{n}\right]^{-1/2} n^{-1/2} \sum_{j\in\mathbb{Z}} \xi_n^{|x\cdot n^{1/2}-j|}\ f(j\cdot n^{-1/2})$$

$$- \frac{\alpha_n(1-u_n)}{\Delta_n(1-\alpha_n)+\alpha_n(1-u_n)} \left[2\lambda + \frac{\lambda^2}{n}\right]^{-1/2} n^{-1/2} \sum_{j\in\mathbb{Z}} \xi_n^{|x\cdot n^{1/2}|+|j|}(f(j\cdot n^{-1/2})-f(0))$$

$$+ \frac{\alpha_n(u_n-1)}{\Delta_n(1-\alpha_n)+\alpha_n(1-u_n)} \left(\frac{\Delta_n}{(u_n-1)\sqrt{n}} + \frac{1+\xi_n}{(1-\xi_n)\sqrt{n}}\right) \left[2\lambda + \frac{\lambda^2}{n}\right]^{-1/2} f(0)\xi_n^{|x\cdot n^{1/2}|}$$

$$= \left[2\lambda + \frac{\lambda^2}{n}\right]^{-1/2} \int_{-\infty}^{+\infty} \xi_n^{|x-y_n|n^{1/2}} f(y_n)dy$$

$$- \frac{\alpha_n(1-u_n)}{\Delta_n(1-\alpha_n)+\alpha_n(1-u_n)} \left[2\lambda+ \frac{\lambda^2}{n}\right]^{-1/2} \int_{-\infty}^{+\infty} \xi_n^{(|x|+|y_n|)n^{1/2}} (f(y_n)-f(0))dy$$

$$+ \frac{\alpha_n(u_n-1)}{\Delta_n(1-\alpha_n)+\alpha_n(1-u_n)} \left[\frac{\Delta_n}{(u_n-1)\sqrt{n}} + \frac{1+\xi_n}{(1-\xi_n)\sqrt{n}}\right] \left[2\lambda+ \frac{\lambda^2}{n}\right]^{-1/2} f(0)\xi_n^{|x\cdot n^{1/2}|},$$

where

$$u_n = (1 + \frac{\lambda}{n})^{-1},$$

$$\Delta_n = (1-u_n^2)^{1/2},$$

$$\xi_n = \frac{1}{u_n}(1-\Delta_n)$$

and $y_n = \frac{[y\sqrt{n}]}{\sqrt{n}}$.

Substitution of $\alpha_n = \frac{\gamma\cdot n^{1/2}}{1+\gamma\cdot n^{1/2}}$ yields

$$\lim_{n\to\infty} \frac{\alpha_n(1-u_n)}{\Delta_n(1-\alpha_n)+\alpha_n(1-u_n)} = \frac{\lambda\gamma}{\sqrt{2\lambda}+\lambda\gamma}$$

and

$$\lim_{n\to\infty} \left\{ \frac{\Delta_n}{(u_n-1)\sqrt{n}} + \frac{1+\xi_n}{(1-\xi_n)\sqrt{n}} \right\} = 0.$$

Since

$$V_\lambda^\gamma f(x)=\frac{1}{\sqrt{2\lambda}}\int_{-\infty}^{+\infty} e^{-\sqrt{2\lambda}|x-y|}f(y)dy-\frac{\lambda\gamma}{\lambda\gamma+\sqrt{2\lambda}}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\lambda}}e^{-\sqrt{2\lambda}(|x|+|y|)}(f(y)-f(0))dx,$$

the result follows from of lemma (4.2.8)(i).□

4.3.5 Theorem. Let $(\nu_n)_{n\geq 1}$ be a sequence of probability measures on $\mathbb{R}$ with $\text{supp}(\nu_n) \subset n^{-1/2}\cdot\mathbb{Z}$ converging weakly to a probability measure m on $\mathbb{R}$. If $\alpha_n = \frac{\gamma\cdot n^{1/2}}{1+\gamma\cdot n^{1/2}}$ then for every $k \geq 1$ and $0 \leq t_1 < \ldots < t_k$ the finite dimensional distributions $\pi_{t_1\ldots t_k} P_{\nu_n}^{(n)}$ converge weakly to the finite dimensional distribution $\pi_{t_1\ldots t_k} P_m^{(\gamma)}$ as $n \to \infty$.

Proof.The result follows from theorem (4.1.1) and proposition (4.3.4).□

4.3.6 Remark. It can be proved that under the assumptions of theorem (4.3.5) the sequence of probability distributions $(P_{\nu_n}^{(n)})$ on $D_{\mathbb{R}}([0,\infty[)$ converges weakly to the probability distribution $P_m^{(\gamma)}$. We suspect that weak convergence of the finite dimensional distributions of the sequence of probability distributions $(P_{\nu_n}^{(n)})$ in theorem (4.2.4) also can be strengthened to weak convergence of the probability distributions if the probability distribution $(p_k, k \in \mathbb{Z})$ has a finite second moment.

APPENDIX

A1 The existence of an $\mathcal{S}$-finite base for the topology.

Let X be a polish space and let $\mathcal{S}$ be a collection of open subsets of X. Denote by $\mathcal{S}'$ the family of all Borel subsets of X contained in some element of $\mathcal{S}$.

Proposition. If $\mathcal{S}$ covers X, then there exists a countable base for the topology consisting entirely of open subsets with closure in $\mathcal{S}'$.

Proof. Let D be a dense subset of X and let d be a metric on X compatible with the topology of X. For each $x \in D$ there is an $A \in \mathcal{S}$ such that $x \in A$. Let $\delta = d(x,A^*) = \inf\{d(x,y): y \notin A\}$; since $\bar{A}^*$ is closed and $x \notin A^*$ we have $\delta > 0$.

If $0 < r < \delta$, then the closure $\overline{B}_x(r)$ of the ball with center x and radius r is contained in A. Indeed

$$y \in \overline{B}_x(r) \Rightarrow d(x,y) \leq r \Rightarrow d(y,A^*) \geq d(x,A^*) - d(x,y) \geq \delta - r > 0$$

$$\Rightarrow y \notin \overline{A^*} \Rightarrow y \in A.$$

So $\overline{B}_x(r) \in \mathcal{S}'$. Define for $x \in D$

$$I_x = \{q \in \mathbb{Q}: \overline{B}_x(q) \in \mathcal{S}'\}.$$

We claim that $\mathcal{U} = \{B_x(q): x \in D, q \in I_x\}$ is a countable base for the topology of X. To see this let $O \subset X$ be open. It is clear that

$$O \supset \cup\{B_x(q): B_x(q) \in \mathcal{U}, B_x(q) \subset O\}. \tag{1}$$

Let $y \in O$. There is an $A \in \mathcal{S}$ such that $y \in A$. Then

$$\epsilon = \min(d(y,O^*), d(y,A^*)) > 0.$$

For $x \in D \cap B_y(\frac{1}{4}\epsilon)$ and $q \in \mathbb{Q} \cap]\frac{1}{4}\epsilon, \frac{3}{4}\epsilon[$ we have

$$d(x,A^*) \geq d(y,A^*) - d(x,y) \geq \epsilon - \frac{1}{4}\epsilon > q.$$

So $q \in I_x$ and $y \in B_x(q)$. Hence

$$O \subset \cup\{B_x(q) : B_x(q) \in \mathcal{U},\ B_x(q) \subset O\}. \tag{2}$$

It follows from (1) and (2) that each open set can be covered by elements of $\mathcal{U}$. □

A2 The Skorohod topology.

Let (X,ρ) be a sample separable metric space. Let $t_o, T \in \mathbb{R}$, $t_o < T$ and let $D_X([t_o,T])$ be the space of functions $u : [t_o,T] \to X$ which are right continuous on $[t_o,T[$ and have left limits on $]t_o,T]$. Let $\Lambda([t_o,T])$ denote the class of strictly increasing, continuous maps $\lambda : [t_o,T] \to [t_o,T]$, such that $\lambda(t_o) = t_o$ and $\lambda(T) = T$. For u and v in $D_X([t_o,T])$, define $d_1(u,v)$ to be the infimum of those positive ϵ's for which there exists a map $\lambda \in \Lambda([t_o,T])$ such that

$$\sup\{|\lambda(t)-t| : t \in [t_o,T]\} \leq \epsilon$$

and

$$\sup\{\rho(u(t),\ v\circ\lambda(t)) : t \in [t_o,T]\} \leq \epsilon.$$

The function d_1 is a metric on $D_X([t_o,T])$. The topology on $D_X([t_o,T])$ induced by d_1 is called Skorohod's J_1 topology. Equipped with the J_1 topology, $D_X([t_o,T])$ is a polish space. See Billingsley [1]. Let U be the space of càdlàg functions of $[0,\infty[$ in X. There are several papers about the extension of the J_1 topology to U, see among others Lindvall [37] and Whitt [57]. We will summarize the theory from Whitt [57]. Let $r_{bc} : U \to D_X([b,c])$ be the restriction to $[b,c]$ defined for any $0 < b < c$ by $(r_{bc}x)(t) = x(t)$, $b \leq t \leq c$. For any $x,y \in U$, let d be defined by

$$d(x,y) = \int_0^\infty dt\ e^{-t} \max\,[d_{ot}(r_{ot}x,\ r_{ot}y),\ 1]$$

where d_{ot} is the metric on $D_X([0,t])$ as defined above. The function d is a metric on U. The topology induced by d is called the Skorohod topology on U. Note that a sequence $(x_n) \subset U$ converges to $x \in U$ iff $d_{ot}(r_{ot}x_n, r_{ot}x) \to 0$ for almost all t. The basic properties of the Skorohod topology are:

(i) the space U equipped with the Skorohod topology is a polish space,

(ii) the Borel σ-algebra on U coincides with the σ-algebra generated by the coordinate evaluations,

(iii) let P_n, $n \geq 1$, and P be probability measures on U, then $P_n \to P$ if and only if $r_{s_k t_k}(P_n) \Rightarrow r_{s_k t_k}(P)$ on $D_X([s_k, t_k])$ for all k and some sequence $\{[s_k, t_k], k \geq 1\}$ with $\bigcup_{k=1}^{\infty} [s_k, t_k] = [0,\infty[$.

Fix $a \in X$ and define the map $\zeta : u \in U \to \zeta_u \in [0,\infty]$ by

$$\zeta_u = \inf \{t > 0: u(t) = a \text{ or } u(t-) = a\}.$$

Lemma. ζ is a lower semi-continuous map.

Proof. It is sufficient to prove that the sets $\{u \in U : \zeta_u \leq k\}$, $k > 0$, are closed. So let $k > 0$ be fixed and let (u_n) be a sequence in $\{u \in U : \zeta_u \leq k\}$ converging to u. Let $\epsilon > 0$. If the restrictions of u_n to $[0,k]$ converge in $D_X([0,k])$ to the restriction of u to $[0,k]$, there exists for every n sufficiently large a function $\lambda \in \Lambda([0,k])$ such that

$$\sup\{|\lambda(t)-t| : t \in [0,k]\} \leq \epsilon$$

and

$$\sup\{\rho(u_n(t), u\circ\lambda(t)) : t \in [0,k]\} \leq \epsilon.$$

So

$$\rho(u\circ\lambda(t),a) \leq \rho(u\circ\lambda(t), u_n(t)) + \rho(u_n(t),a) \leq 2\epsilon$$

for some $t \in [0,k]$. It follows that

$$\forall\epsilon > 0, \exists s \in [0,k], \rho(u(s), a) < \epsilon$$

and this implies that $\zeta_u \leq k$.

If the restrictions of u_n to $[0,k]$ do not converge in $D_X([0,k])$ to the restriction of u, there exists a sequence (k_m) decreasing to k, such that $r_{ok_m}u_n \to r_{ok_m}u$ as $n \to \infty$ for every $m \geq 1$. As above we may conclude that $\zeta_u \leq k_m$ for every $m \geq 1$ and it follows that $\zeta_u \leq k$. □

A3 Some results on real functions.

A3.1 Lemma. (Greenwood & Pitman). For each $n > 0$ let $f_n(t)$ be a positive, nondecreasing function of $t \in [0,\infty[$ and let S be a subset of $[0,\infty[$. Suppose that, for each $s \in S$, $f_n(s)$ converges to a finite limit $f(s)$ as $n \to \infty$, and that the set of limitpoints $\{f(s): s \in S\}$ is dense in $[0,\infty[$. Let $a = \sup S$. Then there is a continuous nondecreasing function f defined on $[0,a[$ such that $\lim_{n\to\infty} f_n(t) = f(t)$ uniformly on bounded sub-intervals of $[0,a[$.

Proof. For every $n > 0$ and $s \in S$ we have $0 \leq f_n(0) \leq f_n(s)$. So

$$0 \leq \underline{\lim}\, f_n(0) \leq \overline{\lim}\, f_n(0) \leq \inf\,\{f(s) : s \in S\} = 0$$

and

$$\lim f_n(0) = 0.$$

Let $x \in\,]0,a[$. If $S \cap [0,x] = \emptyset$, then $\lim f_n(x)=0$. In the remaining case we have

$$\sup\{f(s) : s \in S \cap [0,x]\} \leq \underline{\lim}\, f_n(x) \leq \overline{\lim}\, f_n(x)$$
$$\leq \inf\,\{f(s) : s \in S \cap [x,\infty[\}.$$

Since $\{f(s): s \in S\}$ is dense in $[0,\infty[$, $\lim f_n(x)$ exists. Define the function $f : [0,a[\to [0,\infty[$ as the pointwise limit of the sequence of functions (f_n). It is clear that f is a nondecreasing, continuous function on $[0,a[$. If the convergence of the sequence (f_n) is not uniform on bounded sub-intervals of $[0,a[$, then there exists an $M < a$ and an $\epsilon > 0$ such that

$$\forall n \in \mathbb{N},\ \exists t_n \in [0,M],\ |f_n(t_n) - f(t_n)| > \epsilon.$$

Let $(t_{n'})$ be a convergent subsequence of (t_n), $t_\infty = \lim t_{n'}$. Choose $x_1, x_2 \in [0,a[$ such that

$$x_1 < t_\infty < x_2 \quad \text{and} \quad f(x_2) - f(x_1) < \frac{1}{4}\epsilon.$$

If $t_\infty = 0$, take $x_1 = 0$. Then for n' sufficiently large

$$f_{n'}(x_1) - f(t_{n'}) < f_{n'}(t_{n'}) - f(t_{n'}) < f_{n'}(x_2) - f(t_{n'}),$$

and it follows by letting $n' \to \infty$ that

$$-\frac{1}{4}\epsilon \le \overline{\lim}\,[f_{n'}(t_{n'}) - f(t_{n'})] \le \frac{1}{4}\epsilon$$

which is a contradiction. So the convergence of the sequence (f_n) is uniform on bounded sub-intervals of $[0,a[$. □

A3.2 Let A be a function on $[0,\infty[$, which is nonnegative, nondecreasing and right continuous. Denote by λ the Lebesgue measure on $[0,\infty[$, and let φ be the distribution function of the measure $\nu = A(\lambda)$ on $[0,\infty[$.

Then for $t \ge 0$

$$\begin{aligned} \varphi(t) &= \int 1_{[0,t]} dA(\lambda) \\ &= \int 1_{[0,t]} \circ A \, d\lambda \\ &= \lambda(\{x : A(x) \le t\}) = \sup \{x \in [0,\infty[\ : A(x) \le t\}. \end{aligned}$$

φ is a nonnegative, possibly infinite valued, nondecreasing function on $[0,\infty[$. It is also clear that φ is right continuous. If $\varphi(t) \le y$, then y is an upperbound of the set $\{x : A(x) \le t\}$. It follows that $A(y+\epsilon) > t$ for every $\epsilon > 0$ and this implies that $A(y) = A(y+) \ge t$. So $A(y)$ is an upperbound of the set $\{t : \varphi(t) \le y\}$. On the other hand, if u is an upperbound of the set $\{t : \varphi(t) \le y\}$, then $\varphi(u+\epsilon) > y$ for every $\epsilon > 0$. It follows that $A(y) \le u+\epsilon$ for every $\epsilon > 0$. So

$A(y) = \sup \{t : \varphi(t) \le y\}$ and A is the distribution function of the measure $\varphi(\lambda)$.

φ is called the right continuous inverse of A. We have shown that A is the right continuous inverse of φ.

Let $F \in L^1(\nu)$, then

$$\int F \circ A(x)\lambda(dx) = \int F dA(\lambda) = \int F(y)\, d\varphi(y).$$

A3.3 Let f be a nondecreasing, right continuous function on $[0,\infty[$, such that $f(0) = 0$ and $\lim_{x\to\infty} f(x) = +\infty$.

Define

$$J = \{t \in]0,\infty[\;:\; f(t-) < f(t)\}$$

and

$$R = \text{range } (f) = \{s \in [0,\infty[\;:\; s = f(t) \text{ for some } t \geq 0\}.$$

Lemma. If f is strictly increasing, then

$$[0,\infty[= R + \sum_{t \in J} [f(t-), f(t)[$$

where the union is a disjoint union.

Proof. Let $t \in R$, say $t = f(r)$. Assume that there is an $s \in J$ such that $t \in [f(s-), f(s)[$.

Then

$$f(s-) \leq f(r) < f(s).$$

It follows that $r < s$, so $f(r) = f(s-)$. This can only be the case when f is constant on $[r,s[$ which is impossible since f is strictly increasing. So

$$R \cap \sum_{t \in J} [f(t-), f(t)[= \phi.$$

Let $t \in [0,\infty[\backslash R$. Then for any $s \in [0,\infty[$ we have $f(s) < t$ or $f(s) > t$.

Define

$$u = \inf \{s : f(s) > t\} = \sup\{s : f(s) < t\}.$$

Then

$$f(u) \geq t, \text{ so } f(u) > t$$

and

$$f(u-) \leq t.$$

It follows that

$$t \in [f(u-), f(u)[\subset \bigcup_{t \in J} [f(t-), f(t)[,$$

which completes the proof of the lemma. □

A4 Proof of lemma (4.2.8).

Proof of (i).

Let $x > 0$.

From the mean value theorem it follows that

$$|\xi_n^x - \exp(-\left[\frac{2\lambda}{n}\right]^{½} x)| = |e^{x\, ln\, \xi_n} - \exp(-\left[\frac{2\lambda}{n}\right]^{½} x)|$$

$$= x\, e^{x\, \eta(n,x)} |\, ln\, \xi_n + \left[\frac{2\lambda}{n}\right]^{½}|$$

where $\eta(n,x)$ is a number between $ln\xi_n$ and $-\left[\frac{2\lambda}{n}\right]^{½}$.

$$\xi_n = 1 + \frac{\lambda}{n} - \left[(\frac{\lambda}{n})^2 + \frac{2\lambda}{n}\right]^{½} = 1 - \left[\frac{2\lambda}{n}\right]^{½} + \frac{\lambda}{n} + O(n^{-3/2})$$

$$ln\, \xi_n = -\left[\frac{2\lambda}{n}\right]^{½} + g_\lambda(n), \quad g_\lambda(n) = O(n^{-3/2}).$$

So: $|n^{½}\, \eta(n,x) + (2\lambda)^{½}| \le |\, n^{½}\, g_\lambda(n)|$.

It follows that

$$|\xi_n^x - \exp(-\left[\frac{2\lambda}{n}\right]^{½} x)| \le \sup_{y\ge 0} y\, e^{y\, \eta(n,x)} |g_\lambda(n)|$$

$$\le -\frac{1}{\eta(n,x)} |g_\lambda(n)|$$

$$\le \frac{1}{(2\lambda)^{½} - |g_\lambda(n)n^{½}|} |g_\lambda(n)n^{½}|.$$

We conclude:

$$\sup_{x\ge 0} n^{1-\delta} |\xi_n - \exp(-\left[\frac{2\lambda}{n}\right]^{½} x)| \le \frac{1}{(2\lambda)^{½} - |g_\lambda(n)n^{½}|} |g_\lambda(n)\, n^{3/2-\delta}|.$$

The righthand side of this inequality tends to 0 as $n \to \infty$

since $g_\lambda(n)\, n^{3/2-\delta} = O(n^{-\delta})$ and $\dfrac{1}{(2\lambda)^{½} - |g_\lambda(n)n^{½}|} = O(1)$.

This completes the proof of the first part of lemma (4.2.8).

Proof of (ii).

For the proof of the second part, we first note that

$$\gamma_{[y\cdot n^{½}]} = \frac{\xi_n^{-|[y\cdot n^{½}]|} \sum p_k\, \xi_n^{|k-[y\cdot n^{½}]|} - 1 - \frac{\lambda}{n} + (1 + \frac{\lambda}{n})\Delta 1_{\{y=0\}}}{1 + \frac{\lambda}{n} - \Sigma p_k\, \xi_n^{|k|}}.$$

It is sufficient to give a proof for $y > 0$, since the case $y \leq 0$ can be obtained from the case $y > 0$ by substitution of the distribution $\{q_k\}$ with $q_k = p_{-k}$ for the distribution $\{p_k\}$.

Fix $y > 0$ and $\epsilon > 0$:

A. The first step consists of estimating $1 - \Sigma\, p_k\, \xi_n^{|k|}$.

$$1 - \Sigma\, p_k\, \xi_n^{|k|} = \Sigma\, p_k(1-\xi_n^{|k|}).$$

From lemma 4.2.8 (i) follows the existence of a number N_1 such that

$$|\Sigma\, p_k(1-\xi_n^{|k|}) - \Sigma\, p_k(1 - \exp(-\left[\frac{2\lambda}{n}\right]^{1/2} |k|))|$$
$$= |\Sigma\, p_k(\xi_n^{|k|} - \exp(-\left[\frac{2\lambda}{n}\right]^{1/2}|k|))| \leq \epsilon\cdot\left[\frac{2\lambda}{n}\right]^{1/2} \text{ for all } n \geq N_1.$$

From the mean value theorem follows the existence of a number $\zeta \in \left]0, \left[\frac{2\lambda}{n}\right]^{1/2}\right[$ such that

$$1 - \exp(-\left[\frac{2\lambda}{n}\right]^{1/2}|k|) = \left[\frac{2\lambda}{n}\right]^{1/2} |k|\; e^{-|k|\zeta}.$$

So

$$\left[\frac{2\lambda}{n}\right]^{1/2} \Sigma|k|p_k \exp(- \left[\frac{2\lambda}{n}\right]^{1/2}|k|) \leq \Sigma p_k(1-\exp(-\left[\frac{2\lambda}{n}\right]^{1/2}|k|)) \leq \left[\frac{2\lambda}{n}\right]^{1/2}\Sigma|k|p_k.$$

It is possible to choose K and N_2 such that

$$\sum_{|k|>K} |k|p_k < \epsilon$$

and

$$\exp(- \left[\frac{2\lambda}{n}\right]^{1/2}K) \geq 1-\epsilon \text{ for all } n \geq N_2.$$

This implies:

$$\Sigma|k|p_k \exp(- \left[\frac{2\lambda}{n}\right]^{1/2}|k|) \geq (1-\epsilon) \sum_{|k|\leq K} |k|p_k$$
$$\geq \Sigma|k|p_k + \epsilon\cdot(-1-\Sigma|k|p_k).$$

So for $n \geq \max(N_1,N_2)$

$$\left[\frac{2\lambda}{n}\right]^{1/2} \Sigma\, |k|p_k + \epsilon\cdot\left[\frac{2\lambda}{n}\right]^{1/2} (-2 - \Sigma|k|p_k)$$
$$\leq \Sigma\, p_k(1-\xi_n^{|k|}) \leq \left[\frac{2\lambda}{n}\right]^{1/2} \Sigma\, |k|p_k + \epsilon\cdot\left[\frac{2\lambda}{n}\right]^{1/2}. \qquad (1)$$

B. Consider

$$\xi_n^{-|[y\cdot n^{1/2}]|} \sum p_k(\xi_n^{|k-[y\cdot n^{1/2}]|} - 1)$$

$$= \sum_{k\le[y\cdot n^{1/2}]} p_k(\xi_n^{-k}-1) + \sum_{k>[y\cdot n^{1/2}]} p_k(\xi_n^{k-2|[y\cdot n^{1/2}]|} - 1).$$

In the same way as in A we find:

$$- \left(\frac{2\lambda}{n}\right)^{1/2} \sum_{k\le 0} |k|p_k - \epsilon\cdot\left(\frac{2\lambda}{n}\right)^{1/2}$$

$$\le \sum_{k\le 0} p_k(\xi_n^{-k} -1) \le - \left(\frac{2\lambda}{n}\right)^{1/2} \sum_{k\le 0} |k|p_k + \epsilon\cdot\left(\frac{2\lambda}{n}\right)^{1/2} (2+\Sigma|k|p_k) \qquad (2)$$

and

$$-\left(\frac{2\lambda}{n}\right)^{1/2} \sum_{k>2[y\cdot n^{1/2}]} kp_k + \epsilon\cdot\left(\frac{2\lambda}{n}\right)^{1/2} \le \sum_{k>2[y\cdot n^{1/2}]} p_k(\xi_n^{k-2|[y\cdot n^{1/2}]|} - 1) \le 0. \quad (3)$$

So we only have to estimate

$$\sum_{0<k\le[y\cdot n^{1/2}]} p_k(\xi_n^{-k}- 1) + \sum_{[y\cdot n^{1/2}]<k\le 2[y\cdot n^{1/2}]} p_k(\xi_n^{2|[y\cdot n^{1/2}]|-k} - 1). \qquad (4)$$

We have:

$$\xi_n^{-1}= 1 + \left(\frac{2\lambda}{n}\right)^{1/2} + \frac{\lambda}{n} + O(n^{-3/2}).$$

For $x > 0$

$$|\xi_n^{-x}- \exp \left(\frac{2\lambda}{n}\right)^{1/2}x)| = x\, e^{x\tau(n,x)}|\ln \xi_n^{-1}- \left(\frac{2\lambda}{n}\right)^{1/2}|$$

with $\tau(n,x)$ a number between $\ln\xi_n^{-1}$ and $\left(\frac{2\lambda}{n}\right)^{1/2}$.

So for $0 < k \le [M\cdot n^{1/2}]$ we have:

$$|\xi_n^{-k}- \exp(\left(\frac{2\lambda}{n}\right)^{1/2}k)| \le M\cdot\exp\, [M((2\lambda)^{1/2}+O(n^{-1}))]\, O(n^{-1})$$

and it follows that

$$\lim_{n\to\infty} \{ \sup_{0<k\le[M\cdot n^{1/2}]} \frac{|\xi_n^{-k}- \exp\left(\frac{2\lambda}{n}\right)^{1/2} k)|}{\left(\frac{2\lambda}{n}\right)^{1/2}}\} = 0.$$

Substitution in (4) of $\exp(-\left(\frac{2\lambda}{n}\right)^{1/2})$ for ξ_n gives an expression which, for n sufficiently large, say for $n \ge N_3$, differs no more than $\epsilon\cdot\left(\frac{2\lambda}{n}\right)^{1/2}$ from the original expression.

Applying the mean value theorem again yields:

$$\left(\frac{2\lambda}{n}\right)^{1/2} \sum_{0<k\le[y\cdot n^{1/2}]} kp_k - \epsilon\cdot\left(\frac{2\lambda}{n}\right)^{1/2}$$

$$\le \sum_{0<k\le[y\cdot n^{1/2}]} p_k(\xi_n^{-k} - 1) \le \left(\frac{2\lambda}{n}\right)^{1/2} \sum_{0<k\le[y\cdot n^{1/2}]} kp_k\exp(\left(\frac{2\lambda}{n}\right)^{1/2} k) + \epsilon\cdot\left(\frac{2\lambda}{n}\right)^{1/2}.$$

Choose $\eta > 0$ then

$$[y\cdot n^{1/2}] \ge K \text{ for all } y \in [\eta,M] \text{ if } n \ge (\frac{K}{\eta})^2.$$

There exists also a number N_4 such that $\exp(\left(\frac{2\lambda}{n}\right)^{1/2} K) \le 1+\epsilon$ for all $n \ge N_4$.

This gives for $n \ge \max\left(\left(\frac{K}{\eta}\right)^2, N_4\right)$

$$\sum_{0<k\le[y\cdot n^{1/2}]} kp_k \exp(\left(\frac{2\lambda}{n}\right)^{1/2} k)$$

$$\le \sum_{0<k\le K} kp_k(1+\epsilon) + e^{M\cdot(2\lambda)^{1/2}} \sum_{K\le k\le 2[y\cdot n^{1/2}]} kp_k$$

$$\le \sum_{k>0} kp_k + \epsilon\cdot(\sum_{k>0} kp_k + e^{M\cdot(2\lambda)^{1/2}}), \text{ since } \sum_{|k|>K} |k|p_k < \epsilon.$$

It follows that for all $y \in [\eta,M]$ and $n \ge \max(\left(\frac{K}{\eta}\right)^2, N_3, N_4)$ it is true that

$$\left(\frac{2\lambda}{n}\right)^{1/2} \sum_{k>0} kp_k - 2\epsilon\cdot\left(\frac{2\lambda}{n}\right)^{1/2}$$

$$\le \sum_{0<k\le[y\cdot n^{1/2}]} p_k(\xi_n^{-k} - 1)$$

$$\le \left(\frac{2\lambda}{n}\right)^{1/2} \sum_{k>0} kp_k + \epsilon\cdot\left(\frac{2\lambda}{n}\right)^{1/2}(\sum_{k>0} kp_k + e^{M\cdot(2\lambda)^{1/2}} + 1). \tag{5}$$

Further for all $y \in [\eta,M]$ and $n \ge \max\left(\left(\frac{K}{\eta}\right)^2, N_3, N_4\right)$:

$$0 \le \sum_{[y\cdot n^{1/2}]<k\le 2[y\cdot n^{1/2}]} p_k(\xi_n^{2[y\cdot n^{1/2}]-k} - 1)$$

$$\le \sum p_k \left(\frac{2\lambda}{n}\right)^{1/2} (2[y\cdot n^{1/2}] - k) \exp\left(\left(\frac{2\lambda}{n}\right)^{1/2}(2[y\cdot n^{1/2}]-k)\right) + \epsilon\cdot\left(\frac{2\lambda}{n}\right)^{1/2}$$

$$\le 2\left(\frac{2\lambda}{n}\right)^{1/2} e^{M\cdot(2\lambda)^{1/2}} \sum_{[y\cdot n^{1/2}]<k\le 2[y\cdot n^{1/2}]} kp_k + \epsilon\cdot\left(\frac{2\lambda}{n}\right)^{1/2}$$

$$\le \epsilon\cdot\left(\frac{2\lambda}{n}\right)^{1/2} (2\, e^{M\cdot(2\lambda)^{1/2}} + 1). \tag{6}$$

For a given $\delta > 0$, combination of (2), (3), (4), (5) and (6) gives that for all $y \in [\eta,M]$ and for all $n \ge \max(N_1, N_2, N_3, N_4, \left(\frac{K}{\eta}\right)^2)$ we have

$$|\gamma_{[y\cdot n^{1/2}]} - (\alpha-\beta)| < \delta.$$

From the same inequalities follows the existence of the constant V in lemma 4.2.8 (ii).□

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STOCHASTISCHE PROCESSEN EN PUNTPROCESSEN VAN EXCURSIES

SAMENVATTING

In excursietheorie bestudeert men stochastische processen aan de hand van de eigenschappen van de excursies vanuit een gegeven toestand a. Excursies vanuit toestand a zijn restricties van het proces tot tijdsintervallen tussen twee opeenvolgende bezoeken aan toestand a. Deze methode is in het bijzonder succesvol bij de bestudering van Markov-processen, omdat uit de Markov-eigenschap onafhankelijkheidseigenschappen volgen voor de excursies. In 1970 publiceerde Itô een artikel, waarin hij de excursies van een sterk Markov-proces beschreef als een stochastische puntfunctie.

In het eerste deel van dit proefschrift wordt een theorie ontwikkeld van puntprocessen, die eindig veel punten hebben in de deelverzamelingen die behoren tot een gegeven filterende familie. Vervolgens beschrijven we in dat kader het puntproces van excursies vanuit een gegeven toestand a van een Ray-proces. Omgekeerd construeren we een stochastisch proces uit zo'n excursieproces, waarbij nu gebruik gemaakt kan worden van technieken uit de theorie van de puntprocessen.

In het tweede deel van dit proefschrift construeren we met behulp van deze theorie stochastische bewegingen op een eenvoudige graph. De betekenis van verschillende parameters die optreden in deze constructie wordt nader onderzocht met behulp van Random-Walk benaderingen.

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De schrijver van dit proefschrift werd geboren op 17 maart 1949 te Leeuwarden. In 1967 behaalde hij het diploma Gymnasium β aan het St. Janslyceum te 's-Hertogenbosch, waarna hij ging studeren aan de Rijksuniversiteit Utrecht. In 1973 behaalde hij het doctoraal examen wiskunde. Vervolgens was hij als leraar verbonden aan het Stedelijk Lyceum te Zutphen en het Niels Stensen College te Utrecht, waarna hij in 1976 in dienst trad van de Technische Hogeschool te Delft als wetenschappelijk medewerker in de Afdeling der Algemene Wetenschappen/ Onderafdeling der Wiskunde. Momenteel is hij werkzaam als universitair docent aan de Faculteit der Technische Wiskunde en Informatica van de Technische Universiteit Delft.