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STOCHASTIC PROCESSES AS FOURIER TRANSFORMS OF STOCHASTIC MEASURES

BY

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Preface

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Contents

Introduction

The spectral properties of wide-sense stationary stochastic processes $x(t)$, $t \in R$, have been extensively studied. It is well-known that every wide-sense stationary stochastic process is the Fourier transform of a bounded stochastic measure on R with uncorrelated values for disjoint Borel sets of R (see e.g. Karhunen [11] or Loève [13: p. 482]).

Several generalizations of the class of wide-sense stationary stochastic processes have been presented in literature. Loève [13: p. 474] has introduced the class of harmonizable stochastic processes. Loève showed that the boundedness and the continuity of a stochastic process are necessary conditions for its harmonizability. He brought up the question of the sufficiency of these conditions and the supplementary conditions possibly required for the harmonizability of a stochastic process [13: p. 477]. Harmonizable stochastic processes were also studied by Cramér [6]. Bochner [4: p. 18] introduced another generalization, the class of V-bounded stochastic processes. Bochner proved that every harmonizable stochastic process is V-bounded. One can ask whether every continuous (and bounded) V-bounded stochastic process is harmonizable. Moreover, Rozanov [18] studied stochastic processes which are Fourier transforms of bounded stochastic measures, and called them harmonizable. In the above papers all stochastic and scalar-valued measures are treated as completely additive set functions.

In his recent extensive paper Thomas [20] considered vector measures as Radon measures with values in a topological vector space. The same starting point is also used in his later paper [21] on some related topics. His former study gave us many of the basic ideas for this paper; results proved there will frequently be used by us.

In this paper we shall consider stochastic measures as Radon measures with values in a linear space of stochastic variables. In chapter 2 we develop the integration theory of such stochastic measures. We shall also consider the so called covariance bimeasures of stochastic measures and their integration. In studying these covariance bimeasures we shall use the connection between stochastic mappings and reproducing kernel Hilbert spaces. A short review of this connection is given in chapter 1.

In chapter 3 we give a modified definition of a V-bounded stochastic process and of a harmonizable stochastic process, based on the use of Radon measures. Then we show that a stochastic process $x(t)$, $t \in R$, is the Fourier transform of a bounded stochastic measure if and only if it is V -bounded and weakly continuous. von Bahr [3] has recently constructed an example of a non-harmonizable bounded and continuous stochastic process. We shall use this example to show that there exist continuous and bounded stochastic processes which are not V -bounded.

Furthermore, we shall give a characterization of harmonizable stochastic processes. Modifying an example, due to Edwards [9], we show that there exist continuous (and bounded) V -bounded stochastic processes which are not harmonizable. At the end of the paper we shall give a method of approximating a continuous V -bounded stochastic process by a sequence of harmonizable stochastic processes.

Our results concerning the constructed examples are also valid if harmonizable and V -bounded stochastic processes are defined as in Loève [13: p. 474] and in Bochner [4: p. 18].

Note added in proof: We learned quite recently that Gladyshev [10] has already in 1961 constructed examples of bounded and continuous non-harmonizable stochastic processes. These examples are very similar to those constructed by von Bahr and the author.

1. Stochastic mappings and reproducing kernel Hilbert spaces

1.1. Stochastic mappings

1. In the following we shall consider complex-valued random variables which are defined on a fixed probability space and which have finite second order moments and zero mean value. Let us first introduce the space of all such random variables and its Hilbert space structure.

Let $(\Omega, \mathcal{F}, P)$ be a fixed probability space, i.e., Ω is a set, $\mathcal{F}$ is a σ -algebra of subsets of Ω and P is a completely additive nonnegative set function on $\mathcal{F}$ such that $P(\Omega) = 1$. The complex vector space of all complex-valued random variables ξ defined on $(\Omega, \mathcal{F}, P)$ is denoted by $\mathcal{L}(\Omega, \mathcal{F})$.

Let $\xi \in \mathcal{L}(\Omega, \mathcal{F})$ be a random variable. Its *mean value* is defined by the relation

$$E\,\xi = \int \operatorname{Re} \xi \, dP + \int \operatorname{Im} \xi \, dP ,$$

if the right hand side is well-defined.

The set

$$\mathcal{L}^2(\Omega, \mathcal{F}, P) = \{ \xi \in \mathcal{L}(\Omega, \mathcal{F}) \mid E|\xi|^2 < \infty \}$$

is a linear subspace of $\mathcal{L}(\Omega, \mathcal{F})$. The relation

(1.1.1) $\quad \xi_1 \sim \xi_2 \quad$ if $\quad P\{\omega \in \Omega \mid \xi_1(\omega) = \xi_2(\omega)\} = 1,$

$\xi_1, \xi_2 \in \mathcal{L}^2(\Omega, \mathcal{F}, P)$, is an equivalence relation in the space $\mathcal{L}^2(\Omega, \mathcal{F}, P)$. The quotient space of $\mathcal{L}^2(\Omega, \mathcal{F}, P)$ with respect to the equivalence relation defined in (1.1.1) is denoted by $L^2(\Omega, \mathcal{F}, P)$. In the following we use the same notation for an equivalence class $\xi \in L^2(\Omega, \mathcal{F}, P)$ and for its arbitrary representative $\xi \in \mathcal{L}^2(\Omega, \mathcal{F}, P)$, if no confusion is possible.

The space $L^2(\Omega, \mathcal{F}, P)$ is a Hilbert space if the inner product of $\xi_1, \xi_2 \in L^2(\Omega, \mathcal{F}, P)$ is defined by the relation

(1.1.2) $$(\xi_1 \mid \xi_2) = E\, \xi_1 \overline{\xi}_2 ,$$

where $\overline{\xi}_2$ is the complex conjugate of ξ_2, and the norm of $\xi \in L^2(\Omega, \mathcal{F}, P)$ is

$$\|\xi\| = (\xi \mid \xi)^{1/2} .$$

On the right hand side of (1.1.2), ξ_1 and ξ_2 are arbitrary representatives of ξ_1 and ξ_2. The definition (1.1.2) is independent of the choice of the representatives.

As usual, we consider only random variables $\xi \in \mathcal{L}(\Omega, \mathcal{F})$ for which $E\,\xi = 0$. We denote

$$L_0^2(P) = L_0^2(\Omega, \mathcal{F}, P) = \{ \xi \in L^2(\Omega, \mathcal{F}, P) \mid E\,\xi = 0 \} .$$

The space $L_0^2(P)$ is a closed linear subspace of $L^2(\Omega, \mathcal{F}, P)$. Unless otherwise stated, the topology of $L_0^2(P)$ is the norm topology.

2. In this paper we consider stochastic processes as mappings from a parameter set T into the space $L_0^2(P)$. In the following we use the term *stochastic process* only when the parameter set is R, i.e. the set of all real numbers, and the term *stochastic mapping* in the case of a more general space. In our applications the parameter set is either R or a function space.

Next we give some general definitions. Before stating the definition we remark that in this paper the set of all positive integers is denoted by N and the set of all complex numbers is denoted by C.

Definition 1.1.1. *A stochastic mapping of a set* T *is a mapping* $x : T \to L_0^2(P)$ *and its covariance mapping is the mapping* $r : T \times T \to C$,

$$r(s, t) = (x(s) \mid x(t)) , \qquad s, t \in T .$$

Let F be a topological vector space and let $A \subset F$. Troughout the paper we denote by $\operatorname{sp}\{A\}$ the linear subspace of F, spanned by the set A and by $\overline{\operatorname{sp}}\{A\}$ the closure of the space $\operatorname{sp}\{A\}$.

Let T be a set and let x be a stochastic mapping of T. In the following we use the notation

$$\operatorname{sp}\{x\} = \operatorname{sp}\{\, x(t) \mid t \in T \,\}$$

and denote the closure of $\operatorname{sp}\{x\}$ by $\overline{\operatorname{sp}}\{x\}$ and call it the *Hilbert space spanned by* x.

3. Let T be a topological space and let x be a stochastic mapping of T. We call x *continuous* (resp. *weakly continuous*) if the mapping $x: T \to L_0^2(P)$ is continuous, when $L_0^2(P)$ carries its norm topology (resp. weak topology).

The following results are direct consequences of the above definitions. They can be found in many studies. We refer to Karhunen [11: pp. 27—28].

Lemma 1.1.2. *Let T be a topological space. A stochastic mapping of T is continuous if and only if its covariance mapping r is continuous at all diagonal points $(t, t) \in T \times T$. If r is continuous at all diagonal points $(t, t) \in T \times T$, then it is everywhere continuous.*

Lemma 1.1.3. *Let T be a separable topological space and let x be a continuous stochastic mapping of T. Then the space $\overline{\operatorname{sp}}\{x\}$ is separable.*

1.2. Reproducing kernel Hilbert spaces

1. The use of reproducing kernel Hilbert spaces in studying stochastic processes or mappings is nowadays a well-known method. An extensive presentation of this method is given for example by Parzen [17: pp. 251—382]. Our aim is to give a short review of the connection between reproducing kernel Hilbert spaces and stochastic mappings in section 1.3, and in 2.4 to use this connection in studying stochastic measures and their covariance mappings.

Here we give the definition of a reproducing kernel Hilbert space and also give some of their basic properties. For further results see Aronszajn [2] or Meschkowski [14].

Definition 1.2.1. *Let E be an arbitrary set, and let H be a Hilbert space of mappings $f: E \to C$. The space H is called a reproducing kernel Hilbert space on E or in short an r. k. Hilbert space on E, if there exists a mapping $K: E \times E \to C$ such that*

(i) $K_t \in H$ *(* $K_t(s) = K(s, t)$, $s, t \in E$ *) for all* $t \in E$,

(ii) $f(t) = (f \mid K_t)$, $t \in E$, *for all* $f \in H$.

A mapping $K: E \times E \to C$ *which satisfies the conditions (i) and (ii) of this definition is called a reproducing kernel of the space* H .

An r. k. Hilbert space has one and only one reproducing kernel (Aronszajn [2: p. 343]).

2. Next we give some results concerning the continuity properties of the functions of an r. k. Hilbert space.

Let H be an r. k. Hilbert space on a topological space E and let $f \in H$, then

$$|f(s) - f(t)| = |(f \mid K_s - K_t)| \leq \|f\| \|K_s - K_t\| , \qquad t \in E ,$$

where K is the reproducing kernel of H . Thus the continuity of the mapping $i_K : E \to H$, $i_K(t) = K_t$, $t \in E$, implies the continuity of f . For later use we state this result as a lemma.

Lemma 1.2.2. *Let* H *be an r. k. Hilbert space on a topological space* E *and let* K *be the reproducing kernel of* H . *If the mapping* $i_K : E \to H$, $i_K(t) = K_t$, $t \in E$, *is continuous, then all functions* $f \in H$ *are continuous.*

The next result is an analogue to Lemma 1.1.2.

Lemma 1.2.3. *Let* H *be an r. k. Hilbert space on a topological space* E *and let* K *be its reproducing kernel. The mapping* $i_K : E \to H$, $i_K(t) = K_t$, $t \in E$, *is continuous if and only if* K *is continuous at all diagonal points* $(t , t) \in E \times E$. *If* K *is continuous at diagonal points* $(t , t) \in E \times E$, *then* K *is everywhere continuous.*

Proof. The proof is a repetition of the proof of Lemma 1.1.2.

3. The rest of this section is devoted to a generalization of a theorem by Mercer (see Neveu [16: p. 42—43]). Let H be an r. k. Hilbert space on a topological space E and let K be its reproducing kernel. Let $f_\alpha \in H$, $\lambda \in I$, be an orthonormal basis of the Hilbert space H . We note, that

$$(1.2.1) \qquad K(s , t) = (K_s \mid K_t) = \sum_{\lambda \in I} f_\lambda(s) \overline{f_\lambda(t)} , \qquad s , t \in E .$$

Mercer's theorem states that if K is continuous, then the series (1.2.1) converges uniformly on every set $S \times S$, where $S \subset E$ is compact.

In the proof of our generalization we use the approximation property of the Hilbert space H .

We recall that a locally convex topological vector space F is said to have the *approximation property*, if the identity operator e of F can be approximated uniformly on every precompact set in F by continuous linear operators of finite rank.

Every Hilbert space has the approximation property. First we note that in a Hilbert space precompact and relatively compact sets are the same, thus it suffices to consider compact sets. Let H be a Hilbert space and let $x_\alpha \in H$, $\alpha \in I$, be an orthonormal basis of H. Then

$$x = \sum_{\alpha \in I} (x \mid x_\alpha)\, x_\alpha\,, \qquad x \in H\,.$$

Let $J \subset I$ be finite. Denote

$$e_J(x) = \sum_{\alpha \in J} (x \mid x_\alpha)\, x_\alpha\,, \qquad x \in H\,,$$

then the operators e_J, $J \subset I$, J is finite, are of finite rank, and converge uniformly to e on every compact subset of H, the limit being taken with respect to the filtering increasing family of finite subsets of I (Schaefer [19: p. 108]).

Next we give the generalization of Mercer's theorem.

Theorem 1.2.4. *Let H be an r. k. Hilbert space on a set E and let K be its reproducing kernel. Let $f_\alpha \in H$, $\alpha \in I$, be an orthonormal basis of H. Then the representation*

$$K(s\,, t) = \sum_{\alpha \in I} f_\alpha(s)\, \overline{f_\alpha(t)}\,, \qquad s\,,\ t \in E\,,$$

converges uniformly on every set $A \times B \subset E \times E$, for which $i_K(A)$ is a bounded set and $i_K(B)$ is a compact set of H, where $i_K : E \to H$ is defined by $i_K(t) = K_t$, $t \in E$.

Proof. Denote again

$$e_J(f) = \sum_{\alpha \in J} (f \mid f_\alpha) f_\alpha\,, \qquad f \in H\,,$$

when $J \subset I$ is finite. Let $\varepsilon > 0$. Since $i_K(B)$ is compact in H the approximation property of H implies that there exists a finite $J_0 \subset I$ such that

$$\|e_J(K_t) - K_t\| = \Big\| \sum_{\alpha \in J} \overline{f_\alpha(t)}\, f_\alpha - K_t \Big\| < \varepsilon\,, \qquad t \in B\,,$$

for all finite $J \subset I$ such that $J_0 \subset J$. Since $i_K(A)$ is bounded, we get

$$\Big| K(s\,, t) - \sum_{\alpha \in J} f_\alpha(s)\, \overline{f_\alpha(t)} \Big| = \Big| \Big(K_t - \sum_{\alpha \in J} \overline{f_\alpha(t)}\, f_\alpha \,\Big|\, K_s \Big) \Big|$$

$$\leq \Big\| K_t - \sum_{\alpha \in J} \overline{f_\alpha(t)}\, f_\alpha \Big\|\, \|K_s\| \leq \varepsilon \Big(\sup_{s \in A} \|K_s\| \Big)\,,$$

when $s \in A$, $t \in B$, for every finite $J \subset I$ such that $J_0 \subset J$, which proves the theorem.

1.3. Stochastic mappings and r. k. Hilbert spaces

1. The following theorem gives the connection between r. k. Hilbert spaces and mappings from an arbitrary set into an inner product space.

Theorem 1.3.1. *Let* H *be an inner product space and let* E *be a set. Then for every mapping* $i : E \to H$ *there exists a unique r. k. Hilbert space* $H(K)$ *on* E *such that the mapping* $K : E \times E \to C$, $K(s, t) = (i(s) \mid i(t))$, $s, t \in E$, *is the reproducing kernel of the space* $H(K)$. *The spaces* $H(K)$ *and the strong dual of* $\operatorname{sp}\{i(E)\}$ *are isometrically isomorphic.*

Proof. Let M be the completion of the linear subspace $\operatorname{sp}\{i(E)\}$ of H, then M is a Hilbert space. Let M' be the dual of M equipped with the norm topology, then there exists an isometric anti-linear bijection $j : M \to M'$. The space M' is a Hilbert space if the inner product is defined by the relation

$$(x' \mid y')' = (j^{-1}(y') \mid j^{-1}(x')), \qquad x', y' \in M'.$$

Let $x' \in M'$, we define a mapping $\tilde{x}' : E \to C$ by setting

$$\tilde{x}'(s) = \langle i(s), x' \rangle, \qquad s \in E,$$

where $\langle i(s), x' \rangle$ is the value of $x' \in M'$ at $i(s) \in M$. We show that the linear space $H(K) = \{ \tilde{x}' \mid x' \in M' \}$ is an r. k. Hilbert space with K as its reproducing kernel.

Define a mapping $h : M' \to H(K)$ by setting $h(x') = \tilde{x}'$, $x' \in M'$. The mapping h is linear and well-defined. Furthermore, h is injective. Let $x', y' \in M'$, $x' \neq y'$; since $M = \overline{\operatorname{sp}}\{i(E)\}$, there exists at least one $s \in E$, such that $\langle i(s), x' \rangle \neq \langle i(s), y' \rangle$, thus $\tilde{x}' \neq \tilde{y}'$. The mapping h is onto, since by definition $H(K) = h(M')$, thus h is a bijection.

Define a mapping $B_K : H(K) \times H(K) \to C$ by setting

$$B_K(\tilde{x}', \tilde{y}') = (\tilde{x}' \mid \tilde{y}')_K = (h^{-1}(\tilde{x}') \mid h^{-1}(\tilde{y}'))', \qquad \tilde{x}', \tilde{y}' \in H(K),$$

then B_K is an inner product of $H(K)$ and h is an inner product preserving bijection. Therefore the space $H(K)$, equipped with the topology induced by the inner product B_K, is a Hilbert space, which is isometrically isomorphic to the space M'.

Next we show that the space $H(K)$ is an r. k. Hilbert space on E. First we note that the elements of $H(K)$ are complex-valued mappings defined on E. Set $\tilde{K}_t' = h(j(i(t)))$, $t \in E$, then for every $s \in E$

$$K_t(s) = K(s, t) = (i(s) \mid i(t)) = \langle i(s), j(i(t)) \rangle = \tilde{K}_t'(s),$$

thus $K_t = \tilde{K}_t'$, so $K_t \in H(K)$ for every $t \in E$. Furthermore, for every $\tilde{x}' \in H(K)$ we have

$$\tilde{x}'(t) = \langle i(t) , x' \rangle = (x' \mid j(i(t)))' = (\tilde{x}' \mid \tilde{K}_t')_K = (\tilde{x}' \mid K_t)_K ,$$

$t \in E$, since $K_t = \tilde{K}_t'$. Thus K is the reproducing kernel of the space $H(K)$.

The space $H(K)$ is the only r. k. Hilbert space on E with K as its reproducing kernel as the values of every element of such an r. k. Hilbert space are uniquely determined in all points of E by the reproducing property.

The space $H(K)$ is isometrically isomorphic to the strong dual of the space $\mathrm{sp}\{i(E)\}$, since it is isometrically isomorphic to the strong dual of the completion M of $\mathrm{sp}\{i(E)\}$. The proof of the theorem is complete.

2. Next we consider the connection between positive definite functions and r. k. Hilbert spaces.

Definition 1.3.2. *Let E be a set. A mapping $K : E \times E \to C$ is called positive definite, if*

$$\sum_{j=1}^{n} \sum_{k=1}^{n} a_j \bar{a}_k K(t_j , t_k) \geq 0$$

for all $n \in N$, $t_j \in E$, $a_j \in C$, $j = 1 , \ldots , n$.

The next results are immediate.

Corollary 1.3.3. *Let K be the reproducing kernel of an r. k. Hilbert space. Then K is positive definite.*

Corollary 1.3.4. *Let r be the covariance mapping of a stochastic mapping. Then r is positive definite.*

Aronszajn [1] has shown that the converse of Corollary 1.3.3 is also true, i.e., for a given positive definite mapping $K : E \times E \to C$, where E is an arbitrary set, there exists a unique r. k. Hilbert space $\overline{\mathrm{sp}}\{K\}$ on E , such that K is the reproducing kernel of the space $\overline{\mathrm{sp}}\{K\}$. The space $\overline{\mathrm{sp}}\{K\}$ is called the *r. k. Hilbert space spanned by* K and it consists of the linear space $\mathrm{sp}\{K\} = \mathrm{sp}\{K_t \mid t \in E\}$ and of functions, which are pointwise limits of the Cauchy sequences of the functions of the space $\mathrm{sp}\{K\}$. We state this result as a theorem and give a proof which shows that the result is a consequence of Theorem 1.3.1.

Theorem 1.3.5. *Let E be a set and let $K : E \times E \to C$ be a positive definite mapping. Then there exists a unique r. k. Hilbert space $\overline{\mathrm{sp}}\{K\}$ on E , such that K is the reproducing kernel of the space $\overline{\mathrm{sp}}\{K\}$. The space $\overline{\mathrm{sp}}\{K\}$ consists of the space $\mathrm{sp}\{K\} = \mathrm{sp}\{K_t \mid t \in E\}$ and the pointwise limits of the Cauchy sequences (in the norm topology) of $\mathrm{sp}\{K\}$.*

Proof. Denote by $\mathrm{sp}\{\bar{K}\}$ the linear space spanned by the set $\{\bar{K}_t \mid t \in E\}$. The space $\mathrm{sp}\{\bar{K}\}$ is an inner product space with the inner product defined by the relation

$$(f \mid g)_{\bar{K}} = \sum_{j=1}^{m} \sum_{k=1}^{n} a_j \bar{b}_k K(s_j , t_k)$$

for $f , g \in \mathrm{sp}\{\bar{K}\}$,

$$f = \sum_{j=1}^{m} a_j \bar{K}_{s_j} , \quad g = \sum_{k=1}^{n} b_k \bar{K}_{t_k} .$$

The proof of this fact is a repetition of the proof given by Aronszajn [1: pp. 143—145]. Define a mapping $i : E \to \mathrm{sp}\{\bar{K}\}$, by setting $i(s) = \bar{K}_s$, $s \in E$. Then $K(s , t) = (i(s) \mid i(t))_{\bar{K}}$. Thus, the mappings $i : E \to \mathrm{sp}\{\bar{K}\}$ and K satisfy the conditions of Theorem 1.3.1, which proves the first part of the theorem. Denote $\overline{\mathrm{sp}}\{K\} = H(K)$ (see Theorem 1.3.1).

The second part of the theorem follows from the fact that $\mathrm{sp}\{K\}$ is dense in $\overline{\mathrm{sp}}\{K\}$ and from the inequality

$$|f_m(s) - f_n(s)| = |(f_m - f_n \mid K_s)| \leq \|f_m - f_n\| \|K_s\| , \quad s \in E ,$$

when $f_m , f_n \in \mathrm{sp}\{K\}$.

Remark 1.3.6. Let H be an inner product space and let E be a set. Let $i : E \to H$ be a mapping. Then the mapping $K : E \times E \to C$, $K(s , t) = (i(s) \mid i(t))$, $t \in E$, is positive definite. Let $\overline{\mathrm{sp}}\{K\}$ be the r. k. Hilbert space spanned by K . By Theorem 1.3.1 there exists a unique r. k. Hilbert space $H(K)$ on E , for which K is the reproducing kernel. It follows that $\overline{\mathrm{sp}}\{K\} = H(K)$, since K is the reproducing kernel also of the space $\overline{\mathrm{sp}}\{K\}$.

2. Stochastic measures

In this paper we shall consider stochastic measures as vector measures taking values in the space $L_0^2(P)$. We shall define a vector measure as a continuous linear mapping from a suitable function space into a topological vector space. We shall use the integration theory of vector measures given by Thomas [20]. It is closely related to the one given by Bourbaki [5: Ch. 6, § 2]. In fact, in the case of a stochastic measure the integrable functions are the same. We prefer Thomas's integration theory, because we want to show how the integration of a stochastic measure is related to the integration of its covariance mapping, when the covariance mapping is interpreted as a bimeasure.

We note that one very often defines a vector measure as a completely additive vector-valued set function, defined on an σ -algebra or a σ -ring of subsets of a space S (see for example Dinculeanu [7]).

In the works of Karhunen [11], Loève [13], Cramér [6], Bochner [4] and Rozanov [18], mentioned in the Introduction, a stochastic measure is defined as a completely additive set function taking values in the space $L_0^2(P)$.

Our aim is first to give the definition of a vector measure and of the integral with respect to it. Then we collect some general results about vector measures. After that we consider vector measures taking values in a Hilbert space. In our applications the Hilbert space is either the space $L_0^2(P)$ or an r. k. Hilbert space. As noted above, in the first case the vector measure is called a stochastic measure. In section 2.3 we introduce the notion of a bimeasure. The rest of chapter 2 is devoted to stochastic measures and to their covariance mappings.

We begin with a short review of complex Radon measures. The integration of complex Radon measures is similar to the integration of vector measures taking values in a normed space. We consider only the latter case. For reference on the complex-valued case see Bourbaki [5: Ch. 3—5].

Note that in this paper we consider essential integrals and we use the same terminology as Thomas, that is, we use the terms *integral* and *integrable*. when Bourbaki uses the terms *essential integral* and *essentially integrable*.

2.1. On Radon measures

1. All results given here are proved in Bourbaki [5: Ch. 3]. Let T be a fixed locally compact Hausdorff space. By $\mathcal{K}_C(T)$ we denote the vector space of all continuous mappings $f: T \to C$. for which the support of f, $\operatorname{supp}(f)$, is compact. Let $K \subset T$ be compact. Set

$$\mathcal{K}_C(T\,;K) = \{ f \in \mathcal{K}_C(T) \quad \operatorname{supp}(f) \subset K \} \, .$$

then the spaces $\mathcal{K}_C(T\,;K)$ are Banach spaces if the topology is defined by the supremum norm. The topology of $\mathcal{K}_C(T)$ is the inductive limit of spaces $\mathcal{K}_C(T\,;K)$ relative to the canonical injections $j_K : \mathcal{K}_C(T : K) \to \mathcal{K}_C(T)$, $K \subset T$, K compact. We recall that the space $\mathcal{K}_C(T)$ is barreled.

2. In the following the dual of $\mathcal{K}_C(T)$ is denoted by $\mathcal{M}_C(T)$ and its elements are called *complex Radon measures on* T or in short *Radon measures,* if no confusion with respect to the domain is possible. A Radon measure μ is *real valued* if it coincides with its complex conjugate, that is if $\mu(f) = \overline{\mu(\bar{f})}$ for all $f \in \mathcal{K}_C(T)$. A real valued Radon measure is *positive* if $\mu(f) \geq 0$ for all $f \in \mathcal{K}_C(T)$, $f \geq 0$. The *absolute value* of a Radon measure μ is the positive Radon measure $|\mu|$, for which

$$|\mu|(f) = \sup_{|g| \leq f} |\mu(g)| , \qquad \text{where } g \in \mathcal{K}_C(T) ,$$

for $f \in \mathcal{K}_C(T)$, $f \geq 0$. We note that $|\mu(f)| \leq |\mu|(|f|)$ for all $f \in \mathcal{K}_C(T)$, when $\mu \in \mathcal{M}_C(T)$.

The elements of the dual of $\mathcal{K}_C(T)$, when the space $\mathcal{K}_C(T)$ carries the topology defined by the supremum norm, are called *bounded Radon measures on* T . In the following $\mathcal{M}^1_C(T)$ denotes the space of all bounded Radon measures on T .

Let $G \subset T$ be open, then G is, as a topological subspace of T , a locally compact Hausdorff space. We may identify the space $\mathcal{K}_C(G)$ with a linear subspace of $\mathcal{K}_C(T)$. Let $\mu \in \mathcal{M}_C(T)$, then the restriction of μ to $\mathcal{K}_C(G)$ is an element of $\mathcal{M}_C(G)$.

2.2. On vector measures

1. As mentioned above, we begin this section by giving the definition of a vector measure and of the integral with respect to it. Ours is an obvious generalization of the definition given by Thomas [20], who had the scalar field R . In the following the scalar field of the topological vector spaces in which the vector measures take their values is C .

Let T be a fixed locally compact Hausdorff space.

Definition 2.2.1. *Let F be a locally convex topological vector space. An F-valued vector measure on T or in short a vector measure is a continuous linear mapping $\mu : \mathcal{K}_C(T) \to F$.*

Next we shall define the set of integrable functions for a vector measure. First we consider the case of a normed space F and then give the generalization to the case where F is an arbitrary locally convex vector space.

2. Let F be a fixed normed space. We begin with the definition of the semi-variation of a vector measure. Let us denote the norm of F by $\| \|$ and by $\mathcal{I}_+(T)$ the set of the lower semi-continuous nonnegative functions $f : T \to \bar{R}^+$, where $\bar{R}^+ = \{ x \in R \mid x \geq 0 \} \cup \{ +\infty \}$.

Definition 2.2.2. *Let μ be an F-valued vector measure. We define a nonnegative, not necessarily finite number $\mu^\bullet(f)$ for every nonnegative function $f : T \to \bar{R}^+$ by setting*

(i) $\mu^\bullet(f) = \sup_{|g| \leq f} \|\mu(g)\|$ *where* $g \in \mathcal{K}_C(T)$, *if* $f \in \mathcal{I}_+(T)$,

(ii) $\mu^\bullet(f) = \inf_{g \geq f} \mu^\bullet(g)$ *where* $g \in \mathcal{I}_+(T)$, *if* supp (f) *is compact, and*

(iii) $\mu^\bullet(f) = \sup_{g \leq f} \mu^\bullet(g)$ *where* $g \geq 0$ *is such that* supp (g) *is compact, if f is an arbitrary nonnegative function.*

The mapping $\mu^\bullet$ *is called the semi-variation of* μ *.*

Let $A \subset T$. We denote its characteristic function by χ_A and define $\mu^\bullet(A) = \mu^\bullet(\chi_A)$, when μ is an F-valued vector measure.

Remark 2.2.3. For a complex Radon measure μ, that is, in case $F = C$, the definition of $\mu^\bullet$ coincides with the definition of the essential upper integral of $|\mu|$ given by Bourbaki [5: Ch. 5, p. 2].

Let μ be an F-valued vector measure. In the following we use the notation $\mu_{x'} = x' \circ \mu$, $x' \in F'$. Note that $\mu_{x'} \in \mathcal{M}_C(T)$ for all $x' \in F'$.

Let μ be an F-valued vector measure and let $f \in \mathcal{J}_+(T)$, then

$$\mu^\bullet(f) = \sup_{|g| \leq f} \|\mu(g)\| = \sup_{|g| \leq f} \sup_{\|x'\| \leq 1} |\mu_{x'}(g)|$$
$$= \sup_{\|x'\| \leq 1} \sup_{|g| \leq f} |\mu_{x'}(g)| = \sup_{\|x'\| \leq 1} |\mu_{x'}|^\bullet(f),$$

where $g \in \mathcal{K}_C(T)$. For later use we state this result, due to Thomas, as a lemma.

Lemma 2.2.4. *Let* μ *be an* F*-valued vector measure. Then*

$$\mu^\bullet(f) = \sup_{\|x'\| \leq 1} |\mu_{x'}|^\bullet(f)$$

for all $f \in \mathcal{J}_+(T)$.

Let μ be an F-valued vector measure. Then for all $\lambda \in C$ and for all complex-valued functions f, g and h, defined on T, we have

$$\mu^\bullet(|\lambda f|) = |\lambda|\, \mu^\bullet(|f|), \quad \mu^\bullet(|f+g|) \leq \mu^\bullet(|f|) + \mu^\bullet(|g|),$$
$$\mu^\bullet(|f|) \leq \mu^\bullet(|h|) \text{ if } |f| \leq |h|.$$

The set of all functions $f: T \to C$, for which $\mu^\bullet(|f|) < \infty$ is denoted by $\mathcal{F}_C^\bullet(\mu)$. The mapping $N_1(f) = \mu^\bullet(|f|)$, $f \in \mathcal{F}_C^\bullet(\mu)$, is a semi-norm of $\mathcal{F}_C^\bullet(\mu)$. We define the topology of $\mathcal{F}_C^\bullet(\mu)$ as the locally convex topology defined by the semi-norm N_1. Obviously $\mathcal{K}_C(T) \subset \mathcal{F}_C^\bullet(\mu)$.

Definition 2.2.5. *Let* μ *be an* F*-valued vector measure. The set* $\mathcal{L}_C^1(\mu)$ *of* μ*-integrable functions* $f: T \to C$ *is the closure of* $\mathcal{K}_C(T)$ *in* $\mathcal{F}_C^\bullet(\mu)$.

Let μ be an F-valued vector measure. In order to define the integral of the functions of $\mathcal{L}_C^1(\mu)$ with respect to the vector measure μ, we note that $\|\mu(f)\| \leq \mu^\bullet(|f|)$, when $f \in \mathcal{K}_C(T)$; that is, the mapping $\mu: \mathcal{K}_C(T) \to F$ is continuous, when $\mathcal{K}_C(T)$ carries the topology induced by $\mathcal{L}_C^1(\mu)$.

Definition 2.2.6. *Let* μ *be an* F*-valued vector measure. Then the integral of a function* $f \in \mathcal{L}_C^1(\mu)$ *with respect to* μ *is the value* $\mu(f) \in \hat{F}$, *of the extension by continuity of the mapping* $\mu: \mathcal{K}_C(T) \to F$ *to a mapping* $\mu: \mathcal{L}_C^1(\mu) \to \hat{F}$.

We use the notation

$$\mu(f) = \int f\, d\mu\,, \qquad f \in \mathcal{L}^1_C(\mu)\,.$$

Let $x' \in F'$, then $\mathcal{L}^1_C(\mu) \subset \mathcal{L}^1_C(\mu_{x'})$ and

$$\langle \int f\, d\mu\,,\, x'\rangle = \int f\, d\mu_{x'}\,, \qquad f \in \mathcal{L}^1_C(\mu)\,. \tag{2.2.1}$$

Let F be a Banach space and let $\mu : \mathcal{K}_C(T) \to F$ be a vector measure, then

$$\left\| \int f\, d\mu \right\| \leq \mu^\bullet(|f|)$$

for all $f \in \mathcal{L}^1_C(\mu)$.

3. We go on to the case where F is an arbitrary locally convex vector space.

Let F be a locally convex vector space and let μ be an F-valued vector measure. Let p be a continuous semi-norm of F and let F_p be the quotient space $F / \operatorname{Ker}(p)$, when F carries the locally convex topology defined by the semi-norm p. Set $\mu_p = \pi_p \circ \mu$, where π_p denotes the canonical mapping from F into F_p, then μ_p is an F_p-valued vector measure. Let $\mathcal{P}$ be the collection of all continuous semi-norms of F. We denote

$$\mathcal{F}^\bullet_C(\mu) = \bigcap_{p \in \mathcal{P}} \mathcal{F}^\bullet_C(\mu_p)$$

and define the topology of $\mathcal{F}^\bullet_C(\mu)$ as the projective locally convex topology with respect to the canonical injections $j_p : \mathcal{F}^\bullet_C(\mu) \to \mathcal{F}^\bullet_C(\mu_p)$, $p \in \mathcal{P}$ (the topology of $\mathcal{F}^\bullet_C(\mu_p)$ is defined by the semi-norm $\mu_p^\bullet$ for all $p \in \mathcal{P}$). Note that $\mathcal{K}_C(T) \subset \mathcal{F}^\bullet_C(\mu)$.

As in the case of a normed space, we denote by $\mathcal{L}^1_C(\mu)$ the closure of $\mathcal{K}_C(T)$ in $\mathcal{F}^\bullet_C(\mu)$ and call the functions $f \in \mathcal{L}^1_C(\mu)$ *μ-integrable*. We see that

$$\mathcal{L}^1_C(\mu) = \bigcap_{p \in \mathcal{P}} \mathcal{L}^1_C(\mu_p)\,.$$

Furthermore, the topology of $\mathcal{L}^1_C(\mu)$ is the projective locally convex topology with respect to the canonical injections $j_p : \mathcal{L}^1_C(\mu) \to \mathcal{L}^1_C(\mu_p)$, $p \in \mathcal{P}$.

As in the case of a normed space, we see that the mapping $\mu : \mathcal{K}_C(T) \to F$ is continuous if $\mathcal{K}_C(T)$ carries the topology induced by $\mathcal{L}^1_C(\mu)$. Suppose that the space F is a Hausdorff space, then we define the *integral of* *a function* $f \in \mathcal{L}^1_C(\mu)$ *with respect to the vector measure* μ analogously to the case of a normed space (see Definition 2.2.6). The equation (2.2.1) is still valid.

Let F be a locally convex vector space and let $\mu : \mathcal{K}_C(T) \to F$ be an F-valued vector measure. Denote by F_1 the Hausdorff space $F / \overline{\{0\}}$ associated with F and let $\pi : F \to F_1$ be the canonical mapping, then $\mathcal{L}^1_C(\mu) = \mathcal{L}^1_C(\pi \circ \mu)$. Thus it is no restriction to consider only locally convex Hausdorff spaces.

4. Next we consider how Bourbaki's definition of the integral of a vector measure [5: Ch. 6, § 2] is related to Thomas's.

Let F be a locally convex vector space and let F' be its dual. Let $\mu : \mathcal{K}_C(T) \to F$ be a vector measure, then the mapping μ is still continuous if we equip the space F with the weak topology $\sigma(F, F')$.

Definition 2.2.7. *Let F be a locally convex vector space and let F' be its dual. Let μ be an F-valued vector measure. Then the (continuous) mapping $\mu_w : \mathcal{K}_C(T) \to F$, $\mu_w(f) = \mu(f)$, $f \in \mathcal{K}_C(T)$, with the topology $\sigma(F, F')$ on F, is called the weak vector measure defined by μ.*

The following lemma is needed to show how to get the set of integrable functions of a weak vector measure.

Lemma 2.2.8. *Let F and F_α, $\alpha \in I$, be locally convex vector spaces, such that the topology of F is defined as the projective locally convex topology with respect to the given linear mappings $u_\alpha : F \to F_\alpha$, $\alpha \in I$. Let μ be an F-valued vector measure and let $\mu_\alpha = u_\alpha \circ \mu$, $\alpha \in I$. Then*

$$\mathcal{L}^1_C(\mu) = \bigcap_{\alpha \in I} \mathcal{L}^1_C(\mu_\alpha)$$

and the topology of $\mathcal{L}^1_C(\mu)$ is the projective locally convex topology with respect to the canonical injections $j_\alpha : \mathcal{L}^1_C(\mu) \to \mathcal{L}^1_C(\mu_\alpha)$, $\alpha \in I$.

Proof. The proof given by Thomas [20: pp. 77—78] is also valid, when the scalar field of F is C.

Let F be a locally convex vector space and let $\mu : \mathcal{K}_C(T) \to F$ be a vector measure. By Lemma 2.2.8 we get

$$\mathcal{L}^1_C(\mu_w) = \bigcap_{x' \in F'} \mathcal{L}^1_C(\mu_{x'}) .$$

Since the completion of the space F, when it carries the topology $\sigma(F, F')$, is the space F'^* (i.e. the algebraic dual of F'), we have

$$\int f \, d\mu_w \in F'^* \qquad \text{for all } f \in \mathcal{L}^1_C(\mu_w) .$$

By (2.2.1) we have

$$\langle \int f \, d\mu_w \, , x' \rangle = \int f \, d\mu_{x'} \qquad \text{for all } x' \in F' ,$$

when $f \in \mathcal{L}^1_C(\mu_w)$. Since

$$\mathcal{L}^1_C(\mu) \subset \bigcap_{x' \in F'} \mathcal{L}^1_C(\mu_{x'}) = \mathcal{L}^1_C(\mu_w)$$

we get

$$\int f \, d\mu = \int f \, d\mu_w \qquad \text{if } f \in \mathcal{L}^1_C(\mu) .$$

In the following we call the functions $f \in \mathcal{L}^1_C(\mu_w)$ *weakly* μ *-integrable* and the integral of $f \in \mathcal{L}^1_C(\mu_w)$ with respect to μ_w is called the *weak integral of* f *with respect to* μ .

We note that the above definition of the weak integral is equivalent to Bourbaki's way of defining the integral of a vector measure [5: Ch. 6, § 2].

5. Let F be a locally convex Hausdorff space and let μ be an F -valued vector measure, then in general $\int f \, d\mu \in \hat{F}$, when $f \in \mathcal{L}^1_C(\mu)$, if the space F is not complete. The following theorem gives a sufficient condition for $\int f \, d\mu \in F$ to be valid for all $f \in \mathcal{L}^1_C(\mu)$. The theorem is an obvious generalization of Thomas's corresponding result [20: pp. 80—81].

Before stating the theorem we recall that a topological vector space is said to be *quasi-complete* if its every bounded and closed subset is complete.

Theorem 2.2.9. *Let* F *be a quasi-complete locally convex Hausdorff space and let* μ *be an* F *-valued vector measure. Then* $\int f \, d\mu \in F$ *for all* $f \in \mathcal{L}^1_C(\mu)$.

Remark 2.2.10. Let F be a barreled locally convex Hausdorff space, then the space F' equipped with the topology $\sigma(F' , F)$ is a quasi-complete locally convex Hausdorff space, since in the topology $\sigma(F' , F)$ every bounded set of the space F' is relatively compact (Schaefer [19: p. 141]).

Let F be a semi-reflexive locally convex Hausdorff space, then the space F equipped with the topology $\sigma(F , F')$ is a quasi-complete locally convex Hausdorff space (Schaefer [19: p. 144]).

6. In studying the connection of stochastic measures and their covariance mappings we use a result proved by Thomas [20: pp. 106—107]. The proof given by Thomas is valid also in the case of the complex scalars. Before stating this result as a theorem we give a definition.

First we recall that a function $f : T \to C$ is said to be a *Borel function* if $f^{-1}(G)$ is a Borel set of T for every open set $G \subset C$.

Definition 2.2.11. *Let* F *be a locally convex vector space. An* F *-valued* *vector measure* μ *is said to be extendible if for every bounded Borel function* $f: T \to C$ *with compact support we have* $f \in \mathcal{L}^1_C(\mu)$.

Theorem 2.2.12. *Let* F *be a Banach space and let* μ *be an extendible* F *-valued vector measure. Then a weakly* μ *-integrable function* f *is* μ *-integrable if and only if* $\int f \chi_G \, d\mu_w \in F$ *for all open sets* $G \subset T$.

For later use we still state one lemma concerning a condition for a vector measure to be extendible. As above, the proof given by Thomas [20: p. 101] is valid also in the case of the complex scalars.

Let F be a locally convex Hausdorff space and let $\mu : \mathcal{K}_C(T) \to F$ be an F -valued vector measure. Suppose that $G \subset T$ is an open relatively compact set. As noted above, the space $\mathcal{K}_C(G)$ can be considered as a linear subspace of $\mathcal{K}_C(T)$. If p is a continuous semi-norm of F , then using the notation introduced in 2.2.3, we have

$$\sup_{\substack{|g| \leq 1 \\ g \in \mathcal{K}_C(G)}} p(\mu(g)) = \mu_p^{\bullet}(G) < \infty .$$

Thus the restriction of μ to $\mathcal{K}_C(G)$ is continuous if the space $\mathcal{K}_C(G)$ carries the norm topology defined by the supremum norm.

In the following $C_0(T)$ denotes the space of all continuous functions $f: T \to C$ vanishing at infinity. The topology of $C_0(T)$ is defined by the supremum norm.

Lemma 2.2.13. *Let* F *be a locally convex Hausdorff space and let* μ *be an* F *-valued vector measure. Then* μ *is extendible if and only if the restriction of* μ *to* $\mathcal{K}_C(G)$, *where* G *is an arbitrary relatively compact open set contained in* T , *can be extended by continuity to a weakly compact mapping from* $C_0(G)$ *into* $\hat{F}$.

2.3. Bimeasures

1. In this section we present the basic results on the integration of bimeasures. We note that bimeasures are studied especially in the papers of Morse and Transue [15] and Thomas [20: pp. 144—147].

Let S and T be fixed locally compact Hausdorff spaces.

Definition 2.3.1. *A continuous bilinear mapping* $B : \mathcal{K}_C(S) \times \mathcal{K}_C(T) \to C$ *is called a bimeasure on* $S \times T$.

We remark that every separately continuous bilinear mapping $B : \mathcal{K}_C(S) \times \mathcal{K}_C(T) \to C$ is continuous.

Let B be a bimeasure on $S \times T$ and let $g \in \mathcal{K}_C(T)$ be fixed. We define a linear mapping $B(. , g) : \mathcal{K}_C(S) \to C$ by setting $B(. , g)(f) = B(f , g)$, $f \in \mathcal{K}_C(S)$. Let $f \in \mathcal{K}_C(S)$ be fixed. As above, we define a linear mapping $B(f , .) : \mathcal{K}_C(T) \to C$ by setting $B(f , .)(g) = B(f , g)$,

$g \in \mathcal{K}_C(T)$. Since the bilinear mapping $B : \mathcal{K}_C(S) \times \mathcal{K}_C(T) \to C$ is continuous, we have $B(\,.\,, g) \in \mathcal{M}_C(S)$ and $B(f\,,\,.\,) \in \mathcal{M}_C(T)$ for all $g \in \mathcal{K}_C(T)$, $f \in \mathcal{K}_C(S)$.

Let $\mathcal{M}_C(S)$ carry the topology $\sigma(\mathcal{M}_C(S)\,, \mathcal{K}_C(S))$, then the linear mapping $\mu_B^r : \mathcal{K}_C(T) \to \mathcal{M}_C(S)$,

$$(2.3.1) \qquad \mu_B^r(g) = B(\,.\,, g)\,, \qquad g \in \mathcal{K}_C(T)\,,$$

is continuous, i.e., the linear mapping μ_B^r is an $\mathcal{M}_C(S)$-valued vector measure on T. By Lemma 2.2.8 we get

$$\mathcal{L}_C^1(\mu_B^r) = \bigcap_{\in \mathcal{K}_C(S)} \mathcal{L}_C^1(B(f\,,\,.\,))\,,$$

since $\langle f\,, \mu_B^r(g)\rangle = B(f\,, g)$ for all $f \in \mathcal{K}_C(S)$, $g \in \mathcal{K}_C(T)$.

Similarly, let $\mathcal{M}_C(T)$ carry the topology $\sigma(\mathcal{M}_C(T)\,, \mathcal{K}_C(T))$, then the linear mapping $\mu_B^l : \mathcal{K}_C(S) \to \mathcal{M}_C(T)$,

$$(2.3.2) \qquad \mu_B^l(f) = B(f\,,\,.\,)\,, \qquad f \in \mathcal{K}_C(S)\,,$$

is an $\mathcal{M}_C(T)$-valued vector measure on S. As above, we have

$$\mathcal{L}_C^1(\mu_B^l) = \bigcap_{g \in \mathcal{K}_C(T)} \mathcal{L}_C^1(B(\,.\,, g))\,.$$

Definition 2.3.2. *Let B be a bimeasure on $S \times T$. Then the $\mathcal{M}_C(S)$-valued vector measure μ_B^r on T (defined by (2.3.1)) is called the right measure defined by B and the functions $h : T \to C$, $h \in \mathcal{L}_C^1(\mu_B^r)$ are called right integrable with respect to B. Similarly, the $\mathcal{M}_C(T)$-valued vector measure μ_B^l on S (defined by (2.3.2)) is called the left measure defined by B and the functions $h : S \to C$, $h \in \mathcal{L}_C^1(\mu_B^l)$ are called left integrable with respect to B.*

The following lemma is due to Thomas [20: pp. 144—145].

Lemma 2.3.3. *Let B be a bimeasure on $S \times T$. Then $\int h\, d\mu_B^r \in \mathcal{M}_C(S)$ for all $h : T \to C$, $h \in \mathcal{L}_C^1(\mu_B^r)$ and $\int h\, d\mu_B^l \in \mathcal{M}_C(T)$ for all $h : S \to C$, $h \in \mathcal{L}_C^1(\mu_B^l)$.*

Proof. We consider only the former assertion, the latter's proof being similar. The space $\mathcal{M}_C(S)$ carrying the topology $\sigma(\mathcal{M}_C(S)\,, \mathcal{K}_C(S))$ is a quasi-complete locally convex Hausdorff space, as noted in Remark 2.2.3, since the space $\mathcal{K}_C(S)$ is barreled (Bourbaki [5: Ch. 3, p. 42]). Thus, the lemma follows by Theorem 2.2.1.

Let B be a bimeasure on $S \times T$ and let $h : T \to C$ be such that $h \in \mathcal{L}_C^1(\mu_B^r)$, then by Lemma 2.3.3 $\int h\, d\mu_B^r \in \mathcal{M}_C(S)$. In the following we use the notation $B(\,.\,, h) = \int h\, d\mu_B^r$, that is

$$B(\,.\,, h)(f) = \langle f\,, \int h\, d\mu_B^r\rangle = \int h\, dB(f\,,\,.\,)\,, \qquad f \in \mathcal{K}_C(S)\,.$$

Similarly, let $h : S \to C$ be such that $h \in \mathcal{L}_C^1(\mu_B^l)$. As above, we use the notation $B(h\,,.) = \int h\, d\mu_B^l$, then by Lemma 2.3.3 $B(h\,,.) \in \mathcal{M}_C(T)$.

Definition 2.3.4. *Let B be a bimeasure on $S \times T$. We suppose that the functions $f : S \to C$ and $g : T \to C$ satisfy the conditions*

(i) $f \in \mathcal{L}_C^1(\mu_B^l)\,,\ g \in \mathcal{L}_C^1(\mu_B^r)$

(the Radon measures $B(f\,,.) \in \mathcal{M}_C(T)$ and $B(.\,,g) \in \mathcal{M}_C(S)$ are thus defined),

(ii) $f \in \mathcal{L}_C^1(B(.\,,g))$ *and* $g \in \mathcal{L}_C^1(B(f\,,.))$,

(iii) $\int f\, dB(.\,,g) = \int g\, dB(f\,,.)$;

then we say that the pair $(f\,,g)$ is integrable with respect to the bimeasure B and denote

$$B(f\,,g) = \int g\, dB(f\,,.) = \int f\, dB(.\,,g)\,.$$

2. In the following we call a bimeasure B *bounded* if the bilinear mapping $B : \mathcal{K}_C(S) \times \mathcal{K}_C(T) \to C$ is continuous, when the spaces $\mathcal{K}_C(S)$ and $\mathcal{K}_C(T)$ carry the topology defined by the supremum norm. Let B be a bounded bimeasure. Since the space $\mathcal{K}_C(S)$ (resp. $\mathcal{K}_C(T)$) is dense in $C_0(S)$ (resp. in $C_0(T)$) we can extend the bilinear mapping B by continuity to a continuous bilinear mapping $B : C_0(S) \times C_0(T) \to C$. Obviously, every pair $(f\,,g) \in C_0(S) \times C_0(T)$ is integrable with respect to B and $B(f\,,g) = \hat{B}(f\,,g)$, where $\hat{B}(f\,,g)$ is the value of the extension (by continuity) of B.

As above, we introduce the *(bounded) right measure* $\mu_B^r : C_0(T) \to \mathcal{M}_C^1(S)$ (resp. the *(bounded) left measure* $\mu_B^l : C_0(S) \to \mathcal{M}_C^1(T)$) *defined by* B. Clearly

$$\mathcal{L}_C^1(\mu_B^r) = \bigcap_{f \in C_0(S)} \mathcal{L}_C^1(B(f\,,.))$$

and

$$\mathcal{L}_C^1(\mu_B^l) = \bigcap_{g \in C_0(T)} \mathcal{L}_C^1(B(.\,,g))\,.$$

Analogously to Lemma 2.3.3 we have $\int h\, d\mu_B^r \in \mathcal{M}_C^1(S)$ for all $h \in \mathcal{L}_C^1(\mu_B^r)$ and $\int h\, d\mu_B^l \in \mathcal{M}_C^1(T)$ for all $h \in \mathcal{L}_C^1(\mu_B^l)$. Thus the bounded Radon measures

$$B(.\,,h_1) = \int h_1\, d\mu_B^r \quad \text{and} \quad B(h_2\,,.) = \int h_2\, d\mu_B^l$$

can be defined for all $h_1 \in \mathcal{L}_C^1(\mu_B^r)$, $h_2 \in \mathcal{L}_C^1(\mu_B^l)$, as above.

Let B be a bounded bimeasure on $S \times T$. Suppose that the functions $f: S \to C$, $g: T \to C$ satisfy the conditions

(i′) $f \in \bigcap_{h \in C_0(T)} \mathcal{L}^1_C(B(\,.\,, h))$, $g \in \bigcap_{h \in C_0(S)} \mathcal{L}^1_C(B(h\,,\,.\,))$,

(ii′) $f \in \mathcal{L}^1_C(B(\,.\,, g))$, $g \in \mathcal{L}^1_C(B(f\,,\,.\,))$,

(iii′) $\int f\, dB(\,.\,, g) = \int g\, dB(f\,,\,.\,)$,

then we say that the *pair* (f, g) *is »strongly» integrable with respect to the bounded bimeasure* B.

3. Let E (resp. F) be a linear space of functions $f: U \to C$ (resp. of functions $g: V \to C$), where U (resp. V) is an arbitrary set. We remark that in this paper the *tensor product* $E \otimes F$ of the spaces E and F is considered as the linear space spanned by the functions $f \otimes g: U \times V \to C$,

$$(f \otimes g)(u, v) = f(u)\, g(v)\,, \qquad u \in U\,,\ v \in V\,.$$

Let B be a bimeasure on $S \times T$, then there does not always exist a Radon measure μ_B on $S \times T$ such that

$$B(f, g) = \mu_B(f \otimes g)\,, \qquad f \in \mathcal{K}_C(S)\,,\ g \in \mathcal{K}_C(T) \tag{2.3.3}$$

(see Example 3.3.4). A simple condition for the existence of a unique Radon measure μ_B on $S \times T$ such that (2.3.3) is satisfied, is given in the following lemma.

Let B be a bimeasure on $S \times T$. We denote by $\tilde{B}$ the unique linear mapping $\tilde{B}: \mathcal{K}_C(S) \otimes \mathcal{K}_C(T) \to C$, for which

$$\tilde{B}(f \otimes g) = B(f, g)\,, \qquad f \in \mathcal{K}_C(S)\,,\ g \in \mathcal{K}_C(T)\,. \tag{2.3.4}$$

Lemma 2.3.5. *Let* B *be a bimeasure on* $S \times T$. *If the linear mapping* $\tilde{B}: \mathcal{K}_C(S) \otimes \mathcal{K}_C(T) \to C$ *is continuous, when* $\mathcal{K}_C(S) \otimes \mathcal{K}_C(T)$ *carries the topology induced by* $\mathcal{K}_C(S \times T)$, *then there exists a unique Radon measure* μ_B *on* $S \times T$ *such that (2.3.3) is valid.*

Proof. First we note that the tensor product $\mathcal{K}_C(S) \otimes \mathcal{K}_C(T)$ is dense in $\mathcal{K}_C(S \times T)$ (Bourbaki [5: Ch. 3, p. 83]). The rest of the proof is then immediate.

Definition 2.3.6. *Let* B *be a bimeasure on* $S \times T$ *satisfying the hypothesis of Lemma 2.3.5, then we say that* B *can be extended to a Radon measure and the unique Radon measure* μ_B *satisfying (2.3.3) is called the Radon measure induced by the bimeasure* B.

Remark 2.3.7. Let B be a bounded bimeasure on $S \times T$. Suppose that the linear mapping $\tilde{B} : C_0(S) \otimes C_0(T) \to C$, defined by the extended bilinear mapping $B : C_0(S) \times C_0(T) \to C$, is continuous, when the space $C_0(S) \otimes C_0(T)$ carries the topology induced by the space $C_0(S \times T)$. Then there exists a unique bounded Radon measure μ_B on $S \times T$ such that

$$\mu_B(f \otimes g) = B(f, g), \qquad \text{for all } f \in C_0(S), \; g \in C_0(T),$$

and we say that the bounded bimeasure B *can be extended to a bounded Radon measure.*

2.4. Stochastic measures

1. In this section we shall develop the integration theory of stochastic measures and show how the integration of stochastic measures is related to the integration of their covariance mappings, when the covariance mappings are interpreted as bimeasures.

We begin with the definition of a stochastic measure.

Definition 2.4.1. *A vector measure* $\mu : \mathcal{K}_C(T) \to L_0^2(P)$ *is called a stochastic measure.*

In the following we shall also consider vector measures $\mu : \mathcal{K}_C(T) \to \overline{\mathrm{sp}}\{Q\}$, where the space $\overline{\mathrm{sp}}\{Q\}$ is an r. k. Hilbert space spanned by a positive definite mapping Q. For that reason we here state the results for vector measures $\mu : \mathcal{K}_C(T) \to H$, where H is an arbitrary Hilbert space. Of course, the results are then valid also for stochastic measures.

We recall that in general, if F is an arbitrary locally convex vector space and μ is an F-valued vector measure, we have

$$\mathcal{L}_C^1(\mu) \subset \mathcal{L}_C^1(\mu_w) = \bigcap_{x' \in F'} \mathcal{L}_C^1(\mu_{x'}) .$$

We shall show that for a vector measure μ taking values in a Hilbert space, $\mathcal{L}_C^1(\mu_w) = \mathcal{L}_C^1(\mu)$. For more general results of this type see Thomas [20: pp. 134—139].

We shall also consider the following case. Let H be a Hilbert space and let A be a linear subspace of H'. Let μ be an H-valued vector measure. Then the mapping $\mu : \mathcal{K}_C(T) \to H$ is continuous, also if H carries the topology $\sigma(H, A)$. Define a mapping $\tilde{\mu}_A : \mathcal{K}_C(T) \to H$, when H carries the topology $\sigma(H, A)$, by setting

$$\tilde{\mu}_A(f) = \mu(f), \qquad f \in \mathcal{K}_C(T). \tag{2.4.1}$$

We recall that by Lemma 2.2.8 we get

$$\mathcal{L}_C^1(\tilde{\mu}_A) = \bigcap_{x' \in A} \mathcal{L}_C^1(\mu_{x'}) .$$

Suppose that A is dense in $j(\overline{\mathrm{sp}}\{\mu\}) \subset H'$, in the norm topology of H', where $j: H \to H'$ is the canonical anti-linear bijection. In Theorem 2.4.8 we give a (necessary and) sufficient condition for a function $f \in \mathcal{L}^1_C(\tilde{\mu}_A)$ to be μ-integrable.

2. First we consider weakly integrable functions.

Lemma 2.4.2. *Let H be a Hilbert space and let μ be an H-valued vector measure. Then $\int f \, d\mu_w \in H$ for every $f \in \mathcal{L}^1_C(\mu_w)$.*

Proof. First we note that the weak vector measure μ_w, defined by the H-valued vector measure μ, is an H-valued vector measure, when H carries the topology $\sigma(H, H')$. By Remark 2.2.3 the space H carrying the topology $\sigma(H, H')$ is a quasi-complete locally convex Hausdorff space, since H is as a Hilbert space barreled. Thus, the lemma follows by Theorem 2.2.9.

Lemma 2.4.3. *Let H be a Hilbert space and let μ be an H-valued vector measure, then μ is extendible.*

Proof. Suppose that $G \subset T$ is an open relatively compact set. Denote by μ_G the restriction of μ to $\mathcal{K}_C(G)$ and let $\hat{\mu}_G$ be the extension by continuity of μ_G to a continuous linear mapping $\hat{\mu}_G : C_0(G) \to H$. Then $\hat{\mu}_G$ is a weakly compact mapping as a continuous linear mapping from a normed space to a Hilbert space. Thus, the lemma follows by Lemma 2.2.13.

Theorem 2.4.4. *Let H be a Hilbert space and let μ be an H-valued vector measure, then $\mathcal{L}^1_C(\mu) = \mathcal{L}^1_C(\mu_w)$.*

Proof. Suppose that $G \subset T$ is an open set. Let $f \in \mathcal{L}^1_C(\mu_w)$, then $f \chi_G \in \mathcal{L}^1_C(\mu_{x'})$ for all $x' \in H'$. Thus $f \chi_G \in \mathcal{L}^1_C(\mu_w)$. By Lemma 2.4.2 we have

$$\int f \chi_G \, d\mu_w \in H .$$

Furthermore, by Lemma 2.4.3, the vector measure μ is extendible. Thus, the theorem follows by Theorem 2.2.12.

Let H be a Hilbert space and let μ be an H-valued vector measure. Denote

$$\overline{\mathrm{sp}}\{\mu\} = \overline{\mathrm{sp}}\{ \mu(f) \mid f \in \mathcal{K}_C(T) \},$$

then $\int f \, d\mu \in \overline{\mathrm{sp}}\{\mu\}$ for all $f \in \mathcal{L}^1_C(\mu)$. The following corollary shows that only the Radon measures $\mu_{x'}$, $x' \in \overline{\mathrm{sp}}\{\mu\}'$ are essential in studying the integrability with respect to the vector measure μ.

Corollary 2.4.5. *Let H be a Hilbert space and let μ be an H-valued vector measure, then*

$$\mathcal{L}^1_C(\mu) = \mathcal{L}^1_C(\mu_w) = \bigcap_{x' \in \overline{\mathrm{sp}}\{\mu\}'} \mathcal{L}^1_C(\mu_{x'}) .$$

Proof. The inclusion

$$\mathcal{L}^1_C(\mu) = \mathcal{L}^1_C(\mu_w) = \bigcap_{x' \in H'} \mathcal{L}^1_C(\mu_{x'}) \subset \bigcap_{x' \in \overline{\text{sp}}\{\mu\}'} \mathcal{L}^1_C(\mu_{x'})$$

is immediate. On the other hand, denote by $x'_{\overline{\text{sp}}\{\mu\}}$ the restriction of $x' \in H'$ to the space $\overline{\text{sp}}\{\mu\}$. Since

$$\mu_{x'}(f) = \langle \mu(f), x' \rangle = \mu_{x'_{\overline{\text{sp}}\{\mu\}}}(f), \qquad f \in \mathcal{K}_C(T),$$

for all $x' \in H'$ and since $\overline{\text{sp}}\{\mu\}' = \{ x'_{\overline{\text{sp}}\{\mu\}} \mid x' \in H' \}$ we get

$$\bigcap_{x' \in H'} \mathcal{L}^1_C(\mu_{x'}) = \bigcap_{x' \in \overline{\text{sp}}\{\mu\}'} \mathcal{L}^1_C(\mu_{x'}) .$$

Remark 2.4.6. Let H be a Hilbert space and let μ be an H-valued vector measure. Then by Corollary 2.4.5 and by the remarks made before it we can consider μ as a vector measure with values in the space $\overline{\text{sp}}\{\mu\}$.

3. Let H be a Hilbert space and let μ be an H-valued vector measure. Let $A \subset H'$ be a linear subspace which is dense in $j(\overline{\text{sp}}\{\mu\})$ (in the norm topology), where $j : H \to H'$ is the canonical anti-linear bijection. Let $\tilde{\mu}_A : \mathcal{K}_C(T) \to H$ be defined as in (2.4.1). As noted above, we can consider μ as a vector measure with values in the space $\overline{\text{sp}}\{\mu\}$. Since $A \subset \overline{\text{sp}}\{\mu\}'$ is dense (in the norm topology), the pairing $(\overline{\text{sp}}\{\mu\}, A)$ separates the points of $\overline{\text{sp}}\{\mu\}$, thus the topology $\sigma(\overline{\text{sp}}\{\mu\}, A)$ of the space $\overline{\text{sp}}\{\mu\}$ is a locally convex Hausdorff topology.

In Theorem 2.4.8 we give a (necessary and) sufficient condition for a function $f \in \mathcal{L}^1_C(\tilde{\mu}_A)$ to be μ-integrable. The following example shows that the inclusion

$$\mathcal{L}^1_C(\mu) = \bigcap_{x' \in H'} \mathcal{L}^1_C(\mu_{x'}) \subset \bigcap_{x' \in A} \mathcal{L}^1_C(\mu_{x'}) = \mathcal{L}^1_C(\tilde{\mu}_A)$$

can be strict.

Example 2.4.7. Suppose that the probability space $(\Omega, \mathcal{F}, P)$ is so large that the space $L^2_0(P)$ contains a countable orthonormal set $\{\xi_n\}_{n \in N}$, and let $T = R$. Define a mapping $\mu : \mathcal{K}_C(R) \to L^2_0(P)$ by setting

$$\mu(f) = \sum_{n \in N} f(n)\, \xi_n , \qquad f \in \mathcal{K}_C(R) .$$

The mapping μ is linear. We show that it is continuous. Let $K \subset R$ be a compact set, then $K \cap N$ is a finite set $\{ n_1, \ldots, n_{m_K} \}$ (or empty) and

$$\|\mu(f)\|^2 = \sum_{n \in N} |f(n)|^2 \leq m_K (\sup |f|)^2 ,$$

if $f \in \mathcal{K}_C(R ; K)$. Thus, μ is a stochastic measure on R. Let

$j : L_0^2(P) \to L_0^2(P)'$ be the canonical anti-linear bijection. Denote $A = \mathrm{sp}\{ j(\xi_n) \mid n \in N \}$, then A is dense in the space $j(\overline{\mathrm{sp}}\{\mu\}) = j(\overline{\mathrm{sp}}\{\xi_1, \xi_2, \ldots\})$. We show that for the constant function 1 we have $1 \in \mathcal{L}_C^1(\tilde{\mu}_A)$ but $1 \notin \mathcal{L}_C^1(\mu)$.

As noted above

$$\mathcal{L}_C^1(\tilde{\mu}_A) = \bigcap_{x' \in A} \mathcal{L}_C^1(\mu_{x'}) .$$

Let $x' \in A$, then x' can be represented in the form

$$x' = \sum_{m=1}^{n} a_m j(\xi_{k_m}) , \qquad a_m \in C , \; m = 1, \ldots, n ,$$

thus

$$\mu_{x'} = \sum_{m=1}^{n} a_m j(\xi_{k_m}) \circ \mu .$$

Therefore

$$\mu_{x'}^\bullet(1) = \sup_{|f| \leq 1} |\mu_{x'}(f)| = \sup_{|f| \leq 1} \left| \sum_{m=1}^{n} a_m f(k_m) \right|$$
$$\leq \sum_{m=1}^{n} |a_m| < \infty \quad (\text{where } f \in \mathcal{K}_C(R)) .$$

Thus $1 \in \mathcal{L}_C^1(\mu_{x'})$, since the constant function is continuous, so that $1 \in \mathcal{L}_C^1(\tilde{\mu}_A)$. Furthermore, $\int 1 \, d\tilde{\mu}_A$ is a linear mapping from A into C, for which

$$\langle x' , \int 1 \, d\tilde{\mu}_A \rangle = \sum_{m=1}^{n} a_m ,$$

when

$$x' = \sum_{m=1}^{n} a_m j(\xi_{k_m}) \in A .$$

We see that $\int 1 \, d\tilde{\mu}_A \notin \overline{\mathrm{sp}}\{\mu\}$, so by Lemma 2.4.2 we get $1 \notin \mathcal{L}_C^1(\mu_w) = \mathcal{L}_C^1(\mu)$.

The following theorem gives a (necessary and) sufficient condition for a function $f \in \mathcal{L}_C^1(\tilde{\mu}_A)$ to be μ-integrable. For the proof of the theorem we note that Thomas [20: Lemme 3.14] proved (in the real-valued case, but the proof is valid also in the complex-valued case): For a sequence $\{\nu_n\}_{n \in N} \subset \mathcal{M}(T)$ with the property

$$\sum_{n \in N} |\nu_n(f)| < \infty \qquad \text{for all } f \in \mathcal{K}(T) ,$$

one has $g \in \mathcal{L}^1(\nu)$, if $g \in \mathcal{L}^1(\nu_n)$, $n \in N$, and if

$$\sum_{n \in N} \left| \int g \, \chi_G \, d\nu_n \right| < \infty$$

for every open set $G \subset T$, where

$$\nu = \sum_{n \in N} \nu_n .$$

Theorem 2.4.8. *Let H be a Hilbert space and let μ be an H-valued vector measure. Let $A \subset H'$ be a linear subspace, which is dense in $j(\overline{\mathrm{sp}}\{\mu\})$ (in the norm topology of H'), where $j: H \to H'$ is the canonical anti-linear bijection. If $\tilde{\mu}_A: \mathcal{K}_C(T) \to H$ is defined as in (2.4.1), then a function $g \in \mathcal{L}^1_B(\tilde{\mu}_A)$ is μ-integrable if and only if $\int g\, \chi_G\, d\tilde{\mu}_A \in \overline{\mathrm{sp}}\{\mu\}$ for every open set $G \subset T$.*

Proof. The condition is necessary. Suppose that $g \in \mathcal{L}^1_C(\mu)$, and let $G \subset T$ be open. Then $g\,\chi_G \in \mathcal{L}^1_C(\mu_{x'})$ for all $x' \in H'$. Therefore $g\,\chi_G \in \mathcal{L}^1_C(\mu)$, thus

$$\int g\,\chi_G\, d\tilde{\mu}_A = \int g\,\chi_G\, d\mu \in \overline{\mathrm{sp}}\{\mu\} .$$

Next we show the sufficiency of the condition. Suppose that $g \in \mathcal{L}^1_C(\tilde{\mu}_A)$ is such that $\int g\,\chi_G\, d\tilde{\mu}_A \in \overline{\mathrm{sp}}\{\mu\}$ for every open set $G \subset T$. In proving that

$$g \in \mathcal{L}^1_C(\mu) = \bigcap_{x' \in \overline{\mathrm{sp}}\{\mu\}'} \mathcal{L}^1_C(\mu_{x'})$$

we use an idea due to Thomas [20: pp. 110–111].

Suppose that $y' \in \overline{\mathrm{sp}}\{\mu\}'$ is such that $y' \in A$, then by assumption

$$g \in \mathcal{L}^1_C(\tilde{\mu}_A) = \bigcap_{x' \in A} \mathcal{L}^1_C(\mu_{x'}) \subset \mathcal{L}^1_C(\mu_{y'}) .$$

Suppose that $y' \in \overline{\mathrm{sp}}\{\mu\}'$ is such that $y' \notin A$. Then there exists a sequence $\{x'_n\}_{n \in N} \subset A$ such that

$$\sum_{n \in N} \|x'_n\| < \infty \quad \text{and} \quad y' = \sum_{n \in N} x'_n$$

in the norm topology of H'. The sequence $\{\mu_{x'_n}\}_{n \in N} \subset \mathcal{M}_C(T)$ and the function g satisfy the conditions of Lemme 3.14 of Thomas [20: p. 109], mentioned above, since

$$\sum_{n \in N} |\mu_{x'_n}(f)| \leq \|\mu(f)\| \left(\sum_{n \in N} \|x'_n\| \right) < \infty ,$$

if $f \in \mathcal{K}_C(T)$, and by assumption

$$\sum_{n \in N} \left| \int g\,\chi_G\, d\mu_{x'_n} \right| \leq \left\| \int g\,\chi_G\, d\tilde{\mu}_A \right\| \left(\sum_{n \in N} \|x'_n\| \right) < \infty$$

for all open sets $G \subset T$. Thus $g \in \mathcal{L}^1_C(\mu_{y'})$, where

$$\mu_{y'} = \sum_{n \in N} \mu_{x'_n} .$$

So $g \in \mathcal{L}^1_C(\mu_{x'})$ for all $x' \in \overline{\mathrm{sp}}\,\{\mu\}'$, which proves the theorem.

The next example points out the character of the assumptions of Theorem 2.4.8.

Example 2.4.9. Suppose that the probability space $(\Omega, \mathcal{F}, P)$ is so large that the space $L^2_0(P)$ contains a countable orthonormal sequence $\{\xi_n\}_{n \in N}$, and let $T = R$. Define a linear mapping $\mu : \mathcal{K}_C(R) \to L^2_0(P)$ by setting

$$\mu(f) = \sum_{n \in N} (f(2\,n) - f(2\,n+1))\, \xi_n , \qquad f \in \mathcal{K}_C(R) .$$

As in Example 2.4.8 we see that μ is a stochastic measure on R. Let $j : L^2_0(P) \to L^2_0(P)'$ be the canonical anti-linear bijection. Denote again $A = \mathrm{sp}\,\{ j(\xi_n) \mid n \in N \}$. Then A is dense in $\overline{\mathrm{sp}}\,\{\mu\}'$ (in the norm topology). As in Example 2.4.7 we see that $1 \in \mathcal{L}^1_C(\tilde{\mu}_A)$. Furthermore,

$$\int 1 \, d\tilde{\mu}_A = 0 \in \overline{\mathrm{sp}}\,\{\mu\} ,$$

but using Lemma 2.4.4 we get $\mu^\bullet(1) = \infty$. The assumptions of Theorem 2.4.8 are not satisfied. Choose, for example, an open set $G \subset R$ such that $G \cap N = \{ 2\,n \mid n \in N \}$, then $\int 1\, \chi_G \, d\tilde{\mu}_A \notin \overline{\mathrm{sp}}\,\{\mu\}$.

3. Next we consider the covariance mapping of a stochastic measure.

Let $\mu : \mathcal{K}_C(T) \to L^2_0(P)$ be a stochastic measure, then its covariance mapping $Q : \mathcal{K}_C(T) \times \mathcal{K}_C(T) \to C$,

$$Q(f, g) = (\mu(f) \mid \mu(g)) , \qquad f, g \in \mathcal{K}_C(T) ,$$

is a continuous sesquilinear mapping. We make a small modification, which allows us to interpret the covariance mapping Q as a bimeasure.

Definition 2.4.10. *The covariance bimeasure of a stochastic measure* $\mu : \mathcal{K}_C(T) \to L^2_0(P)$ *is the bilinear mapping* $B : \mathcal{K}_C(T) \times \mathcal{K}_C(T) \to C$ *defined by*

$$B(f, g) = (\mu(f) \mid \mu(\bar{g})) , \qquad f, g \in \mathcal{K}_C(T) .$$

Our aim is to show how the integration of a stochastic measure is related to the integration of its covariance bimeasure.

First we show that the integrability with respect to a stochastic measure implies the integrability with respect to its covariance bimeasure.

Theorem 2.4.11. *Let* $\mu : \mathcal{K}_C(T) \to L^2_0(P)$ *be a stochastic measure and let* B *be its covariance bimeasure. If* $f, g \in \mathcal{L}^1_C(\mu)$, *then the pair* $(f, \bar{g})$ *is integrable with respect to the bimeasure* B *and* $B(f, \bar{g}) = (\mu(f) \mid \mu(g))$.

Proof. Let $f, g \in \mathcal{L}^1_C(\mu)$ and let μ^l_B (resp. μ^r_B) be the left (resp. the right) measure defined by B (see Definition 2.3.2). Then

$$\mathcal{L}^1_C(\mu^l_B) = \bigcap_{h \in \mathcal{K}_C(T)} \mathcal{L}^1_C(B(\,.\,, h)) = \bigcap_{h \in \mathcal{K}_C(T)} \mathcal{L}^1_C(B(\,.\,, \bar{h})) .$$

Since $f \in \mathcal{L}^1_C(\mu)$, we have $f \in \mathcal{L}^1_C(\mu_{x'})$ for all $x' \in \overline{\mathrm{sp}}\,\{\mu\}'$, especially $f \in \mathcal{L}^1_C(\mu_{x'})$ for all $x' = j(\mu(h))$, $h \in \mathcal{K}_C(T)$, where $j: L^2_0(P) \to L^2_0(P)'$ is the canonical anti-linear bijection. It follows that $f \in \mathcal{L}^1_C(\mu^l_B)$, since $\mu_{j(\mu(h))} = B(\,.\,, \bar{h})$, $h \in \mathcal{K}_C(T)$. Furthermore,

$$B(\,.\,, \bar{h})(f) = (\mu(f) \mid \mu(h)) \qquad \text{for all } h \in \mathcal{K}_C(T) .$$

Similarly we get $\bar{g} \in \mathcal{L}^1_C(\mu^r_B)$ and $B(h\,,\,.\,)(\bar{g}) = (\mu(h) \mid \mu(g))$ for all $h \in \mathcal{K}_C(T)$. Thus

$$B(f\,,\,.\,)(\bar{h}) = (\mu(f) \mid \mu(h)) = \overline{\mu_{j(\mu(f))}(h)}$$

and

$$B(\,.\,, \bar{g})(h) = (\mu(h) \mid \mu(g)) = \mu_{j(\mu(g))}(h) ,$$

when $h \in \mathcal{K}_C(T)$. Since $g \in \mathcal{L}^1_C(\mu_{x'})$ for all $x' \in \overline{\mathrm{sp}}\,\{\mu\}'$ we have $\bar{g} \in \mathcal{L}^1_C(B(f\,,\,.\,))$. Similarly $f \in \mathcal{L}^1_C(B(\,.\,, \bar{g}))$. Furthermore,

$$B(f\,,\,.\,)(\bar{g}) = \overline{\mu_{j(\mu(f))}(g)} = (\mu(f) \mid \mu(g)) = \mu_{j(\mu(g))}(f) = B(\,.\,, \bar{g})(f) ,$$

thus $B(f, \bar{g}) = (\mu(f) \mid \mu(g))$.

The converse of Theorem 2.4.11 is not valid as the following example shows.

Example 2.4.12. Consider the stochastic measure μ defined in Example 2.4.7, i.e.,

$$\mu(f) = \sum_{n \in N} f(n)\, \xi_n , \qquad f \in \mathcal{K}_C(R) ,$$

where $\{\xi_n\}_{n \in N} \subset L^2_0(P)$ is an orthonormal sequence. Let $j: L^2_0(P) \to L^2_0(P)'$ be the canonical anti-linear bijection. Then

$$A = \mathrm{sp}\,\{ j(\xi_n) \mid n \in N \} = \{ j(\mu(h)) \mid h \in \mathcal{K}_C(R) \} .$$

Since $1 \in \mathcal{L}^1_C(\mu_{x'})$ for all $x' \in A$, we have

$$1 \in \bigcap_{h \in \mathcal{K}_C(R)} \mathcal{L}^1_C(B(\,.\,, \bar{h})) = \bigcap_{h \in \mathcal{K}_C(R)} \mathcal{L}^1_C(B(h\,,\,.\,)) .$$

Therefore, $1 \in \mathcal{L}^1_C(\mu^r_B)$ and $1 \in \mathcal{L}^1_C(\mu^l_B)$, but $1 \notin \mathcal{L}^1_C(B(1\,,\,.\,))$, $1 \notin \mathcal{L}^1_C(B(\,.\,, 1))$ and $1 \notin \mathcal{L}^1_C(\mu)$.

Consider then the stochastic measure μ defined in Example 2.4.9, i.e.,

$$\mu(f) = \sum_{n \in N} (f(2\,n) - f(2\,n+1))\, \xi_n , \qquad f \in \mathcal{K}_C(R) ,$$

where $\{\xi_n\}_{n\in N} \subset L_0^2(P)$ is an orthonormal sequence. As above, we have $1 \in \mathcal{L}_C^1(\mu_B^r)$ and $1 \in \mathcal{L}_C^1(\mu_B^l)$. Furthermore, $B(1\,,\,.\,) = B(\,.\,,1) = 0$, thus the pair $(1\,,1)$ is integrable with respect to B and $B(1\,,1) = 0$, but $1 \notin \mathcal{L}_C^1(\mu)$.

The following theorem is an analogue to Theorem 2.4.11.

Theorem 2.4.13. *Let* $\mu : \mathcal{K}_C(T) \to L_0^2(P)$ *be a bounded stochastic measure and let* B *be its covariance bimeasure. If* $f\,,g \in \mathcal{L}_C^1(\mu)$, *then the pair* $(f\,,\bar{g})$ *is »strongly» integrable with respect to* B *and* $B(f\,,\bar{g}) = (\mu(f) \mid \mu(g))$.

4. Let $\mu : \mathcal{K}_C(T) \to L_0^2(P)$ be a stochastic measure and let B be its covariance bimeasure. Our aim is to give a (necessary and) sufficient condition for the functions $f : T \to C$ and $g : T \to C$ to be μ-integrable, if the pair $(f\,,g)$ is integrable with respect to B. We have included these considerations for the sake of completeness. The results in this subsection are not used in the sequel.

First we make some preliminary considerations.

Let $\mu : \mathcal{K}_C(T) \to L_0^2(P)$ be a stochastic measure and let Q be its covariance mapping. Define a mapping $\mu_Q : \mathcal{K}_C(T) \to \overline{\text{sp}}\,\{Q\}$ by setting

$$\mu^Q(f) \;=\; Q_{\bar{f}}, \qquad f \in \mathcal{K}_C(T)\,.$$

The mapping μ^Q is linear, since

$$\begin{aligned} Q_{\overline{\alpha f+\beta g}}(h) \;&=\; Q(h\,,\overline{\alpha f} + \overline{\beta g}) \;=\; \alpha\, Q(h\,,\bar{f}) + \beta\, Q(h\,,\bar{g}) \\ &=\; \alpha\, Q_{\bar{f}}(h) + \beta\, Q_{\bar{g}}(h) \qquad \text{for all } h \in \mathcal{K}_C(T)\,, \end{aligned}$$

when $\alpha\,,\beta \in C$ and $f\,,g \in \mathcal{K}_C(T)$. Moreover, μ^Q is continuous, since for every compact $K \subset T$ there exists an $M_K > 0$ such that

$$\|\mu^Q(f)\| \;=\; \|Q_{\bar{f}}\| \;=\; \|\mu(\bar{f})\| \;\le\; M_K \sup |f|\,,$$

if $f \in \mathcal{K}_C(K\,;T)$. Thus μ^Q is a vector measure with values in $\overline{\text{sp}}\,\{Q\}$. In the following μ^Q is called the *r. k. vector measure defined by* μ.

Lemma 2.4.14. *Let* $\mu : \mathcal{K}_C(T) \to L_0^2(P)$ *be a stochastic measure and let* Q *be its covariance mapping, then* $\mathcal{L}_C^1(\mu) = \mathcal{L}_C^1(\mu^Q)$.

Proof. Let $f \in \mathcal{J}_+(T)$, then

$$\begin{aligned} (\mu^Q)^\bullet(f) \;&=\; \sup_{|g| \le f} \|\mu^Q(g)\| \;=\; \sup_{|g| \le f} \|\mu^Q(\bar{g})\| \\ &=\; \sup_{|g| \le f} \|\mu(g)\| \;=\; \mu^\bullet(f)\,, \end{aligned}$$

where $g \in \mathcal{K}_C(T)$, which proves the lemma.

The following lemmas show the connection between the r. k. vector measure μ^Q, defined by a stochastic measure μ, and the covariance bimeasure B of μ.

Lemma 2.4.15. *Let* $\mu : \mathcal{K}_C(T) \to L_0^2(P)$ *be a stochastic measure and let* Q *be its covariance mapping. Then the topology* $\sigma(\overline{\mathrm{sp}}\,\{Q\}\,, j(\mathrm{sp}\,\{Q\}))$ *of the space* $\overline{\mathrm{sp}}\,\{Q\} \subset \mathcal{M}_C(T)$, *where* $j : \overline{\mathrm{sp}}\,\{Q\} \to \overline{\mathrm{sp}}\,\{Q\}'$ *is the canonical anti-linear bijection, is identical with the topology of* $\overline{\mathrm{sp}}\,\{Q\}$ *induced by the topology* $\sigma(\mathcal{M}_C(T)\,, \mathcal{K}_C(T))$ *of the space* $\mathcal{M}_C(T)$.

Proof. The topology $\sigma(\mathcal{M}_C(T)\,, \mathcal{K}_C(T))$ of $\mathcal{M}_C(T)$ is the projective locally convex topology of $\mathcal{M}_C(T)$ with respect to the mappings $u_f : \mathcal{M}_C(T) \to C$, $u_f(\nu) = \nu(f)$, $\nu \in \mathcal{M}_C(T)$, $f \in \mathcal{K}_C(T)$. On the other hand, the topology $\sigma(\overline{\mathrm{sp}}\,\{Q\}\,, j(\mathrm{sp}\,\{Q\}))$ of $\overline{\mathrm{sp}}\,\{Q\}$ is the projective locally convex topology with respect to the mappings $v_{z'} : \overline{\mathrm{sp}}\,\{Q\} \to C$, $v_{z'}(\theta) = \langle \theta\,, z' \rangle$, $\theta \in \overline{\mathrm{sp}}\,\{Q\}$, $z' \in j(\mathrm{sp}\,\{Q\})$.

Let $z' \in j(\mathrm{sp}\,\{Q\})$, then it can be represented in the form

$$z' = j(\sum_{m=1}^{n} a_m\, Q_{f_m})\,, \quad a_m \in C\,,\ f_m \in \mathcal{K}_C(T)\,,\ m = 1\,, \ldots\,, n\,.$$

Suppose that $\theta \in \overline{\mathrm{sp}}\,\{Q\}$, then

$$\langle \theta\,, z' \rangle = \left(\theta \,\Big|\, \sum_{m=1}^{n} a_m\, Q_{f_m}\right) = (\theta \mid Q_g) = \theta(g)\,,$$

where

$$g = \sum_{m=1}^{n} \bar{a}_m f_m\,.$$

Thus $v_{z'}(\theta) = u_g(\theta)$, i.e. the semi-norms defining the topology $\sigma(\overline{\mathrm{sp}}\,\{Q\}\,, j(\mathrm{sp}\,\{Q\}))$ and the topology of $\overline{\mathrm{sp}}\,\{Q\}$, induced by the topology $\sigma(\mathcal{M}_C(T)\,, \mathcal{K}_C(T))$ of $\mathcal{M}_C(T)$, are the same.

Lemma 2.4.16. *Let* $\mu : \mathcal{K}_C(T) \to L_0^2(P)$ *be a stochastic measure and let* Q *be its covariance mapping. Denote* $A = j(\mathrm{sp}\,\{Q\})$, *where* $j : \overline{\mathrm{sp}}\,\{Q\} \to \overline{\mathrm{sp}}\,\{Q\}'$ *is the canonical anti-linear bijection. Then* $\mathcal{L}^1_C(\tilde{\mu}^Q_A) = \mathcal{L}^1_C(\mu^r_B) = \mathcal{L}^1_C(\mu^l_B)$, *where* B *is the covariance bimeasure of* μ . *A function* $f \in \mathcal{L}^1_C(\mu^l_B)$ *is* μ *-integrable if and only if* $B(\,.\,, f\,\chi_G) \in \overline{\mathrm{sp}}\,\{Q\}$ *for all open sets* $G \subset T$.

Proof. The statement $\mathcal{L}^1_C(\mu^r_B) = \mathcal{L}^1_C(\mu^l_B)$ is immediate, since

$$(\mu^r_B(f))(g) = B(\,.\,, f)(g) = \overline{B(\bar{f}\,, \bar{g})} = \overline{B(\bar{f}\,,\,.\,)(\bar{g})} = \overline{(\mu^l_B(\bar{f}))(\bar{g})}$$

for all $f\,, g \in \mathcal{K}_C(T)$. Moreover, for a fixed $f \in \mathcal{K}_C(T)$

$$(\tilde{\mu}^Q_A(f))(g) = (\mu^Q(f))(g) = Q_{\bar{f}}(g) = B(\,.\,, f)(g)\,,\ g \in \mathcal{K}_C(T)\,.$$

Therefore $\tilde{\mu}^Q_A(f) = \mu^r_B(f)$, $f \in \mathcal{K}_C(T)$. Thus, the first part of the lemma follows by Lemma 2.4.15.

If $f \in \mathcal{L}^1_C(\mu^l_B)$ is such that $B(\,.\,, f\,\chi_G) \in \overline{\mathrm{sp}}\,\{Q\}$, then $\tilde{\mu}^Q_A(f\,\chi_G) \in \overline{\mathrm{sp}}\,\{Q\}$. By assumption this is valid for all open sets $G \subset T$, thus by Theorem 2.4.8 $f \in \mathcal{L}^1_C(\mu^Q)$. By Lemma 2.4.14 $\mathcal{L}^1_C(\mu^Q) = \mathcal{L}^1_C(\mu)$, which proves the lemma.

For the sake of completeness we collect the above results into a theorem.

Theorem 2.4.17. *Let* $\mu : \mathcal{K}_C(T) \to L_0^2(P)$ *be a stochastic measure and let* Q *be its covariance mapping. Suppose that the functions* $f : T \to C$, $g : T \to C$ *are such that the pair* (f, g) *is integrable with respect to the covariance bimeasure* B *of* μ. *Then* $f, g \in \mathcal{L}_C^1(\mu)$ *if and only if* $B(\,.\,, f\chi_G) \in \overline{\mathrm{sp}}\,\{Q\}$ *and* $B(\,.\,, g\chi_G) \in \overline{\mathrm{sp}}\,\{Q\}$ *for all open sets* $G \subset T$.

5. Next we consider a special way to define a stochastic measure. Let $\nu : \mathcal{K}_C(T) \to C$ be a Radon measure and let S be a topological space. We recall that a mapping $f : T \to S$ is said to be ν*-measurable* if for every $\varepsilon > 0$ and for every compact set $K \subset T$ there exists a compact set $K_1 \subset K$, such that $|\nu|(\chi_{K \setminus K_1}) < \varepsilon$ and the restriction of f to K_1 is continuous.

Let $\nu : \mathcal{K}_C(T) \to C$ be a Radon measure and let $x : T \to L_0^2(P)$ be a stochastic mapping of T. We call x *scalarly* ν*-measurable* (resp. *scalarly* ν*-integrable*) if the mapping $x_{z'} : T \to C$, $x_{z'}(t) = \langle x(t), z' \rangle$, $t \in T$, is ν-measurable (resp. ν-integrable) for all $z' \in L_0^2(P)'$.

Suppose that $x : T \to L_0^2(P)$ is scalarly ν-measurable and that for all $z' \in L_0^2(P)'$ we have $\nu^\bullet(x_{z'}\,\chi_K) < \infty$ for all compact sets $K \subset T$. Then $x_{z'} f \in \mathcal{L}_C^1(\nu)$ for all $f \in \mathcal{K}_C(T)$ (Bourbaki [5: Ch. 5, pp. 41—42]). Let $f \in \mathcal{K}_C(T)$. If the mapping $\int x f\, d\nu : L_0^2(P)' \to C$,

$$\langle z', \int x f\, d\nu \rangle = \int x_{z'} f\, d\nu\,, \qquad z' \in L_0^2(P)'\,,$$

is continuous, when $L_0^2(P)'$ carries the norm topology, then $\int x f\, d\nu \in L_0^2(P)'' \cong L_0^2(P)$. In the following we consider $\int x f\, d\nu$ as an element of the space $L_0^2(P)$ if $\int x f\, d\nu \in L_0^2(P)''$. Suppose that $\int x f\, d\nu \in L_0^2(P)$ for all $f \in \mathcal{K}_C(T)$. If the mapping $\mu : \mathcal{K}_C(T) \to L_0^2(P)$, $\mu(f) = \int x f\, d\nu$, $f \in \mathcal{K}_C(T)$, is continuous, we call μ the *stochastic measure defined by* x *and* ν.

Lemma 2.4.18. *Let* $\mu : \mathcal{K}_C(T) \to L_0^2(P)$ *be a stochastic measure defined by a stochastic mapping* $x : T \to L_0^2(P)$ *and a Radon measure* $\nu : \mathcal{K}_C(T) \to C$. *Then*

$$\left(\int f\, d\mu \,\Big|\, \int g\, d\mu\right) = \int f(s) \overline{\left(\int g(t)\, (x(t) \,|\, x(s))\, d\nu(t)\right)}\, d\nu(s)$$

for all $f, g \in \mathcal{L}_C^1(\mu)$.

Proof. First we note that $\int f\, d\mu$, $\int g\, d\mu \in L_0^2(P)$ for all $f, g \in \mathcal{L}_C^1(\mu)$. Moreover,

$$\left(\int g\, d\mu \,\Big|\, x(s)\right) = \int g(t)\, (x(t) \,|\, x(s))\, d\nu(t)\,, \qquad s \in T\,.$$

Thus

$$\left(\int f\,d\mu \,\middle|\, \int g\,d\mu\right) = \int f(s)\left(x(s) \,\middle|\, \int g\,d\mu\right) d\nu(s)$$
$$= \int f(s)\left(\int g(t)\,(x(t) \mid x(s))\,d\nu(t)\right) d\nu(s)\,.$$

3. V-bounded and harmonizable stochastic processes

3.1. Classification of stochastic processes

1. Our aim is to characterize the class of (weakly) continuous V-bounded stochastic processes and the class of harmonizable stochastic processes.

We begin with the definitions. The definition of a V-bounded stochastic process is due to Bochner [4: p. 18]. Our definition differs slightly from that of Bochner, because we use the weak integration technique developed in 2.4.5.

Before stating the definition we note that in the following the Lebesgue measure of R is denoted by m. Let $p \in \mathcal{L}^1_C(m)$, then its Fourier transform is denoted by $\mathcal{F} p$, i.e.,

$$(\mathcal{F} p)(t) = \int p(\lambda)\, e^{it\lambda}\, dm(\lambda)\,, \qquad t \in R\,.$$

Furthermore, in the following we call a stochastic process $x : R \to L^2_0(P)$ *bounded,* if there exists a $M > 0$ such that $\|x(t)\| \leq M$ for all $t \in R$.

Let $x : R \to L^2_0(P)$ be a bounded scalarly m-measurable stochastic process. Then one gets

$$\left|\int x_{z'}\, h\, dm\right| \leq M\, \|z'\| \int |h|\, dm \quad \text{for all} \quad z' \in L^2_0(P)'\,, \quad h \in \mathcal{K}_C(R)\,,$$

where $M > 0$ is such that $\|x(t)\| \leq M$ for all $t \in R$. Thus the stochastic measure μ defined by x and m exists and $\mathcal{L}^1_C(m) \subset \mathcal{L}^1_C(\mu)$.

Definition 3.1.1. *A bounded scalarly m-measurable stochastic process $x : R \to L^2_0(P)$ is called V-bounded if there exists a constant $c > 0$ such that*

$$\left\|\int x\, p\, dm\right\| \leq c \sup |\mathcal{F} p| \qquad \textit{for all } p \in \mathcal{L}^1_C(m)\,.$$

The definition of a harmonizable stochastic process (or a harmonizable covariance function) is due to Loève [13: p. 474]. Note that Loève considers stochastic and scalar-valued measures as completely additive set functions.

Definition 3.1.2. *A stochastic process* $x: R \to L_0^2(P)$ *is called harmonizable if its covariance function can be represented in the form*

$$r(s, t) = \nu(e^{is\cdot} \otimes e^{-it\cdot}) = \int e^{is\lambda} e^{-it\theta} \, d\nu(\lambda, \theta), \quad s, t \in R,$$

where ν *is a bounded Radon measure on* $R \times R$, *for which* $\nu(f \otimes \bar{f}) \geq 0$ *for all* $f \in C_0(R)$.

3.2. V-bounded stochastic processes

1. We begin with a result which shows that every weakly continuous V-bounded stochastic process is the Fourier transform of a bounded stochastic measure. The result is analogous to Theorem 2 of Kluvánek [12].

Theorem 3.2.1. *A weakly continuous stochastic process* $x: R \to L_0^2(P)$ *is* V*-bounded if and only if there exists a bounded stochastic measure* μ *on* R *such that*

$$x(t) = \int e^{it\lambda} \, d\mu(\lambda), \qquad t \in R. \tag{3.2.1}$$

If x *can be represented in the form (3.2.1), then it is uniformly continuous.*

Proof. Let $x: R \to L_0^2(P)$ be weakly continuous and V-bounded. Define a stochastic measure μ^x on R by setting

$$\mu^x(h) = \int x \, h \, dm, \qquad h \in \mathcal{K}_C(R).$$

Since x is V-bounded $\mathcal{L}_C^1(m) \subset \mathcal{L}_C^1(\mu^x)$, and there exists a $c > 0$ such that

$$\|\mu^x(p)\| \leq c \sup |\mathcal{F} p| \qquad \text{for all } p \in \mathcal{L}_C^1(m).$$

Define a linear mapping $\mu: C_0(R) \to L_0^2(P)$ by setting $\mu(f) = \mu^x(p)$ if $f \in C_0(R)$ is such that $f = \mathcal{F} p$ for some $p \in \mathcal{L}_C^1(m)$. The definition is unique since for functions $p, q \in \mathcal{L}_C^1(m)$ such that $f = \mathcal{F} p = \mathcal{F} q$, we have

$$\begin{aligned} \langle \mu^x(p), z' \rangle &= \int \langle x(t), z' \rangle \, p(t) \, dm(t) \\ &= \int \langle x(t), z' \rangle \, q(t) \, dm(t) = \langle \mu^x(q), z' \rangle \end{aligned}$$

for all $z' \in L_0^2(P)'$, thus $\mu^x(p) = \mu^x(q)$. The mapping μ is linear on the linear subspace $\mathcal{F}(\mathcal{L}_C^1(m)) = \{ f \in C_0(R) \mid f = \mathcal{F} p, \ p \in \mathcal{L}_C^1(m) \}$ of $C_0(R)$ and $\mu: \mathcal{F}(\mathcal{L}_C^1(m)) \to L_0^2(P)$ is continuous if $\mathcal{F}(\mathcal{L}_C^1(m))$ carries the norm topology induced by $C_0(R)$. Since $L_0^2(P)$ is complete, the mapping μ can be extended by continuity to a continuous linear mapping $\mu : C_0(R) \to L_0^2(P)$, i.e.

$$\|\mu(f)\| \leq c \sup |f| \qquad \text{for all } f \in C_0(R) .$$

Thus μ is a bounded stochastic measure on R. Our aim is to show that x is the Fourier transform of μ. Let $z' \in L_0^2(P)'$ and $p \in \mathcal{L}_C^1(m)$, then

$$\begin{aligned} \int x_{z'}\, p\, dm &= \left\langle \int x\, p\, dm\, ,\, z' \right\rangle = \langle \mu(\mathcal{F} p)\, ,\, z' \rangle = \int \mathcal{F} p\, d\mu_{z'} \\ &= \int \left(\int p(t)\, e^{it\lambda}\, dm(t) \right) d\mu_{z'}(\lambda) \\ &= \int p(t) \left(\int e^{it\lambda}\, d\mu_{z'}(\lambda) \right) dm(t) . \end{aligned}$$

Thus, by the continuity of the functions $x_{z'}$ and $\int e^{i\cdot\lambda}\, d\mu_{z'}(\lambda)$ we get

$$\langle x(t)\, ,\, z' \rangle = \left\langle \int e^{it\lambda}\, d\mu(\lambda)\, ,\, z' \right\rangle \qquad \text{for all } t \in R .$$

Therefore

$$x(t) = \int e^{it\lambda}\, d\mu(\lambda)\, , \qquad t \in R .$$

Suppose then that the stochastic process $x : R \to L_0^2(P)$ is such that it can be represented in the form (3.2.1). Then x is bounded and weakly continuous. Moreover,

$$\begin{aligned} \left| \left\langle \int x\, p\, dm\, ,\, z' \right\rangle \right| &= \left| \int x_{z'}\, p\, dm \right| = \left| \int \left(\int e^{it\lambda}\, d\mu_{z'}(\lambda) \right) p(t)\, dm(t) \right| \\ &= \left| \int \mathcal{F} p\, d\mu_{z'} \right| \leq \mu^{\bullet}(1)\, (\sup |\mathcal{F} p|)\, \|z'\|\, , \quad p \in \mathcal{L}_C^1(m) , \end{aligned}$$

for all $z' \in L_0^2(P)'$, thus x is Γ-bounded.

Next we show that x is uniformly continuous if it is representable in the form (3.2.1), where μ is a bounded stochastic measure on R. First we note that μ can extended to a continuous mapping $\mu : C_0(R) \to L_0^2(P)$. The mapping μ is weakly compact, since $C_0(R)$ is a normed space and $L_0^2(P)$ is a Hilbert space. Then by Gantmacher's theorem (Dunford and Schwartz [8: p. 485]) the transpose $\mu' : L_0^2(P)' \to C_0(R)'$ $(= \mathcal{M}_C^1(R))$ is weakly compact. Therefore the set $\{ \mu_{z'} \in \mathcal{M}_C^1(R) \mid z' \in L_0^2(P)' ,\ \|z'\| \leq 1 \}$ is relatively weakly compact. Let $\varepsilon > 0$. Then, by a criterion concerning weak compactness of sets of bounded Radon measures, due to Grothendieck (see Thomas [20: Condition 4, p. 174]), there exists a compact set $K \subset R$ such that

$$|\mu_{z'}|(\chi_{R \setminus K}) < \varepsilon \qquad \text{for all } z' \in L_0^2(P)' , \ \|z'\| \leq 1 .$$

Let $s , t \in R$ and $\|z'\| \leq 1$. Then

$$\begin{aligned} |\langle x(s) - x(t) , z' \rangle| &= \left| \int (e^{is\lambda} - e^{it\lambda}) \, d\mu_{z'}(\lambda) \right| \\ &\leq \left| \int \chi_K(\lambda) \, (e^{is\lambda} - e^{it\lambda}) \, d\mu_{z'}(\lambda) \right| + \left| \int \chi_{R \setminus K}(\lambda) \, (e^{is\lambda} - e^{it\lambda}) \, d\mu_{z'}(\lambda) \right| \\ &\leq \varepsilon \, \mu^{\bullet}(1) \, \|z'\| + 2 \, \varepsilon , \end{aligned}$$

if $|s - t|$ is small enough, thus x is uniformly continuous.

Remark 3.2.2. There exist V-bounded stochastic processes $x : R \to L_0^2(P)$ which are not (weakly) continuous. Consider, for example, the process x ,

$$x(t) = 0, \ t \neq 0, \quad x(0) = \xi ,$$

where $\xi \in L_0^2(P)$, $|\xi| = 1$.

Definition 3.2.3. *Let $x : R \to L_0^2(P)$ be a (weakly) continuous V-bounded stochastic process. Then the bounded stochastic measure μ appearing in (3.2.1) is called the spectral measure of x .*

Remark 3.2.4. The spectral measure of a (weakly) continuous V-bounded stochastic process is unique.

2. The following characterization of (weakly) continuous V-bounded stochastic processes is similar to the definition of harmonizable stochastic processes.

First we give a preliminary result.

Corollary 3.2.5. *A bounded scalarly m-measurable stochastic process is V-bounded if and only if its covariance function r satisfies the condition*

$$\left| \int p(s) \left(\int r(s , t) \, q(t) \, dm(t) \right) dm(s) \right| \leq c \sup |\mathcal{F} p| \ \sup |\mathcal{F} q|$$

for all $p , q \in \mathcal{L}_C^1(m)$, for some $c > 0$.

Proof. Suppose that x is V-bounded, then

$$\left| \left(\int x \, p \, dm \, \Big| \int x \, q \, dm \right) \right| \leq \left\| \int x \, p \, dm \right\| \left\| \int x \, q \, dm \right\| \leq c \sup |\mathcal{F} p| \ \sup |\mathcal{F} q|$$

for all $p , q \in \mathcal{L}_C^1(m)$, for some $c > 0$. Moreover, by Lemma 2.4.18

$$\left(\int x \, p \, dm \, \Big| \int x \, q \, dm \right) = \int p(s) \left(\int r(s , t) \, \overline{q(t)} \, dm(t) \right) dm(s) ,$$

which proves the first part of the lemma. The second part of the lemma is immediate.

Theorem 3.2.6. *A bounded weakly continuous stochastic process* $x : R \to L_0^2(P)$ *is* V*-bounded if and only if there exists a bounded bimeasure* B *on* $R \times R$ *such that* $B(f, \bar{f}) \geq 0$ *for all* $f \in C_0(R)$, *the pair* $(e^{is\cdot}, e^{-it\cdot})$ *is »strongly» integrable with respect to* B *for all* $s, t \in R$, *and the covariance function* r *of* x *can be represented in the form*

$$r(s, t) = B(e^{is\cdot}, e^{-it\cdot}), \qquad s, t \in R.$$

Proof. Suppose that x is weakly continuous and V-bounded, then by Theorem 3.2.1 there exists a bounded stochastic measure μ on R such that

$$x(t) = \int e^{it\lambda} d\mu(\lambda), \quad t \in R.$$

Let B be the covariance bimeasure of μ, then $B(f, \bar{f}) = (\mu(f) \mid \mu(f)) \geq 0$ for all $f \in C_0(R)$, and by Theorem 2.4.13 the pair $(e^{is\cdot}, e^{-it\cdot})$ is »strongly» integrable with respect to B and

$$r(s, t) = \left(\int e^{is\cdot} d\mu \,\Big|\, \int e^{it\cdot} d\mu\right) = B(e^{is\cdot}, e^{-it\cdot}), \quad s, t \in R.$$

Conversely, if x is weakly continuous and $r(s, t) = B(e^{is\cdot}, e^{-it\cdot})$, $s, t \in R$, where B is a bounded bimeasure on $R \times R$ such that $B(f, \bar{f}) \geq 0$ for all $f \in C_0(R)$. Then

$$\begin{aligned}
&\int p(s)\left(\int r(s, t)\, \overline{q(t)}\, dm(t)\right) dm(s) \\
&= \int p(s)\left(\int B(e^{is\cdot}, e^{-it\cdot})\, \overline{q(t)}\, dm(t)\right) dm(s) \\
&= \int p(s)\left(\int\left(\int e^{-it\cdot}\, \overline{q(t)}\, dm(t)\right) dB(e^{is\cdot}, .\,)\right) dm(s) \\
&= \int\left(\int p(s)\, e^{is\cdot}\, dm(s)\right) dB(\,.\,, \overline{\mathcal{F} q}) = B(\mathcal{F} p, \overline{\mathcal{F} q}),
\end{aligned}$$

thus there exists a $c > 0$ such that

$$\begin{aligned}
&\left|\int p(s)\left(\int r(s, t)\, \overline{q(t)}\, dm(t)\right) dm(s)\right| = |B(\mathcal{F} p, \overline{\mathcal{F} q})| \\
&\leq c \sup |\mathcal{F} p| \sup |\mathcal{F} q|, \qquad p, q \in \mathcal{L}_C^1(m),
\end{aligned}$$

since B is bounded. The theorem then follows by Corollary 3.2.5.

3. Next we consider a stochastic process constructed by von Bahr [3]. He uses it to show that there exist bounded and continuous stochastic processes which are not harmonizable. The same example can be used to show that the class of bounded and continuous stochastic processes is strictly larger than that of the continuous V-bounded stochastic processes.

Example 3.2.7. Let $\xi \in L_0^2(P)$, $\|\xi\| = 1$. Define a function $f\colon R \to R$ by setting

$$f(t) = \sum_{n=2}^{\infty} \frac{\sin n t}{n \log n}, \qquad t \in R .$$

The series is uniformly convergent and f is bounded and continuous (Zygmund [22: pp. 182—183]). Define a stochastic process $x\colon R \to L_0^2(P)$ by setting $x(t) = f(t)\,\xi$, $t \in R$, then x is bounded and continuous. Let us show that x is not V-bounded. If x were V-bounded, there would exist a unique bounded Radon measure ν on R such that

$$(x(t) \mid \xi) = f(t) = \int e^{it\lambda}\, d\nu(\lambda), \qquad t \in R .$$

Moreover, let $g \in \mathcal{L}_C^1(m)$ be continuous and such that $\mathcal{F} g \in \mathcal{L}_C^1(m)$. Then by Parseval's formula

$$\int g(x)\, d\nu_n(x) = \frac{1}{2\pi} \int (\mathcal{F} g)(t)\, \frac{\sin(-n t)}{n \log n}\, dm(t) ,$$

where

$$\nu_n = \frac{1}{2 n i \log n}\, (\delta_n - \delta_{-n})$$

(δ_n is the Dirac measure, i.e. $\delta_n(g) = g(n)$, $g \in \mathcal{K}_C(R)$), since

$$\frac{\sin n t}{n \log n} = \int e^{it\lambda}\, d\nu_n(\lambda), \qquad t \in R, \ n \in N, \ n \geq 2 .$$

Suppose that supp $(\mathcal{F} g)$ is compact. Then, using again Parseval's formula (and since the order of the integration and the summation can be changed)

$$\begin{aligned} \int g(x)\, d\nu(x) &= \frac{1}{2\pi}\, (\mathcal{F} g)(t)\, f(-t)\, dm(t) \\ &= \sum_{n=2}^{\infty} \frac{1}{2\pi} \int (\mathcal{F} g)(t)\, \frac{\sin(-n t)}{n \log n}\, dm(t) \\ &= \sum_{n=2}^{\infty} \int g(x)\, d\nu_n(x) . \end{aligned}$$

Therefore

$$\nu^\bullet(1) \geq \sum_{k=2}^{p} \frac{1}{k \log k} \qquad \text{for all} \quad p \in N\,, \quad p \geq 2\,,$$

thus ν cannot be bounded. This proves that x is not V-bounded.

3.3. Harmonizable stochastic processes

1. Our aim is to show first that every harmonizable stochastic process is V-bounded. Then we shall give a characterization of harmonizable stochastic processes which is related to the definition of V-bounded stochastic processes. After that we shall construct an example of a continous and V-bounded stochastic process which is not harmonizable.

Corollary 3.3.1. *Every harmonizable stochastic process is V-bounded and continuous.*

Proof. Let $x: R \to L_0^2(P)$ be a harmonizable stochastic process. Then the covariance function r of x can be represented in the form

$$r(s\,,t) = \int e^{is\lambda}\, e^{-it\theta}\, d\nu(\lambda\,,\theta)\,, \qquad s\,,\ t \in R\,,$$

where ν is a bounded Radon measure on $R \times R$, for which $\nu(f \otimes \bar{f}) \geq 0$, when $f \in C_0(R)$. Since ν is bounded, r is bounded and continuous. Thus, x is bounded and by Lemma 1.1.2 continuous. By the continuity of r we get

$$\begin{aligned}\left|\int p(s)\left(\int r(s\,,t)\, \overline{q(t)}\, dm(t)\right) dm(s)\right| &= \left|\int r(s\,,t)\, p(s)\, \overline{q(t)}\, d(m \otimes m)(s\,,t)\right| \\ &= \left|\int \left(\int e^{is\lambda}\, e^{-it\theta}\, d\nu(\lambda\,,\theta)\right) p(s)\, \overline{q(t)}\, d(m \otimes m)(s\,,t)\right| \\ &= \left|\int \mathcal{F} p(\lambda)\, \overline{\mathcal{F} q(\theta)}\, d\nu(\lambda\,,\theta)\right| \\ &\leq \nu^\bullet(1) \sup |\mathcal{F} p| \sup |\mathcal{F} q|\end{aligned}$$

for all $p\,, q \in \mathcal{L}_C^1(m)$. Thus, the corollary follows by Corollary 3.2.4.

Theorem 3.3.2. *Let $x: R \to L_0^2(P)$ be bounded and continuous. Then x is harmonizable if and only if there exists a $c > 0$ such that*

$$(3.3.1) \qquad \left|\sum_{k=1}^{n} \left(\int x\, p_k\, dm \,\Big|\, \int x\, q_k\, dm\right)\right| \leq c \sup_{\lambda,\theta \in R} \left|\sum_{k=1}^{n} \mathcal{F} p_k(\lambda)\, \overline{\mathcal{F} q_k(\theta)}\right|,$$

or equivalently

$$(3.3.1')\qquad \left|\sum_{k=1}^{n}\int r(s\,,t)\,p_k(s)\,q_k(t)\,d(m\otimes m)(s\,,t)\right| \le c \sup_{\lambda,\theta\in R}\left|\sum_{k=1}^{n}\mathcal{F}\,p_k(\lambda)\,\mathcal{F}\,q_k(-\theta)\right|,$$

for all $p_k\,,q_k\in\mathcal{L}^1_C(m)$, $k=1\,,\ldots,n$, $n\in N$, *where* r *is the covariance function of* x .

Proof. The equivalence of the conditions (3.3.1) and (3.3.1′) is immediate, since $(\mathcal{F}\,q)(-\lambda)=\overline{(\mathcal{F}\,\bar{q})(\lambda)}$ for all $\lambda\in R$ and $q\in\mathcal{L}^1_C(m)$.

Suppose that x is harmonizable, then the covariance function r of x can be represented in the form

$$r(s\,,t) = \int e^{is\lambda}\,e^{-it\theta}\,d\nu(\lambda\,,\theta)\,,\qquad s\,,\ t\in R\,,$$

where ν is a bounded Radon measure on $R\times R$. Moreover,

$$\left|\sum_{k=1}^{n}\left(\int x\,p_k\,dm\,\Big|\int x\,q_k\,dm\right)\right| = \left|\sum_{k=1}^{n}\int\mathcal{F}\,p_k(\lambda)\,\overline{\mathcal{F}\,q_k(\theta)}\,d\nu(\lambda\,,\theta)\right| \le \nu^{\bullet}(1)\left(\sup_{\lambda,\theta\in R}\left|\sum_{k=1}^{n}\mathcal{F}\,p_k(\lambda)\,\overline{\mathcal{F}\,q_k(\theta)}\right|\right)$$

for all $p_k\,,q_k\in\mathcal{L}^1_C(m)$, $k=1\,,\ldots,n$, $n\in N$, which proves the first part of the theorem.

Suppose then that a bounded and continuous stochastic process $x:\ R\to L^2_0(P)$ satisfies the inequality (3.3.1′). Set $\mathcal{F}\,(\mathcal{L}^1_C(m)) = \{f\in C_0(R)\mid f=\mathcal{F}\,p$ for some $p\in\mathcal{L}^1_C(m)\}$. Define a bilinear mapping $\nu^x:\ \mathcal{F}\,(\mathcal{L}^1_C(m))\times\mathcal{F}\,(\mathcal{L}^1_C(m))\to C$ by setting

$$\nu^x(f\,,\tilde{g}) = \int r(s\,,t)\,p(s)\,q(t)\,d(m\otimes m)(s\,,t)$$

for $f\,,g\in C_0(R)$ to which there exist $p\,,\ q\in\mathcal{L}^1_C(m)$ such that $f=\mathcal{F}\,p$, $g=\mathcal{F}\,q$; here we have written $\tilde{g}(\lambda)=g(-\lambda)$, $\lambda\in R$. The definition of $\nu^x(f\,,\tilde{g})$ is unique. Moreover,

$$|\nu^x(f\,,\tilde{g})| = \left|\int r(s\,,t)\,p(s)\,q(t)\,d(m\otimes m)(s\,,t)\right| \le c\sup_{\lambda,\theta\in R}|\mathcal{F}\,p(\lambda)\,\mathcal{F}\,q(-\theta)| = c\sup|f|\sup|\tilde{g}|\,,$$

thus

$$\nu^x:\ \mathcal{F}\,(\mathcal{L}^1_C(m))\times\mathcal{F}\,(\mathcal{L}^1_C(m))\to C$$

is continuous, when $\mathcal{F}\,(\mathcal{L}^1_C(m))$ carries the norm topology induced by $C_0(R)$. Let

$$\tilde{\nu}^x : \mathcal{F}(\mathcal{L}^1_C(m)) \otimes \mathcal{F}(\mathcal{L}^1_C(m)) \to C$$

be the unique linear mapping defined by the bilinear mapping ν^x for which $\tilde{\nu}^x(f \otimes g) = \nu^x(f, g)$, $f, g \in \mathcal{F}(\mathcal{L}^1_C(m))$. The inequality (3.3.1′) implies that $\tilde{\nu}^x$ is continuous, when the space $\mathcal{F}(\mathcal{L}^1_C(m)) \otimes \mathcal{F}(\mathcal{L}^1_C(m))$ carries the norm topology induced by $C_0(R \times R)$. Furthermore, the space $\mathcal{F}(\mathcal{L}^1_C(m)) \otimes \mathcal{F}(\mathcal{L}^1_C(m))$ is dense in $C_0(R \times R)$, since $C_0(R) \otimes C_0(R)$ is dense in $C_0(R \times R)$ and since $\mathcal{F}(\mathcal{L}^1_C(m))$ is dense in $C_0(R)$. Thus the mapping $\tilde{\nu}^x : \mathcal{F}(\mathcal{L}^1_C(m)) \otimes \mathcal{F}(\mathcal{L}^1_C(m)) \to C$ can be extended by continuity to a continuous linear mapping $\nu : C_0(R \times R) \to C$. Furthermore,

$$\int r(s, t)\, p(s)\, q(t)\, d(m \otimes m)(s, t) = \int \mathcal{F} p(\lambda)\, \mathcal{F} q(-\theta)\, d\nu(\lambda, \theta)$$
$$= \int p(s)\, q(t) \left(\int e^{is\lambda} e^{-it\theta}\, d\nu(\lambda, \theta) \right) d(m \otimes m)(s, t),$$

for all $p, q \in \mathcal{L}^1_C(m)$. Thus by the continuity of the functions r and $\int e^{i\lambda \cdot} e^{-i\theta \cdot}\, d\nu(\lambda, \theta)$ we get

$$r(s, t) = \int e^{is\lambda} e^{-it\theta}\, d\nu(\lambda, \theta), \qquad (s, t) \in R \times R.$$

Moreover, let $f = \mathcal{F} p$, $p \in \mathcal{L}^1_C(m)$; since $\overline{(\mathcal{F} p)(-\lambda)} = (\mathcal{F} \bar{p})(\lambda)$, $\lambda \in R$, we get

$$\nu(f \otimes \bar{f}) = \int r(s, t)\, p(s)\, \overline{p(t)}\, d(m \otimes m)(s, t)$$
$$= \left(\int x\, p\, dm \,\Big|\, \int x\, p\, dm \right) \geq 0,$$

thus $\nu(g \otimes \bar{g}) \geq 0$ for all $g \in C_0(R)$, which proves the theorem.

The following characterization is a direct consequence of Theorem 3.3.2 and Remark 2.3.7.

Theorem 3.3.3. *A (weakly) continuous V-bounded stochastic process $x : R \to L^2_0(P)$ is harmonizable if and only if the covariance bimeasure of the spectral measure of x can be extended to a bounded Radon measure on $R \times R$.*

2. The following example shows that the class of all continuous V-bounded stochastic processes is strictly larger than the class of all harmonizable stochastic processes. The example is a modification of an example due to Edwards [9: pp. 93—94].

Example 3.3.4. We construct a bounded stochastic measure μ on R having the property that its covariance bimeasure cannot be extended to a bounded Radon measure on $R \times R$. Thus the Fourier transform of μ is a continuous V-bounded stochastic process which is not harmonizable.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Denote by $L_0^2(P)_{\mathrm{Re}}$ the real Hilbert space of all real-valued stochastic variables ξ defined on $(\Omega, \mathcal{F}, P)$ such that

$$E\,\xi = 0 \quad \text{and} \quad E|\xi|^2 < \infty , \tag{3.3.2}$$

and denote, as usual, by $L_0^2(P)$ the complex Hilbert space of all complex-valued stochastic variables ξ defined on $(\Omega, \mathcal{F}, P)$ satisfying (3.3.2). Suppose that the probability space $(\Omega, \mathcal{F}, P)$ is such that $L_0^2(P)_{\mathrm{Re}}$ is separable and infinite dimensional.

Edwards [9: pp. 93—94] has shown that there exist real numbers c_{mn}, $m, n \in N$ and a constant $M > 0$ such that

$$\sum_{m \in N} \sum_{n \in N} |c_{mn}| = \infty \tag{3.3.3}$$

and

$$\left| \sum_{j \in J} \sum_{k \in K} a_j \, b_k \, c_{jk} \right| \leq M \tag{3.3.4}$$

for all finite J, $K \subset N$, where $-1 \leq a_j \leq 1$, $-1 \leq b_k \leq 1$, $j \in J$, $k \in K$. In fact

$$c_{nn} = \frac{\pi}{2\,n\,[\log(n+1)]^2}, \quad n \in N,$$

$$c_{mn} = \frac{\sin[\pi(m-n)/2]}{(m-n)\, m^{1/2}\, n^{1/2} \log(m+1)\log(n+1)}, \quad m \neq n, \quad m, \; n \in N.$$

As in the case considered by Edwards, there exists a sequence

$$\{x_n\}_{n \in N} \subset L_0^2(P)_{\mathrm{Re}}$$

such that

$$(x_m \mid x_n) = c_{mn} \qquad m, \; n \in N.$$

Define a stochastic measure $\mu : \mathcal{K}_C(R) \to L_0^2(P)$ by setting

$$\mu(f) = \sum_{n \in N} f(n)\, x_n , \qquad f \in \mathcal{K}_C(R).$$

Using (3.3.4) we see that μ is bounded. Furthermore, by (3.3.3) the covariance bimeasure of μ cannot be extended to a bounded Radon measure on $R \times R$. Thus the stochastic process $x : R \to L_0^2(P)$,

$$x(t) = \int e^{it\lambda}\, d\mu(\lambda), \qquad t \in R,$$

is continuous and V-bounded, but not harmonizable.

3.4. Approximation of continuous V-bounded stochastic processes

1. In this section we shall consider a method of approximating a continuous V-bounded stochastic process by a sequence of harmonizable stochastic processes. For another method of approximating an arbitrary continuous and bounded stochastic process by a sequence of harmonizable stochastic processes see von Bahr [3].

Theorem 3.4.1. *Let $x : R \to L_0^2(P)$ be a continuous V-bounded stochastic process. Then there exists a sequence $x_n : R \to L_0^2(P)$, $n \in N$, of harmonizable stochastic processes such that*

$$x(t) = \lim_{n \to \infty} x_n(t) , \qquad t \in R ,$$

uniformly on every compact set $K \subset R$.

Proof. Let x be a continuous V-bounded stochastic process. By Theorem 3.2.1 there exists a bounded stochastic measure μ on R such that

$$x(t) = \int e^{it\lambda} \, d\mu(\lambda) , \qquad t \in R .$$

Since x is continuous the space $\overline{\mathrm{sp}}\,\{x\}$ is, by Lemma 1.1.3, separable. Let $\{\xi_n\}_{k \in N} \subset \overline{\mathrm{sp}}\,\{x\}$ be an orthonormal basis of $\overline{\mathrm{sp}}\,\{x\}$, then

$$x(t) = \sum_{k \in N} (x(t) \mid \xi_k) \, \xi_k , \qquad t \in R .$$

Denote

$$x_n(t) = \sum_{k=1}^{n} (x(t) \mid \xi_k) \, \xi_k , \qquad t \in R , \ n \in N .$$

Then the process $x_n : R \to L_0^2(P)$ is bounded and continuous for every $n \in N$. Moreover

$$x_n(t) = \int e^{it\lambda} \, d\mu_n(\lambda) , \qquad t \in R ,$$

where

$$\mu_n(f) = \sum_{k=1}^{n} (\mu(f) \mid \xi_k) \, \xi_k , \qquad f \in \mathcal{K}_C(R) , \ n \in N .$$

The stochastic processes $x_n : R \to L_0^2(P)$, $n \in N$, are harmonizable, since the stochastic measures μ_n are bounded, and since for every $n \in N$ the covariance bimeasure B_n,

$$B_n(f , g) = \sum_{k=1}^{n} (\mu(f) \mid \xi_k) \, \overline{(\mu(\bar{g}) \mid \xi_k)} , \qquad f , g \in \mathcal{K}_C(R) ,$$

of μ_n can be extended to a bounded Radon measure on $R \times R$.

Let $K \subset R$ be a compact set, then the set $\{ x(t) \mid t \in K \} \subset \overline{\mathrm{sp}}\{x\}$ is compact. Thus by the approximation property of the space $\overline{\mathrm{sp}}\{x\}$ (see 1.2.3 or Theorem 1.2.4)

$$x(t) = \lim_{n \to \infty} x_n(t)$$

uniformly on K.

Corollary 3.4.2. *Let x be a continuous V-bounded stochastic process and let r be the covariance function of x. Then there exists a sequence $x_n : R \to L_0^2(P)$, $n \in N$, of harmonizable stochastic processes such that*

$$r(s, t) = \lim_{n \to \infty} r_n(s, t)$$

uniformly on every set $K \times R$ (or $R \times K$), where $K \subset R$ is compact and r_n is the covariance function of x_n, $n \in N$.

Proof. The corollary is a direct consequence of Theorem 3.4.1 and Theorem 1.2.4.

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