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Stochastic Processes in Microphysics in Connection with Relativity

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The difficulties of extending to relativity the results obtained in a preceding paper[1] are set forth; in particular the relativistic behaviour of the time and length minimal constants brings us to a well-known problem. The proposed views attempt to deal with these questions. They give a relativistic generalization of the classical diffusion theory, from which the Klein-Gordon equation is deduced. Consistent results are obtained by a relativistic survey of Dirac's fluid model. Subsequent investigations have been carried out concerning the character of relativity and the limitation of its validity in microphysics.

§ 1. Introduction

In a preceding paper,[1] (which will be referred to as (A)), we have shown that, with suitable conventions, Schrödinger's equation can be deduced from the laws of stochastic diffusion. We have seen that the diffusion process involves time and length constants (the time and length mean intervals between two collisions) having the meaning of minimal entities for diffusion, and consequently for quantum mechanics. This investigation had the double aim of providing a concrete basis to microphysics, and answering the need for minimal length and time constants, connected with the experimental data leading to spatial extension of particles.

A further investigation in the field of relativity is advisable both to better generalize the result of the earlier work, and to specify the relativistic behaviour of the constants therein introduced.

But this latter task meets with a well-known difficulty: How can minimal time and length constants be invariant, and if they are not, how can they have a physical meaning? This discrepancy, and the subsequent impediments in the quantum field theory, have already incited several physicists to think that some modification should be found (probably in limiting its extension) to the use of relativistic formalism in microphysics.

This necessity also appears if one attempts to conciliate stochastic processes with relativity. The first step should be to generalize the classical diffusion theory, which is not relativistic; but this seems to lead to a deadlock. In the field of diffusion theory, a stochastic element has no definite velocity (see (A), § 2,2); how then can it be related to a Lorentz frame? On the other hand the Fokker-Planck equation lacks just one term that could render it relativistic; it is thus very unlikely that its relativistic extension should meet with a definitive impossibility.

We will show below how this contradiction can be overcome, together with the aforementioned difficulty concerning the variance of the constants. We will first prove that appropriate postulates permit a relativistic approach to the theory of diffusion, from which relativistic quantum potential and the Klein-Gordon equation can be deduced. We will then examine the consequences of the proposed views on relativity and relativistic microphysics. (When we refer to our thesis,[2] we shall mention it by (T)).

§ 2. Scheme of the present work

2·1 *Recall of the basic assumptions used in the previous work*

We associate to every particle of rest mass m_0 a stochastic fluid, the diffusion constant of which is $k=\hbar/m_0 c$, within the approximation of the classical diffusion theory. The physical character of this fluid is not specified: It may be either a real subquantal fluid, or the fluid of probability concerning the particle itself, submitted to stochastic motions.*) In any case, all the ordinary data concerning continuous media (temperature, entropy, etc.) are macroscopic entities which have no meaning at the scale of our study.

The basic assumptions concerning the laws of diffusion properly, will be recalled in § 3.

When the spin is taken into account, it is considered as representing local (and non aleatory) rotations in the fluid. Dirac's fluid is obtained by addition of two spinned constituents, as recalled in § 5.

2·2 *Extension of the classical theory of diffusion*

We have given in (A) reasons leading to take the velocity of light c as the free travel velocity of the stochastic elements. We now show that, with this choice, the approximation of the classical diffusion theory is identical to the non relativistic approximation.

The extension of the diffusion theory to the relativistic case consequently entails new laws for diffusion, of which the classical laws are an approximation.

The following question then arises: shall we, as it is commonly done, start from the relativistic formalism? If we do so, we fall again into the aforementioned deadlock: impossibility of associating a Lorentz frame to basic elements with no definite velocity. We therefore infer that relativity *should not apply* at the basis, but—the relativistic formalism being not taken into account for the basic elements—it should appear *by itself*, at the higher level where an average velocity can be defined.

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*) In (A), we suggested at the end that the double character of the fluid should be maintained, and should be an aspect of the quantum dualism.

This will be verified in two ways:

a) by searching for the general laws to be laid down for diffusion. We shall see that convenient stochastic assumptions make Lorentz's contraction appear by itself.

b) by working out the equation generalizing the Fokker-Planck equation; after stochastic assumptions equivalent to those of a), relativistic covariance also appears by itself, the space-time formalism having not been used at the start.

But if relativity does not apply at the basis, this implies an inferior limit of its validity in the microscopic field. This point is confirmed by the fact that the covariance of the general diffusion equation is only obtained if the entities involved are the proper ones, although the frame is unique in the whole space.

We defer to the end (see subsection 2·4 hereafter) the close investigation of this limitation.

2·3 *The model of Dirac's fluid*

We now examine the model of Dirac's fluid in the relativistic case; this survey, worked out independently of the former, leads exactly to the same results:

a) Lorentz's contraction appears from itself in the model.

b) The concrete interpretation given by the model implies that relativity is not valid for the motions inside the proper frame (local microscopic motions).

2·4 *Further suggestions*

It still remains to define the limit of the field of relativistic laws. We propose an explanatory principle and show that:

a) it covers all the points dealt with in the present work;

b) it answers several problems or difficulties concerning quantum mechanics and relativity;

c) it is consistent with all the laws of relativity, and allows, with the help of the stochastic basic fluid, a concrete interpretation of these laws.

§ 3. A relativistic approach to the theory of diffusion

3·1 *Preliminary observations*

a) *Recapitulation of the principles of the theory of diffusion*

These principles have been explained in ((A), § 2). A homogeneous and isotropic diffusion process for the positional chance variable X is defined:

α) by the probability $P(\boldsymbol{x}t/(\boldsymbol{X}(0)=\boldsymbol{x}_0))$ of transition from a position $\boldsymbol{x}_0$ given at time zero to a random position $\boldsymbol{x}$ at time t;

β) by the average velocity:

$$\boldsymbol{u}=\lim_{t\to 0}\text{it}: 1/t\cdot\int(\boldsymbol{x}-\boldsymbol{x}_0)P[\boldsymbol{x}t/(\boldsymbol{X}(0)=\boldsymbol{x}_0)]d\boldsymbol{x}, \tag{1}$$

γ) by the diffusion constant k given by

$$k\delta_{ij}=\lim_{t\to 0}\text{it}: 1/t\cdot\int(x_i-x_{i0})(x_j-x_{j0})P[\boldsymbol{x}t/(\boldsymbol{X}(0)=\boldsymbol{x}_0)]d\boldsymbol{x} \tag{2}$$

with the condition for higher orders:

$$\lim_{t\to 0}\text{it}: 1/t\cdot\int(x_i-x_{i0})(x_j-x_{j0})\cdots(x_l-x_{l0})P[\boldsymbol{x}t/(\boldsymbol{X}(0)=\boldsymbol{x}_0)]d\boldsymbol{x}=0, \tag{3}$$

δ) by the probability density $\rho(\boldsymbol{x}t)$ connected with $\boldsymbol{u}$ by the Fokker-Planck equation

$$\partial\rho/\partial t+\partial/\partial x_i\cdot(\rho u_i)=k/2\cdot\Delta\rho. \tag{4}$$

These conditions define a *diffusion representation* in which: the velocity σ_0 of the free travel is infinite; the number of collisions in any finite time is infinite (if the mean free path is finite); the mean time θ between two collisions is infinitesimal; the individual stochastic elements have no definite velocity; only the average velocity $\boldsymbol{u}$ can be defined.

To this representation we have opposed the *kinetic representation* in which σ_0 is finite, θ non infinitesimal, and consequently, between two collisions, the elements have a velocity. To determine a boundary between these two representations, which are not situated on the same scale, we started from the diffusion representation, and assuming the position of an element to be known in M at time zero, we pictured the diffusion as follows:

I. We first assume $u=0$. The diffusion at time t is figured by the sphere of expansion (Σ) of center M and radius $\Delta=\sqrt{kt}$ (standard deviation at time t). The radial expansion velocity of (Σ), $\sigma=\Delta/t=\sqrt{k/t}$, is infinite for $t\to 0$, which confirms that the diffusion representation is not valid for small values of t, for one must have $\sigma\ll\sigma_0$.

We have defined the boundary between the two representations as given by the instant θ' when $\sigma=\sigma_0$, then $\theta'=k/\sigma_0^2$. We identify θ' with the time θ of free travel, and then for the free path Λ, we get $\Lambda=\theta\sigma_0$; θ and Λ take on the meaning of minimal constants.

II. If $u\neq 0$, this picture vanishes, as during the minimal time θ the center M of (Σ) is displaced by $u\theta$; the picture may however be maintained if this displacement is much smaller than the expansion $\Lambda=\theta\sigma_0$ i.e. if

$$u\ll\sigma_0. \tag{5}$$

b) The constant σ_0

In (A) we have taken $\sigma_0=c$; this choice (besides letting us successfully deduce the quantum laws from the stochastic laws), is justified for the following reasons:

α) The condition (5) provides a limit of validity to the above picture. Another limit emerges from the fact that the diffusion theory is not relativistic. It is very likely that the two limits should be the same; if so, $\sigma_0 = c$.

β) If a stochastic fluid is assumed to afford a physical basis to matter, the free travel velocity of this fluid is a structural constant which should appear in matter equations; this also gives $\sigma_0 = c$.

γ) We have suggested in (A) that there should be no separation between the stochastic fluid and the vacuum; the diffusion model extended by this continuity to the vacuum should then give an average velocity $\boldsymbol{u}$ equal to zero in every frame of reference. Now, since $\boldsymbol{u}$ is obtained by addition of velocities of magnitude σ_0, we again find $\sigma_0 = c$.

With $\sigma_0 = c$, θ and Λ become respectively:

$$\tau_0 = \hbar / m_0 c^2, \quad l_0 = \hbar / m_0 c.$$

c) *First consequences*

α) The choice $\sigma_0 = c$ shows by (5) that *the approximation of the classical diffusion theory is identical to the non-relativistic approximation.*

β) This approximation is also characterized by the fact that, during the minimal time τ_0, the translation $l_0 = u\tau_0$ is negligible in comparison with the expansion $\Lambda_0 = l_0$. Now if we call the two entities d and Λ at any arbitrary instant $t \geq \tau_0$, we have $d = ut$, $\Lambda = \sqrt{kt}$. The condition $d \ll \Lambda$ will be satisfied if t is infinitesimal.

Applying this for $t = \tau_0$, we see that *the non-relativistic approximation consists in the fact that τ_0 may be considered infinitesimal.*

It follows that beyond this approximation the definitions of $\boldsymbol{u}$ and k by Eqs. (1) and (2) are no longer valid, for on one hand they are based on the limiting process $t \to 0$, on the other hand they belong to the diffusion representation, in which $t \geq \tau_0$. These two conditions are not consistent if τ_0 is not infinitesimal. We have to replace Eqs. (1) and (2) by new formulae containing, instead of $t \to 0$, the minimal time τ_0, i.e.

$$\boldsymbol{u}^{(1)} = 1/\tau_0 \cdot \int (\boldsymbol{x} - \boldsymbol{x}_0) P[\boldsymbol{x}\tau_0 / (\boldsymbol{X}(0) = \boldsymbol{x}_0)]\, d\boldsymbol{x}, \tag{6}$$

$$k_{ij}^{(1)} = 1/\tau_0 \cdot \int (x_i - x_{i0})(x_j - x_{j0}) P[\boldsymbol{x}\tau_0 / (\boldsymbol{X}(0) = \boldsymbol{x}_0)]\, d\boldsymbol{x}. \tag{7}$$

We may consider $\boldsymbol{u}^{(1)}$ as a new conventional definition of the average velocity (henceforth we write $\boldsymbol{u}$ with the meaning of $\boldsymbol{u}^{(1)}$). As for the $k_{ij}^{(1)}$, we write (with the approximation defined below):

$$k_{ij}^{(1)} \sim k_{ij} = k\,\delta_{ij}. \tag{8}$$

The physical assumption which exists implicitly in Eq. (8) will soon appear.

We finally remark that the Fokker-Planck Eq. (4) does not contain τ_0, k being finite (which actually gives τ_0 infinitesimal, as $\tau_0 = k/c^2$ and c is infinite). In the relativistic case c is finite, then $k = \tau_0 c^2$ is of order 1 in τ_0. We assume that Eq. (8) is valid to the same order.

3·2 *Basic assumptions for the relativistic case*

In the non relativistic case we had considered, during the infinitesimal time t, the average translation motion $\boldsymbol{d} \sim \boldsymbol{u}t$ and the isotropic dispersion $D \sim \sqrt{kt}$, with $|\boldsymbol{d}| \ll D$.

Let $M_i(t)$ be the mean quadratic value of the coordinate x_i, we have

$$M_i^2 = d_i^2 + D^2.$$

The assumption $d_i \ll D$ can be expressed as follows: For $t \to 0$ the mean quadratic value M_i and the isotropic standard deviation $D_i = D$ are equivalent; in other words

$$\lim_{t \to 0} M_i^2/t = \lim_{t \to 0} D^2/t = k.$$

Moreover k is a constant; at the limit $t \to 0$, M_i and D have an invariant character; only the average motion depends on the frame of reference; the structure of the fluid, determined by k, is not modified by the relative motion.

In short, the approximation of the classical diffusion theory may be defined as follows:

The mean quadratic value and the standard deviation of a coordinate, taken in an infinitesimal time interval, are equivalent, and invariant during a change of reference frame.

Outside this approximation, we have to compare the two mean values M_i and D at time τ_0; they remain the same for the coordinates normal to $\boldsymbol{u}$, if the direction of $\boldsymbol{u}$ has not varied sensibly during the time τ_0; but for the coordinate directed along $\boldsymbol{u}$, we have, (calling L the mean quadratic value and l the standard deviation after time τ_0):

$$L^2 = l^2 + u^2 \tau_0^2 = l^2 + (u^2/c^2) l_0^2. \tag{9}$$

Then, not only L and l are different, but the difference of their squares depends on the reference frame. This result can be expressed as follows:

In the relativistic case, the quadratic mean value L and the standard deviation l of the coordinate directed along the average velocity, at the minimal time τ_0,) are no longer the same, and consequently cannot be both invariant in a change of reference frame.*

To lay down new laws for diffusion, we shall keep our choice as simple as possible: We may adopt the invariance of either l or L.

*) They could be, instead, taken at the corrected time $\tau_{0/2}$, as the expansion is linear up to τ_0.

First choice: Invariance of l.

We equate l to the proper value l_0.

Second choice: Invariance of L.

We put $L=l_0$, as the two mean values are the same in the proper frame. Equation (9) yields

$$l^2=L^2-u^2/c^2\cdot l_0^2=l_0^2(1-u^2/c^2).$$

The appearance of the Lorentz contraction with the second choice incites us simultaneously to decide for the second choice and to lay down a connection between the standard deviation of the coordinates (at time τ_0) and the macroscopic lengths.

From the law of lengths the space-time formalism can be deduced.*) As for the law of mass and energy, we could define the mass in motion m by the relation $l=\hbar/mc$, (similar to $l_0=\hbar/m_0c$) whence $m=m_0/\sqrt{1-u^2/c^2}$. But we can proceed less formally, observing that the relation $m_0=\hbar/l_0c$ defines the rest mass as inversely proportional to the frequency of collisions: Thus mass-energy equivalence means that the rest mass may be defined as random agitation energy of the stochastic fluid.

Then, if this significance is conserved for the system in translation, and if the energy of agitation remains proportional to the density ρ of the fluid, which varies inversely as l, we get for the energy W

$$W/m_0c^2=l_0/l=1/\sqrt{1-u^2/c^2}. \qquad (10)$$

Remark. In (A) we have introduced the quantum velocity $\boldsymbol{v}$ connected to $\boldsymbol{u}$ by

$$\boldsymbol{v}=\boldsymbol{u}-cl_0/2\cdot\nabla\log\rho. \qquad (10')$$

As $\boldsymbol{v}$ and $\boldsymbol{u}$ differ by a gradient, they are equivalent in macrophysics, where the relativistic formulae may be written with $\boldsymbol{v}$ instead of $\boldsymbol{u}$.

3·3 *The relativistic equation of diffusion*

We have already mentioned the impossibility of assigning a frame of reference to a stochastic element without definite velocity. A similar fact appears when considering the definitions of the basic diffusion constants: For instance let us take Eq. (6) and ask ourselves in what frame of reference it could be read. Obviously this question is meaningless: The only velocity which could determine a reference frame is that defined by the formula itself. This shows that spacetime formalism cannot be considered as a starting point for the study of the individual elements, but should afterwards appear by itself—in connection with the average translation motion—just as we have seen the Lorentz contraction arising from physical postulates concerning diffusion. This will be verified in

*) We shall hereafter deduce it directly from concrete considerations.

working out the relativistic equation of diffusion.

a) Recapitulation of the proof of the Fokker-Planck equation

We here give the more intuitive proof, on which the principle of relativistic extension appears in the simplest way; we refer to (T) for rigorous demonstrations based on Chapman's equation (Markovian processes) or on more general assumptions (non-Markovian processes), and also for the equation in terms of initial variables.

Let a stochastic element be in $M(\boldsymbol{x})$ at time t; we compute in two ways $Q=\lim_{t'\to t} 1/(t'-t)\cdot\delta\bar{f}$, $\delta\bar{f}$ being the variation, between t and t', of the average value of a time-independent function $f(\boldsymbol{x})$ in its field of existence V.

We first have, $\rho(\boldsymbol{x}\, t)$ being the probability density:

$$Q=\lim_{t'\to t}\,[1/(t'-t)\cdot\{\int_V f(\boldsymbol{x})\rho(\boldsymbol{x}t')\,d\boldsymbol{x}-\int_V f(\boldsymbol{x})\rho(\boldsymbol{x}t)\,d\boldsymbol{x}\}]$$

$$=\int_V f(\boldsymbol{x})\,\partial\rho(\boldsymbol{x}t)/\partial t\cdot d\boldsymbol{x}. \tag{11}$$

On the other hand, to account for the transitions between t and t', we also have, calling (S) the whole space:

$$Q=\int_V \rho(\boldsymbol{x}t)\,d\boldsymbol{x}[\int_{(S)} f(\boldsymbol{x}')\,P[\boldsymbol{x}'t'/(X(t)=\boldsymbol{x})]\,d\boldsymbol{x}'-f(\boldsymbol{x})].$$

Expanding:

$$f(\boldsymbol{x}')=f(\boldsymbol{x})+(x_i'-x_i)\,\partial/\partial x_i\cdot f(\boldsymbol{x})+(x_i'-x_i)\,(x_j'-x_j)\;\partial^2/\partial x_i\partial x_j\cdot f(\boldsymbol{x})+\cdots,$$

we get by Eqs. (1), (2), (3) and $\int_{(S)} P[\boldsymbol{x}'t'/(\boldsymbol{X}(t)=\boldsymbol{x})]\,d\boldsymbol{x}'=1$,

$$Q=\int_V[\rho(\boldsymbol{x}t)\,u_i(\boldsymbol{x}t)\,\partial/\partial x_i\cdot f(\boldsymbol{x})\,d\boldsymbol{x}+k/2\cdot\rho(\boldsymbol{x}t)\,\Delta f(\boldsymbol{x})]\,d\boldsymbol{x}\} \tag{12}$$

$$=\int_V f(\boldsymbol{x})\,[-\partial/\partial x_i\cdot(\rho u_i)+k/2\cdot\Delta\rho]\,d\boldsymbol{x}.$$

Equating Eq. (12) with Eq. (11), we get, V and f being arbitrary, the Fokker-Planck equation

$$\partial/\partial t\cdot\rho+\partial/\partial x_i\cdot(\rho u_i)=k/2\cdot\Delta\rho.$$

b) Adaptation to the relativistic case

The only modification is that, instead of $\lim_{t'\to t} 1/(t'-t)\cdot\delta\bar{f}$, we have to take $1/\tau_0\cdot\delta_1\bar{f}$, $\delta_1\bar{f}$ being the variation of $\bar{f}$ between t and $t+\tau_0$; besides, Eqs. (1) and (2) must be changed into Eqs. (6) and (7).

We now have, instead of Eq. (11)

$$Q=1/\tau_0\cdot[\int_V f(\boldsymbol{x})\rho(\boldsymbol{x},\ t+\tau_0)\,d\boldsymbol{x}-\int_V f(\boldsymbol{x})\rho(\boldsymbol{x}\ t)\,d\boldsymbol{x}], \tag{13}$$

and, to obtain the new first order equation in τ_0, we have to expand $\rho(\boldsymbol{x}, t+\tau_0)$ in Eq. (13) to the second order in τ_0; thus

$$Q=\int f(\boldsymbol{x})\,[\partial\rho/\partial t+\tau_0/2\cdot\partial^2\rho/\partial t^2]\,d\boldsymbol{x}$$

$$=\int f(\boldsymbol{x})\,[\partial\rho/\partial t+k/2c^2\cdot\partial^2\rho/\partial t^2]\,d\boldsymbol{x}. \tag{14}$$

Equating (14) with (12) we get

$$\partial\rho/\partial t+\partial/\partial x_i\cdot(\rho u_i)=k/2\cdot\Box\,\rho. \tag{15}$$

This equation leads to the following remarks:

α) It has space-time symmetry, which, as foreseen, has arisen by itself, no relativistic formalism having been imposed at the beginning.

β) We have worked out this equation without specifying in which reference frame it should read; about this frame two conditions appear.

β_1) The frame (Σ) must be unique, for the calculation takes into account an extended volume where the velocity $\boldsymbol{u}$ is variable.

β_2) Nevertheless the equation is not covariant unless the density ρ is the proper density and the velocity $\boldsymbol{u}$ is defined by means of the proper time.

We seem once more to have fallen upon a contradiction, impossible to reconcile with classical views, and of the same type as before: Taking into account the proper entities means ignoring the formalism at the outset of our reasoning, and yet the results deduced force us into this very formalism.*)

The connection with subsection 3·2 is easy: The postulate now admitted is expressed by Eq. (8); it means, by Eq. (7), the invariance of the mean quadratic values of the coordinates after time τ_0, provided (in accordance with condition β_1) that the random point is related to the fixed and unique frame (Σ).

These points will be confirmed by consistent results concerning the spin particle model (§ 5).

§ 4. Relativistic diffusion and quantum mechanics without spin

4·1 *Relativistic quantum potential and the Klein-Gordon equation*

a) *Quantum potential*

We proceed similarly to the non relativistic case; there we had chosen an action function $\mathcal{T}$ given for $u=0$ by

*) This will be explained below more accurately (§ 6).

$$\rho\mathcal{T} = -\rho_0\, m_0\, c^2, \tag{16}$$

ρ_0 being related to ρ by

$$[\rho_0\ d\boldsymbol{x}]_{(t)} = [\rho\ d\boldsymbol{x}]_{(t-\tau_0/2)}. \tag{17}$$

The entity ρ represented both the stochastic and the quantum densities; only the velocites ($\boldsymbol{u}$ and $\boldsymbol{v}$) were different. In the relativistic case we shall see that the densities also must be distinguished. Therefore, we shall henceforward call p the density of the stochastic fluid, reserving ρ for the density which we shall find in the quantum equations.

We then remark that Eq. (15) reads

$$\partial j_\mu / \partial x_\mu = 0 \tag{18}$$

with

$$j_i = pu_i - c^2/2 \cdot \tau_0\ \partial p/\partial x_i \quad (i=1, 2, 3), \tag{19}$$

$$j_4 = ic(p + 1/2 \cdot \tau_0\ \partial p/\partial t). \tag{20}$$

This allows us to define a fictitious velocity $\boldsymbol{v}$ and a fictitious density ρ forming a conservative stream whose fact leads us to consider them as the velocity and density of the quantum fluid; we take

$$v_i = ic\ j_i/j_4 = (pu_i - c^2\ \tau_0/2 \cdot \partial p/\partial x_i)/(p + \tau_0/2 \cdot \partial p/\partial t), \tag{21}$$

$$\rho = p + \tau_0/2 \cdot \partial p/\partial t. \tag{22}$$

Now, similarly to Eq. (17), we define a density ρ_0 by

$$[\rho_0\ d\boldsymbol{x}]_{(t)} = [p\ d\boldsymbol{x}]_{(t-\tau_0/2)}, \tag{23}$$

and change Eq. (16) into

$$\rho\mathcal{T} = -\mu\rho_0 m_0 c^2 \ \ (\text{for } u=0), \tag{24}$$

where μ may be a function of local entities, equal to 1 for $\tau_0 = 0$, and expressible, to the second order in τ_0, by

$$\mu = 1 + \alpha\tau_0 + \beta\tau_0^2,$$

α and β being indeterminate functions.

We have by Eqs. (23) and (24)

$$\rho\mathcal{T}\, d\boldsymbol{x} = -(1 + \alpha\tau_0 + \beta\tau_0^2) m_0\, c^2 (p\ d\boldsymbol{x})_{(t-\tau_0/2)}.$$

Now, by Eq. (22)

$$\begin{aligned}(p\ d\boldsymbol{x})_{(t-\tau_0/2)} &= (p\ d\boldsymbol{x})_{(t)} - \tau_0/2 \cdot \partial/\partial x_\nu \cdot (pu_\nu)_{(t)} \cdot d\boldsymbol{x}(t) \\ &= [\rho(t) - \tau_0/2 \cdot \partial p/\partial t(t)]\, d\boldsymbol{x}(t) - \tau_0/2 \cdot \partial/\partial x_\nu \cdot (pu_\nu)_{(t)} \cdot d\boldsymbol{x}(t),\end{aligned}$$

whence, by Eq. (15).

$$\rho\mathcal{T} = -m_0\, c^2 (1 + \alpha\tau_0 + \beta\tau_0^2)\, (\rho - \tau_0/2 \cdot \partial p/\partial t - c^2\ \tau_0^2/4 \cdot \Box p]$$

or, as we have, to the second order in τ_0; $\tau_0^2 \Box p \sim \tau_0^2 \Box \rho$:

$$\mathcal{I} = -m_0 c^2 (1+\alpha\tau_0+\beta\tau_0^2)\,[1-\tau_0/2\rho\cdot\partial p/\partial t - c^2\,\tau_0^2/4\rho\cdot\Box\,\rho]. \quad (25)$$

This expression is to be equated to that obtained with the quantum entities ρ and $\boldsymbol{v}$. Now, in relativistic mechanics, a mass point has an action function

$$\mathcal{I}_m = -m_0\, c^2 \sqrt{1-v^2/c^2}.$$

We must then put here

$$\mathcal{I} = -m_0\, c^2 \sqrt{1-v^2/c^2}\cdot K, \quad (26)$$

K accounting for the quantum potential.

By Eqs. (19) and (20), Eq. (26) reads, for $u=0$, to the second order in τ_0:

$$\left.\begin{aligned} \mathcal{I} &= -m_0\, c^2 \sqrt{1-c^2\,\tau_0^2/4\rho^2\cdot(\boldsymbol{\nabla} p)^2}\cdot K \\ &\sim -m_0 c^2 \sqrt{1-c^2\,\tau_0^2/4\rho^2\cdot(\boldsymbol{\nabla}\rho)^2}\cdot K \end{aligned}\right\} \quad (26')$$

Equation (25) must contain no term of first order in τ_0, as the action function does not contain such a term in the non-relativistic approximation [see (A), Eq. (19)]; thus $\alpha = 1/2\rho\cdot\partial p/\partial t$.

Then, identifying Eqs. (25) and (26′), we have

$$K^2 = 1-c^2\,\tau_0^2[1/2\cdot\Box\,\rho/\rho - 1/4\rho^2\cdot\sum(\partial\rho/\partial x_\nu)^2] + [2\beta - 1/4\rho^2\cdot(\partial\rho/\partial t)^2]\tau_0^2$$

$$= 1-c^2\,\tau_0^2\,\Box\,\sqrt{\rho}/\sqrt{\rho} + 2\tau_0^2[\beta - 1/8\rho^2\cdot(\partial\rho/\partial t)^2].$$

Relativistic invariance gives

$$\beta = 1/8\rho^2\cdot(\partial\rho/\partial t)^2 + (C),$$

(C) being a linear combination of relativistic invariants, quadratic in the dx_μ.

The correct expression in the non-relativistic approximation is obtained only if (C) vanishes; then

$$\mathcal{I} = -m_0\, c^2 \sqrt{1-\hbar^2/m_0\, c^2\cdot\Box\sqrt{\rho}/\sqrt{\rho}}\cdot\sqrt{1-v^2/c^2},$$

whence we deduce the quantum potential

$$Q = -\hbar^2/2m_0\cdot\Box\sqrt{\rho}/\sqrt{\rho},$$

i.e. the value known in quantum mechanics.[3)]

b) The Klein-Gordon equation

To get the Klein-Gordon equation for a free particle, we consider a monochromatic wave in the rest system:

$$\psi = \sqrt{\rho(xyz)}\;\exp(im_0 c^2 t)/\hbar$$

for which the quantum potential must vanish, the particle being stationary; this condition yields

$$\square\,\psi/\psi=\square\sqrt{\rho}/\sqrt{\rho}+m_0^2\,c^2/\hbar^2=m_0^2c^2/\hbar^2. \tag{27}$$

Any combination of monochromatic waves satisfies the same equation, and it is possible, as in ((A), § 3, 2, b) to connect the amplitude a and phase φ of a non-monochromatic solution of Eq. (27) to the local entities of the stochastic fluid, (putting here $r_\mu=-\hbar/m_0\cdot\partial\varphi/\partial x_\mu$), which allows us to associate with every non-rotational stochastic fluid a wave function ruled by the Klein-Gordon equation.

c) *Remark*

It is very easy to obtain $\mathcal{T}$ in terms of p, in the frame where $u=0$; Eq. (25) reads to the second order:

$$\mathcal{T}=-m_0\,c^2[1+\tau_0/2\rho\cdot\frac{\partial p}{\partial t}+\tau_0^2/8\rho^2\cdot(\partial\rho/\partial t)^2]\,[1-\tau_0/2\rho\cdot\partial p/\partial t-c^2\,\tau_0^2/4\cdot\square\rho/\rho],$$

or, expanding (with $\rho\sim p$ in the second order term)

$$\mathcal{T}=-m_0\,c^2[1-\tau_0^2/8p^2\cdot(\partial p/\partial t)^2-c^2\tau_0^2/4\cdot\square\,p/p]. \tag{28}$$

4·2 *Survey at higher order of approximation*

In the non-relativistic case, the diffusion equation was worked out to order zero in τ_0. Now we have obtained the diffusion equation to order 1, and drawn from it the Klein-Gordon equation.

We ask ourselves if, with a higher order of approximation, we should obtain not only a new equation for diffusion, but also a new wave equation.

a) *Diffusion equation to the second order in* τ_0

The additional terms should satisfy the following conditions: they should be rational functions of densities, velocity components and their derivatives; they should be covariant in three dimensional space; and they should not include rot $\boldsymbol{u}$, as we are still dealing with a spinless fluid, and we have seen in (A) that spin is connected with rotational properties of the stochastic fluid. We modify our previous conventions as follows:

α) The velocity defined by Eq. (6)—and hereafter called $\boldsymbol{u}^1$—is no longer the correct velocity $\boldsymbol{u}$; we may put $\boldsymbol{u}=\boldsymbol{u}^1+\boldsymbol{u}'$, and because of the aforementioned condition, we have to write, with indeterminate numerical coefficients $\lambda^{(a)}$, $\lambda^{(b)}$, $\lambda^{(c)}$, $\lambda^{(d)}$, κ_1, κ_2 ($\boldsymbol{u}$ and $\boldsymbol{u}^1$ being equivalent in the corrective terms):

$$u_i'=\tau_0^2/p\cdot\{\lambda^{(a)}\partial^2(pu_i)/\partial t^2+\lambda^{(b)}c^2\partial p/\partial x_i\cdot(\mathrm{div}\boldsymbol{u})+\lambda^{(c)}p\partial^2u_i\cdot/\partial t^2$$
$$+\lambda^{(d)}pu_i[(\mathrm{div}\boldsymbol{u})^2+\kappa_1/p^2\cdot\sum_j(\partial p/\partial x_j)^2+\kappa_2/p^2\cdot(\partial p/\partial t)^2]\}.$$

β) Instead of Eq. (8) we write

$$k_{ij}=k\delta_{ij}+\lambda^{(e)}\tau_0^2/p\cdot\partial(pu_iu_j)/\partial t+\lambda^{(f)}\tau_0^2/p\cdot u_iu_j\,\partial p/\partial t.$$

γ) Finally we introduce the third order moments, and put

$$\mu_{ijk}=1/\tau_0\cdot\int(x_i'-x_i)(x_j'-x_j)(x_k'-x_k)P[\boldsymbol{x}',\ t+\tau_0/(\boldsymbol{X}(t)=\boldsymbol{x})]d\boldsymbol{x}$$

$$=\lambda^{(g)}\tau_0^2\,u_i\,u_j\,u_k.$$

We have now to adapt the proof worked out in (§ 3·3, b) for the diffusion equation, and expand to the third order derivatives. In Eq. (13) we get the additional term (with p instead of ρ):

$$1/6\cdot\tau_0^2\,\partial^3p/\partial t^3.$$

In Eq. (12) the integration by parts gives

for α): $\quad -\partial/\partial x_i\cdot(pu_i')$,

for β): $\quad 1/2\,\partial^2/\partial x_i\,\partial x_j\cdot(p\,k_{ij})$,

for γ): $\quad -1/6\,\partial^3/\partial x_i\,\partial x_j\,\partial x_k\cdot(p\mu_{ijk})$.

The addition of the terms

$$-1/6\cdot\tau_0^2\,\partial^3p/\partial t^3,\ -\tau_0^2\lambda^{(a)}\,\partial^3(\mu u_i)/\partial t^2\,\partial x_i,$$

$$1/2\,\lambda^{(e)}\,\tau_0^2\,\partial^3(p\,u_i\,u_j)/\partial x_i\,\partial x_j\,\partial t,$$

$$-1/6\ \lambda^{(g)}\tau_0^2\,\partial^3(p\,u_i\,u_j\,u_k)/\partial x_i\,\partial x_j\,\partial x_k$$

gives, with $\lambda^{(a)}=1/6$, $\lambda^{(e)}=1/3$, $\lambda^{(g)}=-1$, the invariant

$$1/6\cdot\tau_0^2\,\partial^3(p\,u_\mu\,u_\nu\,u_\sigma)/\partial x_\mu\,\partial x_\nu\,\partial x_\sigma$$

(as it is still admitted that space and time variables are proper variables). All other coefficients must vanish. The diffusion equation becomes

$$\partial(pu_\mu)/\partial x_\mu\cdot-c^2/2\cdot\tau_0\Box p-1/6\ \ \tau_0^2\,\partial^3(pu_\mu u_\nu u_\sigma)/\partial x_\mu\,\partial x_\nu\,\partial x_\sigma=0 \tag{29}$$

or $\partial j_\mu/\partial x_\mu=0$, with

$$j_\mu=pu_\mu-c^2\,\tau_0/2\cdot\partial p/\partial x_\mu+j_\mu', \tag{30}$$

putting

$$j_\mu'=-1/6\cdot\tau_0^2\,\partial^2(p\,u_\mu\,u_\nu\,u_\sigma)/\partial x_\nu\,\partial x_\sigma.$$

b) *Quantum potential*

We write

$$\rho=j_4/ic,\ v_i=j_i/\rho$$

and

$$v_i'=j_i'/\rho',\ \rho'=j_4'/ic.$$

In the frame where $u=0$, we have again

$$\mathcal{T} = -m_0 c^2 \sqrt{1-v^2/c^2} \cdot K. \tag{31}$$

To find $\mathcal{T}$, the most convenient way is to start from Eq. (28), with additional terms determined by the following conditions:

α) $\mathcal{T}$ must be invariant in three dimensional space;

β) In Eq. (28) we had (for $\mathcal{T}/m_0 c^2$) a term $\tau_0^2/8p^2 \cdot (\partial p/\partial t)^2$ connected with density variations, with a first order derivative; and a term $c^2 \tau_0^2/4 \cdot \Box p/p$, connected with the first order derivatives of the additional stream: $c^2 \tau_0^2/2 \cdot \partial p/\partial x_\mu$.

We lay down the same conditions, the additional stream being now j_μ'. Thus, using indeterminate coefficients, we have

$$\left.\begin{aligned}\mathcal{T} = -m_0 c^2 [1 - \tau_0^2/8p^2 \cdot (\partial p/\partial t)^2 - c^2 \tau_0^2/4 \cdot \Box\, p/p \\ + \theta\, c^2 \tau_0^3/p^3 \cdot \partial p/\partial t \cdot \sum_i (\partial p/\partial x_i)^2 + \eta \tau_0\, \partial v_i'/\partial x_i \\ + \zeta\, \tau_0/\rho \cdot \partial \rho'/\partial t + \xi \tau_0\, v_i'/\rho \cdot \partial \rho/\partial x_i].\end{aligned}\right\} \tag{32}$$

We have then to write

$$p = \rho - \tau_0/2 \cdot \partial \rho/\partial t + 0 \times \tau_0,$$

$$\partial p/\partial x_\mu = \partial \rho/\partial x_\mu - \tau_0/2 \cdot \partial^2 \rho/\partial t\, \partial x_\mu + 0 \times \tau_0,$$

$$\Box\, p = \Box\, \rho - \tau_0/2 + 0 \times \tau_0.$$

Substituting in Eq. (32), and calculating K [by Eqs. (30) and (31)), with $u=0$] from the expression

$$K = \mathcal{T} + 1/2 \cdot v^2/c^2 = \mathcal{T} + \sum_i j_i^2/2\rho^2 c^2$$

$$= \mathcal{T} + \sum_i \cdot 1/2\rho^2 c^2 \cdot [-c^2 \tau^2_0/2 \cdot \partial p/\partial x_i + j_i']^2,$$

the computation gives [see details in ((T), chap. VII)]:

$$K = 1 - c^2 \tau_0^2/2 \cdot \Box \sqrt{\rho}/\sqrt{\rho} + c^2 \tau_0^3/4 \cdot \partial (\Box \sqrt{\rho}/\sqrt{\rho})/\partial t$$

$$+ (\theta - 1/8) c^2 \tau_0^2/\rho^3 \cdot \partial \rho/\partial t \sum_i (\partial \rho/\partial x_i)^2 + (\xi - 1/2) \tau_0/\rho \cdot v_i'\, \partial \rho/\partial x_i$$

$$+ \eta \tau_0\, \partial v_i'/\partial x_i + \zeta \tau_0/\rho \cdot \partial \rho'/\partial t.$$

Taking, for the sake of covariance, $\theta = 1/8$, $\xi = 1/2$, $\eta = \zeta = 0$ we get

$$K = 1 - c^2 \tau_0^2/2 \cdot \Box \sqrt{\rho}/\sqrt{\rho} + c^2 \tau_0^2/4 \cdot \partial/\partial t \cdot \Box \sqrt{\rho}/\sqrt{\rho}\,.$$

This equation is covariant if, in an arbitrary frame (t_0 being the proper time), we write

$$(\Box \sqrt{\rho_0}/\sqrt{\rho_0})_{(t-\tau_0/2)} = (\Box \sqrt{\rho_0}/\sqrt{\rho_0})_{(t)} - \tau_0/2 \cdot D(\Box \sqrt{\rho_0}/\sqrt{\rho_0})/Dt_0$$

and we may take $D/Dt_0 = v_\mu\, \partial/\partial x_\mu$, as v_μ and u_μ have a difference of first order in τ_0.

c) *Wave equation*

Thus we have obtained a refined expression for the quantum potential, accounting for a time shift $t \to t-\tau_0/2$ in the action function. This shift is a consequence of the theory, for the action function being the basic entity, quantum entities are drawn from it with a relative time shift of $+\tau_0/2$.

To get the Klein-Gordon equation we had stated that for a stationary monochromatic wave the quantum potential vanishes. This condition gives again: $\Box\sqrt{\rho}/\sqrt{\rho}=0$; thus we obtain no equation other than Klein-Gordon equation. It is very likely that the result would be the same at higher orders: One would get only finer corrections of the local entities. But these corrections could be divergent expansions if τ_0 were not infinitesimal on the scale of the variations of the system, and both the equation and the quantum potential would then lose their meaning (see below § 6·1).

§ 5. Spin fluid, model and relativity

5·1 *The model of Dirac's fluid outside the proper frame*

We have shown in (A) that Dirac's fluid is obtained by addition of two constituents Φ and Ψ with opposite screw motions, having velocities equal to c and colinear to their spins.

Calling σ_Φ, $\boldsymbol{v}_\Phi$ and $\boldsymbol{S}_\Phi$ the density, velocity and spin density of the fluid Φ (and similarly for Ψ), and σ, $\boldsymbol{v}$ and $\boldsymbol{S}$ the same entities for the global fluid; one has the following properties:

$\boldsymbol{v}_\Phi$ and $\boldsymbol{v}_\Psi$ are respectively colinear to $\boldsymbol{S}_\Phi$ and $\boldsymbol{S}_\Psi$ ($\boldsymbol{v}_\Phi$ along $\boldsymbol{S}_\Phi$, $\boldsymbol{v}_\Psi$ contrary to $\boldsymbol{S}_\Psi$),

$$\sigma=\sigma_\Phi+\sigma_\Psi,$$

$$\sigma\boldsymbol{v}=\sigma_\Phi\boldsymbol{v}_\Phi+\sigma_\Psi\boldsymbol{v}_\Psi,$$

$$\boldsymbol{S}=\boldsymbol{S}_\Phi+\boldsymbol{S}_\Psi.$$

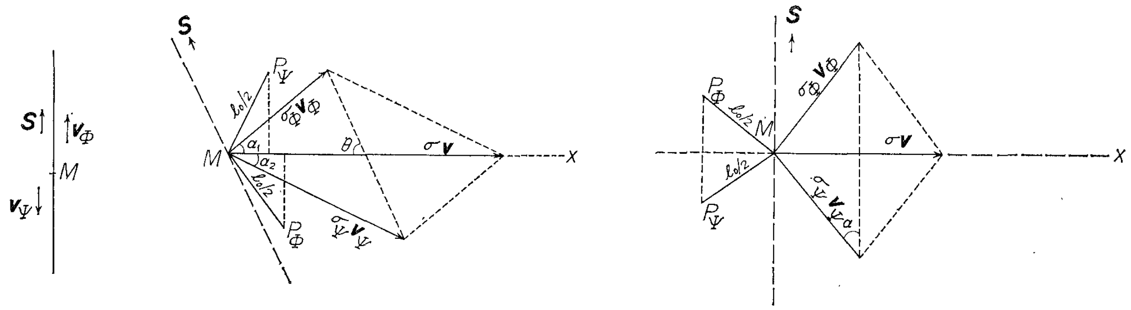

Fig. 1. Fig. 2. Fig. 3.

In the proper frame ($v=0$), $\boldsymbol{v}_\Phi$ and $\boldsymbol{v}_\Psi$ are both colinear to the spin S/σ of the global fluid, $\boldsymbol{v}_\Phi$ along S, $\boldsymbol{v}_\Psi$ contrary to S; as $|\boldsymbol{v}_\Phi|=|\boldsymbol{v}_\Psi|=c$, it results $\sigma_\Phi=\sigma_\Psi=\sigma/2$. The model is depicted by Fig. 1.

Outside the proper frame ($v\neq 0$), the model is represented by Fig. 2.

First case

We first examine the case of $\boldsymbol{S}$ and $\boldsymbol{v}$ being normal (Fig. 3). Then

$$\sigma_\Phi = \sigma_\Psi = \sigma/2.$$

In this case $\sin\alpha = \sigma v/\sigma c = v/c$. Thus the passage from Fig. 1 to Fig. 3 actually corresponds to the passage from the non-relativistic to the relativistic case.

Each component stream is related to the circular orbit (Γ_Φ or Γ_Ψ) defined in ((A), § 4, 2 and 4), having the spin ($\boldsymbol{S}_\Phi/\sigma_\Phi$ or $\boldsymbol{S}_\Psi/\sigma_\Psi$) for its axis, $l_0/2$ for its radius, the orbit being described by a fictitious point of velocity c, so that the spin is the angular moment of this point bearing the mass of the particle. The radius ($MP_\Phi = MP_\Psi = l_0/2$) of the orbit ($\Gamma_\Phi$ or Γ_Ψ) defines the dispersion at time $\tau_0/2$ of a random element (Φ or Ψ) located in M at time zero. The projection of these two orbits on a plane normal to S is the same ellipse ($\mathcal{E}$), of which the diameter of the minor axis, directed along $\boldsymbol{v}$, is

$$l = l_0 \cos\alpha = l_0 \sqrt{1 - v^2/c^2}.$$

In the case of Fig. 1, (Γ_Φ) and (Γ_Ψ) corresponded to the unique orbit (Γ) of which the diameter l_0 was the standard deviation of coordinates in the stochastic fluid after the minimal time τ_0. We may admit that ($\mathcal{E}$) has the same significance; then l represents, for the coordinate x directed along the translation velocity $\boldsymbol{v}$, the standard deviation in the global fluid after time τ_0.

Thus we find again, independently, the result obtained in § 3: *The standard deviations of coordinates after the minimal time τ_0 accord with the Lorentz transformation.*

General case

In the general case (Fig. 2), the projections of MP_Φ and MP_Ψ on $\boldsymbol{v}$ are now different; say $x_\Phi = l_0/2 \cdot \sin\alpha_1$; $x_\Psi = l_0/2 \cdot \sin\alpha_2$. But $\sigma_\Phi x_\Phi = \sigma_\Psi x_\Psi$; then we may imagine a fictitious fluid, such that the standard deviation X of the coordinate x after time τ_0 would be given by

$$\sigma/2 \cdot X = \sigma_\Phi\, x_\Phi = \sigma_\Psi\, x_\Psi.$$

The calculation of X [see (T), chapter X, p. 101] yields, with $\beta = v/c$

$$X^2 = l_0^2/4 \cdot (1-\beta^2)\,[1 - 1/\beta^2 \cdot (\sigma_\Phi - \sigma_\Psi)^2/\sigma^2],$$

or

$$X^2 = l_0^2/4 \cdot (1-\beta^2) \sin^2\theta / (1 - \beta^2 \cos^2\theta),$$

(θ angle of $\boldsymbol{v}$ with $\boldsymbol{S}$).

The difference with the Lorentz transformation is of the second order in $\pi/2 - \theta$, or in $\sigma_\Phi - \sigma_\Psi$. If these entities are small, X can be defined as a conventional free path, the variation of which follows the Lorentz transformation. This law having been obtained in § 3, it leads us to suppose that the normal state of the fluid is the state in which the densities σ_Ψ and σ_Ψ are equal or nearly equal, and the velocity normal or nearly normal to the spin.

5·2 *Relativity and local motions*

We have again arrived at relativity from the model; but we can go a step further. A concrete interpretation of Takabayasi's representation (known to be equivalent to Dirac's equation) has been given in ((T), Appendix I). Now, in working out this interpretation, we have taken no account of relativistic rules.

For instance:

α) The relation ((A), 35) for the angular momentum L

$$\boldsymbol{L}=m_0\, l_0^2/4\cdot\mathbf{rot}\ \boldsymbol{v}$$

which does not violate the relativistic formalism for the global fluid (as $v=0$ in M, the calculation being carried out in the proper system), has been applied to each of the component fluids ($\boldsymbol{L}_\Phi=m_0\, l_0^2/4\cdot\mathbf{rot}\ \boldsymbol{v}_\Phi$, etc.), although $|\boldsymbol{v}_\Phi|$ and $|\boldsymbol{v}_\Psi|$ are equal to c; and this has given the correct term $\mathbf{rot}\ \boldsymbol{S}$ for the momentum.

β) The angular momentum of the fictitious point P moving on the orbit (Γ) with a velocity c has been computed in a Newtonian manner, as $m_0\, l_0/2\cdot c=\hbar/2$, which has given the interpretation of the spin as the value of this momentum.

γ) The tensions have been calculated after the Newtonian formula $t_{ij}=m_0\,\sigma_0 \times v_i v_j$, and the energy as the work done by them on the unit of length, which for the primary motions $\boldsymbol{v}_\Phi$ and $\boldsymbol{v}_\Psi$ has given the proper energy density

$$m_0[\sigma_\Phi|\boldsymbol{v}_\Phi|^2+\sigma_\Psi|\boldsymbol{v}_\Psi|^2]=m_0c^2(\sigma_\Phi+\sigma_\Psi)=m_0\,c^2\sigma.$$

Thus the following conclusions appear:

a) The laws of relativity are included in the model, and concern the global translation motion (velocity $\boldsymbol{v}$) to which alone a proper frame of reference can be related.

b) *The motions inside the proper frame (local microscopic motions) are not submitted to relativistic laws.* In other words, the times and lengths involved in these motions are always the proper times and lengths.

These two conclusions are identical to those drawn in § 3 from a critical survey of stochastic diffusion.

Let us remark that we already adopted implicitly this view in (A), when making out the uncertainty relations (§ 3·3, a): for the dispersion on p_x, (actions inside the proper frame with velocity c) we had taken simply $m_0\, c$. As $\sqrt{\overline{(\Delta x)^2}}$ is the invariant quadratic mean value,*) the relation is invariant.

§ 6. Comments and suggestions

6·1 *The definition of the relativistic case*

*) Because all measurements are exerted from the fixed frame (see § 6·2b).

a) *Principle*

The relativistic case is usually characterized by the magnitude of v/c. But we have obtained a more general definition (see § 3·1, *c*). We enter the field of relativity as soon as τ_0—and consequently $l_0=\sqrt{k}\,\tau_0$—are not infinitesimal at the scale of the times and lengths involved; or, in other words, as soon as the physical entities have variations sensible on τ_0 or l_0.

The condition $u\sim c$ is a particular application of this general rule to the coordinates. But relativity also applies if other local entities, for instance the density ρ, have variations sensible on τ_0 or l_0. This has already appeared in Eq. (10′): if $\nabla\rho/\rho$ has variations sensible on l_0, it gives to $\boldsymbol{v}-\boldsymbol{u}$ variations of order c, so that at least one of the two velocities is relativistic.

A confirmation appears in the interpretation of Takabayasi's relations, which is valid only up to the second order in τ_0 (see (A), § 4·4, *c*).

b) *Consequences*

These views explain two facts that have seemed rather difficult to understand; viz. uncertainty of localization arising with relativity, and connection of spin with relativity.

α) *Uncertainty of localization*

As Newton and Wigner have shown,[4),2)] the positional operator $\boldsymbol{X}=-i\hbar\partial/\partial\boldsymbol{p}$ is hermitian in Schrödinger's equation, and admits $\boldsymbol{x}$ for eigenvalue. In the Klein-Gordon and Dirac equations, $\boldsymbol{X}$ is not hermitian; the eigenfunctions for coordinates are no longer δ functions; the correct eigenfunctions have a field of extension of order l_0; why has this indeterminacy arisen with relativity?

The answer is obvious; the non-relativistic case is the approximation in which l_0 is infinitesimal.

β) *Spin and relativity*

Why is it necessary, in order to obtain a correct wave equation, to introduce simultaneously spin and relativity? For instance, why does Pauli's equation fail to give an exact forecast for hydrogen atom spectroscopy, although in this atom no motion has a relativistic velocity?

Here again the answer is simple. The elementary vibration described in ((A), § 4·2, *b*) involves spin and density variations sensible on τ_0 and l_0. That is why Dirac's equation is consistent, taking into account both spin and relativity. The Klein-Gordon equation is not consistent, for a relativistic equation should contain the spin vibration.*) As to Pauli's equation, which is not relativistic, it cannot provide a complete picture of the elementary local motions;**) it only

*) Here we study particles with spin 1/2, which, according to the theory of fusion, are the basic ones. In that view, an equation "without spin" does not mean that the particle is spinless, but that its spin variations are neglected.

**) This can be verified in following the passage from Dirac's to Pauli's equation.[5)] In the momentum vector $\boldsymbol{G}$ [see (T), chap. IX, Eq. (15)], the term $1/c\cdot\boldsymbol{S}\partial A/\partial t$ is neglected, which is not legitimate if A has rapid temporal variations, as it has in the elementary vibration.

gives a global representation of local structure on a higher scale (see (A), § 4, 3). The only consistent equations are Schrodinger's equation (spin and relativity both neglected) and Dirac's equation (Spin and relativity both admitted).

6·2 *Character and limits of relativistic formalism*

a) First comments

By two independent methods we have obtained the same result; Eqs. (6), (7) and (15), in which the times and lengths involved are at every point the proper time and length (although related to a unique frame in the whole space) as well as the model of Dirac's equation, show that *at the basis relativity does not apply*. Relativity appears with the average translation motion, that allows us to define a physical frame of reference, the "proper" frame; local microscopic motions inside that frame are not submitted to relativistic laws; no physical frame can be assigned to them.

This leads us to think that the classical interpretation of special relativity may be somewhat conventional;*) for instance imagining an observer with rulers and clocks riding on a microscopic element might have but a formal significance. We do not even know if the motion in close contact of two Galilean macroscopic systems with relative velocity approaching c is physically possible:**) relativistic effects concerning galaxies are measured on photons emitted by them.***) In fact, effects of special relativity have been always observed on microscopic elements confronting a macroscopic material system.

Let us now show that the whole formalism of special relativity can be grounded on a concrete basis, admitting only principles sure to be physically significant.

b) Concrete interpretation of the formalism

At the basis is a stochastic fluid (F) moving before a fixed system (Σ); all measurements, all physical actions are exerted from (Σ); the "proper" frame (Σ_0) will only be alluded to in order to prove the practical equivalence of the proposed views with the classical ones.

Instead of comparing measurements in (Σ) and (Σ_0), we compare measurements in two states of (F):

State I: (F) motionless in (Σ),****)

State II: (F) moving before (Σ).

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*) This is not easily recognized because Einstein's views on space and time seem to give a physical basis to the formalism. But they are purely *critical*, and the formalism is directly laid down, without any basis other than the experimental invariance of c.

Another difficulty is that special relativity is commonly considered as a part of general relativity; but the formal unity may cover facts which are physically different (see below remarks on general relativity).

**) See below remarks on artificial satellites.

***) See below for Doppler effect.

****) Or far enough from (Σ) (see below).

In the first state lengths concerning (F) are unperturbed (and also times, as will be shown); in the second state they are perturbed by the motion, in a way that will be investigated later.

Obviously, as far as the measurements are concerned, this view is equivalent to the classical one: Measurements done from (Σ_0) would yield the results of state I, the entities to be measured and the measuring devices being perturbed in the same ratio.

α) Propagation of actions with velocity c. Invariance of this velocity

The maximal propagation velocity of actions in the fluid is the velocity c of free travel (transmission along an axis by successive collisions, the elements concerned having each time their velocities directed along this axis). We call *elementary action**) such an action propagating with velocity c.

Now the propagation of an elementary action exerted from (Σ) on the fluid in M at time t depends on its microscopic local structure at time $t+\tau_0/2$, such as it appears to an observer *in* (Σ). This structure is figured by the quadratic mean values of the distances to (Σ) at time $t+\tau_0/2$. These values being invariant, the local structure is also invariant, and so is the propagation velocity. Thus, the invariance of the maximal wave velocity is another expression of the invariance of the quadratic mean values of the coordinates up to the minimal time τ_0.

β) Measurements of times

We define clocks in (Σ) by the vibration of elementary waves propagating in (Σ) with velocity c.

β_1) Period of waves

If (F) is unperturbed in state I [in which case we call it (F_0)], the wave (W_0) can be identified with the propagation of an elementary periodic action in (F_0): period T_0, dephasing between two elements E_1 and E_2 at distance x_0: $\varphi_0=2\pi\, x_0/T_0$.

If (F) is in state II [moving before (Σ)], we consider a wave (W) of period T, of velocity c in (Σ), [as caused by a physical action exerted from (Σ)], and clocks in (Σ), of which the dephasing between two points M and N at the distance x is $\varphi=2\pi x/cT$.

Taking for E_1 and E_2 the elements of the fluid being at the same time respectively in M and N, we consider that (W) and (W_0) correspond to each other if $\varphi=\varphi_0$. In other words the phase is the intrinsic character defining the wave.

This gives $2\pi x/T=2\pi x_0/T_0$, whence

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*) It might be in fact the basic mircoscopic action, actions propagating with lower velocities being macroscopic averages, such as the average velocity $\boldsymbol{u}$ defined by Eq. (1) or (6).

$$T/T_0 = x/x_0 = \sqrt{1-v^2/c^2}. \tag{33}$$

β_2) *Simultaneousness*

The setting up of the clocks (C) is done by the emission of elementary waves.

Let (E_1) and (E_2) be the elements of (F) passing respectively at time zero before two clocks M and N of (Σ); $(MN=x)$. At this time we launch an elementary wave from M to N; when it reachs N, (E_2) is in N' $(NN'=v\cdot x/c)$, another element (E_3) is in N, having the phase (say zero) that (E_1) had at time zero; (E_1) has now the phase $\varphi=2\pi x/cT$. We choose as simultaneous events in (F) the phase zero for (E_3) and the phase φ for (E_1).

Let us now launch in (F_0) an elementary wave from (E_1) to (E_2); let E_1, E_2, E_3 be the positions of (E_1), (E_2), (E_3); the phase of the vibration of (E_1) when (E_2) has the phase zero is $\varphi_0=\varphi$; the instant at which the phase of (E_1) is φ (wave in E_2) is then separated from the associated instant (wave in E_3) by a delay:

$$t_0 = -E_2E_3/c = -(1/\sqrt{1-v^2/c^2})\cdot NN'/c = -vx/c^2\sqrt{1-v^2/c^2}. \tag{34}$$

β_3) *Lorentz formulae for time*

Let $(\mathcal{E}_1)$ and $(\mathcal{E}_2)$ be two events occurring at the points (A_1) and (A_2) of (Σ) at times zero and t respectively, and concerning elements (E_1) and (E_2) of (F). Let (E_1') be the element arriving in (A_1) at time t; we associate with (E_1') and (E_2) the two events $(\mathcal{E}_1')$ and $(\mathcal{E}_2)$, $(\mathcal{E}_1')$ being the phase of (E_1'). These events being simultaneous in (Σ), are separated in (F_0) by a time $t_0^{(2)}$ given by Eq. (34). $(\mathcal{E}_1)$ and $(\mathcal{E}_1')$ are measured on the clock in (A_1); the corresponding time $t_0^{(1)}$ in (F_0) is given by Eq. (33), i.e.

$$t_0^{(1)} = t/\sqrt{1-v^2/c^2}.$$

The addition gives

$$t_0 = t_0^{(1)} + t_0^{(2)} = (t - vx/c^2)/\sqrt{1-v^2/c^2}. \tag{35}$$

γ) *Addition of velocities*

Let (S_0) be a fluid moving before a fixed system (S) with the velocity v along $0x$. A mass point M has along $0x$ a velocity V relative to (S); let v_0 be the velocity of M relative to (S_0).

In time dt [measured in (S)], the motion of M is dx. Let (E) and (E') be the elements of (S_0) that coincide respectively with M at either end of the interval dt. The distance from (E) to (E') is $(V-v)dt$.

Now let (S_0) be unperturbed; The Lorentz contraction being cancelled, (E) and (E') are now at a distance $dx_0=(V-v)/\sqrt{1-v^2/c^2}\cdot dt$. The time dt_0 measured in (S_0) unperturbed, between the coincidences of M with (E) and (E'), is given in terms of dt and dx by Eq. (35), i.e.

$$dt_0=(dt-v\cdot dx/c^2)/\sqrt{1-v^2/c^2}=dt/\sqrt{1-v^2/c^2}\cdot(1-v\,V/c^2),$$

whence

$$v_0=dx_0/dt_0=(V-v)/(1-vV/c^2).$$

δ) *Kinetic entities*

The mass in motion m having received a stochastic definition, we substitute m for m_0 to get the relativistic momentum $\boldsymbol{G}=m\boldsymbol{v}$; then we consider the force as defined by

$$\boldsymbol{F}=md\boldsymbol{G}/dt=m_0/\sqrt{1-v^2/c^2}\cdot d\cdot(m_0\,\boldsymbol{v}/\sqrt{1-v^2/c^2})/dt.$$

We similarly obtain the canonical tensor; for instance, for the free fluid, the quantity of momentum passing per unit time in direction ν through an unit surface normal to the axis μ is

$$T_{\mu\nu}=\rho v_\mu\,G_\nu=\rho_0/m_0\cdot G_\mu\,G_\nu.$$

The symmetry of the tensor results from the mass-energy equivalence.

ε) *Proper angular momentum and spin*

The classical calculation for the relativistic variance of proper angular momentum applies[6] to the model of a rotator M turning with velocity $\boldsymbol{v}_0$ round a point G. This point is the origin of a reference frame (Σ_0) moving before (Σ) with a velocity $\boldsymbol{v}$ along $0x$; the velocity of M relative to (Σ) is $\boldsymbol{V}$. One takes the moments of the momenta $m_0\,\boldsymbol{V}/\sqrt{1-V^2/c^2}$ and $m_0\,\boldsymbol{v}_0/\sqrt{1-v_0^2/c^2}$. When adding the velocities, the square roots forming these two denominators vanish, and there remains plainly

$$S_x=S_x^0,\ S_y=S_y^0\sqrt{1-v^2/c^2},\ S_z=S_z^0\sqrt{1-v^2/c^2}. \tag{36}$$

This calculation means that one has substituted the mass in motion of M for the rest mass, or its contracted free path l for l_0. But the rotation of M is a motion inside the proper frame (Σ_0); one must take simply in (Σ_0)

$$\boldsymbol{S}_0=m_0\,\boldsymbol{GM}\Lambda\boldsymbol{v}_0. \tag{37}$$

The expression of the spin in (Σ) is deduced from Eq. (37), taking into account the Lorentz contraction in $\boldsymbol{GM}$ and $\boldsymbol{v}_0$; whence directly the relations (36), without having introduced the redundant factors $\sqrt{1-v_0^2/c^2}$ and $\sqrt{1-V^2/c^2}$.

The model of Fig. 3 gives also the result: When passing from Fig. 1 to Fig. 3, the spin component normal to $\boldsymbol{v}$ is multiplied by $\cos\alpha=\sqrt{1-v^2/c^2}$. It may be verified that the definition of the spin as an orbital angular momentum remains valid, the elliptic orbit being substituted for the circular orbit (Γ).

η) *Doppler effect*

Let (S) be the Earth, (S') a star of relative velocity $\boldsymbol{v}$, (Z) a photon emitted by (S') towards (S) in a direction of which the angle with $\boldsymbol{v}$, in (S'), is θ'. Let us recall that the photon frequency has no concrete significance unless its velocity is not strictly c; i.e. its rest mass m_0 not strictly zero [see (A) § 4, 4, b_3]. Then, let w and w' be the velocities of (Z) relatives to (S) and (S'). The action of (S') on (Z) gives to the photon a frequency ν' connected to its proper frequency ν_0 by

$$\nu' = \nu_0/\sqrt{1-w'^2/c^2}.$$

The action of (S) on (Z) gives similarly a frequency

$$\nu = \nu_0/\sqrt{1-w^2/c^2},$$

whence

$$\nu = \nu'\sqrt{1-w'^2/c^2}/\sqrt{1-w^2/c^2} = (\nu'/\sqrt{1-v^2/c^2})\,(1+vw'/c^2\cdot\cos\theta')$$

$$\sim (\nu'/\sqrt{1-v^2/c^2})\cdot(1+v/c\cdot\cos\theta').$$

c) *Additional remarks*

α) *On the consequences of the views proposed*

α_1) They allow us to understand why the velocity of light is included in matter equations; the privileged role of light in Einstein's arguments on space and time had never be clearly justified.

α_2) They solve the aforementioned difficulties and contradictions.

The fact that the diffusion equation is not covariant unless it is read with the proper entities in an unique frame in the whole space, is now easily explained. The proof of this equation deals with an unperturbed fluid: times and lengths involved are the proper entities available for the free fluid. The equation being thus written, (i.e. in the view of the formalism, written in the proper system), carrying out on it a Lorentz transformation means that we now take into account the effects on lengths (and consequently on times, as shown above) of the motion before a macroscopic system.

As for the variance of the length and time constants, it gives rise to no difficulty. The longitudinal standard deviation l, the time $\tau = l/c$ of the longitudinal free path, are average entities of an extended random fluid, and submitted to the laws of relativity: l transforms like a length*) and τ like the period of a wave. The intrinsic conceptions (for instance the "radius" of a particle) concern the unperturbed fluid, and apply to the proper entities. In particular, if the radius of the individual stochastic elements can ever be defined, it will

*) In as much as it is a standard deviation; but in the elementary actions exerted on the fluid the invariant quadratic mean value is involved.

afford the universal and invariant basic length, which has been formally introduced in several recent works.

β) *The physical nature of the Lorentz contraction*

We have considered the Lorentz contraction as a perturbation exerted on the moving fluid; what can be its nature?

If, as suggested, relativistic effects arise by the motion of a stochastic fluid before a macroscopic material system (Σ), it is natural to think that the perturbating effect should consist in a physical action from (Σ). In favour of this assumption we make the following remarks:

β_1) In the model of Dirac's fluid, the two component streams are colinear at rest; motion produces an angle $\alpha(\sin\alpha = v/c)$; thus the model has been distorted, and the distorsion becomes sensible with v/c.

β_2) Relativity has embodied the kinetic energy in the mass, and the mass itself is here identified with stochastic agitation energy; thus, when the fluid is subjected to a motion, the increase in its energy has two aspects: (1) as a *stochastic effect*, it results from the decrease of the free path, due to the Lorentz contraction; (2) as a *kinetic effect*, it results from an external action. The equivalence of these two aspects again suggests the consideration of the Lorentz contraction as a physical action, which besides appears natural once we have admitted the solidarity of macroscopic lengths with the standard deviation in the fluid.

β_3) By this assumption several effects of special relativity become concretely accountable (necessity of a torque to maintain the equilibrium of a cranked lever in translation, etc.).

β_4) A possible objection could be raised in imagining two systems (S) and (Σ) adding their perturbations on a third one. This case would be difficult to conceive; but it does not seem to be concretely possible. If the system (Σ) satisfies both conditions having a relativistic velocity and remaining near enough to (S) so as to undergo its perturbating action, its dimensions should probably be relatively small in comparison to those of (S); then, if another material body is launched from (Σ), when its motion becomes uniform the action of (Σ) will be negligible before that of (S).

Let us point out that the proof given above for the addition of velocities does not involve such cumulative actions; for it does not take into consideration the reference frame connected to the point M in motion; only the action of (S) on (S_0) is accounted for.

β_5) The idea that all relativistic effects are of purely geometrical nature has lost much of its weight since the existence of gravitons has been assumed.

γ) *Connection with general relativity*

If we send an artificial satellite far enough from the Earth, should we expect a decrease in the effects of special relativity?

In practice we also have to take into account effects of general relativity, which similarly consist of a reduction. Are these two effects identical or different? Experience should decide. In the first eventuality the views proposed would bring no limitation to relativity in the macroscopic field; the local action of a reference frame in relative motion would be embodied in the general action of all the masses of the universe.

§ 7. Conclusion

Is it surprising that from the special problem of diffusion two general consequences should arise for relativity: its limitation and its concrete interpretation? We do not think so, as we have suggested in (A) that stochastic diffusion should be considered as a basis for matter; it is then quite natural that it should also afford a concrete basis to relativity. It is commonly considered that, in comparison to microphysics, relativity is macroscopic; but it may seem to be so because it has not yet been investigated at the lower level. In fact, the difficulty of dealing with extended particles within the frame of the relativistic formalism (and the other difficulties mentioned above) should lead to this new investigation.

Let us finally remark that we have changed nothing in relativity, but vague interpretations which are loose transpositions of the formalism: for instance, the common affirmation that "each observer carries his space and time". Some apparent difficulties, such as the variance of the radius of particles, may simply arise from the fact that the relativistic formalism is taken as a physical basis,[*] whilst it is nothing more than a formalism.

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[*] See footnote in (§ 6·1, *a*).