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Stochastic reaction-diffusion systems with multiplicative noise and non-Lipschitz reaction term

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Abstract. We study existence and uniqueness of a *mild* solution in the space of continuous functions and existence of an invariant measure for a class of reaction-diffusion systems on bounded domains of $\mathbb{R}^d$, perturbed by a multiplicative noise. The reaction term is assumed to have polynomial growth and to be locally Lipschitz-continuous and monotone. The noise is white in space and time if $d = 1$ and coloured in space if $d > 1$; in any case the covariance operator is never assumed to be Hilbert-Schmidt. The multiplication term in front of the noise is assumed to be Lipschitz-continuous and no restrictions are given either on its linear growth or on its degenaracy. Our results apply, in particular, to systems of stochastic Ginzburg-Landau equations with multiplicative noise.

1. Introduction

In this paper we are concerned with the study of existence and uniqueness of solutions and existence of an invariant measure for the following class of reaction-diffusion systems perturbed by a multiplicative noise

$$\begin{cases} \dfrac{\partial u_i}{\partial t}(t,\xi) = \mathcal{A}_i\, u_i(t,\xi) + f_i(t,\xi,u_1(t,\xi),\dots,u_r(t,\xi)) \\[2ex] \qquad\qquad + \displaystyle\sum_{j=1}^r g_{ij}(t,\xi,u_1(t,\xi),\dots,u_r(t,\xi))\,Q_j \frac{\partial w_j}{\partial t}(t,\xi),\ t \geq s,\ \xi \in \overline{\mathcal{O}}, \\[2ex] u_i(s,\xi) = x_i(\xi), \quad \xi \in \overline{\mathcal{O}}, \qquad \mathcal{B}_i\, u_i(t,\xi) = 0, \quad t \geq s, \quad \xi \in \partial\mathcal{O}. \end{cases} \tag{1.1}$$

Here $\mathcal{O}$ is a bounded open set of $\mathbb{R}^d$, with $d \geq 1$, having regular boundary. For each $i = 1,\dots,r$, $\mathcal{A}_i$ are second order uniformly elliptic operators with coefficients of class C^1, $\mathcal{B}_i$ are operators acting on the boundary of $\mathcal{O}$, Q_i are bounded linear operators from $L^2(\mathcal{O})$ into itself, which are not assumed to be Hilbert-Schmidt and in the case of space dimension $d = 1$ can be taken equal to identity, and finally

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$\partial w_i(t)/\partial t$ are independent space-time white noises. This means that in dimension $d = 1$ we can consider systems perturbed by white noise and in dimension $d > 1$ we have clearly to colour the noise but we do not assume any trace-class property for its covariance.

Concerning the non linear terms, we assume that

$$f : [0, \infty) \times \overline{\mathcal{O}} \times \mathbb{R}^r \to \mathbb{R}^r, \quad g = [g_{ij}] : [0, \infty) \times \overline{\mathcal{O}} \times \mathbb{R}^r \to \mathcal{L}(\mathbb{R}^r)$$

are measurable and for almost all $t \geq 0$ the mappings $f(t, \cdot, \cdot)$ and $g(t, \cdot, \cdot)$ are continuous on $\overline{\mathcal{O}} \times \mathbb{R}^r$. For almost all $t \geq 0$, the mapping $g(t, \xi, \cdot) : \mathbb{R}^r \to \mathcal{L}(\mathbb{R}^r)$ is Lipschitz-continuous, uniformly with respect to $\xi \in \overline{\mathcal{O}}$, without any restriction on its linear growth or its degeneracy (this means that we can take for example $g_{ij}(t, \xi, u) = \lambda_{ij} u_j$, with $\lambda_{ij} \in \mathbb{R}$ or more general). Moreover, for almost all $t \geq 0$, the mapping $f(t, \xi, \cdot) : \mathbb{R}^r \to \mathbb{R}^r$ has polynomial growth, is locally Lipschitz-continuous and satisfies suitable dissipativity conditions, uniformly with respect to $\xi \in \overline{\mathcal{O}}$. The example that we have in mind is the case of $f_i(t, \xi, \cdot)$'s which are odd-degree polynomials in the space variable having negative leading coefficients.

In this paper we first show that for any initial datum $x \in E := C(\overline{\mathcal{O}}; \mathbb{R}^r)$ problem (1.1) admits a unique *mild* solution u_s^x in $L^p(\Omega; C((s, T]; E) \cap L^\infty(s, T; E))$, for any $p \geq 1$ and $T \geq s$, which depends continuously on the initial datum $x \in E$. Next, in the case of coefficients f and g not depending on time, we introduce the *transition semigroup* P_t associated with system (1.1), by setting for any bounded Borel measurable function $\varphi : E \to \mathbb{R}$

$$P_t\varphi(x) = \mathbb{E}\,\varphi(u^x(t)), \quad t \geq 0, \quad x \in E,$$

where $u^x(t)$ is the solution of (1.1) starting from x at time $s = 0$. By investigating the asymptotic behaviour of the norm of $u^x(t)$ in some spaces of Hölder-continuous functions, we show that for any $a > 0$ the family of probabilities $P_t(x, \cdot)$, $t \geq a$, is tight and then P_t has an invariant measure on $(E, \mathcal{B}(E))$.

Stochastic reaction-diffusion systems with non-Lipschitz reaction terms having polynomial growth and fulfilling some monotonicity assumptions have been studied by several authors, both in the case of additive and multiplicative noise (for the multiplicative case see [1], [11], [13], [14], [15], [3], [16] and the recent [2]; see also [8], [5] and [6] for several applications to Kolmogorov equations, invariant measures and stochastic optimal control problems in the case of additive noise).

In [1], [11], [13], [14], [15] comparison arguments are mainly used, providing the construction of the solution as limit of solutions of auxiliary problems with known bounds. In these papers, with the only exception of [15] where general domains of $\mathbb{R}^d$ are considered, even unbounded, the case of domain $\mathcal{O} = (0, 1)$ and white noise are studied. Instead here, as in [3] and [16], we use semigroup techniques and this enables us to treat also the case of systems of equations for which the use of comparison is much more complicated, as the maximum principle does not hold in general.

It is evident that different approaches to the study of SPDE's involve different settings of hypotheses that are not always easy to compare. In this paper we try to overcome the limits arising from both techniques, which are mainly due to the fact that a multiplicative noise is considered and *pathwise* arguments cannot be used. Thus, concerning the multiplication term in front of the noise, on one hand we do not assume it to be smooth and strictly non degenerate as e.g. in [1], where Malliavin calculus is used in order to get rid of only measurable f, and on the other hand we allow it to have linear growth, unlike in [3] and [16] where a more abstract setting is considered, which in the concrete cases seems to apply only to reaction-diffusion problems in bounded domains of $\mathbb{R}$ with bounded diffusions.

Moreover, the fact that we are dealing with domains of any dimension $d \geq 1$ and looking for solutions in spaces of continuous functions, without having any Itô's formula, makes our work much more complicated from a technical point of view and forces us to work directly on heat kernels and use the factorization formula pointwise (and not in the classical L^2 setting).

In the last section of the paper, the good knowledge of the equation permits us to give a good description of the asymptotic behaviour of solutions in spaces of Hölder continuous functions and to show that the family of probability measures $\{\mathcal{L}(u_0^x(t)),\ t \geq a\}$ is tight in $(E, \mathcal{B}(E))$. Thus, thanks to the classical Krylov-Bogoliubov theorem we get the existence of an invariant measure for system (1.1). We would like to stress that here, as above, the main difficulty is providing good a-priori estimates, which are hard to obtain in the case of non constant diffusion term in front of the noise, as we cannot proceed pathwise and apply directly deterministic techniques.

Finally, we would like to recall that the results proved in this article are applied in [7] to the study of large deviations estimates in the space of paths for the solution of system (1.1).

2. Preliminaries

If X and Y are two Banach spaces, we denote by $\mathcal{L}(X, Y)$ the Banach space of all bounded linear operators $T : X \to Y$, endowed with the *sup-norm*. When $X = Y$ we write $\mathcal{L}(X)$ instead of $\mathcal{L}(X, X)$.

Let H be a separable Hilbert space and let T be a compact linear operator in H. The non negative self-adjoint operator $T^\star T$ is also compact, so that the operator $A = \sqrt{T^\star T}$ is non negative, self-adjoint and compact. The eigenvalues of A are called the *characteristic numbers* of T and are denoted by $\mu_k(T)$, $k \geq 1$.

Definition 2.1. *For any $p \in (0, +\infty)$ we denote by $\mathcal{C}_p(H)$ the set of all compact operators $T \in \mathcal{L}(H)$ such that*

$$\|T\|_p^p := \sum_{k=1}^{\infty} \mu_k(T)^p < \infty.$$

$\mathcal{C}_p(H)$ is a Banach space, endowed with the norm $\|\cdot\|_p$.

If $p = +\infty$ we define $\mathcal{C}_\infty(H) := \mathcal{L}(H)$ and we have

$$\|T\|_\infty := \|T\| = \sup_{k\in\mathbb{N}} \mu_k(T).$$

As shown for example in [12, Lemma XI.9.9], the spaces $\mathcal{C}_p(H)$ fulfill the following properties:

1. if $p < q$, then $\mathcal{C}_p(H) \subset \mathcal{C}_q(H)$ and the mapping $p \mapsto \|T\|_p$ is decreasing;
2. if S is in $\mathcal{C}_p(H)$ and T is in $\mathcal{C}_q(H)$, then ST is in $\mathcal{C}_r(H)$, with $1/r = 1/p+1/q$, and
$$\|ST\|_r \leq 2^{\frac{1}{r}}\, \|S\|_p \|T\|_q, \qquad 0 < r < \infty; \tag{2.1}$$
3. if T is in $\mathcal{C}_p(H)$ and A is bounded, then AT and TA are in $\mathcal{C}_p(H)$ and
$$\|AT\|_p \leq \|A\|\,\|T\|_p, \qquad \|TA\|_p \leq \|A\|\,\|T\|_p;$$
4. if $\{e_i\}$ is a complete orthonormal system in H, then for any $T \in \mathcal{C}_2(H)$
$$\|T\|_2^2 = \sum_{i=1}^{\infty} |Te_i|_H^2.$$

In [12, Lemma XI.9.32] it is also proved that if $\{e_i\}$ is a complete orthonormal set in H and $2 \leq p < \infty$, then for any T in $\mathcal{L}(H)$

$$\sum_{i=1}^{\infty} |Te_i|_H^p < \infty \Longrightarrow \|T\|_p \leq c_p \left(\sum_{i=1}^{\infty} |Te_i|_H^p\right)^{\frac{1}{p}}, \tag{2.2}$$

for some positive constant c_p. In particular $T \in \mathcal{C}_p(H)$.

Next, let $\mathcal{O}$ be a bounded open subset of $\mathbb{R}^d$, having a regular boundary. Throughout the paper we denote by H the separable Hilbert space $L^2(\mathcal{O};\mathbb{R}^r)$, with $r \geq 1$, endowed with the scalar product

$$\langle x, y\rangle_H := \int_{\mathcal{O}} \langle x(\xi), y(\xi)\rangle_{\mathbb{R}^r}\, d\xi = \sum_{i=1}^{r} \int_{\mathcal{O}} x_i(\xi) y_i(\xi)\, d\xi = \sum_{i=1}^{r} \langle x_i, y_i\rangle_{L^2(\mathcal{O})}$$

and the corresponding norm $|\cdot|_H$. For any $p \geq 1$, $p \neq 2$, the usual norm in $L^p(\mathcal{O};\mathbb{R}^r)$ is denoted by $|\cdot|_p$. If $\epsilon > 0$, we denote by $|\cdot|_{\epsilon,p}$ the norm in $W^{\epsilon,p}(\mathcal{O};\mathbb{R}^r)$

$$|x|_{\epsilon,p} := |x|_p + \sum_{i=1}^{r} \int_{\mathcal{O}\times\mathcal{O}} \frac{|x_i(\xi) - x_i(\eta)|^p}{|\xi - \eta|^{\epsilon p + d}}\, d\xi\, d\eta.$$

We denote by E the Banach space $C(\overline{\mathcal{O}};\mathbb{R}^r)$, endowed with the sup-norm

$$|x|_E := \left(\sum_{i=1}^{r} \sup_{\xi\in\overline{\mathcal{O}}} |x_i(\xi)|^2\right)^{\frac{1}{2}}$$

and the duality $\langle \cdot,\cdot\rangle_E$. Finally, for any $\theta \in (0,1)$ we denote by $C^\theta(\overline{\mathcal{O}};\mathbb{R}^r)$ the subspace of θ-Hölder continuous functions, endowed with the norm

$$|x|_{C^\theta(\overline{\mathcal{O}};\mathbb{R}^r)} = |x|_E + [x]_\theta := |x|_E + \sup_{\substack{\xi,\eta\in\overline{\mathcal{O}}\\ \xi\neq\eta}} \frac{|x(\xi) - x(\eta)|}{|\xi - \eta|^\theta} < \infty.$$

Now, if we fix any $x \in E$ there exist $\xi_1, \ldots, \xi_r \in \overline{\mathcal{O}}$ such that $|x_i(\xi_i)| = |x_i|_{C(\overline{\mathcal{O}})}$, for all $i = 1, \ldots, r$. Then, if δ is any element of $E^\star$ having norm equal 1, the element $\delta_x \in E^\star$ defined for any $y \in E$ by

$$\langle \delta_x, y \rangle_E := \begin{cases} \dfrac{1}{|x|_E} \displaystyle\sum_{i=1}^{r} x_i(\xi_i) y_i(\xi_i), & \text{if } x \neq 0 \\ \\ \langle \delta, y \rangle_E, & \text{if } x = 0, \end{cases} \tag{2.3}$$

belongs to $\partial\, |x|_E := \{ h^\star \in E^\star;\ |h^\star|_{E^\star} = 1,\ \langle h, h^\star \rangle_E = |h|_E \}$ (see e.g. [6, Appendix A] for all definitions and details).

2.1. The operator A

For any $i = 1, \ldots, r$ we define

$$\mathcal{A}_i(\xi, D) := \sum_{h,k=1}^{d} a^i_{hk}(\xi) \frac{\partial^2}{\partial \xi_h \partial \xi_k} + \sum_{h=1}^{d} b^i_h(\xi) \frac{\partial}{\partial \xi_h}, \qquad \xi \in \overline{\mathcal{O}}.$$

The coefficients a^i_{hk} are taken of class $C^1(\overline{\mathcal{O}})$ and for any $\xi \in \overline{\mathcal{O}}$ the matrix $[a^i_{hk}(\xi)]$ is non negative and symmetric and the uniform ellipticity condition

$$\inf_{\xi \in \overline{\mathcal{O}}} \sum_{h,k=1}^{d} a^i_{hk}(\xi) \lambda_h \lambda_k \geq \nu\, |\lambda|^2, \qquad \lambda \in \mathbb{R}^d,$$

is fulfilled, for some $\nu > 0$. The coefficients b^i_h are continuous. Moreover, for any $i = 1, \ldots, r$ we define

$$\mathcal{B}_i(\xi, D) := I, \qquad \text{or} \qquad \mathcal{B}_i(\xi, D) := \sum_{h,k=1}^{d} a^i_{hk}(\xi) \nu_h(\xi) \frac{\partial}{\partial \xi_k}, \qquad \xi \in \partial\, \mathcal{O}.$$

Next, we denote by A the realization in H of the differential operator $\mathcal{A} = (\mathcal{A}_1, \ldots, \mathcal{A}_r)$ endowed with the boundary conditions $\mathcal{B} = (\mathcal{B}_1, \ldots, \mathcal{B}_r)$. That is

$$D(A) = \left\{ x \in W^{2,2}(\mathcal{O}; \mathbb{R}^r) \ :\ \mathcal{B}(\cdot, D)x = 0 \text{ in } \partial\, \mathcal{O} \right\}, \qquad Ax = \mathcal{A}(\cdot, D)x.$$

Now, for any $i = 1, \ldots, r$ we set

$$\mathcal{L}_i(\xi, D) := \sum_{h=1}^{d} \left(b^i_h(\xi) - \sum_{k=1}^{d} \frac{\partial}{\partial \xi_k} a^i_{hk}(\xi) \right) \frac{\partial}{\partial \xi_h}, \qquad \xi \in \mathcal{O},$$

and by difference we define the divergence-form operator

$$\mathcal{C}_i := \mathcal{A}_i - \mathcal{L}_i.$$

It is possible to check (see e.g. [10] for all details and proofs) that the realization C in H of the second order elliptic operator $\mathcal{C} = (\mathcal{C}_1, \ldots, \mathcal{C}_r)$, endowed with the boundary conditions $\mathcal{B}$, is a non-positive and self-adjoint operator which generates an analytic semigroup e^{tC} with dense domain given by $e^{tC} = (e^{tC_1}, \dots, e^{tC_r})$, where C_i is the realization in $L^2(\mathcal{O})$ of $\mathcal{C}_i$ with the boundary condition $\mathcal{B}_i$.

In [10, Theorem 1.4.1] it is proved that the space $L^1(\mathcal{O}; \mathbb{R}^r) \cap L^\infty(\mathcal{O}; \mathbb{R}^r)$ is invariant under e^{tC}, so that e^{tC} may be extended to a non-negative one-parameter contraction semigroup $T_p(t)$ on $L^p(\mathcal{O}; \mathbb{R}^r)$, for all $1 \leq p \leq \infty$. These semigroups are strongly continuous for $1 \leq p < \infty$ and are consistent, in the sense that $T_p(t)x = T_q(t)x$, for all $x \in L^p(\mathcal{O}; \mathbb{R}^r) \cap L^q(\mathcal{O}; \mathbb{R}^r)$. This is why we shall denote all $T_p(t)$ by e^{tC}. Moreover, $T_p(t)^\star = T_q(t)$, if $1 \leq p < \infty$ and $1 = p^{-1} + q^{-1}$. Finally, if we consider the part of C in the space of continuous functions E, it generates an analytic semigroup which has no dense domain in general (it clearly depends on the boundary conditions).

For any $t, \epsilon > 0$ and $p \geq 1$, the semigroup e^{tC} maps $L^p(\mathcal{O}; \mathbb{R}^r)$ into $W^{\epsilon,p}(\mathcal{O}; \mathbb{R}^r)$ and by using the semigroup law we easily obtain

$$|e^{tC}x|_{\epsilon,p} \leq c\,(t \wedge 1)^{-\frac{\epsilon}{2}} |x|_p, \qquad x \in L^p(\mathcal{O}; \mathbb{R}^r), \tag{2.4}$$

for some constant c independent of p. Due to the Sobolev embedding theorem, we have that $W^{\epsilon,p}(\mathcal{O}; \mathbb{R}^r)$ embeds into $L^\infty(\mathcal{O}; \mathbb{R}^r)$, for any $\epsilon > d/p$. Then, by easy calculations we have that e^{tC} maps H into $L^\infty(\mathcal{O}; \mathbb{R}^r)$, for any $t > 0$, and

$$|e^{tC}x|_\infty \leq c\,(t \wedge 1)^{-\frac{d}{4}} |x|_H, \qquad x \in H.$$

This means that e^{tC} is *ultracontractive*. As C is self-adjoint, by using the Riesz-Thorin theorem we have that for any $t > 0$ and $1 \leq q \leq p \leq +\infty$ the semigroup e^{tC} maps $L^q(\mathcal{O}; \mathbb{R}^r)$ into $L^p(\mathcal{O}; \mathbb{R}^r)$ and

$$|e^{tC}x|_p \leq c\,(t \wedge 1)^{-\frac{d(p-q)}{2pq}} |x|_q, \qquad x \in L^q(\mathcal{O}; \mathbb{R}^r). \tag{2.5}$$

Finally, as $W^{\epsilon,p}(\mathcal{O}; \mathbb{R}^r)$ embeds continuously into $C^\theta(\overline{\mathcal{O}}; \mathbb{R}^r)$, for any $\theta < \epsilon - d/p$, we easily have

$$|e^{tC}x|_{C^\theta(\overline{\mathcal{O}}; \mathbb{R}^r)} \leq c\,(t \wedge 1)^{-\frac{\theta}{2}} |x|_E. \tag{2.6}$$

Since e^{tC} is ultracontractive, as proved in [10, Lemma 2.1.2] it admits an integral kernel $K : (0, +\infty) \times \mathcal{O} \times \mathcal{O} \to \mathbb{R}^r$. That is

$$e^{tC}x(\xi) = \left(\int_{\mathcal{O}} K_1(t, \xi, \eta) x_1(\eta)\, d\eta\,, \dots, \int_{\mathcal{O}} K_r(t, \xi, \eta) x_r(\eta)\, d\eta \right) \qquad t > 0, \tag{2.7}$$

for any $x \in L^1(\mathcal{O}; \mathbb{R}^r)$ and $\xi \in \overline{\mathcal{O}}$. Moreover, there exist two positive constants c_1 and c_2 such that for each $i = 1, \dots, r$

$$0 \leq K_i(t, \xi, \eta) \leq c_1\, t^{-\frac{d}{2}} \exp\left(-c_2 \frac{|\xi - \eta|^2}{t} \right) \tag{2.8}$$

(see [10, Corollary 3.2.8 and Theorem 3.2.9]).

Furthermore, from the ultracontractivity of e^{tC} and the boundedness of $\mathcal{O}$, as proved in [10, Theorems 2.1.4 and 2.1.5] we have that e^{tC} is compact on $L^p(\mathcal{O}; \mathbb{R}^r)$

for all $1 \leq p \leq \infty$ and $t > 0$. The spectrum $\{-\alpha_k\}_{k\in\mathbb{N}}$ of C is independent of p and e^{tC} is analytic on $L^p(\mathcal{O};\mathbb{R}^r)$, for all $1 \leq p \leq \infty$. Concerning the complete orthonormal system of eigenfunctions $\{e_k\}_{k\in\mathbb{N}}$, we have that $e_k \in L^\infty(\mathcal{O};\mathbb{R}^r)$.

In the sequel we shall assume the following condition on the eigenfunctions e_k.

Hypothesis 1. *The complete orthonormal system of H which diagonalizes C is equibounded in $L^\infty(\mathcal{O};\mathbb{R}^r)$, that is*

$$\sup_{k\in\mathbb{N}} |e_k|_\infty < \infty. \tag{2.9}$$

Remark 2.2. We assume this uniform bound on the L^∞-norm of the eigenfunctions e_k (which is satisfied for example by the Laplace operator on the square with Dirichlet boundary conditions) for the sake of simplicity, even if in several important cases, as for example the disc, it is not satisfied. In fact, what is true in general is that

$$|e_k|_\infty \leq c\, k^\alpha,$$

for some $\alpha \geq 0$. Thus, in order to compensate the terms k^α and get the regularity results of next sections, in general we have to give some further colour to the noise.

Finally, for any $p \geq 1$ we denote by L_p the realization of the operator $\mathcal{L} = (\mathcal{L}_1, \ldots, \mathcal{L}_r)$ in $L^p(\mathcal{O};\mathbb{R}^r)$. It is immediate to verify that the operators L_p are all consistent, in the sense that

$$L_p x = L_q x, \qquad x \in D(L_p) \cap D(L_q).$$

Thus, if no confusion arises, we denote all of them by L. For each $p \in [1,\infty]$ $L : D(L) \subseteq L^p(\mathcal{O};\mathbb{R}^r) \to L^p(\mathcal{O};\mathbb{R}^r)$ is a linear operator with dense domain and the domains of L and $L^\star$ coincide with $D((-C)^{1/2})$, with equivalence of norms. Moreover for any $x \in L^p(\mathcal{O};\mathbb{R}^r)$

$$|L^\star e^{tC} x|_p \leq c\,(t\wedge 1)^{-\frac{1}{2}}|x|_p, \qquad t > 0.$$

3. Regularity of the stochastic convolution

We are here concerned with the study of existence, uniqueness and regularity of solutions for the system

$$\begin{cases} \dfrac{\partial u_i}{\partial t}(t,\xi) = \mathcal{A}_i\, u_i(t,\xi) + \displaystyle\sum_{j=1}^r g_{ij}(t,\xi,u_1(t,\xi),\ldots,u_r(t,\xi))\, Q_j \frac{\partial w_j}{\partial t}(t,\xi), \\ \qquad \xi \in \overline{\mathcal{O}},\ t \geq s \\ \\ u_i(s,\xi) = 0, \quad \xi \in \overline{\mathcal{O}}, \qquad \mathcal{B}_i\, u_i(t,\xi) = 0, \quad \xi \in \partial\mathcal{O},\ t \geq s, \qquad i = 1,\ldots,r. \end{cases}$$

Here $\partial w_j(t)/\partial t$ are independent space-time white noises defined on the same stochastic basis $(\Omega,\mathcal{F},\mathcal{F}_t,\mathbb{P})$. Thus, if we set $w(t) = (w_1(t),\ldots,w_r(t))$, we have that $w(t)$ can be written as

$$w(t) := \sum_{k=1}^\infty e_k \beta_k(t), \qquad t \geq 0,$$

where $\{e_k\}$ is the complete orthonormal system in H which diagonalizes C and $\{\beta_k(t)\}$ is a sequence of mutually independent standard real Brownian motions on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$. As well known, the series above does not converge in H, but it does converge in any Hilbert space U containing H with Hilbert-Schmidt embedding.

If we denote by Q the operator of components $Q_1, \dots, Q_r$ and if we set for any $x, y : \overline{\mathcal{O}} \to \mathbb{R}^r$ and $t \geq 0$

$$(G(t,x)y)(\xi) = g(t,\xi,x(\xi))y(\xi), \qquad \xi \in \mathcal{O},$$

by using the notations introduced in Subsection 2.1 the system above can be rewritten in the following abstract form

$$du(t) = Au(t)\,dt + G(t,u(t))Q\,dw(t), \quad u(s) = 0. \tag{3.1}$$

As we have seen in Subsection 2.1, the operator A can be written as $A = C + L$ and the complete orthonormal system which diagonalizes the operator C fulfills Hypothesis 1. Concerning G and Q we assume the following conditions.

Hypothesis 2. *1. The operator $Q : H \to H$ belongs to $\mathcal{C}_\varrho(H)$, with*

$$\varrho = \infty, \ \text{if } d = 1, \qquad 2 < \varrho < \frac{2d}{d-2}, \ \text{if } d \geq 2. \tag{3.2}$$

2. The mapping $g : [0,\infty) \times \overline{\mathcal{O}} \times \mathbb{R}^r \to \mathcal{L}(\mathbb{R}^r)$ is measurable. Moreover for any $\sigma, \rho \in \mathbb{R}^r$ and almost all $t \geq 0$

$$\sup_{\xi \in \overline{\mathcal{O}}} |g(t,\xi,\sigma) - g(t,\xi,\rho)|_{\mathcal{L}(\mathbb{R}^r)} \leq \Psi(t)\,|\sigma - \rho|, \tag{3.3}$$

for some function $\Psi \in L^\infty_{loc}[0,\infty)$.

As the mapping $g(t,\xi,\cdot) : \mathbb{R}^r \to \mathcal{L}(\mathbb{R}^r)$ is Lipschitz-continuous, uniformly with respect to $\xi \in \overline{\mathcal{O}}$ and t in bounded sets of $[0,\infty)$, the operator $G(t,\cdot)$ is Lipschitz-continuous from H into $\mathcal{L}(H; L^1(\mathcal{O};\mathbb{R}^r))$, uniformly for t in bounded sets of $[0,\infty)$. Indeed, due to (3.3), for any $x, y, v \in H$ and $z \in L^\infty(\mathcal{O};\mathbb{R}^r)$ we have

$$\begin{aligned}\langle (G(t,x) - G(t,y))\,v, z\rangle_H &= \int_{\mathcal{O}} \left\langle v(\xi), \left[g^\star(t,\xi,x(\xi)) - g^\star(t,\xi,y(\xi))\right] z(\xi)\right\rangle d\xi\\ &\leq |z|_\infty \int_{\mathcal{O}} |g^\star(t,\xi,x(\xi)) - g^\star(t,\xi,y(\xi))|_{\mathcal{L}(\mathbb{R}^r)}\,|v(\xi)|\,d\xi\\ &\leq \Psi(t)\,|z|_\infty\,|x-y|_H\,|v|_H.\end{aligned}$$

Thus, we have

$$\|G(t,x) - G(t,y)\|_{\mathcal{L}(H;L^1(\mathcal{O};\mathbb{R}^r))} \leq \Psi(t)\,|x-y|_H. \tag{3.4}$$

In the same way it is possible to show that the operator $G(t,\cdot)$ is Lipschitz-continuous from H into $\mathcal{L}(L^\infty(\mathcal{O};\mathbb{R}^r); H)$ and

$$\|G(t,x) - G(t,y)\|_{\mathcal{L}(L^\infty(\mathcal{O};\mathbb{R}^r);H)} \leq \Psi(t)\,|x-y|_H. \tag{3.5}$$

Note that estimates (3.4) and (3.5) are also verified by the operator

$$(G^\star(t,x)y)(\xi) := g^\star(t,\xi,x(\xi))y(\xi), \qquad \xi \in \overline{\mathcal{O}}.$$

Now, let X be any separable Banach space. For $0 \leq s < T$ and $p \geq 1$, in what follows we shall denote by $L^p(\Omega; C((s,T]; X) \cap L^\infty(s,T;X))$ the set of all predictable X-valued processes u in $C((s,T];X) \cap L^\infty(s,T;X)$, $\mathbb{P}$-a.s., such that

$$|u|^p_{L_{s,T,p}(X)} := \mathbb{E} \sup_{t \in [s,T]} |u(t)|^p_X < \infty.$$

$L^p(\Omega; C((s,T]; X) \cap L^\infty(s,T;X))$ is a Banach space, endowed with the norm $|\cdot|_{L_{s,T,p}(X)}$. Moreover, we shall denote by $L^p(\Omega; C([s,T];X))$ the subspace of processes u which take values in $C([s,T];X)$, $\mathbb{P}$-a.s.

Definition 3.1. *A process $u \in L^p(\Omega; C([s,T]; H))$, with $p \geq 1$, is a* mild *solution for problem (3.1) if*

$$u(t) = \int_s^t e^{(t-r)C} Lu(r)\,dr + \int_s^t e^{(t-r)C} G(r,u(r))Q\,dw(r), \qquad t \in [s,T].$$

For any $p \geq 1$ and $f \in C([s,T]; D(L))$ we define

$$\psi(f)(t) := \int_s^t e^{(t-r)C} Lf(r)\,dr, \qquad t \in [s,T]. \tag{3.6}$$

By proceeding as for example in [6, Lemma 6.1.2] we can prove that if X equals either H or E, then ψ can be extended to a bounded linear operator from $C([s,T];X)$ into itself and for any $t \in [s,T]$

$$|\psi(f)(t)|_X \leq c \int_s^t ((t-r) \wedge 1)^{-\frac{1+\epsilon}{2}} |f(r)|_X\,dr, \tag{3.7}$$

where ϵ is any constant greater than zero. In particular this implies that

$$|\psi(f)(t)|_X \leq c \int_0^{t-s} (r \wedge 1)^{-\frac{1+\epsilon}{2}}\,dr \sup_{r \in [s,t]} |f(r)|_X =: c_s^\psi(t) \sup_{r \in [s,t]} |f(r)|_X, \tag{3.8}$$

so that for any $p \geq 1$ and $u \in L^p(\Omega; C([s,T]; X))$

$$|\psi(u)|_{L_{s,T,p}(X)} \leq c_s^\psi(T)\,|u|_{L_{s,T,p}(X)}. \tag{3.9}$$

Notice that if we fix $\lambda > 0$ and define

$$\psi_\lambda(f)(t) := \int_s^t e^{(t-r)(C-\lambda)} f(r)\,dr, \tag{3.10}$$

due to (2.6), for any $\theta \in (0,1)$

$$\begin{aligned} |\psi_\lambda(f)(t)|_{C^\theta(\overline{\mathcal{O}};\mathbb{R}^r)} &\leq c \int_s^t e^{-\lambda(t-r)} (r \wedge 1)^{-\frac{1+\theta}{2}}\,dr \sup_{r \in [s,T]} |f(r)|_E \\ &:= c^{\psi,\lambda}_{s,\theta}(t) \sup_{r \in [s,T]} |f(r)|_E. \end{aligned} \tag{3.11}$$

Since $\lambda > 0$, it is immediate to check that $c^{\psi,\lambda}_{\theta,s} \in L^\infty[s,\infty)$.

Proposition 3.2. *Under Hypotheses 1 and 2, problem (3.1) admits a unique mild solution in $L^p(\Omega; C([s,T]; H))$, for any $T > s$ and $p \geq 1$.*

Proof. Clearly, a process u is a mild solution of problem (3.1) if it is a fixed point of the mapping

$$\psi + \gamma : L^p(\Omega; C([s,T]; H)) \to L^p(\Omega; C([s,T]; H)),$$

where for any $u \in L^p(\Omega; C([s,T]; H))$ we have defined

$$\gamma(u)(t) := \int_s^t e^{(t-r)C} G(r, u(r)) Q \, dw(r), \qquad t \in [s,T]. \tag{3.12}$$

Now, if we show that there exists some $p_\star \geq 1$ and $T_0 \in (s,T]$ such that γ is a contraction in $L^p(\Omega; C([s,T_0]; H))$, for any $p \geq p_\star$, then, due to (3.9) for $X = H$, we have that the mapping $\psi + \gamma$ is a contraction, for some T_0 possibly smaller. Thus, by a fixed point argument we can conclude that problem (3.1) admits a unique mild solution in $L^p(\Omega; C([s,T]; H))$, for any $T \geq s$ and $p \geq 1$.

By adapting some arguments based on the factorization method described for example in [8, proof of Theorem 8.3], we can prove that if there exists some $\delta > 0$ and some continuous increasing function $c_s(t)$ vanishing at $t = s$ such that for any $u, v \in L^p(\Omega; C([s,T]; H))$

$$\int_s^t (t-r)^{-2\delta} \left\| e^{(t-r)C} \left[G(r,u(r)) - G(r,v(r))\right] Q \right\|_2^2 dr \\ \leq c_s(t) \sup_{r \in [s,T]} |u(r) - v(r)|_H^2, \tag{3.13}$$

then there exists $p_\star \geq 1$ such that γ is a contraction in $L^p(\Omega; C([s,T_0]; H))$, for T_0 sufficiently close to s and $p \geq p_\star$. More precisely, for any $p \geq p_\star$ there exists a continuous increasing function $c_{s,p}(t)$, with $c_{s,p}(s) = 0$, such that for any $u, v \in L^p(\Omega; C([s,T]; H))$

$$|\gamma(u) - \gamma(v)|_{L_{s,T,p}(H)} \leq c_{s,p}(T) |u - v|_{L_{s,T,p}(H)}.$$

Next lemma shows how from Hypotheses 1 and 2 estimate (3.13) can be established.

Lemma 3.3. *Assume Hypotheses 1 and 2. Then the operator $e^{tC} G(r,x) Q$ is in $\mathcal{L}_2(H)$, for any $r \geq 0$, $t > 0$ and $x \in H$. Moreover, for any $x, y \in H$*

$$\left\| e^{tC} \left[G(r,x) - G(r,y)\right] Q \right\|_2 \leq c \, \|Q\|_\varrho \, \Psi(r) \, |x - y|_H \, t^{-\frac{d(\varrho-2)}{4\varrho}}, \tag{3.14}$$

where ϱ is the constant introduced in (3.2) and Ψ is the function introduced in (3.3).

Proof. If we fix ϱ as in (3.2) and $\varsigma := (1 - 2/\varrho)^{-1} = \varrho/(\varrho - 2)$, according to (2.1) we get

$$\|e^{tC} \left[G(r,x) - G(r,y)\right] Q\|_2 \leq \|Q\|_\varrho \|e^{tC} \left[G(r,x) - G(r,y)\right]\|_{2\varsigma}.$$

Due to (2.2), if $\{e_k\}$ is any complete orthonormal system in H, we have

$$\|e^{tC}\,[G(r,x)-G(r,y)]\,\|_{2\varsigma} \le c\left(\sum_{k=1}^{\infty}|e^{tC}\,[G(r,x)-G(r,y)]\,e_k|_H^{2\varsigma}\right)^{\frac{1}{2\varsigma}},$$

and then

$$\|e^{tC}\,[G(r,x)-G(r,y)]\,Q\|_2 \le c\,\|Q\|_\varrho$$

$$\times \sup_{k\in\mathbb{N}}|e^{tC}\,[G(r,x)-G(r,y)]\,e_k|_H^{\frac{\varsigma-1}{\varsigma}}\left(\sum_{k=1}^{\infty}|e^{tC}\,[G(r,x)-G(r,y)]\,e_k|_H^2\right)^{\frac{1}{2\varsigma}}.$$

If $\{e_k\}$ is the basis of H which diagonalizes C (see Hypothesis 1), as the L^∞-norms of e_k are assumed to be equi-bounded, by using (3.5) we easily obtain

$$\sup_{k\in\mathbb{N}}|e^{tC}\,[G(r,x)-G(r,y)]\,e_k|_H \le c\Psi(r)\,|x-y|_H$$

and this yields

$$\|e^{tC}\,[G(r,x)-G(r,y)]\,Q\|_2$$

$$\le c\,\|Q\|_\varrho\Psi(r)^{\frac{\varsigma-1}{\varsigma}}|x-y|_H^{\frac{\varsigma-1}{\varsigma}}\left(\sum_{k=1}^{\infty}|e^{tC}\,[G(r,x)-G(r,y)]\,e_k|_H^2\right)^{\frac{1}{2\varsigma}}. \tag{3.15}$$

Now, for any $r\ge 0$, $t>0$ and $x\in H$ we have

$$\sum_{k=1}^{\infty}|e^{tC}\,[G(r,x)-G(r,y)]\,e_k|_H^2 = \sum_{k=1}^{\infty}\sum_{h=1}^{\infty}\left|\left\langle e^{tC}\,[G(r,x)-G(r,y)]\,e_k, e_h\right\rangle_H\right|^2$$

$$=\sum_{h=1}^{\infty}\sum_{k=1}^{\infty}\left|\left\langle e_k, \left[G^\star(r,x)-G^\star(r,y)\right]e^{tC}e_h\right\rangle_H\right|^2$$

$$=\sum_{h=1}^{\infty}|\left[G^\star(r,x)-G^\star(r,y)\right]e_h|_H^2 e^{-2t\,\alpha_h}.$$

In [10, section 1.9] it is proved that for any $t>0$

$$\sum_{h=1}^{\infty}e^{-t\,\alpha_h} = \mathrm{Tr}\,[e^{tC}] = \sum_{i=1}^{r}\mathrm{Tr}\,[e^{tC_i}] \le c\,|\mathcal{O}|^2 t^{-\frac{d}{2}},$$

where $|\mathcal{O}|$ denotes the Lebesgue measure of the open set $\mathcal{O}$. Thus, by using (3.5) for $G^\star$, we obtain

$$\sum_{k=1}^{\infty}|e^{tC}\,[G(r,x)-G(r,y)]\,e_k|_H^2$$

$$\le \sup_{k\in\mathbb{N}}|\left[G^\star(r,x)-G^\star(r,y)\right]e_k|_H^2\,\mathrm{Tr}\,[e^{2tC}] \le c\,\Psi(r)^2|x-y|_H^2 t^{-\frac{d}{2}}.$$

Due to (3.15), recalling that $\varsigma=\rho/(\rho-2)$ this implies (3.14). □

Now we can conclude the proof of the proposition. Actually, as we are assuming that $\varrho = \infty$, if $d = 1$, and $\varrho < 2d/(d-2)$, if $d \geq 2$, for any $d \geq 1$ we have $(\varrho - 2)/\varrho < 2/d$, and then we can assume that

$$\frac{d(\varrho - 2)}{4\varrho} < \frac{1}{2}.$$

Thus, as $\Psi \in L^\infty_{\text{loc}}[0, \infty)$, due to (3.14) it is possible to find some $\delta > 0$ such that (3.13) is verified. □

Remark 3.4. Assume that there exist some $\kappa \in [0, 1]$ and $\Psi \in L^\infty_{\text{loc}}[0, \infty)$ such that for any $t \in [0, \infty)$ and $\sigma \in \mathbb{R}^r$

$$\sup_{\xi \in \overline{\mathcal{O}}} |g(t, \xi, \sigma)|_{\mathcal{L}(\mathbb{R}^r)} \leq \Psi(t) \left(1 + |\sigma|^\kappa\right).$$

By repeating the arguments used in the proof of Lemma 3.3, it is easy to show that in this case

$$\|e^{t\,C} G(r, x) Q\|_2 \leq c\, \|Q\|_\varrho\, \Psi(r)\, \left(1 + |x|_H^\kappa\right) t^{-\frac{d(\varrho-2)}{4\varrho}},$$

so that for any $p \geq p_\star$

$$|\gamma(u)|^p_{L_{s,T,p}(H)} \leq c_{s,p}(T) \left(1 + \mathbb{E} \sup_{t \in [s,T]} |u(t)|_H^{\kappa p}\right) = c_{s,p}(T) \left(1 + |u|^{\kappa p}_{L_{s,T,\kappa p}(H)}\right).$$

In particular, if $\kappa = 0$ then $\gamma(u)$ is bounded with respect to u.

4. Continuity in space of the solution of problem (3.1)

In this section we show that under more restrictive conditions on the mapping g and the operator Q the solution of problem (3.1) is continuous in space and time, $\mathbb{P}$-a.s.

Hypothesis 3. *1. The bounded linear operator $Q : H \to H$ is non negative and diagonal with respect to the complete orthonormal system $\{e_k\}$ which diagonalizes C, with eigenvalues $\{\lambda_k\}$.*
2. For almost all $t \geq 0$, the mapping $g(t, \cdot, \cdot) : \overline{\mathcal{O}} \times \mathbb{R}^r \to \mathcal{L}(\mathbb{R}^r)$ is continuous.

We first prove the following preliminary result.

Lemma 4.1. *Assume Hypotheses 1, 2 and 3 and fix $x \in H$, $r \geq 0$, $t > 0$ and $\varsigma \geq 1$. Then, for any $\xi \in \overline{\mathcal{O}}$*

$$\sum_{k=1}^\infty |e^{tC}\,[G(r, x) e_k]\,(\xi)|^{2\varsigma} \leq c\, \left|e^{tC} \theta(r, \cdot, x)(\xi)\right|^{2(\varsigma-1)} \times \sum_{i,j=1}^r |K_i(t, \xi, \cdot) g_{ij}(r, \cdot, x)|^2_{L^2(\mathcal{O})}, \tag{4.1}$$

where the function $\theta = (\theta_1, \ldots, \theta_r) : [0, \infty) \times \overline{\mathcal{O}} \times \mathbb{R}^r \to \mathbb{R}^r$ *is defined by*

$$\theta_i(t, \xi, \sigma) := \sum_{j=1}^{r} |g_{ij}(t, \xi, \sigma)|, \qquad i = 1, \ldots, r. \tag{4.2}$$

Proof. If we denote by e_k^j the j-th component of the eigenfunction e_k, thanks to (2.7) for any $\xi \in \overline{\mathcal{O}}$ we have

$$|e^{tC}[G(r, x)e_k](\xi)|^2 = \sum_{i=1}^{r} \left(\int_{\mathcal{O}} K_i(t, \xi, \eta) \sum_{j=1}^{r} g_{ij}(r, \eta, x(\eta)) e_k^j(\eta)\, d\eta \right)^2. \tag{4.3}$$

Thus, as the L^∞-norms of the eigenfunctions e_k^j are equibounded, we easily have

$$\begin{aligned} |e^{tC}[G(r, x)e_k](\xi)|^2 &\leq c \sum_{i=1}^{r} \left(\int_{\mathcal{O}} K_i(t, \xi, \eta) \sum_{j=1}^{r} |g_{ij}(r, \eta, x(\eta))|\, d\eta \right)^2 \\ &= c\, |e^{tC}\theta(r, \cdot, x)(\xi)|^2, \end{aligned}$$

where θ is the function defined in (4.2). Moreover, by using again (4.3)

$$\begin{aligned} \sum_{k=1}^{\infty} |e^{tC}[G(r, x)e_k](\xi)|^2 &= \sum_{k=1}^{\infty} \sum_{i=1}^{r} \left| \sum_{j=1}^{r} \left\langle K_i(t, \xi, \cdot) g_{ij}(r, \cdot, x), e_k^j \right\rangle_{L^2(\mathcal{O})} \right|^2 \\ &\leq c \sum_{i,j=1}^{r} \sum_{k=1}^{\infty} \left| \left\langle K_i(t, \xi, \cdot) g_{ij}(r, \cdot, x), e_k^j \right\rangle_{L^2(\mathcal{O})} \right|^2 \\ &= c \sum_{i,j=1}^{r} |K_i(t, \xi, \cdot) g_{ij}(r, \cdot, x)|^2_{L^2(\mathcal{O})}. \end{aligned}$$

The last equality is due to the fact that for each $j = 1, \ldots, r$ the system $\{e_k^j\}_{k \in \mathbb{N}}$ is complete in $L^2(\mathcal{O})$. Then, as we have

$$\begin{aligned} \sum_{k=1}^{\infty} |e^{tC}[G(r, x)e_k](\xi)|^{2\varsigma} &\leq \sup_{k \in \mathbb{N}} |e^{tC}[G(r, x)e_k](\xi)|^{2(\varsigma - 1)} \\ &\quad \times \sum_{k=1}^{\infty} |e^{tC}[G(r, x)e_k](\xi)|^2, \end{aligned}$$

(4.1) immediately follows. □

By proceeding in the same way, for any $x, y \in H$, $r \geq 0$, $t > 0$ and $\varsigma \geq 1$ we have

$$\sum_{k=1}^{\infty} |e^{tC}\,[G(r,x) - G(r,y)]\,e_k(\xi)|^{2\varsigma}$$
$$\leq c\,\left|e^{tC}\zeta(r,\cdot,x,y)(\xi)\right|^{2(\varsigma-1)} \sum_{i,j=1}^{r} |K_i(t,\xi,\cdot)\left(g_{ij}(r,\cdot,x) - g_{ij}(r,\cdot,y)\right)|^2_{L^2(\mathcal{O})}, \tag{4.4}$$

where the function $\zeta = (\zeta_1, \ldots, \zeta_r) : [0,\infty) \times \overline{\mathcal{O}} \times \mathbb{R}^r \times \mathbb{R}^r \to \mathbb{R}^r$ is defined by

$$\zeta_i(t,\xi,\sigma,\rho) = \sum_{j=1}^{r} |g_{ij}(t,\xi,\sigma) - g_{ij}(t,\xi,\rho)|, \qquad i = 1, \ldots, r. \tag{4.5}$$

We have already seen that the mapping

$$\gamma(u)(t) = \int_s^t e^{(t-r)C} G(r, u(r)) Q\,dw(r),$$

is a contraction in $L^p(\Omega; C([s,T]; H))$, for T sufficiently close to s and p sufficiently large. Now we show that the same is true replacing $L^p(\Omega; C([s,T]; H))$ with $L^p(\Omega; C([s,T]; E))$.

Theorem 4.2. *Under Hypotheses 1, 2 and 3, there exists $p_\star \geq 1$ such that γ maps $L^p(\Omega; C([s,T]; E))$ into itself for any $p \geq p_\star$ and for any $u, v \in L^p(\Omega; C([s,T]; E))$*

$$|\gamma(u) - \gamma(v)|_{L_{s,T,p}(E)} \leq c^{\gamma}_{s,p}(T)|u - v|_{L_{s,T,p}(E)}, \tag{4.6}$$

for some continuous increasing function $c^{\gamma}_{s,p}$ such that $c^{\gamma}_{s,p}(s) = 0$.

Proof. By using a factorization argument (see e.g. [8, Theorem 8.3]), we have

$$\gamma(u)(t) - \gamma(v)(t) = \frac{\sin \pi\alpha}{\pi} \int_s^t (t-r)^{\alpha-1} e^{(t-r)C} \upsilon_\alpha(u,v)(r)\,dr,$$

where

$$\upsilon_\alpha(u,v)(r) := \int_s^r (r-r')^{-\alpha} e^{(r-r')C} \left[G(r', u(r')) - G(r', v(r'))\right] Q\,dw(r'),$$

and $\alpha \in (0, 1/2)$. If we show that $\upsilon_\alpha(u,v) \in L^p([s,T] \times \mathcal{O}; \mathbb{R}^r)$, for $\alpha > 1/p$, then, according to (2.4) and to the Hölder inequality, for any $\epsilon < 2(\alpha - 1/p)$ we have

$$|\gamma(u)(t) - \gamma(v)(t)|_{\epsilon,p} \leq c_\alpha \int_s^t ((t-r)\wedge 1)^{\alpha - \frac{\epsilon}{2} - 1}\,|\upsilon_\alpha(u,v)(r)|_p\,dr$$
$$\leq c_\alpha \left(\int_0^{t-s} (r \wedge 1)^{\frac{p}{p-1}(\alpha - \frac{\epsilon}{2} - 1)}\,dr\right)^{\frac{p-1}{p}} |\upsilon_\alpha(u,v)|_{L^p([s,T]\times\mathcal{O};\mathbb{R}^r)}, \tag{4.7}$$

so that $\gamma(u)-\gamma(v) \in C([s,T]; W^{\epsilon,p}(\mathcal{O};\mathbb{R}^r))$, $\mathbb{P}$-a.s. Moreover, if $\epsilon > d/p$, that is if $\alpha > (d+2)/2p$, by the Sobolev embedding theorem we have $\gamma(u)-\gamma(v) \in C([s,T]\times\overline{\mathcal{O}};\mathbb{R}^r)$, $\mathbb{P}$-a.s.

As we are assuming that the constant ϱ introduced in Hypotheses 2 and 3 fulfills condition (3.2), we can find $p_\star \geq 1$ such that for any $p \geq p_\star$

$$\frac{d+2}{p}+\frac{d(\varrho-2)}{2\varrho}<1.$$

This implies that we can find some $\alpha_\star \in (0,1/2)$ such that for any $p \geq p_\star$

$$\alpha_\star > \frac{d+2}{2p} \quad \text{and} \quad 2\alpha_\star+\frac{d(\varrho-2)}{2\varrho}<1. \tag{4.8}$$

Whence, in correspondence of such $\alpha_\star$ the process $\upsilon_{\alpha_\star}(u,v)$ defined above belongs to $L^p([s,T]\times\mathcal{O};\mathbb{R}^r)$, $\mathbb{P}$-a.s. Actually, for $(r,\xi)\in[s,T]\times\overline{\mathcal{O}}$ we have $\mathbb{P}$-a.s.

$$\begin{aligned}\upsilon_{\alpha_\star}(u,v)(r,\xi) &= \int_s^r (r-r')^{-\alpha_\star}\sum_{k=1}^\infty e^{(r-r')C}\\ &\quad\times\left(\left[G(r',u(r'))-G(r',v(r'))\right]Qe_k\right)(\xi)\,d\beta_k(r').\end{aligned}$$

Then, if $p=2q$, from the Burkholder inequality we get

$$\begin{aligned}\mathbb{E}\,|\upsilon_{\alpha_\star}(u,v)(r,\xi)|^p &\leq c\,\mathbb{E}\Bigg(\int_s^r (r-r')^{-2\alpha_\star}\sum_{k=1}^\infty \lambda_k^2\,|e^{(r-r')C}\\ &\qquad\times\left(\left[G(r',u(r'))-G(r',v(r'))\right]e_k\right)(\xi)|^2\,dr'\Bigg)^{\frac{p}{2}},\end{aligned}$$

and, due to (2.1), this implies

$$\begin{aligned}&\mathbb{E}\,|\upsilon_{\alpha_\star}(u,v)(r,\xi)|^p \leq c\,\|Q\|_\varrho^p\,\mathbb{E}\,\Bigg(\int_s^r (r-r')^{-2\alpha_\star}\\ &\quad\times\Big(\textstyle\sum_{k=1}^\infty |e^{(r-r')C}\left(\left[G(r',u(r'))-G(r',v(r'))\right]e_k\right)(\xi)|^{2\varsigma}\Big)^{\frac{1}{\varsigma}}dr'\Bigg)^{\frac{p}{2}}\end{aligned}$$

where $\varsigma=\varrho/(\varrho-2)$ and $\varrho=+\infty$, if $d=1$, or $\varrho<2d/(d-2)$, if $d\geq 2$. According to (4.4), it follows

$$\begin{aligned}&\mathbb{E}\,|\upsilon_{\alpha_\star}(u,v)(r,\xi)|^p \leq c\,\mathbb{E}\Bigg(\int_s^r (r-r')^{-2\alpha_\star}\left|e^{(r-r')C}\zeta(r',\cdot,u(r'),v(r'))(\xi)\right|^{\frac{2(\varsigma-1)}{\varsigma}}\\ &\times\left(\sum_{i,j=1}^r |K_i(r-r',\xi,\cdot)\left[g_{ij}(r',\cdot,u(r'))-g_{ij}(r',\cdot,v(r'))\right]|^2_{L^2(\mathcal{O})}\right)^{\frac{1}{\varsigma}}dr'\Bigg)^{\frac{p}{2}},\end{aligned}$$

where ζ is the function defined in (4.5). For any $r \geq 0, t > 0, \xi \in \overline{\mathcal{O}}$ and $x, y \in H$, we have

$$\left|K_i(t,\xi,\cdot)\left[g_{ij}(r,\cdot,x)-g_{ij}(r,\cdot,y)\right]\right|^2_{L^2(\mathcal{O})} = \int_{\mathcal{O}} \left|K_i(t,\xi,\eta)\left[g_{ij}(r,\eta,x(\eta))-g_{ij}(r,\eta,y(\eta))\right]\right|^2\,d\eta.$$

Thus, if we define

$$\bar{\zeta}_i(r,\xi,\sigma,\rho) := \sum_{j=1}^r \left|g_{ij}(r,\xi,\sigma)-g_{ij}(r,\xi,\rho)\right|^2,$$

due to (2.8) we have

$$\sum_{j=1}^r |K_i(t,\xi,\cdot)\left[g_{ij}(r,\cdot,x)-g_{ij}(r,\cdot,y)\right]|^2_{L^2(\mathcal{O})} \leq c\,t^{-\frac{d}{2}}\int_{\mathcal{O}} K_i(t,\xi,\eta)\bar{\zeta}_i(r,\eta,x(\eta),y(\eta))\,d\eta = c\,t^{-\frac{d}{2}}e^{tC_i}\bar{\zeta}_i(r,\cdot,x,y)(\xi).$$

This implies that

$$\mathbb{E}\,|\upsilon_{\alpha_\star}(u,v)(r,\xi)|^p \leq c\,\mathbb{E}\Bigg(\int_s^r (r-r')^{-(2\alpha_\star+\frac{d}{2\varsigma})}|e^{(r-r')C}\zeta(r',\cdot,u(r'),\,v(r'))|_\infty^{\frac{2(\varsigma-1)}{\varsigma}} \times |e^{(r-r')C}\bar{\zeta}(r',\cdot,u(r'),v(r'))|_\infty^{\frac{1}{\varsigma}}\,dr'\Bigg)^{\frac{p}{2}}.$$

Now, since

$$|\zeta(r',\cdot,u(r'),v(r'))|_\infty \leq c\,\Psi(r')|u(r')-v(r')|_E \tag{4.9}$$

and

$$|\bar{\zeta}(r',\cdot,u(r'),v(r'))|_\infty \leq c\,\Psi^2(r')|u(r')-v(r')|^2_E, \tag{4.10}$$

from (2.5) by some calculations we get

$$\mathbb{E}\,|\upsilon_{\alpha_\star}(u,v)|^p_{L^p([s,T]\times\mathcal{O};\mathbb{R}^r)} = \mathbb{E}\int_s^T\int_{\mathcal{O}} |\upsilon_{\alpha_\star}(u,v)(r,\xi)|^p\,d\xi\,dr$$

$$\leq c\,|u-v|^p_{L_{s,T,p}(E)}\int_s^T\left(\int_s^r (r-r')^{-(2\alpha_\star+\frac{d}{2\varsigma})}\Psi^2(r')\,dr'\right)^{\frac{p}{2}}dr.$$

Then, as (4.8) holds, from the Young inequality and (4.7) we have that γ maps the space $L^p(\Omega;C([s,T];\mathbb{R}^r))$ into itself for any $p \geq p_\star$ and (4.6) is verified, with

$$c^\gamma_{s,p}(t) := c_\alpha\left(\int_s^t\left(\int_s^r (r-r')^{-(2\alpha_\star+\frac{d}{2\varsigma})}\Psi^2(r')\,dr'\right)^{\frac{p}{2}}dr\right)^{\frac{1}{p}} \times\left(\int_0^{t-s}(r\wedge 1)^{\frac{p}{p-1}(\alpha-\frac{\epsilon}{2}-1)}dr\right)^{\frac{p-1}{p}}. \tag{4.11}$$

□

Remark 4.3. According to (4.9) and (4.10), it is easy to check that if we assume

$$\sup_{\xi\in\overline{\mathcal{O}}} |g(t,\xi,\sigma)|_{\mathcal{L}(\mathbb{R}^r)} \leq \Psi(t)\left(1+|\sigma|^\kappa\right), \qquad t\in[0,\infty), \quad \sigma\in\mathbb{R}^r,$$

for some $\kappa \in [0,1]$ and $\Psi \in L^\infty_{\text{loc}}[0,\infty)$, then for any $p \geq p_\star$

$$|\gamma(u)|^p_{L_{s,T,p}(E)} \leq c^\gamma_{s,p}(T)\left(1+\mathbb{E}\sup_{t\in[s,T]}|u(t)|^{\kappa p}_E\right), \tag{4.12}$$

for some continuous increasing function $c^\gamma_{s,p}(t)$ vanishing at $t=s$.

From Theorem 4.2 we obtain the following regularity result for the solution of (3.1).

Corollary 4.4. *The solution of problem (3.1) belongs to* $L^p(\Omega; C([s,T];E))$*, for any* $p \geq 1$.

Proof. If we fix $t_0 > 0$ such that

$$c^\psi_s(s+t_0) + c^\gamma_{s,p}(s+t_0) \leq \frac{1}{2},$$

due to (3.9) for $X = E$ and to (4.6) the mapping $\psi + \gamma$ is a contraction on $L^p(\Omega; C([s,s+t_0];E))$ and then it admits a unique fixed point. As we can repeat these arguments in the intervals $[s+t_0, s+2t_0]$, $[s+2t_0, s+3t_0]$ and so on, we have a unique mild solution for problem (3.1) in $L^p(\Omega; C([s,T];E))$. □

Now, for any $\lambda > 0$ and $u \in L^p(\Omega; C([s,T];E))$ define

$$\gamma_\lambda(u)(t) := \int_s^t e^{(t-r)(C-\lambda)} G(r,u(r))Q\,dw(r). \tag{4.13}$$

Due to the previous theorem we have that there exists $p_\star \geq 1$ such that γ_λ maps $L^p(\Omega; C([s,T];E))$ into itself for any $p \geq p_\star$ and

$$|\gamma_\lambda(u)|_{L_{s,T,p}(E)} \leq c^{\gamma,\lambda}_{s,p}(T)\left(1+|u|_{L_{s,T,p}(E)}\right), \tag{4.14}$$

for some continuous increasing function $c^{\gamma,\lambda}_{s,p}$ such that $c^{\gamma,\lambda}_{s,p}(s)=0$.

In next proposition we study the asymptotic behaviour of the function $c^{\gamma,\lambda}_{s,p}$.

Proposition 4.5. *If* $\lambda > 0$ *and the function* Ψ *in Hypothesis 2-2 is in* $L^\infty(0,\infty)$*, then* $c^{\gamma,\lambda}_{s,p} \in L^\infty[s,\infty)$*. Moreover,*

$$\lim_{\lambda\to\infty}\sup_{t\geq s} c^{\gamma,\lambda}_{s,p}(t) = 0. \tag{4.15}$$

Proof. Let $p_\star$ and $\alpha_\star$ as in the proof of Theorem 4.2 and fix any $p \geq p_\star$. If we set

$$\upsilon_{\alpha_\star}(u)(r) := \int_s^r (r-\sigma)^{\alpha_\star - 1} e^{(r-\sigma)(C-\lambda)} G(\sigma, u(\sigma)) Q\, dw(\sigma),$$

and if we take $\epsilon < 2(\alpha_\star - 1/p)$, we have

$$|\gamma_\lambda(u)|_E \leq c\, |\gamma_\lambda(u)|_{\epsilon,p} \leq c_{\alpha_\star} \int_s^t e^{-\lambda(t-r)}((t-r)\wedge 1)^{\alpha_\star - \frac{\epsilon}{2} - 1} |\upsilon_{\alpha_\star}(u)(r)|_p\, dr$$

$$\leq c_{\alpha_\star} \left(\int_0^{t-s} e^{-\frac{\lambda p}{2(p-1)} r} (r \wedge 1)^{\frac{p}{p-1}(\alpha_\star - \frac{\epsilon}{2} - 1)}\, dr \right)^{\frac{p-1}{p}}$$

$$\times \left(\int_s^t e^{-\frac{\lambda p}{2}(t-r)} |\upsilon_{\alpha_\star}(u)(r)|_p^p\, dr \right)^{\frac{1}{p}}.$$

As in the proof of Theorem 4.2, for $(r,\xi) \in [s,t] \times \overline{\mathcal{O}}$ we have

$$\mathbb{E}\, |\upsilon_{\alpha_\star}(u)(r,\xi)|^p \leq c\, \mathbb{E} \left(\int_s^r (r-\sigma)^{-\left(2\alpha_\star + \frac{d}{2\varsigma}\right)} \right.$$

$$\left. \times \left| e^{(r-\sigma)(C-\lambda)} \theta(\sigma, \cdot, u(\sigma)) \right|_\infty^{\frac{2(\varsigma-1)}{\varsigma}} \left| e^{(r-\sigma)(C-\lambda)} \bar{\theta}(\sigma, \cdot, u(\sigma)) \right|_\infty^{\frac{1}{\varsigma}}\, d\sigma \right)^{\frac{p}{2}},$$

with ς as in the proof of Theorem 4.2, θ defined by (4.2) and

$$\bar{\theta}_i(s,\xi,u) := \sum_{j=1}^r |g_{ij}(s,\xi,u)|^2.$$

Therefore,

$$\int_s^t e^{-\frac{\lambda p}{2}(t-r)} |\upsilon_{\alpha_\star}(u)(r)|_p^p\, dr$$

$$\leq \|\Psi\|_\infty^p \int_s^t e^{-\frac{\lambda p}{2}(t-r)} \left(\int_0^{r-s} e^{-\lambda(2-\frac{1}{\varsigma})\sigma} \sigma^{-\left(2\alpha_\star + \frac{d}{2\varsigma}\right)} \left(1 + |u(\sigma)|_E^2\right) d\sigma \right)^{\frac{p}{2}} dr.$$

This means that

$$|\gamma_\lambda(u)|_{L_{s,t,p}(E)} \leq c_{s,p}^{\gamma,\lambda}(t) \left(1 + |u|_{L_{s,t,p}(E)}\right),$$

with

$$c_{s,p}^{\gamma,\lambda}(t) := c_{\alpha_\star} \|\Psi\|_\infty \left(\int_s^t e^{-\frac{\lambda p}{2}(t-r)} \left(\int_0^{r-s} e^{-\lambda(2-\frac{1}{\varsigma})\sigma} \sigma^{-\left(2\alpha_\star + \frac{d}{2\varsigma}\right)} d\sigma \right)^{\frac{p}{2}} dr \right)^{\frac{1}{p}}$$

$$\times \left(\int_0^{t-s} e^{-\frac{\lambda p}{2(p-1)} r} (r \wedge 1)^{\frac{p}{p-1}(\alpha_\star - \frac{\epsilon}{2} - 1)}\, dr \right)^{\frac{p-1}{p}}.$$

For any $\eta, \beta > 0$, $\delta < 1$ and $r > \eta$ we have

$$\begin{aligned}\int_0^r e^{-\beta\sigma}\sigma^{-\delta}\,d\sigma &= \int_0^\eta e^{-\beta\sigma}\sigma^{-\delta}\,d\sigma + \int_\eta^r e^{-\beta\sigma}\sigma^{-\delta}\,d\sigma\\ &\leq \frac{1}{1-\delta}\eta^{1-\delta} + \eta^{-\delta}\frac{e^{-\beta\eta}-e^{-\beta r}}{\beta}.\end{aligned}$$

Due to the way we have chosen $p_\star$ and $\alpha_\star$, we have that $\delta_p := p(1+\epsilon/2-\alpha_\star)/(p-1) < 1$, for any $p \geq p_\star$ and then if we take any $t > s+1$ we have

$$\int_0^{t-s} e^{-\frac{\lambda p}{2(p-1)}r}(r\wedge 1)^{\frac{p}{p-1}(\alpha_\star-\frac{\epsilon}{2}-1)}\,dr$$

$$\leq \frac{1}{1-\delta_p}+\frac{2(p-1)}{\lambda p}\left(e^{-\frac{\lambda p}{2(p-1)}} - e^{-\frac{\lambda p}{2(p-1)}(t-s)}\right)\leq \frac{1}{1-\delta_p}+\frac{2(p-1)}{p\lambda} =: c^1_{\lambda\,p}.$$

In the same way, if we set $\delta := 2\alpha_\star + d/2\varsigma$, for any $t > s+1$ we have

$$\int_s^t e^{-\frac{\lambda p}{2}(t-r)}\left(\int_0^{r-s} e^{-\lambda(2-\frac{1}{\varsigma})\sigma}\sigma^{-\left(2\alpha_\star+\frac{d}{2\varsigma}\right)}\,d\sigma\right)^{\frac{p}{2}}dr$$

$$\leq \int_s^t e^{-\frac{\lambda p}{2}(t-r)}dr\left(\frac{1}{1-\delta}+\frac{2}{\lambda(2-\frac{1}{\varsigma})}\right)^{\frac{p}{2}}$$

$$= \frac{2-2\,e^{-\frac{\lambda p}{2}(t-s)}}{\lambda p}\left(\frac{1}{1-\delta}+\frac{2}{\lambda(2-\frac{1}{\varsigma})}\right)^{\frac{p}{2}} \leq \frac{2}{\lambda p}\left(\frac{1}{1-\delta}+\frac{2}{\lambda(2-\frac{1}{\varsigma})}\right)^{\frac{p}{2}} =: c^2_{p,\lambda}.$$

This implies that for any $t \geq s$

$$c^{\gamma,\lambda}_{s,p}(t) \leq c_{\alpha^\star}\,\|\Psi\|_\infty (c^2_{p,\lambda})^{\frac{1}{p}}\left(c^1_{p,\lambda}\right)^{\frac{p-1}{p}},$$

so that $c^{\gamma,\lambda}_{s,p} \in L^\infty(0,\infty)$ and (4.15) holds true. □

Remark 4.6. In the proof of Proposition 4.5 we have seen that if $u \in L^p(\Omega; C([s,T];E))$, with $p \geq p_\star$, then $\gamma_\lambda(u) \in L^p(\Omega; C([s,T]; W^{\epsilon,p}(\overline{\mathcal{O}};\mathbb{R}^r)))$, for any $\epsilon < 2(\alpha_\star - 1/p)$. As $W^{\epsilon,p}(\overline{\mathcal{O}};\mathbb{R}^r)$ continuously embeds into $C^\theta(\overline{\mathcal{O}};\mathbb{R}^r)$, for any $\theta < \epsilon - d/p$, we have that $\gamma_\lambda(u)$ takes values in $C^\theta(\overline{\mathcal{O}};\mathbb{R}^r)$ for any $\theta < \theta_\star := 2\alpha_\star - (2+d)/p_\star$ and for any $T > 0$

$$\mathbb{E}\sup_{t\in[s,T]}|\gamma_\lambda(u)(t)|^p_{C^\theta(\overline{\mathcal{O}};\mathbb{R}^r)} \leq |c^{\gamma,\lambda}_{s,p}|^p_\infty\left(1+|u|^p_{L_{s,T,p}(E)}\right). \tag{4.16}$$

5. The stochastic reaction-diffusion system

Our aim is to apply the results of the previous section to the study of existence, uniqueness and regularity of solutions for the following class of reaction-diffusion systems perturbed by a multiplicative noise

$$\left\{\begin{array}{l} \dfrac{\partial u_i}{\partial t}(t,\xi) = \mathcal{A}_i\, u_i(t,\xi) + f_i(t,\xi,u_1(t,\xi),\dots,u_r(t,\xi)) \\[2ex] \qquad\qquad + \displaystyle\sum_{j=1}^r g_{ij}(t,\xi,u_1(t,\xi),\dots,u_r(t,\xi))\,Q_j\,\frac{\partial w_j}{\partial t}(t,\xi),\ t\geq s,\ \xi\in\overline{\mathcal{O}}, \\[2ex] u_i(s,\xi)=x_i(\xi),\quad \xi\in\overline{\mathcal{O}},\qquad \mathcal{B}_i\,u_i(t,\xi)=0,\quad t\geq s,\quad \xi\in\partial\mathcal{O}. \end{array}\right. \tag{5.1}$$

The operators $\mathcal{A}_i$ with the boundary conditions $\mathcal{B}_i$ fulfill the conditions described in Subsection 2.1. The matrix valued function $g=[g_{ij}]:[0,\infty)\times\overline{\mathcal{O}}\times\mathbb{R}^r\to\mathcal{L}(\mathbb{R}^r)$ and the operator $Q=(Q_1,\dots,Q_r):H\to H$ have been introduced in Section 3. For the non linear term $f=(f_1,\dots,f_r)$ we assume that

$$f_i(t,\xi,\sigma_1,\dots,\sigma_r)=k_i(t,\xi,\sigma_i)+h_i(t,\xi,\sigma_1,\dots,\sigma_r),\qquad i=1,\dots,r,$$

for some measurable mappings $k_i:[0,\infty)\times\overline{\mathcal{O}}\times\mathbb{R}\to\mathbb{R}$ and $h_i:[0,\infty)\times\overline{\mathcal{O}}\times\mathbb{R}^r\to\mathbb{R}$. For almost all $t\geq 0$, the functions $k_i(t,\cdot,\cdot):\overline{\mathcal{O}}\times\mathbb{R}\to\mathbb{R}$ and $h_i(t,\cdot,\cdot):\overline{\mathcal{O}}\times\mathbb{R}^r\to\mathbb{R}$ are continuous and the following further conditions are assumed.

Hypothesis 4. *1. The function $h_i(t,\xi,\cdot):\mathbb{R}^r\to\mathbb{R}$ is locally Lipschitz-continuous with linear growth, uniformly with respect to $\xi\in\overline{\mathcal{O}}$ and t in bounded sets of* $[0,\infty)$. *This means that there exists $\Phi_1\in L^\infty_{loc}[0,\infty)$ such that for any $t\geq 0$*

$$\sup_{\xi\in\overline{\mathcal{O}}}|h_i(t,\xi,\sigma)|\leq\Phi_1(t)\,(1+|\sigma|)\,,\qquad \sigma\in\mathbb{R}^r,$$

and for any $R>0$ there exists $L_R\in L^\infty_{loc}[0,\infty)$ such that

$$|\sigma|,\ |\rho|\leq R\Longrightarrow\sup_{\xi\in\overline{\mathcal{O}}}|h_i(t,\xi,\sigma)-h_i(t,\xi,\rho)|\leq L_R(t)\,|\sigma-\rho|.$$

2. There exist $m\geq 1$ and $\Phi_2\in L^\infty_{loc}[0,\infty)$ such that for any $t\geq 0$

$$\sup_{\xi\in\overline{\mathcal{O}}}|k_i(t,\xi,\sigma_i)|\leq\Phi_2(t)\,(1+|\sigma_i|^m),\qquad\sigma_i\in\mathbb{R}.$$

3. For any $\xi\in\overline{\mathcal{O}}$, $t\geq 0$ and $\sigma_i,\rho_i\in\mathbb{R}$

$$k_i(t,\xi,\sigma_i)-k_i(t,\xi,\rho_i)=\lambda_i(t,\xi,\sigma_i,\rho_i)(\sigma_i-\rho_i), \tag{5.2}$$

for some locally bounded function $\lambda_i:[0,\infty)\times\overline{\mathcal{O}}\times\mathbb{R}^2\to\mathbb{R}$ such that

$$\sup_{\substack{\xi\in\overline{\mathcal{O}}\\ \sigma_i,\rho_i\in\mathbb{R},\,t\geq 0}}\lambda_i(t,\xi,\sigma_i,\rho_i)<\infty.$$

Now, we define the operator F by setting for any $x : \overline{\mathcal{O}} \to \mathbb{R}^r$ and $t \geq 0$

$$F(t,x)(\xi) := f(t,\xi,x(\xi)), \qquad \xi \in \overline{\mathcal{O}}.$$

It is immediate to check that $F(t,\cdot)$ is well defined and continuous from $L^p(\mathcal{O};\mathbb{R}^r)$ into $L^q(\mathcal{O};\mathbb{R}^r)$, for any $p,q \geq 1$ such that $p/q \geq m$. In particular, if $m > 1 F(t,\cdot)$ is not defined from H into itself. Moreover, due to (5.2) for $x,h \in L^{2m}(\mathcal{O};\mathbb{R}^r)$ and $t \geq 0$

$$\langle F(t,x+h) - F(t,x), h\rangle_H \leq \Lambda(t)\left(1 + |h|_H^2 + |x|_H^2\right), \tag{5.3}$$

for some $\Lambda \in L^\infty_{\text{loc}}[0,\infty)$.

In the same way, it is possible to show that the functional $F(t,\cdot)$ is well defined and continuous from E into itself and there exists $\phi \in L^\infty_{\text{loc}}[0,\infty)$ such that for any $t \geq 0$

$$|F(t,x)|_E \leq \Phi(t)\,(1 + |x|_E^m), \quad x \in E. \tag{5.4}$$

Moreover, by using assumption 3 in Hypothesis 4, it is not difficult to check that there exists $\Lambda \in L^\infty_{\text{loc}}[0,\infty)$ such that for any $x,h \in E$

$$\langle F(t,x+h) - F(t,x), \delta_h\rangle_E \leq \Lambda(t)\,(1 + |h|_E + |x|_E), \tag{5.5}$$

where δ_h is the element of $\partial\,|h|_E$ defined in (2.3) (for more details on the properties of F and of subdifferentials we refer to [6, Chapters 4 and 6 and Appendix A]).

Remark 5.1. 1. In the proof of the main results of this paper what is important is that (5.3), (5.4) and (5.5) holds, for any $x,h \in E$ and for some $\delta_h \in \partial|h|_E$, not necessarily given by (2.3). Thus we could consider a more general class of non-linearities f such that the corresponding composition operator F fulfills (5.3), (5.4) and (5.5).

2. If c_i and $c_{i,j}$ are continuous functions from $[0,\infty)\times\overline{\mathcal{O}}$ into $\mathbb{R}$, for $i = 1,\dots,r$ and $j = 1,\dots,2k$, and if

$$\inf_{\substack{\xi\in\overline{\mathcal{O}}\\ t\in[0,\infty)}} c_i(t,\xi) > 0,$$

then, the function

$$k_i(t,\xi,\sigma_i) := -c_i(t,\xi)\,\sigma_i^{2k+1} + \sum_{j=1}^{2k} c_{i,j}(t,\xi)\,\sigma_i^j.$$

fulfills the conditions of Hypothesis 4, with $m = 2k+1$.

3. In Hypothesis 4 we assume that the functions h_i and k_i fulfill conditions 1, 2 and 3, uniformly for t in bounded sets of $[0,\infty)$. This is only for the sake of simplicity. Actually, it is possible to check that in several cases uniformity with respect to t in bounded sets of $[0,\infty)$ can be replaced by some bounds in $L^p_{\text{loc}}[0,\infty)$, for some $p \geq 2$.

If we denote by C the operator $(C_1, \dots, C_r)$ and by L the operator $(L_1, \dots, L_r)$, with the notations introduced above (see also Subsection 2.1 and Section 3) the system (5.1) can be rewritten as

$$du(t) = [Cu(t) + Lu(t) + F(t, u(t))]\, dt + G(t, u(t))Q\, dw(t), \qquad u(s) = x. \tag{5.6}$$

Definition 5.2. *Let X denote both H and E and fix $x \in X$. A X-valued predictable process $u_s^x(t)$ is a* mild *solution of (5.6) if*

$$\begin{aligned} u_s^x(t) &= e^{(t-s)C}x + \int_s^t e^{(t-r)C}\left[Lu_s^x(r) + F(r, u_s^x(r))\right] dr \\ &\quad + \int_s^t e^{(t-r)C} G(r, u_s^x(r))Q\, dw(r). \end{aligned}$$

We prove a first existence and uniqueness result.

Theorem 5.3. *Assume that for any $t \geq 0$*

$$\sup_{\xi \in \overline{\mathcal{O}}} |g(t, \xi, \sigma)|_{\mathcal{L}(\mathbb{R}^r)} \leq \Psi(t) \left(1 + |\sigma|^{\frac{1}{m}}\right), \qquad \sigma \in \mathbb{R}^r, \tag{5.7}$$

where $\Psi \in L^\infty_{loc}[0, \infty)$ and m is the constant introduced in Hypothesis 4. Then, under Hypotheses 1 to 4 for any $x \in E$ there exists a unique mild solution u_s^x of (5.6) which belongs to $L^p(\Omega; C((s, T]; E) \cap L^\infty(s, T; E))$, for any $p \geq 1$ and $T > s$. Moreover

$$|u_s^x|_{L_{s,T,p}(E)} \leq c_{s,p}(T)\ (1 + |x|_E), \qquad x \in E, \tag{5.8}$$

for some continuous increasing function $c_{s,p}$.

Notice that in the case $m = 1$ no further condition on the growth of g is assumed.

Proof. If Hypothesis 4 holds, for some function k which fulfills conditions 2 and 3, $F(t, \cdot)$ is not even defined in H, in general. If we consider $F(t, \cdot)$ restricted to E, it is continuous and due to (5.4) bounded on bounded subsets. Unfortunately it is not Lipschitz-continuous and then we can not proceed by using a contraction argument.

For any $n \in \mathbb{N}$, $i = 1, \dots, r$ and $(t, \xi) \in [0, \infty) \times \overline{\mathcal{O}}$ we define

$$h_{n,i}(t, \xi, \sigma) := \begin{cases} h_i(t, \xi, \sigma) & \text{if } |\sigma| \leq n, \\ h_i(t, \xi, n\sigma/|\sigma|) & \text{if } |\sigma| > n, \end{cases}$$

and

$$k_{n,i}(t, \xi, \sigma_i) := \begin{cases} k_i(t, \xi, \sigma_i) & \text{if } |\sigma_i| \leq n, \\ k_i(t, \xi, n\sigma_i/|\sigma_i|) & \text{if } |\sigma_i| > n. \end{cases}$$

It is immediate to check that $f_n(t, \xi, \cdot) := h_n(t, \xi, \cdot) + k_n(t, \xi, \cdot) : \mathbb{R}^r \to \mathbb{R}^r$ is Lipschitz-continuous for any $n \in \mathbb{N}$, uniformly with respect to $\xi \in \overline{\mathcal{O}}$ and t in bounded sets of $[0, \infty)$, so that the composition operator $F_n(t, \cdot)$ associated with f_n

$$F_n(t, x)(\xi) := f_n(t, \xi, x(\xi)), \qquad \xi \in \overline{\mathcal{O}},$$

is Lipschitz-continuous, both in H and in E. Moreover, it is easy to check that all the functions k_n fulfill conditions 2 and 3 in Hypothesis 4, for some functions $\lambda_i(t, \xi, \sigma_i, \rho_i)$ which do not depend on $n \in \mathbb{N}$. This implies that all F_n satisfy (5.3), (5.4) and (5.5), for common functions Λ and Φ in $L^\infty_{\text{loc}}[0, \infty)$. Finally, if $n < m$ and $t \geq 0$ we have

$$|x|_E \leq n \Longrightarrow F_n(t, x) = F_m(t, x) = F(t, x). \tag{5.9}$$

Now, if we consider the problem

$$du(t) = [Au(t) + F_n(t, u(t))]\, dt + G(t, u(t))Q\, dw(t), \qquad u(s) = x, \tag{5.10}$$

as $F_n(t, \cdot)$ is Lipschitz-continuous both in H and in E, for any x belonging either to H or to E there exists a unique mild solution u_n^x (for simplicity of notations we do not stress its dependence on s) which belongs respectively either to $L^p(\Omega; C([s, T]; H))$ or to $L^p(\Omega; C((s, T]; E) \cap L^\infty(s, T; E))$, for any $p \geq 1$ and $T > s$. Actually, the mapping Λ_n defined by

$$\Lambda_n(u)(t) := e^{(t-s)C}x + \int_s^t e^{(t-r)C} F_n(r, u(r))\, dr$$

is Lipschitz-continuous in $L^p(\Omega; C([s, T]; H))$ and in $L^p(\Omega; C((s, T]; E) \cap L^\infty(s, T; E))$ and then, due to the results proved in Sections 3 and 4, the mild solution u_n^x is easily obtained as the unique fixed point of the mapping

$$u \mapsto \Lambda_n(u) + \psi(u) + \gamma(u),$$

with ψ and γ defined respectively in (3.6) and (3.12).

Next, we show that the sequence $\{u_n^x\}$ is bounded in $L^p(\Omega; C((s, T]; E) \cap L^\infty(s, T; E))$.

Lemma 5.4. *There exists a continuous increasing function $c_{s,p}(t)$ which is independent of $n \in \mathbb{N}$ such that*

$$|u_n^x|_{L_{s,T,p}(E)} \leq c_{s,p}(T)\,(1 + |x|_E)\,. \tag{5.11}$$

Proof. If we denote by $\Gamma(u_n^x)$ the solution of the problem

$$dv(t) = Av(t)\, dt + G(t, u_n^x(t))Q\, dw(t), \qquad v(s) = 0, \tag{5.12}$$

we have that $\Gamma(u_n^x)$ is the unique fixed point in $L^p(\Omega; C([s, T]; E))$ of the mapping

$$\begin{aligned} v(t) &\mapsto \int_s^t e^{(t-r)C} Lv(r)\, dr + \int_s^t e^{(t-r)C} G(r, u_n^x(r))Q\, dw(r) \\ &= \psi(v)(t) + \gamma(u_n^x)(t). \end{aligned}$$

According to (3.7) for $X = E$, if ϵ is any strictly positive constant we have

$$|\Gamma(u_n^x)(t)|_E \leq c \int_s^t ((t-r) \wedge 1)^{-\frac{\epsilon+1}{2}} |\Gamma(u_n^x)(r)|_E \, dr + |\gamma(u_n^x)(t)|_E.$$

If we consider the general integral problem

$$f(t) = c \int_s^t g(t-r) f(r) \, dr + h(t), \quad f(s) = 0,$$

with $h \in C([s, \infty))$ and $g \in L^1(s, T)$ non-negative, it is possible to check that for any $T > s$ the unique solution is given by

$$f(t) = h(t) + \int_s^t h(r) \, g(t-r) \exp \int_0^{t-r} g(\sigma) \, d\sigma \, dr.$$

Thus, if we take in our case

$$h(t) := |\gamma(u_n^x)(t)|_E, \quad g(t) := c \, (t \wedge 1)^{-\frac{1+\epsilon}{2}},$$

from a comparison argument we obtain

$$|\Gamma(u_n^x)(t)|_E \leq |\gamma(u_n^x)(t)|_E$$
$$+ c \int_s^t ((t-r) \wedge 1)^{-\frac{1+\epsilon}{2}} |\gamma(u_n^x)(r)|_E \, \exp \, c \int_0^{t-r} (\sigma \wedge 1)^{-\frac{1+\epsilon}{2}} \, d\sigma \, dr.$$

This means that

$$|\Gamma(u_n^x)(t)|_E \leq |\gamma(u_n^x)(t)|_E + \sup_{r \in [s,t]} |\gamma(u_n^x)(r)|_E \times \left(\exp \, c \int_0^{t-s} (\sigma \wedge 1)^{-\frac{1+\epsilon}{2}} \, d\sigma - 1 \right),$$

so that

$$|\Gamma(u_n^x)(t)|_E \leq \exp \int_0^{t-s} (\sigma \wedge 1)^{-\frac{1+\epsilon}{2}} \, d\sigma \sup_{r \in [s,t]} |\gamma(u_n^x)(r)|_E =: c_s(t) \sup_{r \in [s,t]} |\gamma(u_n^x)(r)|_E. \tag{5.13}$$

Next, if we set $v_n(t) := u_n^x(t) - \Gamma(u_n^x)(t)$, we have that v_n solves the problem

$$\frac{dv_n}{dt}(t) = Av_n(t) + F_n(t, v_n(t) + \Gamma(u_n^x)(t)), \quad v_n(s) = x.$$

We may assume without any loss of generality that v_n is a strict solution, otherwise we can approximate it by means of strict solutions of suitable approximating problems, see for example [6, Proposition 6.2.2]. Hence we obtain

$$\frac{d}{dt}^{-} |v_n(t)|_E \leq \left\langle Av_n(t), \delta_{v_n} \right\rangle_E + \left\langle F_n(t, v_n(t) + \Gamma(u_n^x)(t)), \delta_{v_n} \right\rangle_E = \left\langle Av_n(t), \delta_{v_n} \right\rangle_E$$
$$+ \left\langle F_n(t, v_n(t) + \Gamma(u_n^x)(t)) - F_n(t, \Gamma(u_n^x)(t)), \delta_{v_n} \right\rangle_E + \left\langle F_n(t, \Gamma(u_n^x)(t)), \delta_{v_n} \right\rangle_E,$$

where δ_{v_n} is the element of $\partial |v_n(t)|_E$ introduced in (2.3).

Thus, as F_n satisfies (5.4) and (5.5) for some $\Lambda(t)$ and $\Phi(t)$ independent of $n \in \mathbb{N}$, thanks to the Young inequality we get

$$\frac{d^-}{dt}|v_n(t)|_E \le \Lambda(t)|v_n(t)|_E + (\Lambda(t)+\Phi(t))\left(1+|\Gamma(u_n^x)(t)|_E^m\right).$$

Recalling that $u_n^x(t) = v_n(t) + \Gamma(u_n^x)(t)$, by a comparison argument this yields

$$|u_n^x(t)|_E \le |v_n(t)|_E+|\Gamma(u_n^x)(r)|_E \le c_s(t)\,|x|_E+c_s(t)\left(1+\sup_{r\in[s,t]}|\Gamma(u_n^x)(r)|_E^m\right),$$

so that, according to (4.12) with $\kappa = 1/m$ and to (5.13), for any $p \ge p_\star$ we have

$$\mathbb{E}\sup_{r\in[s,t]}|u_n^x(r)|_E^p \le c_{s,p}(t)\left(1+|x|_E^p + c_{s,p}^\gamma(t)\,\mathbb{E}\sup_{r\in[s,t]}|u_n^x(r)|_E^p\right). \tag{5.14}$$

Now, recalling that as shown in Theorem 4.2 and Remark 4.3 $c_{s,p}^\gamma(s) = 0$ and $c_{s,p}^\gamma$ is continuous, it follows that $c_{s,p}(s+t_0)c_{s,p}^\gamma(s+t_0) \le 1/2$, for some $t_0 > 0$, so that we obtain (5.11) for $T = t_0 + s$. Next we can repeat the same arguments in the intervals $[s+t_0, s+2t_0]$, $[s+2t_0, s+3t_0]$ and so on and (5.11) follows for any $T > s$ and $p \ge p_\star$. Finally, if $p < p_\star$ we have

$$|u_n^x|_{L_{s,T,p}(E)} \le |u_n^x|_{L_{s,T,p_\star}(E)} \le c_{s,p_\star}(T)\,(1+|x|_E). \qquad \square$$

Now we can conclude the proof of Theorem 5.3. For any $n \in \mathbb{N}$ and $x \in E$ we define

$$\tau_n^x := \inf\left\{\, t \ge s \,:\, |u_n^x(t)|_E \ge n \,\right\},$$

with the usual convention that $\inf\emptyset = +\infty$. As the sequence of stopping times $\{\tau_n^x\}$ is non-decreasing, we can define $\tau^x := \sup_{n\in\mathbb{N}}\tau_n^x$. Thanks to (5.11) we have clearly that $\mathbb{P}(\tau^x = \infty) = 1$. Indeed,

$$\mathbb{P}(\tau^x < \infty) = \lim_{T\to\infty}\mathbb{P}(\tau^x \le T)$$

and for each $T > s$

$$\mathbb{P}(\tau^x \le T) = \lim_{n\to\infty}\mathbb{P}(\tau_n^x \le T).$$

Now, for any fixed $n \in \mathbb{N}$ and $T > s$ we have

$$\mathbb{P}(\tau_n^x \le T) = \mathbb{P}\left(\sup_{t\in[s,T]}|u_n^x(t)|_E \ge n\right) \le \frac{1}{n}\,\mathbb{E}\sup_{t\in[s,T]}|u_n^x(t)|_E \le \frac{c_{s,1}(T)}{n}(1+|x|_E),$$

so that $\mathbb{P}(\tau_n^x \le T)$ goes to zero, as n goes to infinity, and $\mathbb{P}(\tau^x = \infty) = 1$.

Therefore, for any $t \ge s$ and $\omega \in \{\,\tau^x = \infty\,\}$ there exists $n \in \mathbb{N}$ such that $t \le \tau_n^x(\omega)$ and then we can define

$$u_s^x(t)(\omega) := u_n^x(t)(\omega).$$

Notice that this is a good definition, as for any $t \leq \tau_n^x \wedge \tau_m^x$ we have $u_n^x(t) = u_m^x(t)$, $\mathbb{P}$-a.s. Actually, if we set $\eta := \tau_n^x \wedge \tau_m^x$, with $n \leq m$, thanks to (5.9) we have

$$u_n^x(t\wedge\eta) - u_m^x(t\wedge\eta) = \psi(u_n^x - u_m^x)(t\wedge\eta) + \gamma(u_n^x)(t\wedge\eta) - \gamma(u_m^x)(t\wedge\eta)$$

$$+\int_s^{t\wedge\eta} e^{(t\wedge\eta - r)C}\left[F_n(r, u_n^x(r)) - F_m(r, u_m^x(r))\right] dr$$

$$= \psi(u_n^x - u_m^x)(t\wedge\eta) + \gamma(u_n^x)(t\wedge\eta) - \gamma(u_m^x)(t\wedge\eta)$$

$$+\int_s^t I_{\{r\leq\eta\}} e^{(t\wedge\eta - r)C}\left[F_m(r\wedge\eta, u_n^x(r\wedge\eta)) - F_m(r\wedge\eta, u_m^x(r\wedge\eta))\right] dr.$$

Then, recalling that $F_m(t,\cdot)$ is Lipschitz-continuous uniformly with respect to $t \in [s, T]$, for any t we easily get

$$\begin{aligned}\sup_{r\in[s,t]}\left|u_n^x(r\wedge\eta) - u_m^x(r\wedge\eta)\right|_E &\leq \sup_{r\in[s,t]}\left|\psi(u_n^x - u_m^x)(r\wedge\eta)\right|_E\\ &+\sup_{r\in[s,t]}\left|\gamma(u_n^x)(r\wedge\eta) - \gamma(u_m^x)(r\wedge\eta)\right|_E\\ &+c_m(T)\int_s^t \sup_{r'\in[s,r]}\left|u_n^x(r'\wedge\eta) - u_m^x(r'\wedge\eta)\right|_E dr'. \end{aligned} \tag{5.15}$$

According to (3.8) with $X = E$ we have

$$\left|\psi(u_n^x - u_m^x)(r\wedge\eta)\right|_E \leq c_s^\psi(r\wedge\eta)\sup_{r'\in[s,r\wedge\eta]}|(u_n^x - u_m^x)(r')|_E$$

$$\leq c_s^\psi(r\wedge\eta)\sup_{r'\in[s,r]}|(u_n^x - u_m^x)(r'\wedge\eta)|_E.$$

Moreover, as shown in the proof of Theorem 4.2, $[\gamma(u_n^x) - \gamma(u_m^x)](r\wedge\eta)$ can be rewritten as

$$\begin{aligned}&\left[\gamma(u_n^x) - \gamma(u_m^x)\right](r\wedge\eta)\\ &= c_{\alpha_\star}\int_s^r (r\wedge\eta - r')^{\alpha_\star - 1} e^{(r\wedge\eta - r')C} I_{\{r'\leq\eta\}} v_{\alpha_\star}(u_n^x, u_m^x)(r'\wedge\eta)dr',\end{aligned}$$

where $\alpha_\star$ is the constant introduced in (4.8), $c_{\alpha_\star} = \sin\pi\alpha_\star/\pi$ and

$$v_{\alpha_\star}(u_n^x, u_m^x)(r'\wedge\eta) = \int_s^{r'} (r'\wedge\eta - \rho)^{-\alpha_\star}$$

$$e^{(r'\wedge\eta - \rho)C} I_{\{\rho\leq\eta\}}\left[G(\rho\wedge\eta, u_n^x(\rho\wedge\eta)) - G(\rho\wedge\eta, u_m^x(\rho\wedge\eta))\right] Q\, dw(\rho).$$

Therefore, we can repeat the same arguments used in the proof of Theorem 4.2 (by substituting the processes u_s^x and v_s^x respectively by the processes $u_n^x(\cdot\wedge\eta)$ and $u_m^x(\cdot\wedge\eta)$) and we obtain

$$\mathbb{E}\sup_{r\in[s,t]}|\left[\gamma(u_n^x) - \gamma(u_m^x)\right](r\wedge\eta)|_E \leq c_{s,1}^\gamma(t)\,\mathbb{E}\sup_{r\in[s,t]}|\left(u_n^x - u_m^x\right)(r\wedge\eta)|_E.$$

Then, collecting all terms, from (5.15) it follows

$$\mathbb{E}\sup_{r\in[s,t]}\left|(u_n^x-u_m^x)(r\wedge\eta)\right|_E\leq\left(c_s^\psi(t)+c_{s,1}^\gamma(t)\right)\mathbb{E}\sup_{r\in[s,t]}\left|(u_n^x-u_m^x)(r\wedge\eta)\right|_E$$

$$+c_m(T)\int_s^t\mathbb{E}\sup_{r'\in[s,r]}\left|(u_n^x-u_m^x)(r'\wedge\eta)\right|_E\,dr'.$$

If we take $t_0>0$ such that $c_s^\psi(s+t_0)+c_{s,1}^\gamma(s+t_0)\leq 1/2$, this yields

$$\begin{aligned}\mathbb{E}\sup_{r\in[s,s+t_0]}\left|(u_n^x-u_m^x)(r\wedge\eta)\right|_E&\leq 2\,c_m(T)\\&\times\int_s^{s+t_0}\mathbb{E}\sup_{r'\in[s,r]}\left|(u_n^x-u_m^x)(r'\wedge\eta)\right|_E\,dr',\end{aligned}$$

so that $u_n^x(t\wedge\eta)=u_m^x(t\wedge\eta)$, for any $t\in[s,s+t_0]$. As $u_n^x((s+t_0)\wedge\eta)-u_m^x((s+t_0)\wedge\eta)=0$, we can repeat the same argument in the interval $[s+t_0,s+2t_0]$ and so on and we get

$$u_n^x(t)=u_m^x(t),\qquad t\leq\tau_n^x\wedge\tau_m^x. \tag{5.16}$$

Now, recalling that $u_s^x(t)$ is taken equal $u_n^x(t)$, if $\omega\in\{\tau^x=+\infty\}$ and $t\leq\tau_n^x$, thanks to (5.9) we have

$$\begin{aligned}u_s^x(t)\;&=e^{(t-s)C}x+\int_s^t e^{(t-r)C}\left[Lu_s^x(r)+F(r,u_s^x(r))\right]dr\\&+\int_s^t e^{(t-r)C}G(r,u_s^x(r))Q\,dw(r),\end{aligned}$$

$\mathbb{P}$-a.s., so that u_s^x is a mild solution of problem (5.6).

Next, we show that such solution is unique. If v_s^x is another solution of system (5.6), by proceeding as in the proof of (5.16) it is possible to show that for any $n\in\mathbb{N}$

$$u_s^x(t\wedge\tau_n^x)=v_s^x(t\wedge\tau_n^x),\qquad t\geq s,$$

and then, as $\{\tau_n^x\leq T\}\downarrow\{\tau^x\leq T\}$ for any $T>s$, we have that $u_s^x=v_s^x$.

Finally, we have to show that $u_s^x\in L^p(\Omega;C((s,T];E)\cap L^\infty(s,T;E))$, for any $p\geq 1$ and $T>s$. Since

$$\sup_{t\in[s,T]}|u_s^x(t)|_E^p=\lim_{n\to+\infty}\sup_{t\in[s,T]}|u_s^x(t)|_E^p I_{\{T\leq\tau_n^x\}}=\lim_{n\to+\infty}\sup_{t\in[s,T]}|u_n^x(t)|_E^p I_{\{T\leq\tau_n^x\}},$$

from estimate (5.11) and the Fatou lemma we obtain (5.8). Continuity of trajectories follows from arguments analogous to those used in Theorem 4.2. □

Next, we show that without assuming condition (5.7) on the growth of g and assuming a more restrictive condition on the drift f, it is possible to prove an existence and uniqueness result analogous to Theorem 5.3.

Theorem 5.5. *Assume that the constant m in Hypothesis 4-2 is greater than* 1 *and that there exist* $a > 0$ *and* $\beta \in L^\infty_{loc}[0,\infty)$ *such that for each* $i = 1, \dots, r$

$$(k_i(t,\xi,\sigma_i+\rho_i) - k_i(t,\xi,\sigma_i))\,\rho_i \leq -a\,|\rho_i|^{m+1} + \beta(t)\left(1+|\sigma_i|^{m+1}\right), \quad (5.17)$$

for any $\xi \in \overline{\mathcal{O}}$ *and* $t, \sigma_i, \rho_i \in \mathbb{R}$. *Then, under Hypotheses 1 to 4, for any* $x \in E$ *system (5.6) admits a unique mild solution* $u_s^x \in L^p(\Omega; C((s,T];E) \cap L^\infty(s,T;E))$, *with* $p \geq 1$ *and* $T > s$. *Moreover,* u_s^x *satisfies estimate (5.8).*

Notice that condition (5.17) is fulfilled for example by the functions k_i described at point 2 in Remark 5.1.

Proof. For any $n \in \mathbb{N}$ and $x \in E$ we define

$$(G_n(t,x)y)(\xi) := g_n(t,\xi,x(\xi))y(\xi), \qquad (t,\xi) \in [0,\infty) \times \overline{\mathcal{O}},$$

where

$$g_n(t,\xi,\sigma) := \begin{cases} g(t,\xi,\sigma) & \text{if } |\sigma| \leq n \\ g(t,\xi,n\sigma/|\sigma|) & \text{if } |\sigma| > n. \end{cases}$$

It is immediate to check that if $n < m$ and $t \geq 0$

$$|x|_E \leq n \Longrightarrow G_n(t,x) = G_m(t,x) = G(t,x)$$

and

$$|G_n(t,x)| \leq \Psi(t)\,(1+|x|_E), \qquad x \in E,$$

for some $\Psi \in L^\infty_{loc}[0,\infty)$ independent of $n \in \mathbb{N}$. Moreover, $g_n(t,\xi,\cdot)$ is Lipschitz-continuous and bounded. Then due to Theorem 5.3 for any $n \in \mathbb{N}$ and $x \in E$ the problem

$$du(t) = [Au(t) + F(t,u(t))]\,dt + G_n(t,u(t))Q\,dw(t), \qquad u(s) = x, \quad (5.18)$$

admits a unique solution u_n^x. If we show that an estimate analogous to (5.11) is verified in this case, then as in the proof of Theorem 5.3 we obtain the existence of a unique solution u^x for system (5.6) which fulfill estimate (5.8).

If $\Gamma_n(u_n^x)$ is the solution of the problem

$$dv(t) = Av(t)\,dt + G_n(t,u_n^x(t))\,dw(t), \qquad v(s) = 0,$$

and $v_n(t) := u_n^x(t) - \Gamma_n(u_n^x)(t)$, by proceeding as in the proof of Lemma 5.4 we have

$$\frac{d^-}{dt}|v_n(t)|_E \leq \left\langle Av_n(t), \delta_{v_n}\right\rangle_E$$

$$+\left\langle F(t,v_n(t)+\Gamma(u_n^x)(t)) - F(t,\Gamma(u_n^x)(t)), \delta_{v_n}\right\rangle_E + \left\langle F(t,\Gamma(u_n^x)(t)), \delta_{v_n}\right\rangle_E.$$

Thanks to (5.17), it is not difficult to show that for any $x, h \in E$ and $t \geq 0$

$$\langle F(t,x+h) - F(t,x), \delta_h\rangle_E \leq -a\,|h|_E^m + \beta(t)\left(1+|x|_E^m\right), \quad (5.19)$$

where δ_h is the element of $\partial|h|_E$ described in (2.3). Thus, by using the Young inequality we have

$$\frac{d}{dt}^{-}|v_n(t)|_E \leq -\frac{a}{2}|v_n(t)|_E^m + c\left(1+|\Gamma_n(u_n^x)(t)|_E^m\right).$$

By a comparison argument, as $u_n^x(t) = v_n(t) + \Gamma_n(u_n^x)(t)$, we obtain

$$|u_n^x(t)|_E \leq |v_n(t)|_E + |\Gamma_n(u_n^x)(r)|_E \leq |x|_E + c\left(1+\sup_{r\in[s,t]}|\Gamma_n(u_n^x)(r)|_E\right).$$

Thanks to (4.12) and (5.13) we get (5.14) and as in the proof of Lemma 5.4 this allows to conclude that

$$|u_n^x|_{L_{s,T,p}(E)} \leq c_{s,p}(T)\,(1+|x|_E),$$

for any $p \geq 1$ and the theorem follows. □

We conclude this section by showing that the solution u_s^x depends continuously on the initial datum $x \in E$.

Proposition 5.6. *Under the same hypotheses of Theorems 5.3 and 5.5, for any* $0 \leq s < T$ *and* $p \geq 1$ *the mapping*

$$x \in E \mapsto u_s^x \in L_{s,T,p}(E)$$

is continuous, uniformly on bounded sets of E.

Proof. For each $n \in \mathbb{N}$ the mapping F_n introduced in the proof of Theorem 5.3 is Lipschitz-continuous. Then, if u_n^x is the solution of problem (5.10) it is not difficult to show that for any $x, y \in E$

$$|u_n^x - u_n^y|_{L_{s,T,p}(E)} \leq c_{n,s,p}(T)\,|x-y|_E. \tag{5.20}$$

Now, let τ_n^x and τ_n^y as in the proof of Theorem 5.3 and assume that g fulfills condition (5.7). For any $p \geq 1$ and $0 \leq s < T$ we have

$$|u_s^x - u_s^y|_{L_{s,T,p}(E)}^p$$

$$= \mathbb{E}\sup_{r\in[s,T]}|u_s^x(r)-u_s^y(r)|_E^p I_{\{\tau_n^x\wedge\tau_n^y>T\}} + \mathbb{E}\sup_{r\in[s,T]}|u_s^x(r)-u_s^y(r)|_E^p I_{\{\tau_n^x\wedge\tau_n^y\leq T\}}$$

$$\leq |u_n^x - u_n^y|_{L_{s,T,p}(E)}^p + \mathbb{E}\sup_{r\in[s,T]}|u_s^x(r)-u_s^y(r)|_E^p I_{\{\tau_n^x\wedge\tau_n^y\leq T\}}$$

$$\leq |u_n^x - u_n^y|_{L_{s,T,p}(E)}^p + c_p\left(1+|u_s^x|_{L_{s,T,2p}(E)}^p + |u_s^y|_{L_{s,T,2p}(E)}^p\right)\left(\mathbb{P}\left(\tau_n^x\wedge\tau_n^y \leq T\right)\right)^{\frac{1}{2}}.$$

Due to (5.11) we have

$$\mathbb{P}\left(\tau_n^x \wedge \tau_n^y \leq T\right) \leq \mathbb{P}\left(\sup_{r\in[s,T]} |u_n^x(r)|_E \geq n\right) + \mathbb{P}\left(\sup_{r\in[s,T]} |u_n^y(r)|_E \geq n\right)$$

$$\leq \frac{1}{n^2}\left(|u_n^x|^2_{L_{s,T,2}(E)} + |u_n^y|^2_{L_{s,T,2}(E)}\right) \leq \frac{c^2_{s,2}(T)}{n^2}\left(1 + |x|^2_E + |y|^2_E\right)$$

and then, by using once more (5.11), due to (5.20) we obtain

$$|u_s^x - u_s^y|^p_{L_{s,T,p}(E)} \leq c^p_{n,s,p}(T)|x-y|^p_E + \frac{\tilde{c}_{s,2}(T)}{n}\left(1 + |x|^{p+1}_E + |y|^{p+1}_E\right).$$

Therefore, once fixed $\epsilon > 0$ and $R > 0$ we first find $\bar{n} \in \mathbb{N}$ such that

$$\sup_{|x|_E, |y|_E \leq R} \frac{\tilde{c}_{s,2}(T)}{\bar{n}}\left(1 + |x|^{p+1}_E + |y|^{p+1}_E\right) \leq \frac{\epsilon}{2}$$

and then, in correspondence of such $\bar{n}$, we determine $\delta > 0$ such that

$$|x-y|_E \leq \delta \Longrightarrow c^p_{\bar{n},s,p}(T)|x-y|^p_E \leq \frac{\epsilon}{2}.$$

This concludes our proof under condition (5.7).

Now, we assume that the conditions of Theorem 5.5 are satisfied. We use for the solution u_n^x of problem (5.18) what we have just proved above and we have

$$|u_n^x - u_n^y|_{L_{s,T,p}(E)} \leq c_{n,s,p}(T)\,|x-y|_E.$$

Then, as the processes u_n^x satisfy the a-priori estimates (5.11), we introduce the stopping times τ_n^x and τ_n^y and we conclude our proof. □

6. Existence of an invariant measure

In what follows we shall denote by $B_b(E)$ the Banach space of bounded Borel measurable functions $\varphi : E \to \mathbb{R}$, endowed with the *sup-norm*

$$\|\varphi\|_0 = \sup_{x\in E} |\varphi(x)|.$$

Moreover, we shall denote by $C_b(E)$ the subspace of continuous functions. Finally, we shall denote by $C^1_b(E)$ the Banach space of differentiable functions $\varphi : E \to \mathbb{R}$, having continuous and bounded derivative, endowed with the norm

$$\|\varphi\|_1 = \|\varphi\|_0 + \sup_{x\in E} |D\varphi(x)|_{E^\star}.$$

Throughout this section we assume that the coefficients F and G in problem (5.6) do not depend on t. For any $x \in E$, $t \geq 0$ and $\varphi \in B_b(E)$ we define

$$P_t\varphi(x) = \mathbb{E}\,\varphi(u^x(t)),$$

where u^x is the solution of (5.6) starting from x at time $s = 0$. P_t is the *transition semigroup* associated with system (5.6). Due to Proposition 5.6 P_t is a *Feller* semigroup, that is it maps $C_b(E)$ into itself. Actually, if φ belongs to $C_b^1(E)$, for any $x, y \in E$ we have

$$|P_t\varphi(x) - P_t\varphi(y)| \leq \mathbb{E}\left|\varphi(u^x(t)) - \varphi(u^y(t))\right| \leq \|\varphi\|_1 \mathbb{E}\,|u^x(t) - u^y(t)|_E \to 0,$$

as $|x - y|_E \to 0$. If $\varphi \in C_b(E)$ we can approximate it in the sup-norm by a sequence $\{\varphi_n\} \subset C_b^1(E)$ and then, as P_t is a contraction semigroup, we obtain that $P_t\varphi \in C_b(E)$.

Our aim here is to prove that P_t has an invariant measure, that is there exists a probability measure μ on $(E, \mathcal{B}(E))$ such that for any $t \geq 0$ and $\varphi \in C_b(E)$

$$\int_E P_t\varphi(x)\,d\mu(x) = \int_E \varphi(x)\,d\mu(x).$$

As the embedding of $C^\theta(\overline{\mathcal{O}};\mathbb{R}^r)$ into E is compact for any $\theta > 0$, once we prove that for some $\theta > 0$ and $a \geq 0$

$$\sup_{t\geq a} \mathbb{E}\,|u^x(t)|_{C^\theta(\overline{\mathcal{O}};\mathbb{R}^r)} < \infty,$$

we have that the family of probability measures $\{P_t(x,\cdot)\}_{t\geq a}$ is tight. Due to the Krylov-Bogoliubov theorem (for all details and proofs see e.g. [9]), this implies the existence of an invariant measure.

To this purpose, we start with the following result.

Proposition 6.1. *Under the same Hypotheses of Theorem 5.5, for any $p \geq 1$*

$$\mathbb{E}\sup_{t\geq 0} |u^x(t)|_E^p \leq c_p(1 + |x|_E^p).$$

Proof. For any $\lambda > 0$ we consider the problem

$$dv(t) = (A - \lambda)v(t)\,dt + G(u^x(t))Q\,dw(t), \qquad v(0) = 0$$

and we denote by $\Gamma_\lambda(u^x)$ its solution. Clearly, $\Gamma_\lambda(u^x)$ is the unique fixed point of the mapping $\psi_\lambda + \gamma_\lambda$, with ψ_λ and γ_λ defined respectively in (3.10) and (4.13). By proceeding as in the proof of Lemma 5.4 we obtain

$$|\Gamma_\lambda(u^x)(t)|_E \leq c_\lambda \sup_{r\in[0,t]} |\gamma_\lambda(u^x)(r)|_E,$$

where

$$c_\lambda := \exp\int_0^\infty e^{-\lambda s}(s \wedge 1)^{-\frac{1+\epsilon}{2}}\,ds.$$

Then, due to (4.14) and to Proposition 4.5, for any $p \geq p_\star$ and $T > 0$

$$\mathbb{E}\sup_{t\in[0,T]} |\Gamma_\lambda(u^x)(t)|_E^p \leq c_\lambda^p\,|c_{0,p}^{\gamma,\lambda}|_\infty^p \left(1 + |u^x|_{L_{0,T,p}(E)}^p\right). \tag{6.1}$$

Now, if we set $v := u^x - \Gamma_\lambda(u^x)$, we have that v solves the problem

$$\frac{dv}{dt}(t) = (A-\lambda)v(t) + \lambda\, u^x(t) + F(u^x(t)), \qquad v(0) = x.$$

Therefore, due to (5.19)

$$\begin{aligned}\tfrac{d^-}{dt}|v(t)|_E &\le \left\langle (A-\lambda)v(t), \delta_{v(t)}\right\rangle_E + \left\langle F(v(t)+\Gamma_\lambda(u^x)(t))\right.\\ &\quad \left. - F(\Gamma_\lambda(u^x)(t)), \delta_{v(t)}\right\rangle_E + \left\langle F(\Gamma_\lambda(u^x)(t)) + \lambda u^x(t), \delta_{v(t)}\right\rangle_E\\ &\le -\tfrac{a}{2}|v(t)|_E^m + c\left(1 + |\Gamma_\lambda(u^x)(t)|_E^m\right) + \lambda\,|u^x(t)|_E.\end{aligned}$$

By comparison, as in the proof of Lemma 5.4, this yields

$$|u^x(t)|_E \le |x|_E + c\left(1 + \sup_{r\in[0,t]}|\Gamma_\lambda(u^x)(r)|_E + \lambda^{\frac{1}{m}}\sup_{r\in[0,t]}|u^x(r)|_E^{\frac{1}{m}}\right),$$

so that, thanks to (6.1) and to the Young inequality

$$|u^x|^p_{L_{0,T,p}(E)} \le |x|_E^p + \frac{1}{4}|u^x|^p_{L_{0,T,p}(E)} + c\,c_\lambda^p|c_{0,p}^{\gamma,\lambda}|_\infty^p\left(1 + |u^x|^p_{L_{0,T,p}(E)}\right) + c + \lambda^{\frac{p}{m-1}}.$$

According to (4.15) we can find $\bar\lambda > 0$ such that

$$c\,c_{\bar\lambda}^p|c_{0,p}^{\gamma,\bar\lambda}|_\infty^p \le \frac{1}{4}$$

and then we get

$$|u^x|^p_{L_{0,T,p}(E)} \le 2\,|x|_E^p + 2\left(c\,c_{\bar\lambda}^p|c_{0,p}^{\gamma,\bar\lambda}|_\infty^p + c + \lambda^{\frac{p}{m-1}}\right),$$

which immediately implies our thesis. □

Now we can prove that the semigroup P_t has an invariant measure.

Theorem 6.2. *Under the same hypotheses of Theorem 5.5 there exists an invariant measure for the transition semigroup P_t.*

Proof. As we have already seen, we have only to show that there exist $\bar\theta > 0$ and $a > 0$ such that

$$\sup_{t\ge a}\mathbb{E}\,|u^x(t)|_{C^{\bar\theta}(\overline{\mathcal{O}};\mathbb{R}^r)} < \infty. \tag{6.2}$$

Due to (3.11), for any $\lambda > 0$ and $p \ge p_\star$, if $\xi, \eta \in \overline{\mathcal{O}}$ and $\theta < 1$

$$\begin{aligned}&\mathbb{E}\sup_{r\in[0,t]}\left|\psi_\lambda(u^x)(r,\xi) - \psi_\lambda(u^x)(r,\eta)\right|^p\\ &\le \mathbb{E}\sup_{r\in[0,t]}\left|\psi_\lambda(u^x)\right|^p_{C^\theta(\overline{\mathcal{O}};\mathbb{R}^r)}|\xi-\eta|^{\theta p} \le |c_{0,\theta}^{\psi,\lambda}|_\infty^p|u^x|^p_{L_{0,t,p}(E)}|\xi-\eta|^{\theta p}.\end{aligned}$$

Similarly, if $\theta < \theta_\star$, with $\theta_\star$ as in Remark 4.6, due to (4.16) we have

$$\mathbb{E}\sup_{r\in[0,t]}\left|\gamma_\lambda(u^x)(r,\xi) - \gamma_\lambda(u^x)(r,\eta)\right|^p \le |c_{0,p}^{\gamma,\lambda}|_\infty^p\left(1 + |u^x|^p_{L_{0,t,p}(E)}\right)|\xi-\eta|^{\theta p}.$$

Moreover, by using again (2.6), we have

$$\left|\int_0^t e^{(t-r)(C-\lambda)}\left(F(u^x(r))+\lambda u^x(r)\right)\,dr\right|_{C^\theta(\overline{\mathcal{O}};\mathbb{R}^r)}$$
$$\leq c_\lambda \int_0^t e^{-\lambda(t-r)}\,((t-r)\wedge 1)^{-\frac{\theta}{2}}\left(1+|u^x(r)|_E^m\right)\,dr$$

and then

$$\mathbb{E}\sup_{r\in[0,t]}\left|\int_0^r e^{(r-\sigma)(C-\lambda)}\left(F(u^x(\sigma))+\lambda u^x(\sigma)\right)\,d\sigma\right|^p_{C^\theta(\overline{\mathcal{O}};\mathbb{R}^r)}$$
$$\leq c_{\lambda,p}\left(1+|u^x|^{mp}_{L_{0,t,mp}(E)}\right)\left(\int_0^r e^{-\lambda\sigma}\,(\sigma\wedge 1)^{-\frac{\theta}{2}}\,d\sigma\right)^p$$
$$=: \widetilde{c}_{\lambda,p}\left(1+|u^x|^{mp}_{L_{0,t,mp}(E)}\right).$$

As

$$u^x(t) = e^{t(C-\lambda)}x+\int_0^t e^{(t-r)(C-\lambda)}\left(F(u^x(r))+\lambda u^x(r)\right)\,dr+\psi_\lambda(u^x)(t)$$
$$+\gamma_\lambda(u^x)(t),$$

thanks to (2.6) and to Proposition 6.1, collecting all terms we find some constant $c_{\lambda,p}>0$ such that for any $\xi,\eta\in\overline{\mathcal{O}}$, $a\in(0,1)$ and $\theta<\theta_\star\wedge 1$

$$\mathbb{E}\sup_{r\in[a,t]}\left|u^x(r,\xi)-u^x(r,\eta)\right|^p\leq c_{\lambda,p}\left(a^{-\frac{\theta p}{2}}|x|_E^p+1+|x|_E^{mp}\right)|\xi-\eta|^{\theta p}.$$

This means that if we take $\alpha<\theta$ we obtain

$$\mathbb{E}\iint_{\overline{\mathcal{O}}\times\overline{\mathcal{O}}}\sup_{r\in[a,t]}\frac{|u^x(r,\xi)-u^x(r,\eta)|^p}{|\xi-\eta|^{d+\alpha p}}\,d\xi d\eta$$
$$\leq c\iint_{\overline{\mathcal{O}}\times\overline{\mathcal{O}}}|\xi-\eta|^{\theta p-d-\alpha p}\,d\xi d\eta<\infty.$$

Therefore, $u^x(t)\in W^{\alpha,p}(\overline{\mathcal{O}};\mathbb{R}^r)$, $\mathbb{P}$-a.s. and due to the Fatou lemma

$$\mathbb{E}\sup_{r\in[a,\infty)}|u^x(r)|_{W^{\alpha,p}(\overline{\mathcal{O}};\mathbb{R}^r)}=: c_{a,|x|_E}<\infty.$$

Thanks to the Sobolev embedding theorem this implies that there exists some $\bar{\theta}>0$ such that u^x takes values in $C^{\bar{\theta}}(\overline{\mathcal{O}};\mathbb{R}^r)$ and (6.2) holds. By using the Krylov-Bogoliubov theorem, this allows us to conclude that there exists an invariant measure. □

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