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Stochastic signaling: information substitutes and complements

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Abstract

I develop a model of stochastic costly signaling in the presence of exogenous imperfect information, and study whether equilibrium signaling decreases ('information substitutes') or increases ('information complements') if the accuracy of exogenous information increases. A stochastic pure costly signaling model is shown to have a unique sequential equilibrium in which at least one type (and possibly all) engages in costly signaling. In the presence of exogenous information, a unique threshold level of prior beliefs generically exists which separates the cases of information complements and substitutes. More accurate exogenous information can induce a less informative signaling equilibrium, and can result in a lower expected accuracy of the uninformed party's equilibrium beliefs. An application to signaling in networks, in which a social network is the source of exogenous information, qualifies the relation between network characteristics (size, density, centrality, component size) and equilibrium signaling.

Keywords: Monotonic Costly Signaling, Stochastic Signaling, Noisy Signaling, Networks, Advertising, Job Market Signaling, Conspicuous Consumption

JEL: C72, D82

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1 Introduction

Costly signaling, pioneered by a.o. Veblen (1899) and Spence (1973),[1] explains ostentatious wasteful behavior by humans, animals and organizations as a credible way of communicating private information to uninformed others. In recent decades, costly signaling was applied to a broad variety of problems.[2] Yet, we observe considerable variation in the degree to which informed parties engage in costly signaling. Veblen (1899(1994), pp.53-55) observes that "*Conspicuous consumption claims a relatively larger portion of the income of the urban than of the rural population, and the claim is also more imperative.* [...] *So it comes, for instance, that the American farmer and his wife and daughters are notoriously less modish in their dress, as well as less urbane in their manners, than the city artisan's family with equal income.* [...] *And in the struggle to outdo one another the city population push their normal standard of conspicuous consumption to a higher point* [...]." Veblen suggests the availability of exogenous information next to costly signaling as an explanation for these variations. "*The means of communication and the mobility of the population now expose the individual to the observation of many persons who have no other means of judging of his reputability than the display of goods [...]. One's neighbors, mechanically speaking, often are socially not one's neighbors, or even acquaintances; and still their transient good opinion has a high degree of utility.*"

In real world signaling problems, costly signaling is usually an imperfect source of information, while uninformed parties often have other information about an informed party's private information. In a job market example, an academic degree can imperfectly reflect an potential employee's productivity, because of luck with the exam questions or rather a bad day during the exams. Employers can have trouble judging a program's difficulty. Moreover, they may have other information about the job candidate in the form of tests in the recruitment stage, know the candidate's reputation through social relations or observe certain ethnic or cultural markers. For advertising, consumers may imperfectly observe the actual amount of advertising or have difficulty judging a firm's advertising costs, and can also rely on the reputation of a firm or its country of origin, or on product tests in consumer magazines.

Veblen's observations suggest the hypothesis that a better availability and quality of exogenous information already provides more information

[1] See Riley (2001) for a survey.

[2] Examples include labor economics (Spence, 1973), advertising (Milgrom and Roberts, 1986), finance (Myers and Majluf 1984, John and Williams, 1985, Bhattacharya 1979), animal behavior and morphology (Zahavi, 1975, Grafen, 1990a,b) consumption (Frank, 1999; or Truyts (2010) for a recent survey).

to uninformed parties to distinguish between different sender types, and thus reduces the need for informed parties to engage in costly signaling. In this case, costly signaling and exogenous information can be called 'information substitutes', in the sense that more accurate exogenous information reduces the incentives to engage in costly signaling. The natural counterpart of information substitutes would be 'information complements', in which case more accurate exogenous information increases the incentives to engage in costly signaling. We distinguish between two kinds of exogenous information. For ethnic markers or brand reputation, the informed party likely knows the realization of the imperfect exogenous signal when choosing a signaling strategy, such that this information results in equilibrium in a change in prior beliefs. For tests at the recruitment stage, product tests or social reputation, the informed party may likely not know the realization of the exogenous signal when choosing a signaling strategy. This last case is the main focus of this paper.

To study the dependence of costly signaling on the accuracy of exogenous information in a smooth and nontrivial way, this paper develops a model of stochastic signaling, in which both a strategically chosen costly signal and exogenous information are imperfect signals concerning an informed party's private information. The model relates to a small literature on stochastic or noisy costly signaling, initiated by Matthews and Mirman (1983). Matthews and Mirman introduced noise in terms of demand shocks in a limit pricing model and demonstrate a number of advantages of stochastic signaling games over their typical non-stochastic counterparts (e.g. Spence, 1973, Riley, 1979): a limited number of equilibria, smooth comparative statics and a solution which depends on prior beliefs.[3] Carlsson and Dasgupta (1997) exploit the first property and use a stochastic game with vanishing noise as an equilibrium selection criterion for non-stochastic costly signaling games. De Haan et al. (2011) and Jeitschko and Norman (2009) formulate a model of stochastic costly signaling and test its implications experimentally. Hertzendorf (1993) formulates a stochastic version of Milgrom and Roberts' (1986) model of advertising as costly signaling, in which a monopolist with private information about product quality has two instruments, price and advertising, to signal quality to consumers. Hertzendorf shows that the imperfectly observed signal (advertising) is only used in equilibrium if the two producer types pool in terms of the perfectly observed price signal. Feltovich et al. (2002) consider a model of non-stochastic costly

[3] See e.g. Mailath et al. (1993) for a critique of this last feature of non-stochastic costly signaling games.

signaling in the presence of imperfect exogenous information. For three types, they show the possibility of an equilibrium in which the middle type engages in costly signaling and the high type pools with the low type at zero signaling if she is sufficiently confident the second exogenous signal will separate her from the lowest type ('countersignaling'). Regarding status consumption as an imperfect signal of ability in the presence of exogenous imperfect information, Frank (1985) concludes that if uninformed parties aggregate both information sources linearly by means of a minimum variance unbiased estimator, "*the ability-signaling rationale* [...] *suggests that incentives to distort consumption in favor of observable goods will be inversely related to the amount and reliability of independent information that exists concerning individual abilities*", such that conspicuous consumption and exogenous information are information substitutes.

We assume that an informed player (Sender) with binary private information sends a signal to an uninformed player (Receiver). This signal is distorted by random noise. After observing a distorted signal, Receiver forms beliefs about Sender's private information and chooses a best reply. In the above stochastic signaling models, Receiver has a binary choice, which results in combination with a monotonicity condition on the error distribution in a cut-off strategy as best reply.[4] We rather consider Receiver having a continuous choice set, and this qualitatively alters the signaling equilibrium. The resulting stochastic signaling game highlights the zero sum nature of the signaling game: in equilibrium, the expected gains in terms of Receiver's action to an increase in signaling efforts of one Sender type are exactly the expected losses of the other type (when compensated for relative frequency). The stochastic signaling game has, under mild regularity conditions, at most two equilibria. A pooling equilibrium always exists and if the high Sender type's signaling costs are not prohibitively high, then a unique 'informative' equilibrium exists, in which the high Sender type signals strictly more than the low type. As in the above stochastic games, the signaling equilibrium is never separating in the sense that Receiver's equilibrium beliefs are never degenerate, such that Receiver never chooses the best action given the true type of Sender. Contrary to the nonstochastic and stochastic pure costly signaling games of respectively Spence (1973) and de Haan et al. (2011), both the low and high Sender type can in equilibrium engage in costly signaling.[5] This 'informative' equilibrium allows

[4]Cut-off rule: Receiver chooses the action most preferred by Sender if $y \geq y^*$, with y^* an optimally chosen threshold.

[5]In the stochastic signaling models of Matthews and Mirman (1983), Carlsson and

for smooth comparative statics of equilibrium signaling levels and of the expected accuracy of Receiver's beliefs with respect to prior beliefs and the costs and benefits of signaling. The availability of imperfect exogenous information about Sender's private information to Receiver does not alter these equilibrium properties qualitatively. Moreover, the effect of a marginal increase in the accuracy of exogenous information is non-monotonic. Generically, a unique threshold level of prior beliefs exists which separates the cases of information complements and substitutes. Moreover, a marginal increase in the accuracy of exogenous information can in some cases result in a lower expected accuracy of Receiver's equilibrium beliefs, due to the changes in signaling strategies this induces. These results are then applied to signaling in networks, where the source of the exogenous information is a social network.

The paper is structured as follows. The second section introduces the formal setting and suggests some specific examples. The third section analyzes equilibrium signaling in a baseline game without exogenous information. In the fourth section, Receiver observes exogenous imperfect information about Sender's private information. The fifth section applies our results to signaling in networks. The last section concludes.

2 Setting

A player, Sender, has private information about a quality parameter θ ('her type'), which is either high or low: $\theta \in \{\theta_H, \theta_L\}$. She cares about the beliefs of an uninformed player, Receiver, about θ. Receiver has prior belief $p \in (0,1)$ that θ is high, and deems θ low with probability $1-p$. Sender sends a costly signal $s \in \bar{\mathbb{R}}_+$. As in Carlsson and Dasgupta (1997),[6] Receiver observes this signal imperfectly as y, the sum of s and random error term ε:

$$y = s + \varepsilon. \tag{1}$$

Error term ε is independently distributed according to a density function φ, with $E(\varepsilon) = 0$ and a variance which is finite and bounded away from zero. Assume that φ satisfies the following properties.

Condition 1 *Let φ be a C^2 probability density function which*

1. (symmetry) *is symmetric around the mean,*

Dasgupta (1997), Hertzendorf (1993) and Jeitschko and Norman (2009), Sender's optimal choice in absence of signaling concerns depends on her type.

[6] Carlsson and Dasgupta (1997) demonstrate how this additive technology encompasses a.o. the demand shocks model of Matthews and Mirman (1983).

2. (MLR) *satisfies the strict monotone likelihood ratio property,*[7]

3. (support) *has full support on* $\mathbb{R}$*.*

Prominent examples of distributions satisfying condition 1 are the normal and logistic distributions. Continuous differentiability, full support and MLR are in line with Matthews and Mirman (1983), Carlsson and Dasgupta (1997) and de Haan et al. (2011). Full support on $\mathbb{R}$ implies that all y have an equilibrium interpretation. As such, the stochastic signaling game does not have the difficulty of specifying out-of-equilibrium beliefs and the resulting multitude of equilibria, typical for non-stochastic signaling games.

Sender's preferences are represented by a utility function

$$u(s, y|\theta, \beta) = v(s|\theta) + \kappa\beta(y) \tag{2}$$

in which $\beta(y)$ represents Receiver's posterior 'believed' probability of Sender being a high type, given y. Parameter $\kappa > 0$ represents Sender's constant marginal utility of β.

Condition 2 *Let v be such that $v_1(.) < 0$ and $v_{12}(.) > 0$.*

Condition 2 means that signals are costly and imposes a standard Spence-Mirrlees single crossing (or supermodularity) condition, and is strengthened later by an additional strict concavity condition. The utility function in (2) departs from the utility function in Matthews and Mirman (1983), Carlsson and Dasgupta (1997) and de Haan et al. (2011) in two respects. First, it takes Receiver's beliefs directly as an argument. This either represents a problem in which Sender cares about Receiver's beliefs directly, or is shorthand notation by omitting an explicit analysis of Receiver's optimal choice of action in function of her beliefs. Receiver's choice of action is easily introduced explicitly into this game, as the examples at the end of this section illustrate. Second, whereas above authors assume a Receiver with a binary action choice, which in combination with MLR implies that Receiver's best reply is a cut off rule, (2) assumes that Sender's utility is linear in $\beta(y)$. Hence, we assume that Receiver's choice set is a continuum, while the linearity most plausibly means that Receiver's best choice is linear in her beliefs

[7] For two means $\mu > \mu'$, a density function $\varphi(\varepsilon|\mu)$ satisfies the strict monotone likelihood ratio property (MLR) if the ratio $\frac{\varphi(\varepsilon|\mu)}{\varphi(\varepsilon|\mu')}$ is everywhere strictly increasing with ε. Note that this is equivalent to log-supermodularity of φ w.r.t. ε and μ, i.e. that for $\varepsilon > \varepsilon'$ and $\mu > \mu'$: $\varphi(\varepsilon|\mu)\varphi(\varepsilon'|\mu') > \varphi(\varepsilon'|\mu)\varphi(\varepsilon|\mu')$. See a.o. Karlin and Rubin (1956) or Athey (2002).

and that Sender's is utility linear in Receiver's action. Although restrictive at first sight, this linearity guarantees analytic tractability, but also keeps information issues well separated from risk issues in the decision problems of Sender and Receiver.

Using (1), we write the density function of y given s as $\varphi(y|s)$. Sender maximizes expected utility, considering all possible realizations of ε, which is for a given interpretation of all distorted signals $\beta(y)$:

$$\mathrm{E}u(s,y|\theta,\beta) = v(s|\theta) + \kappa B(s), \tag{3}$$

with

$$B(s) \equiv \int \beta(y)\,\varphi(y|s)dy.$$

Receiver's beliefs are consistent with a given signaling strategy profile $\mathbf{s} = (s_L, s_H)$ and denoted $\overset{\circ}{\beta}_s(y)$ if they satisfy for each y Bayes' rule:

$$\begin{aligned}\overset{\circ}{\beta}_s(y) &= \frac{p\varphi(y|s_H)}{(1-p)\varphi(y|s_L) + p\varphi(y|s_H)} \\ &= \left(1 + \frac{1-p}{p}\frac{\varphi(y|s_L)}{\varphi(y|s_H)}\right)^{-1}.\end{aligned} \tag{4}$$

The last formulation in (4) illustrates that MLR implies for $s_L < s_H$ strictly increasing consistent posterior beliefs $\overset{\circ}{\beta}_s(y)$. We consider pure strategy sequential equilibria of the stochastic signaling game.[8] A number of standard examples in the literature fit this setting naturally.

Example 1 (Status Signaling) *Sender types differ ex ante only in income $0 < m_L < m_H < \infty$, and Sender cares directly about Receiver's beliefs. Sender spends her income on intrinsically valuable rest consumption or ostentatious waste. Utility is represented by $v^{SS}(m_\theta - s) + \kappa\beta(y)$, with $v^{SS}(.)$ the utility of rest consumption and $v_1^{SS}(.) > 0$ and $v_{11}^{SS}(.) < 0$.*

Example 2 (Job Market Signaling) *As in Spence (1973), Sender is a job candidate of high or low productivity θ, and invests in otherwise useless education s at cost $\frac{-v^E(s)}{\theta}$ with $v_1^E(.) > 0$ and $v_{11}^E(.) > 0$. Receiver*

[8] A Sequential Equilibrium (S.E.) is described by a pair $(\mathbf{s}, \beta)$ of strategy profile and posterior beliefs, such that:
1. $\mathbf{s}$ maximizes expected utility of each type given $\beta(y)$:

$$s_\theta \in \arg\max_s v(s|\theta) + \kappa \int \beta(y)\,\varphi(y|s)dy.$$

2. Beliefs $\beta(y)$ are Bayesian consistent with equilibrium strategies $\mathbf{s}$ as in (4).

is an employer in a competitive job market, who sees a noisy educational score y, and offers in equilibrium a contract with wage $w(y) = \theta_L + \beta(y)(\theta_H - \theta_L)$. Given this wage $w(y)$, expected utility of Sender is $\theta_L - \frac{v^E(s)}{\theta} + (\theta_H - \theta_L) B(s)$. Note that w never equals the true productivity of Sender in this stochastic job market signaling game. The return to education thus only concerns a period needed by employers to learn about Sender's true productivity and to change a possibly rigid contract.

Example 3 (Advertising) *For simplicity, we take the price exogenously fixed. As in Milgrom and Roberts (1986) and Hertzendorf (1993), Sender is a monopolist, selling a product of high or low quality θ to a continuum of consumers, distributed uniformly on $[0,1]$. Sender can invest in advertising s at strictly increasing and strictly convex costs $v^{AD}(s)$ in a first period, after which consumers decide to buy the product or not. Only consumers who buy the product observe true quality θ, and can buy the product again in a second period (they all do if θ is high). Consumers buy the product if they deem the probability of a high quality product higher than their position on $[0,1]$.*[9] *If each consumer draws an independent y, and profits per unit sold are π_θ (with $2\pi_H > \pi_L$), then profits of a high and low quality monopolist are respectively $-v^{AD}(s) + 2\pi_H B(s)$ and $-v^{AD}(s) + \pi_L B(s)$, such that Sender's preferences can be written as $B(s) - \frac{v^{AD}(s)}{2\pi_H}$ and $B(s) - \frac{v^{AD}(s)}{\pi_L}$.*

3 Baseline Signaling Game

Consider beliefs consistent (as in (4)) with an arbitrary pure strategy profile. Three features of such consistent beliefs facilitate the equilibrium analysis. First, Bayesian consistency of beliefs $\overset{\circ}{\beta}_s(y)$ implies that the weighted average of $B(s_\theta)$ for both types, weighted by prior p, always equals p

$$\begin{aligned} &pB(s_H) + (1-p) B(s_L) \\ &= \int \overset{\circ}{\beta}_s(y) \left[p\varphi(y|s_H) + (1-p)\varphi(y|s_L)\right] dy = p. \end{aligned}$$

Bayesian consistency thus implies that the stochastic signaling game is in equilibrium a zero sum game in $B(s_\theta)$ between both Sender types, such that

$$B(s_L) = \frac{p}{1-p}(1 - B(s_H)). \tag{5}$$

[9] If a consumer's willingness to pay for a high and low quality product is resp. $\lambda_H > \lambda_L > 0$, she buys at price γ if $\beta(y) 2(\lambda_H - \gamma) + (1 - \beta(y))(\lambda_L - \gamma) \geq 0$, i.e. if $\beta(y) \geq \frac{\gamma - \lambda_L}{(\lambda_H - \gamma) + (\lambda_H - \lambda_L)} = \zeta$. We thus assume ζ uniformly distributed on $[0,1]$. Consumers with ζ negative or greater than 1 never and always buy, respectively. See also Milgrom and Roberts (1986) and Hertzendorf (1993).

Second, because the distribution of ε is independent of $\mathbf{s}$, we can define $\Delta \equiv s_H - s_L$ and write

$$B(\Delta) \equiv \int \frac{p(\varphi(y|\Delta))^2}{(1-p)\varphi(y|0) + p\varphi(y|\Delta)} dy = \int \beta^{\circ}(y)\,\varphi(y|\Delta)dy = B(s_H).^{10} \tag{6}$$

And third, $B(\Delta)$ strictly increases with Δ in two respects. First, let

$$B'(\Delta) \equiv \int \beta^{\circ}(y)\left(-\varphi'(y|\Delta)\right)dy$$

represent the marginal increase in the expected value of $\beta^{\circ}(y)$ for a high Sender type, who takes beliefs $\beta^{\circ}(y)$ as given.[11] Second, let $\frac{\partial B(\Delta)}{\partial \Delta}$ be the effect of a marginal increase in Δ which also changes the consistent beliefs $\beta^{\circ}(y)$.

Lemma 1 *If Receiver's beliefs are consistent with strategy profile* $\mathbf{s}$*:*

1. *the stochastic signaling game is zero sum in* B*:*

$$B(s_L) = \frac{p}{1-p}(1 - B(\Delta)),$$

2. B *depends only on* $\Delta = s_H - s_L$ *and not on actual levels of* $\mathbf{s}$

3. *the average expected utility of Sender*

$$U(\mathbf{s}) \equiv p \cdot Eu(s, y|\theta_H, \beta) + (1-p)\,Eu(s, y|\theta_L, \beta)$$

is such that $\mathbf{s} < \mathbf{s}' \Rightarrow U(\mathbf{s}) > U(\mathbf{s}')$ *and* U *is maximal at* $\mathbf{s} = \mathbf{0}$.

4. $B'(\Delta) > 0$ *and* $\frac{\partial B(\Delta)}{\partial \Delta} > 0$ *if* $\Delta > 0$, *while* $B'(0) = \frac{\partial B(0)}{\partial \Delta} = 0$.

Proof. In appendix. ■

For now, assume $\Delta \geq 0$ and consistent beliefs (as in (4)). Because the distribution of error terms φ has full support on $\mathbb{R}$, all distorted signals y can always originate from both Sender types. Therefore, no S.E. of this stochastic signaling game can ever be separating in the sense of degenerate equilibrium beliefs, as stressed by Carlsson and Dasgupta (1997). As a result, Receiver's equilibrium choice of action is always suboptimal with respect to Sender's true type. Yet, different strategy profiles $\mathbf{s}$

[11] We denote $-\frac{\partial \varphi(y|\Delta)}{\partial \Delta} = \frac{\partial \varphi(y|\Delta)}{\partial y} \equiv \varphi'(y|\Delta)$.

can communicate in expectation different amounts of information to Receiver, such that the expected accuracy of Receiver's consistent beliefs is strictly increasing with Δ for $\Delta > 0$. Moreover, both Sender types are fighting an arms race for a fixed amount of expected equilibrium beliefs of Receiver. By increasing Δ, the high Sender type increases expected impression $B(s_H)$ by being confused less with the low type. This implies an equal decrease in the low type's expected impression on Receiver (compensated for priors), because of being confused less with the high type. The low type achieves maximal equilibrium expected belief at $s_L = s_H$. The high Sender type can move away from this outcome by engaging in more signaling. The low Sender type can move closer again by also engaging in costly signaling. The division of expected impression on Receiver depends only on distance Δ, such that the amount of signaling s_L of each Sender type is purely 'wasted': exogenously taking a quantity s_L from both Sender types' signal (and adapting Receiver's beliefs) and burning it leaves both Sender types' utility unaffected. For the example of status signaling, where Receiver forms an opinion about Sender without further interests, utilitarian welfare (taking prior beliefs as true population frequencies) is independent of the distribution of B, but strictly decreasing with the signaling efforts of either Sender type. Welfare then is maximal when all consumers spend income only on rest consumption.

Taking the interpretation of distorted signals $\beta^{\circ}(y)$ as given, the Kuhn-Tucker conditions for the high and low Sender types are

$$s_H \left(v_1 \left(s_H|\theta_H\right) + \kappa B'\left(\Delta\right)\right) = 0 \tag{7}$$

$$v_1 \left(s_H|\theta_H\right) + \kappa B'\left(\Delta\right) \leq 0 \text{ and } s_H \geq 0$$

and

$$s_L \left(v_1 \left(s_L|\theta_L\right) + \kappa \frac{p}{1-p} B'\left(\Delta\right)\right) = 0 \tag{8}$$

$$v_1 \left(s_L|\theta_L\right) + \kappa \frac{p}{1-p} B'\left(\Delta\right) \leq 0 \text{ and } s_L \geq 0,$$

while equilibrium beliefs are characterized by (4). Since $B'(0) = 0$ and $v_1(.) < 0$, a pooling equilibrium with $s_H = s_L = 0$ and $\beta^{\circ}(.) = p$ always exists. If Sender's problem is strictly concave for all $\mathbf{s}$,[12] one can construct for each Sender type a function similar to best response functions

[12] An extensive characterization of strict concavity of Sender's problem in terms of the fundamentals is provided in section A.2 of the appendix.

in e.g. Cournot games. For the high Sender type, such a function indicates the unique level of s_H which satisfies (7) for consistent beliefs (4) for each level of s_L under the restriction that $\Delta \geq 0$ (as $\Delta < 0$ cannot be an equilibrium, cfr. infra). After constructing a similar function for the low Sender type, any crossing of both functions constitutes a S.E. We call such S.E. with $\Delta > 0$ 'informative' and demonstrate that such an informative equilibrium is unique if it exists.[13]

Proposition 1 *If φ and v satisfy respectively conditions 1 and 2 and if Sender's problem is strictly concave, then*

1. *At most two S.E. in pure strategies exist:*
   - *a pooling equilibrium with $s_H = s_L = 0$ and $\beta^{\circ}(.) = p$ always exists,*
   - *a unique 'informative' equilibrium with $s_H \neq s_L$ exists if* (7) *has an interior solution.*
2. *In the informative S.E. in pure strategies, $\Delta > 0$.*
3. *The informative S.E. in pure strategies is asymptotically stable on $\mathbb{R}^2_{++}$, given an interpretation of distorted signals according to* (4)*.*
4. *In the informative S.E., $s_L > 0$ if* (8) *has an interior solution.*

Proof. In appendix ■

Hence, a unique informative S.E. in pure strategies exists if (7) has an interior solution. The existence and uniqueness of such an informative equilibrium was already noted in the previous stochastic signaling models, such as Matthews and Mirman (1983), Carlsson and Dasgupta (1997) and de Haan et al. (2011). However, contrary to the non-stochastic or stochastic pure costly signaling models of respectively Spence (1973) and de Haan et al. (2011), the low Sender type does engage in costly signaling in the informative equilibrium unless her optimal choice is a corner solution. Nothing prevents the low Sender type from wasting means in order to appear more often as a high Sender type with greater likelihood. The proof of the existence and uniqueness of the informative S.E. is essentially a contraction mapping argument, such that we can allow for $\bar{\mathbb{R}}^2_+$ as the strategy space. The contraction mapping argument makes the informative S.E. interesting in two respects: it is the unique S.E. in which information is transferred from Sender to Receiver, and it has attractive

[13] 'Informative' distinguishes this S.E. from separating S.E. in non-stochastic games, which imply degenerate equilibrium beliefs (cfr. supra).

convergence properties. If we conceive a learning dynamic in which we start with an arbitrary $\mathbf{s}$ with $\Delta > 0$ and with Receiver having beliefs consistent with $\mathbf{s}$, and then let each Sender type subsequently choose a best reply (according to (7) and (8)) while Receiver adapts her beliefs to ensure consistency with the new strategy profile after every adaptation of a Sender type, then Sender's strategy profile and Receiver's beliefs converge in this process to the informative S.E.

As underlined by a.o. Matthews and Mirman (1983), Carlsson and Dasgupta (1997) and de Haan et al. (2011), an important advantage of non-stochastic costly signaling games concerns smooth comparative statics. Depending on the different parameters of the model, the informative S.E. is characterized by different levels of signaling as well as a different expected accuracy of Receiver's equilibrium beliefs.

Proposition 2 *Let the conditions of proposition 1 apply and* (7) *and* (8) *have interior solutions. If* $B''(\Delta) \equiv \frac{\partial B'(\Delta)}{\partial \Delta}$, *then in the informative S.E.:*

1. $\frac{\partial s_L}{\partial \kappa} > 0$ *and* $\frac{\partial s_H}{\partial \kappa} > 0$,
2. $\frac{\partial s_H}{\partial \theta_H} > 0$ *and* $\frac{\partial s_L}{\partial \theta_L} > 0$
3. $\frac{\partial s_L}{\partial \theta_H} B''(\Delta) > 0$ *and* $\frac{\partial s_H}{\partial \theta_L} B''(\Delta) < 0$,
4. $\frac{\partial s_H}{\partial p}$ *and* $\frac{\partial s_L}{\partial p}$ *depend on* p *and* $B''(\Delta)$ *such that for a threshold* $\tilde{p} < \frac{1}{2}$*:*

	$p < \tilde{p}$	$p > \tilde{p}$
$B''(\Delta) > 0$	$\frac{\partial s_L}{\partial p} > 0, \frac{\partial s_H}{\partial p} \gtrless 0$	$\frac{\partial s_L}{\partial p} \gtrless 0, \frac{\partial s_H}{\partial p} < 0$
$B''(\Delta) < 0$	$\frac{\partial s_L}{\partial p} > 0, \frac{\partial s_H}{\partial p} > 0$	$\frac{\partial s_L}{\partial p} > 0, \frac{\partial s_H}{\partial p} \gtrless 0.$

Proof. In appendix ■

First, a marginal increase in Sender's marginal utility from Receiver's beliefs induces *ceteris paribus* an increase in the signaling levels of both Sender types.[14] Second, a marginal quality increase of one Sender type (i.e. marginal decrease in signaling costs) implies an increase in equilibrium signaling of that Sender type. Third, the effect of such a quality increase on the other Sender type's signaling crucially depends on whether $B'(\Delta)$ increases or decreases with Δ at equilibrium $\mathbf{s}$. A marginal increase in θ_L increases equilibrium s_H if $B'(\Delta)$ decreases with Δ at

[14] Or equivalently: a marginal increase in Receiver's (linear) choice of action and/or Sender's marginal utility from this action.

equilibrium $\mathbf{s}$, because increasing θ_L implies an increase in s_L and thus a decrease in Δ. Similarly, a marginal increase in θ_H increases equilibrium s_L if $B'(\Delta)$ increases with Δ at equilibrium $\mathbf{s}$.

A marginal increase in the accuracy of an exogenous information source of which the realization is known to Sender while choosing a signaling strategy affects equilibrium signaling through a marginal change in prior beliefs p. Examples include family reputation, cultural or ethnic markers and brand reputation. Such a marginal change of p affects signaling incentives of a Sender type directly, and through a changed signaling strategy of the other type. The direct effect of a marginal increase of p always increases signaling incentives for the low Sender type, while for the high Sender type a threshold $\tilde{p}$ exists such that equilibrium signaling increases with p for p below $\tilde{p}$ and decreases for p higher than $\tilde{p}$. If p is very low, $B(\Delta)$ is very low before and after a marginal increase in Δ, and a marginally higher prior likelihood that θ is high induces that a marginal increase in Δ makes more of a difference. If p is already high, a marginal increase in p diminishes the incentives to engage in signaling for the high type, as even $B(0)$ is already fairly high. For the low Sender type, a marginal increase in p also implies that by (5) the costs of marginal increase in Δ must be carried by 'less' low types, such that this raises the stakes of the signaling game and always induces higher signaling for the low Sender type as a direct effect.
As a result, a marginal increase in prior beliefs increases equilibrium s_L if $p < \tilde{p}$, such that both types signal more or if $p > \tilde{p}$ and $B''(\Delta) < 0$, such that a decrease in s_H increases $B'(\Delta)$. If $p > \tilde{p}$ and $B''(\Delta) > 0$, a decrease in s_H decreases $B'(\Delta)$, and this effect can offset the direct effect of a marginal increase of p for the low type, so that the overall effect on s_L is undetermined. For the high Sender type, both the direct effect and effect through increased signaling by the low Sender type increase signaling incentives if $p < \tilde{p}$ and $B''(\Delta) < 0$ (such that an increase in s_L implies higher $B'(\Delta)$), and both effects are negative if $p > \tilde{p}$ and $B''(\Delta) > 0$. For the other cases, the direct and indirect effects counteract, such that $\frac{\partial s_H}{\partial p}$ can take both signs.

How is Receiver affected by such marginal changes in the informative S.E.? As stated above, the expected accuracy of Receiver's beliefs strictly increases with Δ in the informative S.E. (with $\Delta > 0$). The difference in both Sender types' changes in signaling strategies as a consequence of a given marginal change in signaling incentives crucially depends on how the marginal utility cost of signaling marginally changes. First, note that (7) and (8) imply that signaling in an interior informative S.E. is characterized by

$$\frac{p}{1-p}v_1\left(s_H|\theta_H\right)=v_1\left(s_L|\theta_L\right)=-\kappa\frac{p}{1-p}B'\left(\Delta\right),$$

and define

$$h\left(\Delta\right)\equiv v_{11}\left(s_L|\theta_L\right)-\frac{p}{(1-p)}v_{11}\left(s_H|\theta_H\right)$$

as the difference in the rate at which the marginal utility costs of signaling increase for both types, relative to prior beliefs p.

Proposition 3 *Let φ and v satisfy conditions 1 and 2, the consumer problem be strictly concave and* (7) *and* (8) *have interior solutions. Then the equilibrium difference in signaling levels Δ is such that*

1. $\frac{\partial\Delta}{\partial\kappa}$ *has the opposite sign of* $h\left(\Delta\right)$,
2. $\frac{\partial\Delta}{\partial\theta_H}>0$ *and* $\frac{\partial\Delta}{\partial\theta_L}<0$ *and*
3. $\frac{\partial\Delta}{\partial p}<0$ *if* $p>\tilde{p}$ *or if* $h\left(\Delta\right)>0$.
   $\frac{\partial\Delta}{\partial p}>0$ *is only possible if* $p<\tilde{p}$ *and* $h\left(\Delta\right)<0$ *and* $-v_{11}\left(s_L|\theta_L\right)$ *is sufficiently large.*

Hence, if the marginal utility cost of signaling increases relatively faster for the high than for the low Sender type (i.e. relative to prior beliefs p such that $h\left(\Delta\right)>0$), then a marginal increase in the marginal utility from Receiver's beliefs induces a decrease in Δ, and in the expected accuracy of Receiver's beliefs. A decrease in the marginal costs of signaling of the high Sender type increases *ceteris paribus* the expected accuracy of Receiver's equilibrium beliefs, while a decrease in the marginal costs of signaling of the low Sender type achieves the opposite. Finally, a marginal increase in prior beliefs induces a worsening in the expected accuracy of Receiver's beliefs (conditional on p) if prior beliefs are sufficiently high $(p>\tilde{p})$, such that a marginal increase in prior beliefs induces a decrease in direct signaling incentives for the high Sender type $\left(\frac{\partial B'(\Delta)}{\partial p}<0\right)$, or if the marginal utility cost of signaling increases if relatively faster for the high than for the low Sender type (i.e. relative to prior beliefs p such that $h\left(\Delta\right)>0$). A marginal increase in prior beliefs can only lead to a marginal improvement in the expected accuracy of Receiver's beliefs if this increases the high Sender type's direct incentives to engage in signaling $\left(\frac{\partial B'(\Delta)}{\partial p}>0\right)$ and if the marginal utility cost of signaling increases sufficiently fast for the low Sender type.

4 Information Substitutes and Complements

In many real world settings, Receiver has other information about Sender's type, of which Sender does not know the realization when choosing a signaling strategy. Examples include gossip for the case of status signaling, information through social connections or tests during the recruitment process for job market signaling and product tests in magazines for advertising. Receiver then has two (or more) imperfect sources of information at her disposal which should be aggregated. The accuracy of exogenous information crucially determines Sender's signaling incentives. If a marginal increase in the accuracy of exogenous information decreases signaling levels in the informative S.E., we call costly signaling and exogenous information 'information substitutes'. If a marginal increase in accuracy increases equilibrium signaling, we call both information sources 'information complements'.

Assume that Nature sends Receiver an exogenous imperfect signal about θ, of which the distribution and realization are independent of Sender's signaling. Assume for simplicity a binary exogenous signal

$$\omega \in \{L, H\},$$

of which the accuracy is represented by

$$q \equiv \Pr(\omega = H|\theta_H) = \Pr(\omega = L|\theta_L),$$

with $\frac{1}{2} < q < 1$. After observing a pair of imperfect signals (y, ω), Receiver's consistent beliefs are

$$\tilde{\beta}^{\circ}(y, H) = \left(1 + \frac{(1-q)}{q}\frac{(1-p)}{p}\frac{\varphi(y|0)}{\varphi(y|\Delta)}\right)^{-1} \tag{9}$$

$$\tilde{\beta}^{\circ}(y, L) = \left(1 + \frac{q}{(1-q)}\frac{(1-p)}{p}\frac{\varphi(y|0)}{\varphi(y|\Delta)}\right)^{-1}. \tag{10}$$

If Receiver has consistent beliefs, the expected value of Receiver's beliefs is for the high Sender type

$$\tilde{B}(\Delta) = \int \left(q\tilde{\beta}^{\circ}(y, H) + (1-q)\tilde{\beta}^{\circ}(y, L)\right)\varphi(y|\Delta)\, dy,$$

while by the same zero sum property as before:

$$\tilde{B}(s_L) = \frac{p}{1-p}\left(1 - \tilde{B}(\Delta)\right).$$

An interior informative S.E. satisfies first order conditions

$$v_1(s_H|\theta_H) + \kappa \tilde{B}'(\Delta) = 0 \tag{11}$$

$$v_1(s_L|\theta_L) + \kappa \frac{p}{1-p} \tilde{B}'(\Delta) = 0,$$

with

$$\tilde{B}'(\Delta) \equiv \int \left(q\tilde{\beta}^{\circ}(y,H) + (1-q)\tilde{\beta}^{\circ}(y,L) \right) (-\varphi'(y|\Delta))\, dy.$$

This stochastic signaling game with exogenous imperfect information satisfies the same properties as the baseline game, and has, under a similar strict concavity condition, at most one informative S.E.

Theorem 1 *If φ and v satisfy, respectively, conditions 1 and 2 and if Sender's problem is strictly concave, then*

1. *the stochastic signaling game is such that:a pooling equilibrium always exists, and a unique 'informative' sequential equilibrium with $\Delta > 0$ exists if (11) has an interior solution*

2. *in the informative S.E., for all $q \in \left(\frac{1}{2}, \bar{q}\right)$, with $\bar{q} = \frac{2+\sqrt{3}}{4} \cong 0.93301$, a unique level $\bar{p}(q)$ exists such that for*

$$p < \bar{p}(q) \Longrightarrow \frac{\partial s_H}{\partial q} > 0 \text{ and if } s_L > 0 \text{ then } \frac{\partial s_L}{\partial q} > 0$$

$$p > \bar{p}(q) \Longrightarrow \frac{\partial s_H}{\partial q} < 0 \text{ and if } s_L > 0 \text{ then } \frac{\partial s_L}{\partial q} < 0$$

Moreover, $\bar{p}(q)$ is a continuous function of q on $\left(\frac{1}{2}, \bar{q}\right)$.

Proof. In appendix. ■

Figure 1 illustrates $\bar{p}(q)$ for φ the normal density function with $\sigma = 2$ and for three values of Δ. Below $\bar{p}(q)$, both imperfect information sources are information complements, while both are information substitutes above $\bar{p}(q)$.

How should we understand this result? Note that $\frac{\partial \tilde{B}'(\Delta)}{\partial q}$ consists of four terms:

$$\frac{\partial \tilde{B}'(\Delta)}{\partial q} = \int \tilde{\beta}^{\circ}(y,H)(-\varphi'(y|\Delta))\, dy - \int \tilde{\beta}^{\circ}(y,L)(-\varphi'(y|\Delta))\, dy$$

$$+q \int \frac{\partial \tilde{\beta}^{\circ}(y,H)}{\partial q}(-\varphi'(y|\Delta))\, dy + (1-q) \int \frac{\partial \tilde{\beta}^{\circ}(y,L)}{\partial q}(-\varphi'(y|\Delta))\, dy.$$

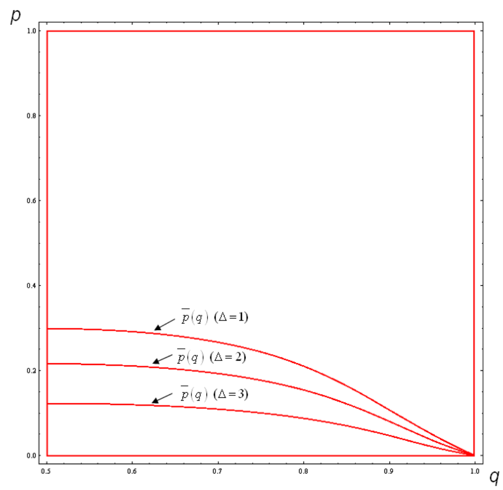


Figure 1: $\bar{p}\,(q)$ for φ the normal density function with $\sigma = 2$ at Δ equal to 1, 2 and 3.

The first two terms capture changes in the probability density of imperfect signal ω, while the last two terms capture changes in Receiver's consistent beliefs because of a marginal increase in the accuracy of ω.

The direction of the first effect of increasing q, which has high Sender types drawing $\omega = H$ more often, on signaling incentives depends on two elements in

$$\int \left(\tilde{\beta}^{\circ}\,(y,H) - \tilde{\beta}^{\circ}\,(y,L)\right)(-\varphi'\,(y|\Delta))\,dy,$$

the difference which ω makes for $\tilde{\beta}^{\circ}$ at a given y, i.e. $\tilde{\beta}^{\circ}\,(y,H)-\tilde{\beta}^{\circ}\,(y,L)\,,$ and the change in the occurrence of each y which a marginal increase in Δ induces: $-\varphi'\,(y|\Delta)\,.$ First, note that for sufficiently low and for sufficiently high y, both $\tilde{\beta}^{\circ}\,(y,H)$ and $\tilde{\beta}^{\circ}\,(y,L)$ are respectively close to zero and to one, such that an increase in ω from L to H or a marginal increase in y makes relatively little difference. For 'intermediate' values of y, which can originate from both high and low Sender types with nontrivial probabilities, a change in alternative signal ω or a marginal increase in y matter most. Which y are intermediate in this sense, depends on prior beliefs p. If p is sufficiently low, a marginal increase in y generates the highest increases in $\tilde{\beta}^{\circ}\,(y,H)$ for values of y which are mostly above Δ. Since $\tilde{\beta}^{\circ}\,(y,H) - \tilde{\beta}^{\circ}\,(y,L) > 0$, the average benefits of a marginal increase in Δ are greater if $\omega = H$ then if $\omega = L$, such that a more frequent occurrence of $\omega = H$ increases the marginal benefits of increasing Δ. If prior beliefs p are high, the values of y for which a marginal increase in y generates the greatest increase in β are largely below Δ if $\omega = H$. Thus, the benefits of a marginal increase in Δ are relatively small if $\omega = H$, and greater if $\omega = L$, such that a more frequent occurrence of $\omega = H$ decreases the average marginal benefits of increasing Δ. One can show that a unique threshold level of prior beliefs p exists such that a greater occurrence of $\omega = H$ increases the marginal benefits of increasing Δ for p below this threshold, and decreases marginal benefits if p is above this threshold.

The second effect concerns the change in Receiver's consistent beliefs. Note in (9) and (10) that, conditional on a realization of ω, the effect of a marginal increase in accuracy q much resembles a change in prior beliefs p. A marginal increase in q changes consistent beliefs given $\omega = H$ much like a marginal increase in p. As such, a unique threshold level of p exists (which decreases with q) such that $\int \frac{\partial \tilde{\beta}^{\circ}(y,H)}{\partial q} \left(-\varphi'(y|\Delta)\right) dy$ is positive for prior beliefs below this threshold, and negative for prior beliefs higher than this threshold. Similarly, if $\omega = L$, a marginal increase in q resembles a marginal decrease in p, such that another (lower) threshold p exists such that $\int \frac{\partial \tilde{\beta}^{\circ}(y,L)}{\partial q} \left(-\varphi'(y|\Delta)\right) dy$ is negative for prior beliefs below this threshold, and positive for prior beliefs higher than this threshold.

For the overall effect, theorem 1 states that a unique threshold level of prior beliefs $\bar{p}(q)$ exists, such that the two information sources y and ω are information complements for prior beliefs p smaller than $\bar{p}(q)$, and information substitutes for p above $\bar{p}(q)$. Generically, an open interval of prior beliefs always exists for which signaling and exogenous information are information complements. If prior beliefs p are sufficiently low, then more accurate exogenous information brings imperfect signals (y, ω) which are attributed by Receiver to a high Sender type with a higher probability more within reach of Sender. Because a marginal increase in y makes the greatest difference in Receiver's beliefs for these intermediate imperfect signals, equilibrium signaling of the high Sender type increases with the accuracy of exogenous information q if prior beliefs are sufficiently low. Given the zero sum nature of this stochastic signaling game, this increased potential for the high Sender type to distinguish herself also raises the stakes for the low Sender type. Therefore, an increase in q also increases equilibrium signaling of the low Sender type. Note that if Δ is higher, such that ceteris paribus more information is on average communicated by y, then the threshold level of prior beliefs $\bar{p}(q)$ below which a quality increase of ω increases signaling is lower, as illustrated in figure 1. The restriction to $q < \bar{q}$ is not a necessary condition, but reflects a limitation of our method of proof. Figure 1 illustrates that the result typically extends beyond $\bar{q}$.

Imagine, for example, a monopolist who wants to market a new product, about which specialized media will publish product tests to distinguish between a true innovation and a marketing scam. If the chance that the new product is truly an improvement is deemed sufficiently low by customers, then an improvement in the reliability of the product test will increase the advertising budget of both true innovators (high types) and would-be innovators selling junk. The reason is that for a sufficiently low prior probability of true innovation, more reliable product tests can more often convince customers that a truly good product could indeed be a true innovation. Without these product tests, the advertising investments needed to convince customers that the product is a true innovation with reasonable probability are prohibitively high. By enhancing the chances of true innovators to distinguish themselves from low quality imitators, more reliable product tests also raise the stakes for the latter, who accordingly react by increasing their advertising efforts to restore confusion with true innovators.

A marginal increase in the accuracy of exogenous information affects the expected accuracy of Receiver's equilibrium beliefs in two ways: directly and by changing equilibrium signaling:

$$\underbrace{\frac{\partial \tilde{B}(\Delta)}{\partial q}}_{\substack{\text{direct} \\ \text{effect}}} + \underbrace{\frac{\partial \tilde{B}(\Delta)}{\partial \Delta}\frac{\partial \Delta}{\partial q}}_{\substack{\text{indirect effect} \\ \text{through } \Delta}}.$$

Clearly, the first effect is always positive. For a given Δ, more accurate exogenous information allows Receiver to form in expectation more accurate equilibrium beliefs about Sender's type. What about the indirect effect through equilibrium Δ? As in the baseline case, the effect of a marginal increase in q on Δ depends crucially on $h(\Delta)$, the difference in the rate at which marginal utility costs of signaling increase for both types, relative to prior beliefs p. It turns out that an open interval of intermediate prior beliefs generically exists at which a marginal increase in q induces a decrease in Δ in an interior informative S.E., and thus a decrease in the amount of information which Receiver can in expectation extract from y. But what about the overall effect of a marginal increase in the accuracy of exogenous information? Can changes in equilibrium signaling outweigh the positive direct effect of increasing q, such that a marginal improvement in the accuracy of exogenous information eventually results in a lower expected accuracy of Receiver's equilibrium beliefs? The following proposition shows that this can be the case.

Proposition 4 *If the conditions of theorem 1 apply and if $q \in \left(\frac{1}{2}, \bar{q}\right)$, then in the informative S.E.:*

1. $\frac{\partial \Delta}{\partial q}$ *takes the opposite sign of* $h(\Delta) \frac{\partial s_\theta}{\partial q}$
2. *for information substitutes and φ the normal distribution, a point in parameter space exists such that*

$$\frac{\partial \tilde{B}(\Delta)}{\partial q} + \frac{\partial \tilde{B}(\Delta)}{\partial \Delta} \frac{\partial \Delta}{\partial q} < 0. \tag{12}$$

If we denote by $\hat{p}$ the level of prior beliefs for which $h(\Delta) = 0$, such that $\hat{p} = \frac{v_{11}(s_L|\theta_L)}{v_{11}(s_L|\theta_L)+v_{11}(s_H|\theta_H)}$, then the first part of proposition 4 shows that $\frac{\partial \Delta}{\partial q} < 0$ only in an open interval between $\bar{p}$ and $\hat{p}$. The second part shows that at least for φ the normal distribution, a negative indirect effect can outweigh the direct effect for information substitutes. This only occurs for very high $-v_{11}(s_L|\theta_L)$, such that a decrease in signaling incentives $\frac{\partial \tilde{B}'(\Delta)}{\partial q}$ results in a negligible reduction in signaling efforts of the low Sender type, or for $s_L = 0$ in equilibrium, while the high Sender type reacts much more to a marginal improvement in the accuracy of exogenous information. Note that because $\tilde{B}(\Delta)$ is continuous in all parameters under the conditions of theorem 1, the strict inequality in (12) implies that an infinite set of parameter profiles can be found for which the same inequality is true.

5 Application: Stochastic Signaling in Networks

Veblen's examples suggest social networks as a typical source of exogenous information ω. If the quality of the exogenous information decays as it travels through a social network, then a simple extension of the previous proposition allows for a characterization of equilibrium signaling in function of network characteristics. For the example of status signaling, Sender draws a Receiver (another consumer) from the population to judge her type. Receiver relies on conspicuous consumption and on gossip to judge Sender, and gossip is less reliable if Receiver is more distant to Sender in the social network. For the job market example, employer draws a consumer from the population to ask about the qualities of a job candidate, Sender, and this information is less reliable if its source is more remote to Sender in the social network.

A social network $g = (I, E)$ is defined by a set of N consumers I and a set of bilateral relations or links between these consumers $E \subseteq I^2$.[15] The geodesic distance between i and another consumer j, denoted $d(i, j|g)$, is the minimal number of links in E which form a path from i to j such that $\{(i, k), (k, l), ..., (m, j)\} \subseteq E$. If no such path between i and j exists, the geodesic distance is set to infinity. The geodesic distribution of network g for consumer i, denoted $D(k|i, g)$ indicates the share of the other $N-1$ consumers who are at most at geodesic distance k from i in g:

$$D(k|i, g) \equiv \frac{1}{N-1} \sum_{l=1}^{k} \left| n^l(i|g) \right|,$$

with $n^l(i|g)$ the l-th neighborhood of i in g, i.e. the set of consumers at geodesic distance l of i. If the accuracy of the exogenous signal of consumer j about i (Sender) is $q(d(i, j|g))$, then information decay means $q_1(.) < 0$. If i and j are not connected, then j receives no exogenous information about i through the network, and $\lim_{d(i,j)\to\infty} q(d(i, j|g)) = \frac{1}{2}$.

Upon observing a pair of imperfect signals (y, ω), a Receiver at distance k of Sender i forms beliefs

$$\begin{aligned}
\tilde{\beta}^{\circ}_k(y, H) &= \left(1 + \frac{1-q(k)}{q(k)} \frac{(1-p)}{p} \frac{\varphi(y|0)}{\varphi(y|\Delta_i)}\right)^{-1} \\
\tilde{\beta}^{\circ}_k(y, L) &= \left(1 + \frac{q(k)}{1-q(k)} \frac{(1-p)}{p} \frac{\varphi(y|0)}{\varphi(y|\Delta_i)}\right)^{-1},
\end{aligned}$$

and if each other consumer in I is drawn with equal probability, the expected value of Receiver's beliefs is

$$\bar{B}(\Delta|i, g) \equiv \sum_k D(k|i, g) \int \left(q(k)\tilde{\beta}^{\circ}_k(y, H) + (1-q(k))\tilde{\beta}^{\circ}_k(y, L)\right)\varphi(y|\Delta)\, dy.$$

An interior informative S.E. satisfies first order conditions

$$v_1(s_H|\theta_H) + \kappa \bar{B}'(\Delta|i, g) = 0 \tag{13}$$

$$v_1(s_L|\theta_L) + \kappa \frac{p}{1-p} \bar{B}'(\Delta|i, g) = 0.$$

Finally, define $Q(p)$ for a given p as the set of all q for which ω and s are information substitutes, i.e. $Q(p) \equiv \{q|\bar{p}(q) < p\}$, and let $\bar{d}(i, g)$ be the maximal geodesic distance to i in g. Note that the signaling incentives of Sender i now depend on (her position in) g. The signaling of both Sender i types is denoted $s_H(i, g)$ and $s_L(i, g)$.

[15] We exclude self-loops for technical reasons: $(i, i) \notin E$ for all $i \in I$.

Proposition 5 *If φ and v satisfy, respectively, conditions 1 and 2 and if Sender's problem is strictly concave, then*

1. *a pooling equilibrium always exists, and a unique 'informative' S.E. with $\Delta_i > 0$ exists if (13) has an interior solution, and this S.E. is asymptotically stable on the interior of $\mathbb{R}^2_+$.*

2. *if for Sender i and two networks g and g' all relevant q characterize information substitutes, i.e. $\left\{q(1), ..., q\left(\max\left\{\bar{d}(i,g), \bar{d}(i,g')\right\}\right)\right\} \subset Q(p)$, then in the informative S.E.*

$$D(k|i,g) \succ_{FOSD} D(k|i,g) \Longrightarrow \begin{cases} s_H(i,g) < s_H(i,g') \\ s_L(i,g) < s_L(i,g') \text{ if } s_L(i,g') > 0. \end{cases} \tag{14}$$

Proof. In appendix ■

Clearly, the dependence of signaling incentives on Sender's position in the social network makes the informative S.E. cognitively demanding. Receiver's equilibrium beliefs $\overset{\circ}{\tilde{\beta}}_k(y,\omega)$ depend on the distance to i in the network (k). Sender takes all these possible equilibrium beliefs into account when deciding on $\mathbf{s}$. Thus, Receiver's interpretation of (y,ω) also depends, through Δ_i, on the equilibrium beliefs given the other possible distances which ω can travel through the network. The uniqueness and asymptotic stability of the informative S.E. can counterbalance these concerns to some degree. Note also that for sufficiently low p, for which all relevant degrees of accuracy of ω characterize information complements $\left(\text{i.e. are in } \left(\frac{1}{2}, 1\right) \backslash Q(p)\right)$, the opposite result of (14) is true such that signaling is higher in the network with the first order stochastically dominant geodesic distribution. Also, note that this result is easily generalized to deal with different probabilities of drawing a source of ω from $I \backslash \{i\}$. If consumers closer to i are drawn more frequently by Receiver, this result should be reinterpreted for $D(k|i,g)$ reflecting the probability that a source of ω with accuracy $q(k)$ is drawn by Receiver. For the case of status signaling, Sender giving more weight to the opinion of other consumers closer to her in the social network can be accommodated for in the same way.

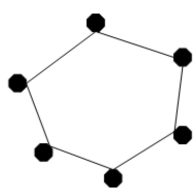
This result relates equilibrium signaling to some network characteristics for simple stylized networks.

Example 4 (Circular networks) *In a circular network $C(N,2)$, all consumers maintain two relations, such that without loss of generality $E = \{(1,2), (2,3), ..., (N-1,N), (N,1)\}$. If consumers form a relation to the neighbors of their neighbors, and to the neighbors of their neighbors' neighbors (in $C(N,2)$) etc., the circular network is generalized to*

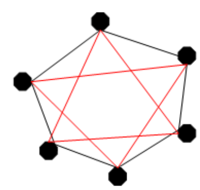
$C(N,k)$, *with* $k \in 2\mathbb{N}$ *the number of relations maintained by consumers such that (with* $0 < k \leq \frac{N-1}{2}$*).*

Example 5 (Nested Star) *In a nested star* $S(k,l)$, *one central consumer relates to* $k \geq 2$ *other consumers. These* k *consumers form relations with* k *others, and this is repeated* l *times, such that a unique path exists between any two distinct consumers* i *and* j *and that the central consumer maintains* k *relations, the least central consumers maintain 1 relation, and all the others maintain* $k+1$ *relations.*

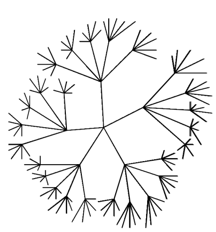
Example 6 (Two circular components) *The population is partitioned in two nonempty sets of respective cardinalities* N_1 *and* N_2, *which both form a circular network amongst each other,* $C(N_1,k)$ *and* $C(N_2,k)$



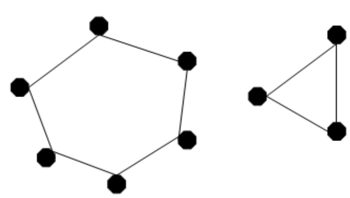
Circular network $C(6,2)$



Circular network $C(6,4)$



Nested star $S(5,3)$



Two components $C(6,2)$ and $C(3,2)$

These very stylized networks illustrate how equilibrium signaling depends on a number of network characteristics.

Corollary 1 *Under the conditions of proposition 5 and for the case of information substitutes, the equilibrium signaling of Sender* i *in the informative S.E.*

1. *increases with* N *and decreases with* k *in circular network* $C(N,k)$,

2. *is lower for more central consumers in nested star $S(k,l)$*

3. *is higher in smaller circular network components*

4. *increases if a new unconnected consumer (a 'stranger') is added to an arbitrary network g, if Sender i maintains at least one relation in g.*

5. *decreases with the addition of a new link (j,k) to an arbitrary network g iff this new link strictly decreases the geodesic distance of i to at least one other consumer, i.e. if $|d(i,j|g) - d(i,k|g)| \geq 2$.*

The first result in corollary 1 reflects Veblen's intuition about towns and villages. If prior beliefs are sufficiently high (such that ω and y are information substitutes), signaling is *ceteris paribus* higher in larger communities. Equilibrium signaling decreases *ceteris paribus* if consumers maintain more social relations, such that the social network is more dense. In the circular social networks, all consumers of the same Sender type face the same signaling incentives. This is no longer the case for the nested star network. Receiver draws on average more precise information about more central consumers in this network. If prior beliefs are sufficiently high to warrant information substitutes, then more central consumers waste (conditional on type) in equilibrium less on signaling than others who are socially more secluded. The size of network components produces similar asymmetries. Members of a small segregated minority, who nevertheless care about the good opinion of the whole population, spend in equilibrium more on costly signaling if prior beliefs are sufficiently high. This is consistent with repeated empirical findings of higher conspicuous consumption in ethnic minorities (e.g. Caplovitz (1967), Van Kempen (2007)). Finally, the two last results in corollary 1 provide some simple dominance relations which apply to arbitrary social networks.

6 Conclusions

Humans, animals and organizations often engage in ostentatious waste to communicate private information which can otherwise not be credibly communicated. Throughout history, economists and policy makers have stressed the welfare losses implied by conspicuous waste (e.g. Frank (1999) for status signaling). At first sight, a better availability of other information seems an effective remedy to ostentatious waste. If exogenous information perfectly reveals Sender's private information, this is obviously true. But if both exogenous information and costly signaling are imperfect information sources, an improvement in the accuracy of exogenous information does not necessarily decrease the amount of means wasted on costly signaling. For all but very high accuracy levels of exogenous information, a threshold level of prior beliefs exists such that if Receiver's prior believed probability that Sender is a high type is below this threshold, more accurate exogenous information induces higher equilibrium signaling. For prior beliefs above this threshold, better exogenous information reduces equilibrium signaling of both Sender types. Moreover, more accurate exogenous information does not always imply a higher expected accuracy of Receiver's equilibrium beliefs. The information gains due to an enhanced accuracy of exogenous information can be dominate by a loss in information due to changes in the equilibrium signaling strategies of both Sender types, induced by the former. In most cases however, Receiver does benefit in expectation from better exogenous information.

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A Mathematical Appendix: Proofs

A.1 Proof of lemma 1

Proof. For part 4, use (5) to write $B(\Delta) = 1 - \frac{1-p}{p}\int \beta^{\circ}(y)\,\varphi(y|0)dy$, such that $\frac{\partial B(\Delta)}{\partial \Delta} = -\int \left(1-\beta^{\circ}(y)\right)^2 \left(-\varphi'(y|\Delta)\right)dy$. Then use condition

1,1 to write $B'(\Delta) = \int_{\Delta}^{+\infty} \left(\beta^{\circ}(y) - \beta^{\circ}(2\Delta - y)\right) |\varphi'(y|\Delta)|\, dy$ and $\frac{\partial B(\Delta)}{\partial \Delta} = \int_{\Delta}^{+\infty} \left(\left(1 - \beta^{\circ}(2\Delta - y)\right)^2 - \left(1 - \beta^{\circ}(y)\right)^2\right) |\varphi'(y|\Delta)|\, dy$, and note that condition 1,2 implies that $\left(\beta^{\circ}(y) - \beta^{\circ}(2\Delta - y)\right) > 0$ and $\left(1 - \beta^{\circ}(2\Delta - y)\right)^2 - \left(1 - \beta^{\circ}(y)\right)^2 > 0$ for all $y \in (\Delta, +\infty)$ if $\Delta > 0$. $B'(0) = 0$ and $\frac{\partial B(0)}{\partial \Delta} = 0$ follow from the same equation. ■

A.2 Proof of proposition 1

Define

$$B''(\Delta) \equiv \int \frac{p(1-p)\varphi(y|0)\left(-\varphi'(y|\Delta)\right)^2}{\left((1-p)\varphi(y|0) + p\varphi(y|\Delta)\right)^2} dy + \int \beta^{\circ}(y)\, \varphi''(y|\Delta)\, dy \tag{15}$$

$$= \int \frac{p(1-p)\varphi(y|\Delta)}{\left((1-p)\varphi(y|0) + p\varphi(y|\Delta)\right)^2} \left(-\varphi'(y|0)\right)\left(-\varphi'(y|\Delta)\right) dy \tag{16}$$

where the last equation uses partial integration and

$$\frac{\partial \beta^{\circ}(y)}{\partial y} = \frac{p\varphi'(y|\Delta)(1-p)\varphi(y|0) - p\varphi(y|\Delta)(1-p)\varphi'(y|0)}{\left((1-p)\varphi(y|0) + p\varphi(y|\Delta)\right)^2}.$$

Lemma 2 *$B''(\Delta)$ changes sign at least once.*

Proof. $\lim_{\Delta \downarrow 0} B''(\Delta) > 0$ because the integrand in (16) is positive everywhere except on $(0, \Delta)$ and $Var(\varepsilon)$ is bounded away from zero. Thus, $B''(\Delta) > 0$ for Δ sufficiently close to 0. However, $\beta(y) \in [0,1]$ and $B(\Delta) \in [0,1]$, and this bound on B implies together with $B'(0) = 0$ that $B''(\Delta) < 0$ at some $\Delta \in \bar{\mathbb{R}}^{+}$. ■

Assume that the following second order conditions are satisfied

Condition 3 *Let u and φ be such that for all* **s**

$$v_{11}(s_H|\theta_H) + \kappa B''(\Delta) < 0$$
$$v_{11}(s_L|\theta_L) - \kappa \frac{p}{1-p} \int \beta^{\circ}(y)\, \varphi''(y|\Delta)\, dy < 0.$$

An interior solution to (7) and (8) is a unique utility maximum for each type, taking $\beta^{\circ}(y)$ as given, if

$$v_{11}(s_H|\theta_H) + \kappa \int \beta^{\circ}(y)\, \varphi''(y|\Delta)\, dy < 0 \tag{17}$$

$$v_{11}(s_L|\theta_L) - \kappa \frac{p}{1-p} \int \beta^{\circ}(y)\, \varphi''(y|\Delta)\, dy < 0. \tag{18}$$

On the other hand, for a given level of signaling of the other type, an interior solution to (7) and (8) defines a unique interior level of signaling consistent with a S.E. if

$$S_H \equiv v_{11}\left(s_H|\theta_H\right) + \kappa B''\left(\Delta\right) < 0 \tag{19}$$

$$S_L \equiv v_{11}\left(s_L|\theta_L\right) - \kappa \frac{p}{1-p} B''\left(\Delta\right) < 0 \tag{20}$$

Note then that the first term on the RHS of (15) is always positive, such that (17) is implied by (19) and (20) is implied by (18). This motivates condition 3. Note that by lemma 2, $B''(.)$ takes positive and negative values, such that condition 3 implies $v_{11}(.) < 0$ and κ sufficiently small.
Proof. Any S.E. must satisfy (7) and (8) and (4). First, $B'(0) < 0$ and $v_1(.) < 0$ imply that $s_H = s_H = 0$ and $\overset{\circ}{\beta}(.) = p$ always satisfies (7) and (8). A pooling S.E. always exists. An informative equilibrium satisfies (7) and (8) for $s_H > 0$ while beliefs satisfy (4). Let $b_L(s_H)$ represent for any s_H the (by condition 3 unique) level of s_L which is consistent with (8) and (4). Similarly, let $b_H(s_L)$ represent for each level s_L the (by condition 3 unique) level of s_H consistent with (7) and (4). If conditions 1, 2 and 3 are satisfied, then $b_L(s_H)$ and $b_H(s_L)$ are continuously differentiable on $\mathbb{R}_+$. Define $g(s_H) \equiv b_H(b_L(s_H))$, and note that any informative S.E. is a fixed point of g. The signal space of the high Sender type is a complete space, and g is a contraction mapping if $|g'(.)| < 1$ everywhere, i.e. if

$$\left|\frac{\partial b_H}{\partial s_L}\right|\left|\frac{\partial b_L}{\partial s_H}\right| = \left|\frac{\frac{\partial F_H}{\partial s_L}}{S_H}\right|\left|\frac{\frac{\partial F_L}{\partial s_H}}{S_L}\right| < 1$$

$$\Leftrightarrow \frac{\partial F_H}{\partial s_L}\frac{\partial F_L}{\partial s_H} - S_L S_H = -\frac{p}{1-p}\left(B''\left(\Delta\right)\right)^2 - S_L S_H < 0,$$

which is always satisfied under condition 3. If g is continuous on $\mathbb{R}_+$ and a contraction, then the informative equilibrium exists and is unique, and is asymptotically stable on $\mathbb{R}_+$ w.r.t. a best reply dynamic of g.
Finally, suppose a S.E. with $\Delta < 0$. Then Receiver takes, in the S.E., advantage of this fact to partially distinguish between the low and high Sender type, such that $B_L(s_L) < B_L(s_H) = p$, while $v(s_L|\theta_L) < v(s_H|\theta_L)$, such that $s_L > s_H$ cannot be the low income consumer's best reply to s_H, and a strategy profile with $\Delta < 0$ cannot be a S.E. ■

A.3 Proof of proposition 2

Signaling levels in the informative S.E. satisfies equations (7) and (8). Write the first order conditions for an interior S.E. in matrix form as

$A * \frac{ds}{dx} = b$, with $\frac{ds}{dx} \equiv \begin{pmatrix} \frac{\partial s_L}{\partial x} \\ \frac{\partial s_H}{\partial x} \end{pmatrix}$ for x the exogenous parameter of interest and $A = \begin{pmatrix} -\kappa B''(\Delta) & v_{11}(s_H|\theta_H) + \kappa B''(\Delta) \\ v_{11}(s_L|\theta_L) - \frac{\kappa p}{(1-p)} B''(\Delta) & \frac{\kappa p}{(1-p)} B''(\Delta) \end{pmatrix}$. This system has a unique solution if $|A| \neq 0$ everywhere, which is always satisfied if condition 3 holds: $|A| = -\frac{p}{(1-p)}(\kappa B''(\Delta))^2 - S_L S_H < 0$. The system is solved by Cramer's rule:

$$\frac{\partial s_L}{\partial x} = \frac{\begin{vmatrix} b_1 & A_{1,2} \\ b_2 & A_{2,2} \end{vmatrix}}{|A|} \quad \text{and} \quad \frac{\partial s_H}{\partial x} = \frac{\begin{vmatrix} A_{1,1} & b_1 \\ A_{2,1} & b_2 \end{vmatrix}}{|A|}.$$

For $\frac{ds}{d\kappa}$, $b = \begin{pmatrix} -B'(\Delta) \\ -\frac{p}{1-p} B'(\Delta) \end{pmatrix}$ such that $\frac{\partial s_L}{\partial \kappa} > 0$ because $b_1 A_{2,2} - b_2 A_{1,2} = \frac{p}{1-p} v_{11}(s_H|\theta_H) B'(\Delta) < 0$. Similarly, $\frac{\partial s_H}{\partial \kappa} > 0$ because $b_2 A_{1,1} - b_1 A_{2,1} = B'(\Delta) v_{11}(s_L|\theta_L) < 0$.

For $\frac{ds}{d\theta_H}$, $b = \begin{pmatrix} -v_{12}(s_H|\theta_H) \\ 0 \end{pmatrix}$ such that $\frac{\partial s_L}{\partial \theta_H} B''(\Delta) > 0$ because $b_1 A_{2,2} - b_2 A_{1,2} = -v_{12}(s_H|\theta_H) \frac{\kappa p}{(1-p)} B''(\Delta)$, while $\frac{\partial s_H}{\partial \theta_H} > 0$ because $b_2 A_{1,1} - b_1 A_{2,1} = v_{12}(s_H|\theta_H) S_L < 0$.

For $\frac{ds}{d\theta_H}$, $b = \begin{pmatrix} 0 \\ -v_{12}(s_L|\theta_L) \end{pmatrix}$ such that $\frac{\partial s_L}{\partial \theta_L} > 0$ because $b_1 A_{2,2} - b_2 A_{1,2} = v_{12}(s_L|\theta_L) S_H < 0$, while $\frac{\partial s_H}{\partial \theta_L} B''(\Delta) < 0$ because $b_2 A_{1,1} - b_1 A_{2,1} = v_{12}(s_L|\theta_L) \kappa B''(\Delta)$.

For $\frac{ds}{dp}$, $b = \begin{pmatrix} -\kappa \frac{\partial B'(\Delta)}{\partial p} \\ -\frac{\kappa}{(1-p)^2} B'(\Delta) - \frac{\kappa p}{(1-p)} \frac{\partial B'(\Delta)}{\partial p} \end{pmatrix}$. Define $P^1(y) \equiv \left(1 - \beta^{\circ}(y)\right) \beta^{\circ}(y)$ and $P^2(y) \equiv \left(1 - \beta^{\circ}(y)\right) \left(\beta^{\circ}(y)\right)^2$. Then $\frac{\partial \beta^{\circ}(y)}{\partial p} = \frac{P^1(y)}{p(1-p)}$ implies $\frac{\partial B'(\Delta)}{\partial p} = \int \frac{P^1(y)}{p(1-p)} (-\varphi'(y|\Delta))\, dy$.

First, note that if $p = \frac{1}{2}$, $P^1(y)$ is symmetric around $\frac{\Delta}{2}$, such that $\frac{\partial B'(\Delta)}{\partial p} < 0$ if $\Delta > 0$.

Second, $\frac{\partial^2 B'(\Delta)}{\partial^2 p} = \int \frac{P^1(y) - 2P^2(y)}{p^2(1-p)^2} (-\varphi'(y|\Delta))\, dy$, such that if at a $\tilde{p}$ we have $\frac{\partial B'(\Delta)}{\partial p} = 0$, then $\frac{\partial^2 B'(\Delta)}{\partial^2 p} = -2 \int \frac{P^2(y)}{p^2(1-p)^2} (-\varphi'(y|\Delta))\, dy < 0$ because the ratio $\frac{P^2(y)}{P^1(y)} = \beta^{\circ}(y)$ is strictly increasing with y, such that if $\frac{\partial B'(\Delta)}{\partial p} = \int_{\Delta}^{+\infty} \left[-P^1(2\Delta - y) - P^1(y)\right] |\varphi'(y|\Delta)|\, dy = 0$, then $\frac{\partial^2 B'(\Delta)}{\partial^2 p} = \int_{\Delta}^{+\infty} \left[-P^2(2\Delta - y) - P^2(y)\right] |\varphi'(y|\Delta)|\, dy > 0$ as $\left[-P^2(2\Delta - y) - P^2(y)\right] > \left[-P^1(2\Delta - y) - P^1(y)\right]$ for all $y \in (\Delta, \infty)$.

Hence, a unique $\tilde{p}$ exists such that $\frac{\partial B'(\Delta)}{\partial p} > 0$ for $p < \tilde{p}$ and $\frac{\partial B'(\Delta)}{\partial p} < 0$ for $p > \tilde{p}$.

Third, $b_2 = \frac{-\kappa}{1-p} \left(\frac{1}{(1-p} B'(\Delta) + p \frac{\partial B'(\Delta)}{\partial p} \right) = \frac{-\kappa \int \left(2\beta^{\circ}(y) - \left(\beta^{\circ}(y)\right)^2\right) (-\varphi'(y|\Delta)) dy}{(1-p)^2}$.

Define $z(y,p) \equiv \frac{(1-p)}{p}\frac{\varphi(y|0)}{\varphi(y|\Delta)}$, such that $z(y,p) \in (0,\infty)$ and $z_1(y,p) < 0$ for all y. For notational simplicity, we omit the arguments of z unless necessary. Then $2\beta^{\circ}(y) - \left(\beta^{\circ}(y)\right)^2 = \frac{2z+1}{(z+1)^2}$ and $\frac{\partial}{\partial z}\left(\frac{2z+1}{(z+1)^2}\right) = \frac{-2z}{(z+1)^3} < 0$, such that $2\beta^{\circ}(y) - \left(\beta^{\circ}(y)\right)^2$ is strictly increasing with y, which implies $\int\left(2\beta^{\circ}(y) - \left(\beta^{\circ}(y)\right)^2\right)(-\varphi'(y|\Delta))\,dy > 0$.

Finally $b_1A_{2,2} - b_2A_{1,2} = \frac{\kappa}{(1-p)^2}B'(\Delta)S_H + v_{11}(s_H|\theta_H)\frac{\kappa p}{(1-p)}\frac{\partial B'(\Delta)}{\partial p}$, such that $\frac{\partial s_L}{\partial p} > 0$ for $p < \tilde{p}$ and if $p > \tilde{p}$ and $B''(\Delta) < 0$. If $p > \tilde{p}$ and $B''(\Delta) > 0$, then possibly $\frac{\partial s_L}{\partial p} < 0$.

Moreover, $b_2A_{1,1} - b_1A_{2,1} = \kappa B''(\Delta)\frac{\kappa}{(1-p)^2}B'(\Delta) + \kappa\frac{\partial B'(\Delta)}{\partial p}v_{11}(s_L|\theta_L)$, such that $\frac{\partial s_H}{\partial p} > 0$ for $p < \tilde{p}$ and $B''(\Delta) < 0$ and $\frac{\partial s_H}{\partial p} < 0$ for $p > \tilde{p}$ and $B''(\Delta) > 0$, while both signs are possible if $p < \tilde{p}$ and $B''(\Delta) > 0$ or $p > \tilde{p}$ and $B''(\Delta) < 0$.

A.4 Proof of proposition 3

First, $\frac{\partial\Delta}{\partial\kappa} = \frac{\begin{vmatrix}A_{1,1} & b_1\\ A_{2,1} & b_2\end{vmatrix} - \begin{vmatrix}b_1 & A_{1,2}\\ b_2 & A_{2,2}\end{vmatrix}}{|A|} = \frac{B'(\Delta)h(\Delta)}{|A|}$, such that $\frac{\partial\Delta}{\partial\kappa}$ has the opposite sign of $h(\Delta)$. Similarly, $\frac{\partial\Delta}{\partial\theta_H} = \frac{v_{12}(s_H|\theta_H)v_{11}(s_L|\theta_L)}{|A|} > 0$ and $\frac{\partial\Delta}{\partial\theta_L} = -\frac{v_{12}(s_L|\theta_L)v_{11}(s_H|\theta_H)}{|A|} < 0$. Finally, $\frac{\partial\Delta}{\partial p} = \frac{\kappa}{p(1-p)}\frac{Q_p}{|A|}$, with
$Q_p \equiv -\frac{p}{1-p}v_{11}(s_H|\theta_H)B'(\Delta) + h(\Delta)\frac{\partial B'(\Delta)}{\partial p} = v_{11}(s_L|\theta_L)\int P^1(y)(-\varphi'(y|\Delta))\,dy - v_{11}(s_H|\theta_H)\int\frac{p}{(1-p}\left(2\beta^{\circ}(y) - \left(\beta^{\circ}(y)\right)^2\right)(-\varphi'(y|\Delta))\,dy$, where
$\int\left(2\beta^{\circ}(y) - \left(\beta^{\circ}(y)\right)^2\right)(-\varphi'(y|\Delta))\,dy > 0$.

Then $\int P^1(y)(-\varphi'(y|\Delta))\,dy < 0$ if $p > \tilde{p}$, such that both terms in Q_p are positive and $\frac{\partial\Delta}{\partial p} < 0$. If $h(\Delta) > 0$ and $p < \tilde{p}$, then all terms in the first equation are positive and $\frac{\partial\Delta}{\partial p} < 0$. Thus, $Q_p < 0$ is only possible if $-v_{11}(s_L|\theta_L)$ is sufficiently high.

A.5 Proof of theorem 1

Define again $\tilde{S}_H \equiv v_{11}(s_H|\theta_H) + \kappa\tilde{B}''(\Delta) < 0$ and $\tilde{S}_L \equiv v_{11}(s_L|\theta_L) - \kappa\frac{p}{1-p}\tilde{B}''(\Delta) < 0$, with

$$\tilde{B}''(\Delta) = \int\begin{pmatrix} q\tilde{\beta}^{\circ}(y|H)\left(1-\tilde{\beta}^{\circ}(y|H)\right)\\ +(1-q)\tilde{\beta}^{\circ}(y|L)\left(1-\tilde{\beta}^{\circ}(y|L)\right)\end{pmatrix}\frac{(-\varphi'(y|0))(-\varphi'(y|\Delta))}{\varphi(y|0)}dy.$$

Again, assume the following second order condition:

Condition 4 *Let u and φ be such that everywhere*

$$v_{11}(s_H|\theta_H) + \kappa\tilde{B}''(\Delta) < 0$$

$$v_{11}(s_L|\theta_L) - \kappa\frac{p}{1-p}\int\left(q\tilde{\beta}^{\circ}(y,H) + (1-q)\tilde{\beta}^{\circ}(y,L)\right)\varphi''(y|\Delta)\,dy < 0$$

If u and φ satisfy conditions 1, 2 and 4, the proof of part 1 of theorem 1 is analogous to that of proposition 1. For part 2, we proceed in 3 steps, and use, as before, $z(y,p) \equiv \frac{(1-p)}{p}\frac{\varphi(y|0)}{\varphi(y|\Delta)}$, such that $z(y,p) \in \mathbb{R}^+$ and $z_1(y,p) < 0$, and define $\tilde{P}^1(y,\omega) \equiv \left(1-\tilde{\beta}^{\circ}(y,\omega)\right)\tilde{\beta}^{\circ}(y,\omega)$ and $\tilde{P}^2(y,\omega) \equiv \left(1-\tilde{\beta}^{\circ}(y,\omega)\right)\left(\tilde{\beta}^{\circ}(y,\omega)\right)^2$.

Claim 1 *At any interior S.E., $\frac{\partial s_H}{\partial q}$ and $\frac{\partial s_L}{\partial q}$ (if $s_L > 0$) have the same sign as $\frac{\partial\tilde{B}'(\Delta)}{\partial q}$.*

Proof. Write $A = \begin{pmatrix} -\kappa\tilde{B}''(\Delta) & \tilde{S}_H \\ \tilde{S}_L & \frac{\kappa p}{(1-p)}\tilde{B}''(\Delta)\end{pmatrix}$ and $b = \begin{pmatrix} -\kappa\frac{\partial\tilde{B}'(\Delta)}{\partial q} \\ -\kappa\frac{p}{(1-p)}\frac{\partial\tilde{B}'(\Delta)}{\partial q}\end{pmatrix}$.
Then $b_2A_{1,1} - b_1A_{2,1} = \kappa\frac{\partial\tilde{B}'(\Delta)}{\partial q}v_{11}(s_L|\theta_L)$, and
$b_1A_{2,2} - b_2A_{1,2} = \kappa\frac{p}{(1-p)}\frac{\partial\tilde{B}'(\Delta)}{\partial q}v_{11}(s_H|\theta_H)$, such that $\frac{\partial s_H}{\partial q}$ and $\frac{\partial s_L}{\partial q}$ take the same sign as $\frac{\partial\tilde{B}'(\Delta)}{\partial q}$. ∎

Claim 2 *$\tilde{B}'(\Delta)$ is continuously differentiable w.r.t. q for $q \in \left(\frac{1}{2},\bar{q}\right)$. $\frac{\partial\tilde{B}'(\Delta)}{\partial q}$ is strictly positive for p sufficiently close to 0, and strictly negative for p sufficiently close to 1.*

Proof. We write $\frac{\partial\tilde{B}'(\Delta)}{\partial q} = \int f(z,q)\left(-\varphi'(y|\Delta)\right)dy$ with

$$f(z,q) \equiv \tilde{\beta}^{\circ}(y,H) + \frac{\tilde{P}^1(y,H)}{1-q} - \tilde{\beta}^{\circ}(y,L) - \frac{\tilde{P}^1(y,L)}{q} = \frac{z^2(2q-1)(z+1)}{\left((q+(1-q)z)\left((1-q)+qz\right)\right)^2}.$$

and note that $f(z,q) > 0$ for all $z \in \mathbb{R}^+$. By condition 1, $f(z,q)$ is continuous and bounded, such that $\tilde{B}'(\Delta)$ is differentiable w.r.t. q. Moreover, $f(0,q) = 0$ and $\lim_{z\to+\infty} f(z,q) = 0$, and $f(z,q)$ has a unique extremum in terms of z, a maximum, as $f_1(z,q) = z(2q-1)\frac{\left(-z^3a+z^2(1-4a)+3za+2a\right)}{\left((q+(1-q)z)((1-q)+qz)\right)^3}$ with $a \equiv q(1-q)$ has for $q \in \left(\frac{1}{2},1\right)$ and $z > 0$ a strictly positive denominator which is finite for finite z. Then $f_1(z,q) = 0$ where $-z^3a + z^2(1-4a) + 3za + 2a = 0$, which occurs at a unique strictly positive finite real root $\xi \equiv -\frac{(4a-1)}{3a} + \frac{-\frac{8}{9a}+\frac{1}{9a^2}+\frac{25}{9}}{X} + X$, with

$X \equiv \sqrt[3]{\sqrt{\frac{47}{9a} - \frac{35}{36a^2} + \frac{2}{27a^3} - \frac{272}{27}} + \frac{41}{18a} - \frac{4}{9a^2} + \frac{1}{27a^3} - \frac{91}{27}}.$

Let $y_\xi(p)$ solve $z(y_\xi(p), p) = \xi$, such that $f_1(z,q) < 0$ at $y < y_\xi(p)$ and $f_1(z,q) > 0$ at $y > y_\xi(p)$ (remember that $z_1(y,p) < 0$).

Note that ξ is independent of p, and that by taking p sufficiently close to 0, $f(z,q)$ is strictly increasing with y for almost all mass under $|\varphi'(y|\Delta)|$ such that $\frac{\partial \tilde{B}'(\Delta)}{\partial q} = \int f(z,q)(-\varphi'(y|\Delta))\,dy > 0$. Similarly, for p sufficiently close to 1, $f(z,q)$ is strictly decreasing with y for almost all mass under $|\varphi'(y|\Delta)|$ such that $\frac{\partial \tilde{B}'(\Delta)}{\partial q} < 0$. ■

Claim 3 $\frac{\partial \tilde{B}'(\Delta)}{\partial q}$ *is continuous w.r.t.* p*, and at the* $\bar{p}$ *where* $\frac{\partial \tilde{B}'(\Delta)}{\partial q} = 0$*, it must be that* $\frac{\partial^2 \tilde{B}'(\Delta)}{\partial q \partial p} < 0$*.*

Proof. Note that $\frac{\partial^2 \tilde{B}'(\Delta)}{\partial p \partial q} = \frac{1}{p(1-p)} \int (-z f_1(z,q))(-\varphi'(y|\Delta))\,dy$ exists everywhere because $z f_1(z,q)$ is continuous w.r.t. y and bounded for $q \in \left(\frac{1}{2}, \bar{q}\right)$, and note that $\lim_{z \to 0} z f_1(z,q) = 0$ and $\lim_{z \to +\infty} z f_1(z,q) = 0$. Consider then $\frac{\partial^2 \tilde{B}'(\Delta)}{\partial p \partial q} = \int \frac{\partial f(z,q)}{\partial p}(-\varphi'(y|\Delta))\,dy$ with $\frac{\partial \overset{\circ}{\tilde{\beta}}(y,\omega)}{\partial p} = \frac{\tilde{P}^1(y,\omega)}{p(1-p)}$ such that

$p(1-p)\frac{\partial f(z,q)}{\partial p} = \tilde{P}^1(y,H) - \tilde{P}^1(y,L) + \frac{\tilde{P}^1(y,H) - 2\tilde{P}^2(y,H)}{1-q} - \frac{\tilde{P}^1(y,L) - 2\tilde{P}^2(y,L)}{q}$
$= f(z,q) - g(z,q)$, with

$$\begin{aligned} g(z,q) &\equiv \left(\overset{\circ}{\tilde{\beta}}(y,H)\right)^2 - \left(\overset{\circ}{\tilde{\beta}}(y,H)\right)^2 + 2\left(\frac{\tilde{P}^2(y,H)}{1-q} - \frac{\tilde{P}^2(y,L)}{q}\right) \\ &= \frac{z^2(2q-1)(5q^2z^2 - 2q^2z - 3q^2 - 5qz^2 + 2qz + 3q + 2z^2 + z)}{((q + (1-q)z)((1-q) + qz))^3} \end{aligned}$$

At $\bar{p}$, $\frac{\partial \tilde{B}'(\Delta)}{\partial q} = \int f(z,q)(-\varphi'(y|\Delta))\,dy = 0$ such that $p(1-p)\frac{\partial^2 \tilde{B}'(\Delta)}{\partial p \partial q} = -\int g(z,q)(-\varphi'(y|\Delta))\,dy$. Note then that

$$\frac{g(z,q)}{f(z,q)} = \frac{(5q^2z^2 - 2q^2z - 3q^2 - 5qz^2 + 2qz + 3q + 2z^2 + z)}{((q + (1-q)z)((1-q) + qz))(z+1)}.$$

One can verify that $\frac{\partial}{\partial z}\left(\frac{g(z,q)}{f(z,q)}\right) = -2(1-q)q\left((q + (1-q)z)^{-2} + (q(z-1)+1)^{-2}\right) + (z+1)^{-2} < 0$ for all $q < \bar{q} = \frac{2+\sqrt{3}}{4} \cong 0.93301$. At $\bar{p}$ we have by definition $\frac{\partial \tilde{B}'(\Delta)}{\partial q} = \int_\Delta^{+\infty} \left[-f(z(2\Delta - y, p)q) + f(z(y,p), q)\right] |\varphi'(y|\Delta)|\,dy = 0$, which implies for all $q < \bar{q}$ that

$$-p(1-p)\frac{\partial^2 \tilde{B}'(\Delta)}{\partial q \partial p} = \int_\Delta^{+\infty} \left[-g(z(2\Delta - y, p)q) + g(z(y,p), q)\right] |\varphi'(y|\Delta)|\,dy > 0,$$

because $\left[-g\left(z\left(2\Delta-y,p\right)q\right)+g\left(z\left(y,p\right),q\right)\right]>\left[-f\left(z\left(2\Delta-y,p\right)q\right)+f\left(z\left(y,p\right),q\right)\right]$ for all $y\in(\Delta,\infty)$. Hence, $\frac{\partial^2\tilde{B}'(\Delta)}{\partial q\partial p}<0$ at $\bar{p}$, and this implies that $\bar{p}(q)$ is unique for all $q\in\left(\frac{1}{2},\bar{q}\right)$. ■

Claims 1,2 and 3 together imply the second part of theorem 1.

A.6 Proof of proposition 4

First, obtain $\frac{\partial\Delta}{\partial q}=\frac{\begin{vmatrix}A_{1,1}&b_1\\A_{2,1}&b_2\end{vmatrix}-\begin{vmatrix}b_1&A_{1,2}\\b_2&A_{2,2}\end{vmatrix}}{|A|}=\frac{\kappa\frac{\partial\tilde{B}'(\Delta)}{\partial q}h(\Delta)}{|A|}$ from the proof of claim 1, which establishes the first part of proposition 4.

For the second part, we provide a numerical example for which $\frac{\partial\tilde{B}(\Delta)}{\partial q}+\frac{\partial\tilde{B}(\Delta)}{\partial\Delta}\frac{\partial\Delta}{\partial q}<0$. Consider $\Delta=\frac{3}{2}$, $p=0.9$, $q=0.91$ and φ the normal distribution with $\sigma=2$. In this case $\frac{\frac{\partial\tilde{B}(\Delta)}{\partial q}}{\frac{\partial\tilde{B}(\Delta)}{\partial\Delta}\frac{\partial B'(\Delta)}{\partial q}}\simeq-395.095$, such that we aim to construct an example for which $-395.095>\frac{\kappa h(\Delta)}{|A|}$. For these parameter values we also have $\max_{\Delta}\left\{\left|\tilde{B}''(\Delta)\right|\right\}<0.008052\equiv C$, $\max_{\Delta}\left\{\left|\int\left(q\tilde{\beta}^{\circ}(y,H)+(1-q)\tilde{\beta}^{\circ}(y,L)\right)\varphi''(y|\Delta)\,dy\right|\right\}<0.0104\equiv D$ and $\tilde{B}''\left(\frac{3}{2}\right)\simeq0.00612221$. First, assume that $-v_{11}(.|\theta_H)$ is minimal at s_H and that for all s_L we have $-v_{11}(.|\theta_L)>\frac{9D}{C}\left(-v_{11}(s_H|\theta_H)\right)=15.289\left(-v_{11}(s_H|\theta_H)\right)$. Note that this implies $h(\Delta)<0$ and $\frac{\partial\Delta}{\partial q}<0$ and that condition (4) is satisfied if we choose $\kappa=\frac{-v_{11}(s_H|\theta_H)}{C}$. Thus, we obtain

$$\frac{\kappa h(\Delta)}{|A|}=\left(\frac{-C}{\left(1-\frac{p}{(1-p)}\frac{v_{11}(s_H|\theta_H)}{v_{11}(s_L|\theta_L)}\right)}+\tilde{B}''(\Delta)\right)^{-1}=\left(\frac{-0.008052}{\left(1-9\frac{v_{11}(s_H|\theta_H)}{v_{11}(s_L|\theta_L)}\right)}+0.00612221\right)^{-1}$$

which is smaller than -395.095 if $\frac{v_{11}(s_H|\theta_H)}{v_{11}(s_L|\theta_L)}\lessapprox7.720\,2\times10^{-3}$, i.e. if $-v_{11}(s_L|\theta_L)>129.53\left(-v_{11}(s_H|\theta_H)\right)$.

Finally, we have $B'(\Delta)\simeq0.0110702$ for the given parameter values, such that the first order conditions for an interior informative S.E. are $\frac{v_1(s_H|\theta_H)}{v_{11}(s_H|\theta_H)}=\frac{0.0110702}{0.008052}=1.3748$ and $\frac{v_1(s_L|\theta_L)}{v_{11}(s_H|\theta_H)}=9\frac{(0.0110702)}{0.008052}=12.374$. Such function v can be constructed.

A.7 Proof of proposition 5

Strict concavity of Sender's problem means imposing the following condition.

Condition 5 *Let u and φ be such that everywhere*

$$v_{11}(s_H|\theta_H) + \kappa \bar{B}''(\Delta|i,g) < 0$$

$$v_{11}(s_L|\theta_L) - \kappa \frac{p}{1-p}\left(\sum_k D(k|i,g)\int \begin{pmatrix} q(k)\,\overset{\circ}{\tilde{\beta}}_k(y,H) + \\ (1-q(k))\,\overset{\circ}{\tilde{\beta}}_k(y,L) \end{pmatrix} \varphi''(y|\Delta)\,dy\right) < 0$$

The proof of the first part of proposition 5 proceeds along the lines of the proof of proposition 1. For part 2, condition 5 implies $v_{11}(s_H|\theta_H) < 0$ and $v_{11}(s_L|\theta_L) < 0$. If $\left\{q(1), ..., q\left(\max\left\{\bar{d}(i,g), \bar{d}(i,g')\right\}\right)\right\} \subset Q(p)$ and $D(k|i,g) \succ_{FOSD} D(k|i,g)$, then $\bar{B}'(\Delta|i,g) < \bar{B}'(\Delta|i,g)$ such that $v_1(s_H(i,g)|\theta_H) + \kappa \bar{B}'(\Delta|i,g) = 0$ and $0 = s_L(i,g)\left(v_1(s_L(i,g)|\theta_L) + \kappa\frac{p}{1-p}\bar{B}'(\Delta|i,g)\right)$ imply $s_H(i,g) < s_H(i,g')$ and $s_L(i,g) < s_L(i,g')$ if $s_L(i,g') > 0$.