# Stochastic Sparse-grid Collocation Algorithm (SSCA) for Periodic Steady-State Analysis of Nonlinear System with Process Variations

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**Abstract—In this paper, Stochastic Collocation Algorithm combined with Sparse Grid technique (SSCA) is proposed to deal with the periodic steady-state analysis for nonlinear systems with process variations. Compared to the existing approaches, SSCA has several considerable merits. Firstly, compared with the moment-matching parameterized model order reduction (PMOR), which equally treats the circuit response on process variables and frequency parameter by Taylor approximation, SSCA employs Homogeneous Chaos to capture the impact of process variations with exponential convergence rate and adopts Fourier series or Wavelet Bases to model the steady-state behavior in time domain. Secondly, contrary to Stochastic Galerkin Algorithm (SGA), which is efficient for stochastic linear system analysis, the complexity of SSCA is much smaller than that of SGA for nonlinear case. Thirdly, different from Efficient Collocation Method, the heuristic approach which may results in "Rank deficient problem" and "Runge phenomenon", Sparse Grid technique is developed to select the collocation points in SSCA in order to reduce the complexity while guaranteeing the approximation accuracy. Furthermore, though SSCA is proposed for the stochastic nonlinear steady-state analysis, it can be applied for any other kinds of nonlinear system simulation with process variations, such as transient analysis, etc..**

## I. Introduction

As IC technology is scaled down to deep sub-micron region, the circuit performance is increasingly less predictable for the indetermination in the manufacturing process. Therefore, it is quite essential to deal with the sensitivity analysis about how process parameters influence the performance of a design[1]. Contrary to the linear system, investigation of nonlinear circuit behavior in the presence of process variations is increasingly more crucial and urgent, especially for RF and Mixed-Signal design. In fact, many important building blocks in RF or Mixed-Signal system, such as mixers and oscillators, are fundamentally dependent on nonlinear effect to operate[2]. In these nonlinear circuits, impact of process variations could be much more severe than that on the linear circuits, since the parameter variations could be amplified exponentially by nonlinear behavior. Therefore, variation aware steady-state simulation for nonlinear circuits is quite demanding.

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Up to now, the majority of variational circuit analysis methods use Taylor series to study the impact of process variations on circuit performance, such as CORE[3] and PMOR[4] for linear system and TPWL-PMOR[5] for nonlinear circuit. In [4] and [5], the same expansion order is employed for both random process parameters and frequency parameter. However, in today's IC technology, the process parameters varies at most around $0 \sim 30\%$, while the frequency parameter may extend in extremely large range from 0 to $10GHz$ . Therefore, the more reasonable choice should be high order moments for frequency parameter and low order moments for process variables. Moreover, since "Taylor expansion" do not consider the randomness property of process variations, all these methods do not result in convergent expansion and can only be applied to study the systems with relatively small variations. "Taylor expansion" methods are not efficient for variational nonlinear system analysis in which case small variations are always amplified significantly.

When considering the random process variables, the nonlinear circuits could be formulated as stochastic ordinary differential equation (ODE). A natural choice for the solution of stochastic nonlinear ODE is the stochastic spectral methods, such as Stochastic Galerkin Algorithm (SGA) and Stochastic Collocation Algorithm (SCA). Previously, SGA has been introduced in [6] and [7] to deal with the stochastic analysis of linear interconnect and Power Grid circuits. However, SGA is not a potential method for the analysis of nonlinear system when considering the process variations even though it is very useful for linear circuit. This is because the forms of resulting equation by SGA for parameterized nonlinear systems are always coupled and much more complicated than the equations generated from linear case, and this could result in unacceptable computational complexity. On the other hand, for SCA, since the high dimension of random variable space makes the collocation points explode with direct tensor product and thus increases the computation cost exponentially, a simple heuristic approach, also named as Efficient Collocation Method(ECM), is previously developed in [8] to choose the fewest collocation points for stochastic analysis in the chemical and biochemical engineering field. Later, ECM is employed by [9] for function approximation in the stochastic gate delay modeling with process variations. However, ECM is an ad-hoc approach, and this approach will result in "Runge phenomenon" and fail to convergence for high order polynomial chaos expansion, i.e., suffer from "Rank deficient problem" attributed to its inherent instability[8][10] discussed detailedly in Section III.B.

In this paper, Stochastic Sparse-grid Collocation Algorithm(SSCA) is developed to analyze periodic steady-state response of the nonlinear circuits with respect to process variations. SSCA has several important advantages compared with the existing methods. Firstly, different from [4] and [5], which expand the different value range parameters, i.e., the process parameters and frequency parameter, with the same Taylor order, SSCA adopts disparate orthogonal bases for different kinds of parameters, i.e., Homogeneous Chaos for process variables and Fourier series or Wavelet basis in time(frequency) domain with much higher convergence rate than

"Taylor expansion". Secondly, contrary to the SGA which always results in high complexity for nonlinear case, SSCA has much smaller CPU cost since it is only required to solve the deterministic nonlinear system with the same order as original ones on several selected collocation points in the process variational space. In addition, Sparse Grid technique, which is well developed in the mathematics and mechanics field[11], is introduced in this paper to reduce the complexity of SCA. Sparse Grid technique could avoid the exponential increase of the number of collocation points generated by direct tensor product scheme in the high dimensional random space. Moreover, compared with the ECM developed in [8][9], which is lack of theoretical background and easily suffer from "Rank deficient problem" and "Runge phenomenon", Sparse Grid technique has the solid mathematical support and it could preserve the accuracy while reducing the number of collocation points remarkably.

The rest of the paper is organized as follows. In Section II, the problems for steady-state analysis of deterministic and stochastic nonlinear systems are formulated. In Section III, background of SCA is reviewed. Stochastic Collocation Algorithm combined with Sparse Grid technique (SSCA) is proposed in Section IV. Numerical results are given in Section V to demonstrate the advantages of the proposed algorithm. Finally, conclusions are drawn in Section VI.

## II. Problem Definition

Without loss of generality, the traditional steady state analysis for the deterministic nonlinear system is trying to find out the exact solution of (1),

$$\frac{dx(t)}{dt} = f(x,t,u(t)) \qquad x(t+T) = x(t) \tag{1}$$

where $x(t) = [x_1(t), x_2(t), \cdots, x_P(t)]^T \in R^P$ are the $P$ unknown state variables, $f(\cdot) = [f_1(\cdot), f_2(\cdot), \cdots, f_P(\cdot)]^T$ denotes the nonlinear vector function and $u(t)$ means a periodic signal with frequency $f_0 = \frac{1}{T}$. There are two main steady-state analysis algorithms, Harmonic Balance Method (HBM)[2] and Wavelet Balance Method (WBM)[12], both of which assume that the solution of (1) can be approximated by a finite series of orthogonal bases as (2),

$$x(t) = \sum_{l=1}^{L} X_l \cdot \psi_l(t) \tag{2}$$

where $\psi_l(t)$ is the Fourier function for HBM or Wavelet bases for WBM. $X_l$ are unknown coefficients. Based on (2), (1) can be transformed to a nonlinear algebraic equation with respect to$X_l$, which can be solved by the classical iterative techniques, such as Newton-Raphson method.

When considering process variations, let $\vec{\xi} = [\xi_1, \xi_2, \cdots, \xi_M]^T \in \Theta^M$ denote series of interested independent random variables with probability density function $p(\vec{\xi})$, where $\Theta^M$ is the $M$-dimensional random space. Then, (1) should be reformulated to (3).

$$\frac{dx(t,\vec{\xi})}{dt} = f(x,t,\vec{\xi},u(t)) \qquad x(t+T,\vec{\xi}) = x(t,\vec{\xi}) \tag{3}$$

The solution of (3) can be regarded as a continuous stochastic process. A good choice to model a stochastic process is Polynomial Chaos expansion. Therefore, in the following, basic concepts of Polynomial Chaos will be reviewed briefly.

## III. Background

In this section, Polynomial Chaos[13], the basic principle of stochastic spectral methods, is introduced at first. Then, Stochastic Collocation Algorithm (SCA) combined with two kinds of collocation points selection approaches, direct tensor product scheme[11] and ECM[8], are reviewed briefly by taking gate delay modeling as an example. In addition, Gaussian Quadrature used in SCA is introduced briefly.

### A. Polynomial Chaos Expansion

For the solution of deterministic systems (1), orthogonal polynomials as Fourier basis or Wavelet basis expansion in the deterministic space can result in high approximation accuracy. Contrarily, for the solution of stochastic ODE as (3), which should be regarded as stochastic process, polynomials chaos (Askey Scheme), i.e., the orthogonal basis defined in the random space according to the corresponding probability density function (*pdf*), are always the best choice for its exponential convergence rate[13]. For instance, Laguerre-chaos can be applied for Gamma process approximation and Charlier-chaos can be employed for Poisson process[13]. In this paper, all the process parameters are assumed to be Gaussian variables, and the best choice should be Hermite polynomials[13]. The Hermite polynomials form a set of orthogonal bases in Hilbert space based on the inner product definition (4).

$$\langle f(\vec{\xi}), g(\vec{\xi}) \rangle = \int_{\vec{\xi}} f(\vec{\xi}) g(\vec{\xi}) p(\vec{\xi}) d\vec{\xi} \tag{4}$$

where $\vec{\xi} = [\xi_1, \xi_2, \cdots, \xi_M]^T$ is a set of independent random variables with standard Gaussian *pdf* as $p(\vec{\xi})$.

According to the orthogonal property of Hermite polynomials, any second order Gaussian process can be approximated by a series of Hermite polynomials in the norm sense[14].

### B. Stochastic Collocation Algorithm

In this section, taken gate delay modeling as an example, SCA is introduced briefly by function approximation in the forms of a truncated series of Homogeneous Chaos with respect to process parameters $\vec{\xi}$,

$$y = f(\vec{\xi}) \approx \sum_{n=1}^{N} c_n H_n(\vec{\xi}) \tag{5}$$

where $H_n(\vec{\xi})$ is the $n$-th Hermite polynomial. The unknown coefficients $c_n$ are estimated by equating delays $y$ and the corresponding polynomial chaos (5) at a set of collocation points in the parameter space. There are two algorithms for the selection of collocation points and calculation of $c_n$, Gaussian Quadrature with Tensor Product Scheme discussed in Section III.B.1 and Efficient Collocation Method presented in Section III.B.2.

#### 1). Gaussian Quadrature with Tensor Product Scheme

With Gaussian Quadrature, tensor product scheme can be applied easily to obtain $c_n$.

**Gaussian Quadrature[15]**

Gaussian Quadrature presented in Theorem 1[15] for one-dimension case is always regarded as an efficient numerical integration technique for nonlinear function.

**Theorem 1** *If $a < t_1 < t_2 < \cdots < t_N < b$ are the real roots of $N$ order orthogonal polynomial $P_N(x) = k_N \Pi_{i=1}^{N}(x - t_i)$, $k_N > 0$ defined in the region $[a, b]$ and with weight function $w(x)$, the quadrature function (6) is exact for all polynomials of degree $\leq 2N-1$.[15]*

$$\int_a^b w(x) f(x) dx \approx \sum_{j=1}^{N} w_j f(t_j) \tag{6}$$

*where $w_j$ is the weight for point $t_j$, which is also called as the Gaussian collocation point for $N-1$-level Gaussian Quadrature.*

With regard to the Gaussian-Hermite Quadrature for multi-dimensional cases, it has the similar conclusion as Theorem 1. For instance, let $g(\vec{\xi})$ denotes a nonlinear function for independent Gaussian random variables $\vec{\xi} = [\xi_1, \xi_2, \cdots, \xi_M]^T$ with probability density function $p(\vec{\xi})$, then,

$$\int_{-\infty}^{\infty} g(\vec{\xi}) p(\vec{\xi}) d\vec{\xi} \approx \sum_{i=1}^{P_t} w_i^t g(\vec{\xi}_i^t) \tag{7}$$

where the Gaussian collocation points $\vec{\xi}_i^t$ and correlated weight $w_i^t$ can be obtained by direct tensor product scheme.

**Tensor Product Scheme**

Let $\Theta_1^q = \{\xi_1^1, \cdots, \xi_1^{q+1}\}$ and $W_1^q = \{w_1^1, \cdots, w_1^{q+1}\}$ denote the set of collocation points and weights for one-dimensional $q$-level accuracy Gaussian-Hermite quadrature rule. According to the direct

tensor product scheme[11], the space of collocation points for $M$-dimensional integration of $Q$-level is[11],

$$\Theta_M^Q = \Theta_1^{i_1} \times \Theta_1^{i_2} \times \cdots \times \Theta_1^{i_M} \tag{8}$$

where $i_1 = i_2 = \cdots = i_M = Q$ and $\Theta_1^Q$ denotes the collocation points for one-dimension with $Q$-level. For each collocation point $\vec{\xi}_j^t = [\xi_1^{j_1}, \cdots, \xi_M^{j_M}]^T \in \Theta_M^Q$, $1 \le j_1, \cdots, j_M \le Q+1$, where $\xi_1^{j_k}$, $k = 1, \cdots, M$, is any one chosen from $\Theta_1^Q$. The corresponding weight $w_j^t$ is defined as,

$$w_j^t = w_1^{j_1} \cdot w_1^{j_2} \cdots w_1^{j_M} \tag{9}$$

where $w_1^{j_k}$ is the weight for collocation point $\xi_1^{j_k}$ in one-dimensional case. Obviously, the number of collocation points is $P_t = (Q+1)^M$, which is always much larger than the number of Hermite polynomials up to $Q$-order in $M$ dimensional space, as $N = \binom{M+Q}{M}$[13]. When the collocation points $\vec{\xi}_j$ and corresponding weights $w_j, j = 1, \cdots, P_t$ have been obtained, the coefficients $c_n$ can be calculated based on SCA and Gaussian Quadrature as,

$$c_n = \frac{\langle y, H_n(\vec{\xi})\rangle}{a_n} = \frac{\sum_{j=1}^{P_t} w_j^t y_j^t H_n(\vec{\xi}_j^t)}{a_n} \tag{10}$$

where $y_j^t$ is the delay on point $\vec{\xi}_j^t$, and $a_n = \langle H_n(\vec{\xi}), H_n(\vec{\xi})\rangle = \int_{\vec{\xi}} H_n^2(\vec{\xi}) p(\vec{\xi}) d\vec{\xi}$.

### 2). Efficient Collocation Method (ECM)

In order to avoid exponential increase of the number of collocation points by applying tensor product scheme, a simple heuristic approach, ECM is proposed by [8] for stochastic analysis in chemical field. Then it is used for delay modeling[9].

In ECM, one "corresponding" collocation point is selected for each basis applied in (5). The collocation point for the constant is the origin. The collocation points for terms involving only one variable are selected by setting all other $\xi$'s to zero and the corresponding variable to the roots of the higher order one-dimension Hermite polynomial. For terms involving two or more random variables, the values of the corresponding variables are set to the roots of the higher order one-dimension polynomial. If more "corresponding" points are available than needed, the points which are closer to the origin and make the overall distribution of selected points symmetric with respect to the origin are preferred. The number of collocation points obtained by ECM is the same as the number of Hermite basis employed in (5). Therefore, $c_n$ can be calculated exactly according to (11).

$$H \times C = Y \tag{11}$$

$$H = \begin{bmatrix} H_1(\vec{\xi}_1)\, H_2(\vec{\xi}_1) \cdots H_N(\vec{\xi}_1) \\ \vdots \\ H_1(\vec{\xi}_K) H_2(\vec{\xi}_K) \cdots H_N(\vec{\xi}_K) \end{bmatrix}, C = \begin{bmatrix} c_1 \\ \vdots \\ c_N \end{bmatrix}, Y = \begin{bmatrix} y_1 \\ \vdots \\ y_K \end{bmatrix}$$

By applying ECM, the number of selected collocation points is reduced remarkably than that by tensor product scheme, and the behavior of the model is captured reasonably well at points corresponding to regions of high probability. However, since the function approximation can also be regarded as interpolation problem and the collocation points selected by ECM are not all exactly the Gaussian collocation points introduced in the above section, the approximation accuracy of $p+1$ order Hermite expansion with more sampling collocation points may not be higher than that of $p$ order Hermite expansion with less collocation points. This "Runge phenomenon" of interpolation problem is observed in the numerical results of ECM as given in Section V. Moreover, ECM will fail to converge for high order polynomial chaos approximation attributed to its inherent instability[8][10], i.e., the rank of matrix $H$ in (11) will be smaller than its size, which is called as "Rank Deficient Problem". Then, there exist infinite solutions for (11) and Least Square method (LSM) is always used to obtain the minimum 2-norm solution[16]. However, since it is impossible to demonstrate whether this minimum 2-norm solution is the exact solution for (5) or not, LSM may result in unacceptable approximation results as shown by numerical results in Section V. Therefore, as a simple heuristic approach, ECM is infeasible for high order polynomial chaos expansion in stochastic analysis of strong nonlinear system .

## IV. Stochastic Sparse-grid Collocation Algorithm

In this section, firstly, Stochastic Galerkin Algorithm (SGA), an efficient method for linear system analysis, is proven to have high complexity for nonlinear case. Then, Stochastic Collocation Algorithm (SCA) is proposed for the parameterized steady-state analysis of nonlinear circuit with much lower CPU cost than SGA. In addition, Sparse Grid technique is developed to further decrease the complexity.

### A. Spectral Stochastic Methods

The two most important components in the spectral methods are trial functions and test functions[14]. Since all the interested variables $\vec{\xi}$ in (3) are normalized Gaussian random variables, according to spectral methods, Hermite polynomials can be chosen as the trial functions to approximate the solution of (3). Then based on the principles of stochastic spectral methods and HBM/WBM, the response of (3) can be approximated by $x_d(t, \vec{\xi})$,

$$\begin{aligned} x_d(t,\vec{\xi}) &= \sum_{n=1}^{N}\sum_{l=1}^{L} c_{nl} H_n(\vec{\xi})\psi_l(t) \\ &= \sum_{n=1}^{N} \beta_n(t) H_n(\vec{\xi}) = \sum_{l=1}^{L} X_l(\vec{\xi})\psi_l(t) \end{aligned} \tag{12}$$

$$\beta_n(t) = \sum_{l=1}^{L} c_{nl}\psi_l(t) \qquad X_l(\vec{\xi}) = \sum_{n=1}^{N} c_{nl} H_n(\vec{\xi}) \tag{13}$$

where $H_n(\vec{\xi})$ is the $n$-th Hermite polynomials, $\psi_l(t)$ is the Fourier basis or Wavelet basis, and $c_{nl}$ are expansion coefficients.

Test functions are used to minimize the residue given in (14) in the norm sense. In other words, the inner product of each test function and $E_n(\vec{\xi})$ in (14) should be enforced to zero. Different choice of test function will result in different spectral methods.

$$E_n(\vec{\xi}) = \frac{dx_d(t,\vec{\xi})}{dt} - f(x_d(t,\vec{\xi}), t, \vec{\xi}, u(t)) \tag{14}$$

In Stochastic Galerkin Algorithm (SGA), test functions are set to be the same as trial functions, i.e., (15) should be satisfied,

$$\langle E_n(\vec{\xi}), H_k(\vec{\xi})\rangle = 0 \tag{15}$$

where $k = 1, \cdots, N$. However, in SCA, test functions are set to be the Dirac delta functions $\delta(\vec{\xi} - \vec{\xi}_j)$ whose value is only non-zero at some special collocation points $\vec{\xi}_j$, $j = 1, 2, \cdots, J$, i.e., (16) should be met,

$$\langle E_n(\vec{\xi}), \delta(\vec{\xi} - \vec{\xi}_j)\rangle = 0 \tag{16}$$

This means that equation (3) is only required to be satisfied exactly at some collocation points in process variable space.

### B. Limitations of Stochastic Galerkin Algorithm (SGA)

According to the principle of SGA (15), (17) should be satisfied.

$$\langle \frac{dx_d(t,\vec{\xi})}{dt}, H_k(\vec{\xi})\rangle = \langle f(x_d(t,\vec{\xi}), t, \vec{\xi}, u(t)), H_k(\vec{\xi})\rangle \tag{17}$$

$$\langle \frac{dx_d}{dt}, H_k(\vec{\xi})\rangle = \sum_{n=0}^{N} \dot{\beta}_n(t) \int_{\vec{\xi}} H_n(\vec{\xi}) H_k(\vec{\xi}) p(\vec{\xi}) d\vec{\xi} = a_k \cdot \dot{\beta}_k(t)$$

$$\langle f(x_d, u(t)), H_k(\vec{\xi})\rangle = \int_{\vec{\xi}} f(x_d, u(t)) H_k(\vec{\xi}) p(\vec{\xi}) d(\vec{\xi})$$

$$= \int_{\vec{\xi}} f(\sum_{n=1}^{N} \beta_n(t) H_n(\vec{\xi}), u(t)) H_k(\vec{\xi}) p(\vec{\xi}) d(\vec{\xi}) = F_k(\cdot)$$

where $a_k = \langle H_k(\vec{\xi}), H_k(\vec{\xi})\rangle = \int_{\vec{\xi}} H_k^2(\vec{\xi})p(\vec{\xi})d\vec{\xi}$. and $F_k(\cdot), k = 1, \cdots, N$ are the functions of unknown coefficients $\beta_1(t), \cdots, \beta_N(t)$. Therefore, a large deterministic coupled system such as (18) should be solved.

$$\begin{bmatrix} \dot{\beta}_1(t) \\ \vdots \\ \dot{\beta}_N(t) \end{bmatrix} = \begin{bmatrix} F_1(\cdot)/a_1 \\ \vdots \\ F_N(\cdot)/a_N \end{bmatrix} \tag{18}$$

Obviously, the complexity of solving (18) is bottleneck of SGA. Let $P$ denote the order of the nonlinear system to be considered. Since the complexity analysis for nonlinear system solution is always very difficult, and in each iteration step of nonlinear solution, the complexity for nonlinear system analysis is at least of $O(P^3)$. Hence, the complexity for solving (18) is at least of $O(P^3 \times N^3)$. However, since small parameter variations may induce large output variations in the nonlinear system, high order polynomial chaos expansion are always required to guarantee the approximation accuracy. The complexity of SGA for nonlinear system will become unacceptable especially for high order polynomial chaos expansion in the high dimension random space. As a result, SGA is not a suitable algorithm for stochastic nonlinear system analysis, though it is efficient for linear case where low order polynomial chaos approximation is enough.

### C. Stochastic Collocation Algorithm

In this section, Stochastic Collocation Algorithm is proposed to calculate the unknown coefficients $c_{nl}$ in (12) by the following three steps.

***Step 1*** *Selection of Collocation Points for* $\vec{\xi}$. Direct tensor product scheme can be adopted to generate the required collocation points and the corresponding weight. However, the number of collocation points $P_t$ will increase exponentially with regard to the space dimension[11].

***Step 2*** *Calculation of Steady-State Response at each Collocation Point.* At each collocation point $\vec{\xi}_k^t, k = 1, \cdots, P_t$, (3) is transformed to a deterministic nonlinear equation (19), following the SCM principle in (16).

$$\frac{dx(t, \vec{\xi}_k^t)}{dt} = f(x(t, \vec{\xi}_k^t), t, \vec{\xi}_k^t, u(t)) \tag{19}$$

Then, classical algorithms, such as HBM or WBM can be applied to calculate the steady-state behavior of (19), which is approximated by a truncated series of Fourier basis or Wavelet basis as,

$$x_d(t, \vec{\xi}_k^t) = \sum_{l=1}^{L} X_{kl}\psi_l(t) \tag{20}$$

where $\psi_l(t)$ is the Fourier basis or Wavelet basis. On the other hand, according to (12), the steady state response $x(t, \vec{\xi})$ of (3) at each collocation point $\vec{\xi}_k^t$ can be approximated by $x_d(t, \vec{\xi}_k^t)$ as,

$$x_d(t, \vec{\xi}_k^t) = \sum_{l=1}^{L} X_l(\vec{\xi}_k^t)\psi_l(t) \tag{21}$$

Therefore,

$$X_l(\vec{\xi}_k^t) = X_{kl} \tag{22}$$

***Step 3*** *Computation of Steady-state Behavior for Nonlinear System with Process Variations.* According to (13), $X_l(\vec{\xi})$ can be approximated by a series of Hermite polynomials. The expansion coefficients $c_{nl}$ can be obtained by Weighted Least Square Method as setting,

$$\langle E_c, H_k(\vec{\xi})\rangle = 0, \qquad k = 1, \cdots, N \tag{23}$$

where $E_c$ is the residue defined as,

$$E_c = X_l(\vec{\xi}) - \sum_{n=0}^{N} c_{nl}H_n(\vec{\xi}) \tag{24}$$

Based on the Gaussian-Hermite Quadrature and direct tensor product scheme, (23) can be rewritten as (25),

$$\begin{aligned}\langle E_c, H_k(\vec{\xi})\rangle &= \int_{\vec{\xi}} E_c H_k(\vec{\xi})p(\vec{\xi})d\xi \\ &= \sum_{i=1}^{P_t} w_i^t X_l(\vec{\xi}_i^t)H_k(\vec{\xi}_i^t) - a_k c_{kl} = 0\end{aligned} \tag{25}$$

Then,

$$c_{kl} = \frac{\sum_{i=1}^{P_t} w_i^t X_l(\vec{\xi}_i^t)H_k(\vec{\xi}_i^t)}{a_k} \tag{26}$$

Obviously, when $c_{kl}$ is computed, the steady-state response of (3) can be obtained by (12). Then it is very easy to calculate the stochastic properties, such as the mean value $u(x(t, \vec{\xi}))$ and variance $\delta^2(x(t, \vec{\xi}))$ given in (27).

$$u(x(t, \vec{\xi})) = \sum_{l=1}^{L} c_{0l}\psi_l(t), \delta^2(x(t, \vec{\xi})) = \sum_{n=1}^{N}\sum_{l=1}^{L} c_{nl}^2 a_n \psi_l^2(t) \tag{27}$$

Since the number of collocation points selected by tensor product scheme increases exponentially according to the expansion order, the complexity of SCA is still considerable. In order to further reduce the CPU consumption, Sparse Grid technique is investigated in the next section.

### D. Sparse Grid Technique

In this section, Sparse Grid technique[11], which is developed very well in the mathematical field, is proposed to decrease the number of required collocation points in SCA, then, reduce the computational complexity remarkably compared with direct tensor product scheme.

Let $\Theta_1^q$ and $W_1^q$ denote the collocation points and weights for one-dimensional $q$-level accuracy Gaussian quadrature rule. Thus the collocation points space generated by Sparse Grid technique for $M$-dimensional space with $Q$-level accuracy is[11],

$$\bar{\Theta}_M^Q = \bigcup_{Q+1 \le |\vec{i}| \le M+Q} \Theta_1^{i_1} \times \Theta_1^{i_2} \times \cdots \times \Theta_1^{i_M} \tag{28}$$

where $|\vec{i}| = i_1 + i_2 + \cdots + i_M$. And the number, denoted by $P_s$, of the generated collocation points $\vec{\xi}_k^s \in \bar{\Theta}_M^Q$ is,

$$P_s = dim(\bar{\Theta}_M^Q) \sim \frac{2^Q}{Q!}M^Q \sim 2^Q N \tag{29}$$

where $N$ is the number of Hermite polynomials for $M$-dimensional space of order at most $Q$. Table I lists the amount of collocation points generated by different kinds of approaches, where $P_t$ and $P_s$ denote the number of points generated by direct tensor scheme and Sparse Grid technique respectively. It can be found that the direct tensor product scheme can result in the exponential increasing of the number of collocation points with respect to dimensionality, while Sparse Grid technique could reduce the number of required collocation points remarkably. Furthermore, the weight corresponding to the collocation point $\vec{\xi}_i^s = [\xi_{j_{i_1}}^{i_1}, \cdots, \xi_{j_{i_M}}^{i_M}]^T \in \bar{\Theta}_M^Q$ [11] is,

$$w_i^s = (-1)^{M+Q-|\vec{i}|}\binom{M-1}{M+Q-|\vec{i}|}(w_{j_{i_1}}^{i_1} \cdots w_{j_{i_M}}^{i_M}). \tag{30}$$

Then, the integration of nonlinear function $f(\vec{\xi})$ in terms of Gaussian random variables $\vec{\xi}$ can be computed as (31) up to $(2Q+1)$-level accuracy according to Theorem 2,

$$\int_{\vec{\xi} \in \bar{\Theta}_M^Q} f(\vec{\xi})p(\vec{\xi})d\vec{\xi} \approx \sum_{i=1}^{P_s} w_i^s f(\vec{\xi}_i^s) \tag{31}$$

**Theorem 2** *Equation (31) is exact for all $M$-variables polynomials of order at most $2Q+1$[11].*

TABLE I
THE NUMBER OF GENERATED COLLOCATION POINTS

| $M$ | $Q$ | $P_s$ | $P_t$ | $Q$ | $P_s$ | $P_t$ |
|---|---|---|---|---|---|---|
| 4 | 1 | 9 | 16 | 2 | 41 | 81 |
| 6 | 1 | 13 | 64 | 2 | 85 | 729 |
| 10 | 1 | 21 | 1024 | 2 | 221 | $5.9 \times 10^4$ |

For SCM proposed in the last section, Sparse Grid technique can be applied in ***Step 1*** to generate collocation points instead of direct tensor product scheme and the number of required collocation points will be reduced remarkably. Then in ***Step 3***, (25) should be transformed to (32),

$$\langle E_c, H_k(\vec{\xi})\rangle = \int_{\vec{\xi}} E_c H_k(\vec{\xi}) p(\vec{\xi}) d\xi = \sum_{i=1}^{P_s} w_i^s X_l(\vec{\xi_i}) H_k(\vec{\xi_i^s}) - a_k c_{lk} = 0 \qquad (32)$$

where $k = 0, 1, \cdots, N$ and $c_{kl}$ can be obtained by (33).

$$c_{kl} = \frac{\sum_{i=1}^{P_s} w_i^s X_l(\vec{\xi_i^s}) H_k(\vec{\xi_i^s})}{a_k} \qquad (33)$$

### E. Summary

Compared with existing algorithms, the proposed Stochastic Sparse-grid Collocation Algorithm(SSCA) has three remarkable advantages.

- Compared to the moment-matching method[5] for nonlinear system analysis, SSCA employs the Homogeneous Chaos rather than Taylor expansion for variation analysis. From the Askey principle[13], Homogeneous chaos has the optimal(exponential) convergence rate for its corresponding stochastic process by solving the stochastic ODE. If for a certain process the optimal Askey polynomial chaos expansion is not employed, the solution also converges but the rate is clearly slower as demonstrated by numerical results in Section V. Furthermore, contrary to the "Taylor" expansion method[5], which approximate the process variables and frequency parameter with different variational range by the same Taylor order, stochastic spectral methods adopt Homogeneous Chaos to capture the impact of process variations and employ Fourier series or Wavelet basis to model the steady-state behavior in time domain in order to improve the efficiency for variational nonlinear system analysis.
- For stochastic nonlinear analysis, the complexity of SCA is much smaller than that of SGA. According to Section IV, the time consumption of SCA with Sparse Grid technique includes two parts, solving (3) at each selected collocation points and calculation of $c_{kl}$ by (33). For a $P$ order system, the complexity of the first part is at least of $P_s \cdot O(P^3) \approx 2^Q N \cdot O(P^3)$, and the complexity of the second part is $O(PP_sNL)$. Since $P_s \cdot O(P^3) \approx 2^Q N \cdot O(P^3) \gg O(PP_sNL)$, the complexity of SCA could be regarded as $2^Q N \cdot O(P^3)$, which is much lower than that of SGA as $O(P^3 \times N^3)$.
- Sparse Grid Technique is developed in this paper to decrease the number of required collocation points in SCA in order to reduce the complexity and guarantee the approximation accuracy. Compared with the direct tensor product scheme, Sparse Grid technique avoids the exponential increase of the number of collocation points in terms of random space dimension. Furthermore, Sparse Grid technique is much more efficient than ECM for strong nonlinear stochastic analysis. As the heuristic approach, ECM will result in "Runge phenomenon" since it does not generate the exact Gaussian collocation points. With high order Hermite expansion, ECM always suffers from "Rank deficient problem" which can not be solved by least square method to obtain the exact solution for the original system. According to Theorem 2, Sparse Grid technique could guarantee the high approximation accuracy and the convergence rate while reducing the number of required collocation points as demonstrated by numerical examples given in Section V.

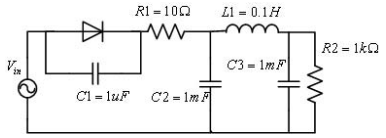


Fig. 1. Schematic of the DC Power Supply

## V. NUMERICAL RESULTS

In this section, a DC power supply (Fig.1), which is widely used in the testing of deterministic steady-state analysis algorithms[12], is applied to demonstrate the properties of proposed SSCA. The mean value of devices parameters are also given in Fig 1. Harmonic Balance Method is taken as the solution engine for deterministic nonlinear system. The voltage on $C1, C2, C3$ (denoted as $v_1, v_2, v_3$), and the current across $L1$(denoted as $i_L$) are taken as the state variables. The current across the diode which has exponential relationship with $v_1$, is denoted as $i_d$. The input for this circuit is a sinusoidal voltage with frequency $100Hz$ and amplitude $10V$. This circuit can be formulated by a series ODE as,

$$i_d = i_s(e^{\frac{v_1}{v_{th}}} - 1)$$

$$C_1\frac{dv_1}{dt} = \frac{v_{in} - v_1 - v_2}{R_1} - i_d \qquad C_3\frac{dv_3}{dt} = i_L - \frac{v_3}{R_2}$$

$$C_2\frac{dv_2}{dt} = \frac{v_{in} - v_1 - v_2}{R_1} - i_L \qquad L_1\frac{di_L}{dt} = v_2 - v_3$$

where $i_s, v_{th}, R_1, R_2$ are considered as random variables with Gaussian variations $\leq 7\%$. $i_L$ is taken as the testing signal. Obviously, any small variations of process parameters will result in exponential variations for the state variables. This is a typical strong nonlinear system where high order polynomial chaos expansion should be applied. The response obtained by Monte Carlo HSPICE simulation with $10^4$ sampling points is taken as the standard result.

### A. Comparison with Taylor Expansion

By applying "Taylor expansion", the response is approximated by a truncated Taylor series in terms of process parameters, and the expansion coefficients can be obtained by moment matching scheme combined with Harmonic Balance algorithm. However, when four parameters $i_s, v_{th}, R_1, R_2$ are considered as random variables and the expansion order is 3, "Taylor expansion" will suffer into "out of memory". In order to study the convergence rate of SSCA and "Taylor expansion", only two parameters, $R_1$ and $R_2$ are regarded as random parameters with Gaussian variations $\leq 7\%$. Fig.2 shows the relative errors of the mean values and variances obtained by "Taylor" scheme and SSCA. The relative errors of "Taylor" method for order 3 and order 4 are almost the same, while SSCA with order 4 has much higher accuracy than that with order 3. Evidently, both the convergence rate and accuracy of SSCA are much higher than those of "Taylor" methods for the analysis of nonlinear system with process variations. This example demonstrates the exponential convergence rate of Homogeneous Chaos approximation for stochastic process, especially for the analysis of strong nonlinear circuit with process variations.

### B. Comparison with SGA

Fig.3 (a) shows the relative errors of the mean values for the steady state response obtained by SGA with Sparse Grid technique and SSCA, and (b) shows the computational costs for these two algorithms. It is clear that even though with same order expansion, the accuracy of SGA and SSCA are almost at the same order, the CPU consumption of SGM is much higher than SSCM. SGM is most likely to suffer from "out of memory" when the order of employed Hermite polynomial is higher than 3, while SSCA could give the results with high accuracy with acceptable complexity. The numerical results demonstrate that SSCM is more efficient than SGM for nonlinear system analysis with the similar accuracy and much lower CPU cost.

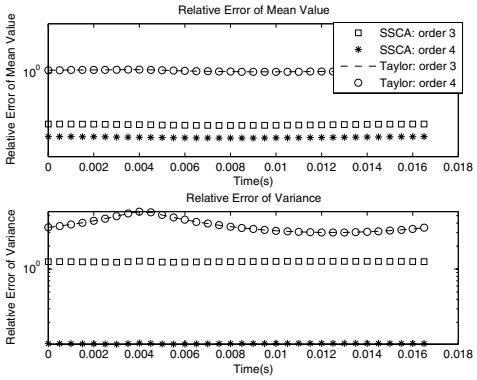


Fig. 2. Results of Taylor expansion method and SSCA

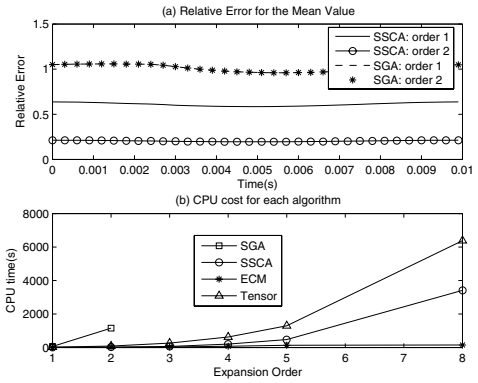


Fig. 3. Results of SGA and SSCA

### C. Comparison with ECM

Fig.4 shows the relative errors of mean values and variances for the steady state response obtained by ECM, SSCA and Tensor product scheme, while Fig.3(b) gives the CPU cost for these three algorithms. Firstly, though Tensor scheme could result in much high approximation accuracy as shown in Fig.4, its computational complexity is always very high. Secondly, though the CPU time cost by ECM is much smaller than SSCA as shown in Fig 3(b), ECM can not obtain the acceptable approximation accuracy for strong nonlinear stochastic analysis when increasing Homogeneous expansion order. Furthermore, according to Fig.4, the relative error of ECM for order 2 is smaller than those for order 1 and order 3, i.e., the accuracy could not be improved by increasing the expansion polynomial chaos order and the number of collocation points for ECM, which is called as the Runge phenomenon in interpolation problem. Moreover, when the expansion order of Homogeneous Chaos is larger than 3, ECM suffers from "Rank deficient problem". For instance, for matrix H, the rank is 60 and the size is 70 for order 4, the rank is 99 and the size is 126 for order 5. Though Least square method in Matlab is applied here to obtain the minimum 2-norm result [16] for order 4 with ECM, the relative error is up to $10^4$ as given in Fig 4. Therefore, ECM is infeasible for strong nonlinear stochastic analysis because of its lack of theorem for error control. On the other hand, by applying SSCA, the response are convergent to Tensor product scheme results with the increase of expansion order, while the CPU cost is much lower than that of Tensor scheme as shown in Fig.3. Therefore, SSCA, which has the solid mathematical background, is a practical and efficient method for stochastic analysis of the strong nonlinear system.

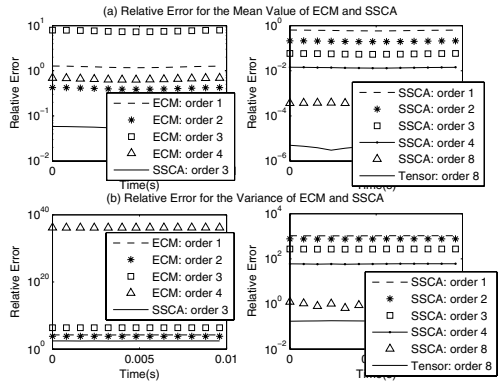


Fig. 4. Results of ECM and SSCA

## VI. Conclusion

In this paper, Stochastic Sparse-grid Collocation Algorithm (SSCA) is proposed to analyze the stochastic steady-state performance of nonlinear circuit with process variations. Homogeneous Chaos is applied in SSCA to capture the variation behavior and Fourier series or Wavelet bases is used to model the circuit behavior in time domain. Both theoretical analysis and numerical results have shown that, by applying SSCA, the accuracy is much higher than "Taylor expansion" methods and the time consumption is much lower than Stochastic Galerkin Algorithm. Furthermore, Sparse Grid technique is introduced to reduce the complexity of SSCA while guaranteeing the accuracy and avoiding "Rank deficient problem" and "Runge phenomenon" appeared in the heuristic approach ECM.

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