The covariance of $e(k) = [e_1(k)e_2(k)]^T$ is

$$E\{e(k)e^T(k)\} = P(k) = \left(\begin{array}{c|c} P_1(k) & P_{12}(k) \\ \hline P_{12}^T(k) & P_2(k) \end{array}\right). \tag{21}$$

The covariance, $P(k)$, evolves as

$$P(k+1/k+1) = \Psi P(k/k)\Psi^T + \Gamma Y(k+1)\Gamma^T \tag{22}$$

where

$$\Psi = \left(\begin{array}{c|c} [I-G(k+1)H(k+1)]\Phi(k+1, k) & 0 \\ \hline 0 & \Phi(k+1, k) \end{array}\right)$$

$$\Gamma = \left(\begin{array}{c|c} -[I-G(k+1)H(k+1)]B(k+1) & G(k+1) \\ \hline -B(k+1) & 0 \end{array}\right)$$

$$Y(k+1) = \left(\begin{array}{c|c} Q(k+1) & 0 \\ \hline 0 & R(k+1) \end{array}\right).$$

The initial conditions for each estimate are

$$\hat{x}_1(0) = \hat{x}_2(0) = E\{x(0)\} = x_0$$

and the initial covariance $P(0)$ is

$$P(0) = \left(\begin{array}{c|c} P_0 & P_0 \\ \hline P_0 & P_0 \end{array}\right). \tag{23}$$

Equation (22) can be expanded to yield the transition equations for the submatrices of $P(k)$ shown in (21). For the optimal filter gain, identical state models for the two estimates $\hat{x}_1$ and $\hat{x}_2$, and the initial covariance of (23) it can then be shown that

$$P_{12}(k) = P_1(k), \qquad k = 0, 1, 2, \cdots. \tag{24}$$

Substituting (24) into (13) yields

$$B(k) = P_2(k) - P_1(k). \tag{25}$$

Only $P_1(k)$ and $P_2(k)$ must be computed in order for the chi-square test statistic $\lambda(k)$ to account for the correlation between the errors $e_1(k)$ and $e_2(k)$. Two $n$-state covariance models can be computed instead of the single $2n$-state covariance model of (22).

The test statistic $\lambda(k)$ is defined by (15). However, it can be computed using the Cholesky decomposition [7],

$$B(k) = LL^T$$

by solving a triangular system of equations, and computing $\lambda(k)$ as an inner product of a vector with itself.

## IV. Conclusions

The two-estimate technique developed by Kerr is suitable for use as a local-filter self-test that determines the validity of the filter estimate and covariance model by computing two estimates of the system state, one based on on line data and another based solely on *a priori* information. This technique is a long-term test, since at each check-time all integrated effects since system start-time are considered. The long-term aspect suggests that the two-estimate technique is well-suited to failure detection with sensors that are subject to soft failures, such as instrument bias shifts that require time to integrate into the measured variables. The test can also be used with suboptimal filters since it does not depend on the residuals being white in the unfailed case.

For the particular case where the two estimates are computed based on the same model of the system state, and assuming the filter is optimal, the two covariances completely define the joint covariance of the errors in the two estimates. The threshold for the chi-square test can be obtained from tables of the chi-square distribution, chosen according to the number of degrees of freedom in the chi-square variable. The chi-square test yields a closed form for two and higher dimensions, and is computationally straightforward.

Finally, it is important to note that the chi-square test requires knowledge of the cross-covariance of the errors in the two estimates. For the case where the two estimates are the Kalman filter estimate and the time-extrapolated *a priori* estimate, both computed assuming the same state model, the cross-covariance is the same as the filter covariance (and, hence, does not require additional computation). An interesting application is the comparison of two Kalman filter estimates, each based on different measurements, to determine if they agree within the confidence limits for both. It is possible to compute the cross-covariance of the errors in these two estimates, however, Kerr's two-ellipsoid overlap test may require less computational effort and provide a framework for approximation methods.

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# Stochastic Teams with Nonclassical Information Revisited: When is an Affine Law Optimal?

RAJESH BANSAL AND TAMER BAŞAR


***Abstract*—In this note we consider a parameterized family of two-stage stochastic control problems with nonclassical information patterns, which includes the well-known 1968 counterexample of Witsenhausen. We show that whenever the performance index does not contain a product term between the decision variables, the optimal solution is linear in the observation variables. The parameter space can be partitioned into two regions in one of which the optimal solution is linear, whereas in the other it is inherently nonlinear. Extensive computations using two-point piecewise constant policies and linear plus piecewise constant policies provide numerical evidence that nonlinear policies may indeed outperform linear policies when the product term is present.**


## I. Introduction

In the context of team decision theory it is now customary to distinguish problems on the basis of their information structure, which may be of the classical, quasi-classical, or nonclassical type.


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The authors are with the Coordinated Science Laboratory, University of Illinois, Urbana, IL 61801.



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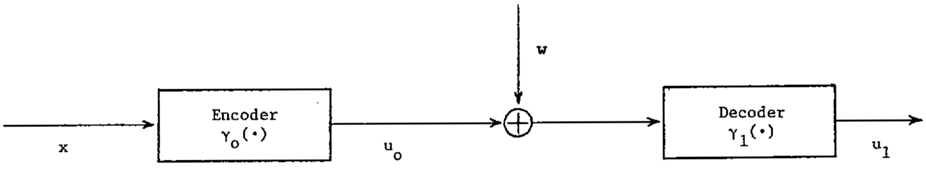


Fig. 1. The Gaussian test channel.

By the classical or quasi-classical information structure, we refer to the situation of either static teams, where there is no explicit causal relationship between the control and information of different members, or dynamic teams with partially nested information structure [3]. In case of classical or quasi-classical information structures, the LQG team problem is known to admit linear optimal solutions.

However, if the decision maker $j$'s action affects the information of $i$ and there is no way in which $i$ can infer the information available to $j$, the information structure is of the *nonclassical type*. Such problems have defied all attempts for the development of realistic algorithms or representations of their solution. Recently, Papadimitriou and Tsitsiklis [5] have studied the example provided by Witsenhausen [6] and have shown that the discrete version of the problem is NP complete, thus explaining the failures in the literature to attack the problem computationally. Here we are interested in identifying the precise causes that contribute to the intractability of the problem (NP complete or worse).

Hitherto, two basic problems with quadratic cost structure and nonclassical information patterns have been identified in the literature, one of which admits a nonlinear optimal solution whose exact form is yet not known, whereas the other one is quite tractable and admits a linear optimal solution.

The former is the problem investigated by Witsenhausen, which we call W1:

*Problem W1:*

$$\text{minimize } J=E\{k_0u_0^2+(x+u_0-u_1)^2\}$$

$$\text{with } u_0=\gamma_0(x) \text{ and } u_1=\gamma_1(x+u_0+w)$$

over all Borel measurable maps $\gamma_0$ and $\gamma_1$. Here $k_0$ is a positive constant, and $x$ and $w$ are independent Gaussian random variables with zero mean and variances $\sigma_x^2$ and $\sigma_w^2$, respectively. □

It has been established by Witsenhausen [6] that an optimal design exists for this problem, and an affine design is not necessarily optimal.

The latter is the Gaussian test channel arising in communications, which, when viewed as a stochastic team problem, shows an information pattern of the nonclassical type. To further describe this problem, consider the communication system depicted in Fig. 1, where a Gaussian message $x$ is to be transmitted reliably over a noisy Gaussian channel. The problem here is to minimize the expected value of distortion $(u_1 - x)^2$ under a given power constraint $E[u_0^2] \leqslant P^2$. Using Shannon's capacity theorem, it can be shown [2] that the optimal design for this problem is linear with $u_0 = \lambda x$ and

$$u_1=E[x|u_0+w]$$

where $\lambda^2\sigma_x^2 = P^2$.

If we replace the 'hard' constraint on the power in the Gaussian test channel above by a soft constraint, the problem may be viewed in the control theoretic framework as Problem C below.

*Problem C:*

$$\text{minimize } J=E[k_0u_0^2+(u_1-x)^2]$$

$$\text{with } u_0=\gamma_0(x) \text{ and } u_1=\gamma_1(u_0+w)$$

over all Borel measurable maps $\gamma_0$ and $\gamma_1$. □

The optimal solution for Problem C, which may again be obtained using information theoretic bounds, is also linear [7], [4].

The Witsenhausen problem W1 may be rewritten as W2 below.

*Problem W2:*

$$\text{minimize } J=E[k_0(u_0-x)^2+(u_0-u_1)^2]$$

$$\text{with } u_0=\gamma_0(x) \text{ and } u_1=\gamma_1(u_0+w)$$

over all Borel measurable $\gamma_0$ and $\gamma_1$. □

Both Problems C and W2 have identical information structures, and yet only Problem C admits linear optimal solutions. We would like to identify the sources of difficulty for problems endowed with this type of nonclassical information structure. Towards this end we will consider a general class of problems which contains both Problems C and W2 as special cases, and then determine the conditions under which this class will admit linear optimal solutions.

The organization of this note is as follows. In Section II we formulate the basic LQG team problem, which includes the two well-known problems with nonclassical information structure as special cases. In Section III we solve this problem when the product term between the decision variables is absent, using some results from information theory. Section IV contains numerical results on optimal policies, and provides numerical evidence that nonlinear policies may indeed outperform the optimal linear policy when the product term between the decision variables is present in the performance index. The note ends with a Conclusion section which indicates some directions for further work, and an Appendix.

## II. Problem Statement

Towards the goal identified in Section I, we first consider the general stochastic team problem P1 below. We then restrict the coefficient of the $u_0u_1$ term to be zero and rewrite P1 in a form appropriate for our use.

Note that both W2 and C can be viewed as special cases of P1 below.

*Problem P1:*

$$\text{minimize } J=E[k_0'u_0''^2+k_1'u_1''^2+k_{01}'u_0''u_1''+s_{01}'u_0''x+s_{11}'u_1''x]$$

$$k_0'>0,\ k_1'>0,$$

$$\text{with } u_0''=\gamma_0''(x) \text{ and } u_1''=\gamma_1''(u_0''+bx+w). \quad □$$

We first observe that the parameter $b$ can, without loss of generality, be assumed to be zero, since by substituting $u_0' = u_0'' + bx$ and appropriately redefining $s_{01}'$ and $s_{11}'$ we get the equivalent problem P2 below.

*Problem P2:*

$$\text{minimize } J=E[k_0'u_0'^2+k_1'u_1'^2+k_{01}'u_0'u_1'+s_{01}'u_0'x+s_{11}'u_1'x]$$

$$k_0'>0,\ k_1'>0$$

$$\text{with } u_0'=\gamma_0'(x) \text{ and } u_1'=\gamma_1'(u_0'+w). \quad □$$

If $k_{01}'$ is zero, we have two possibilities: either $s_{11}' = 0$ or $s_{11}' \neq 0$. For the former we arrive at the equivalent problem P3 below.

*Problem P3:*

$$\text{minimize } J=E[k_0u_0^2+k_1u_1^2+s_{01}u_0x]$$

$$\text{with } u_0=\gamma_0(x) \text{ and } u_1=\gamma_1(u_0+w). \quad □$$

This problem trivially admits the solution

$$u_0=-\frac{s_{01}}{2k_0}x \text{ and } u_1=0.$$

We therefore restrict ourselves to the latter case (i.e., $k_{01}' = 0$ and $s_{11}'$

$\neq 0$) where the cost function for P2 may be rewritten as

$$J=E\left[\frac{4k_1'k_0'}{s_{11}'^2}u_0'^2+\frac{4k_1's_{01}'}{s_{11}'^2}u_0'x+\left(\frac{2k_1'u_1'}{s_{11}'}+x\right)^2\right]$$

and problem P2 may be rewritten as problem P4 below.

*Problem P4:*

$$\text{minimize } J=E[k_0u_0^2+s_{01}u_0x+(u_1-x)^2]$$

$$\text{with } u_0=\gamma_0(x) \text{ and } u_1=\gamma_1(u_0+w)$$

$$\text{where } k_0=\frac{4k_1'k_0'}{s_{11}'^2},\ s_{01}=\frac{4k_1's_{01}'}{s_{11}'^2} \text{ and } u_1=-\frac{2k_1'}{s_{11}'}u_1'. \qquad \square$$

The reason for rewriting the problem as P4 is that this is the appropriate form for the application of Shannon's capacity theorem which we will utilize to prove the existence and optimality of linear solutions and thus establish our claim that the complexity of the class of problems considered is due to the presence of the $u_0u_1$ term, coupled with the nonclassical information structure.

## III. Optimal Solution for P4

Viewing problem P4 from an information theoretic viewpoint, we can say that the problem is to minimize the expected value of a generalized distortion measure given by $k_0u_0^2 + s_{01}u_0x + (u_1 - x)^2$; i.e., in addition to the distortion $(u_1 - x)^2$ of a standard Gaussian test channel, we have soft constraints on $u_0^2$ and $u_0x$. The encoder decision $u_0$ is based on the random variable $x$ and the decoder output is based on the channel output $y = u_0 + w$, which is a noise corrupted version of the encoder output.

Fig. 1 represents the team decision problem P4 from an information theoretic viewpoint. Using a basic property of mutual information, we have

$$I\{x;u_1\}\leqslant I\{u_0;y\} \tag{3.1}$$

where $I\{x; u_1\}$ represents the mutual information of the random variables $x$ and $u_1$, and $I\{u_0; y\}$ denotes that of $u_0$ and $y$.

Now, under the constraint $E[u_0^2] \leqslant P^2$, we have the inequality [8]

$$I\{u_0;y\}\leqslant\frac{1}{2}\log\frac{P^2+\sigma_w^2}{\sigma_w^2} \tag{3.2}$$

where the expression on the right-hand side is the channel capacity. Furthermore, we have

$$I\{x;u_1\}=H\{x\}-H\{x|u_1\}\geqslant\frac{1}{2}\log 2\pi e\sigma_x^2-\frac{1}{2}\log 2\pi e\Delta^2 \tag{3.3a}$$

$$=\frac{1}{2}\log\frac{\sigma_x^2}{\Delta^2} \tag{3.3b}$$

where $\Delta^2 = E[(u_1 - x)^2]$. Here $H\{x\}$ denotes the entropy of the random variable $x$.

From (3.1)–(3.3) we obtain

$$\frac{1}{2}\log\frac{\sigma_x^2}{\Delta^2}\leqslant\frac{1}{2}\log\frac{P^2+\sigma_w^2}{\sigma_w^2} \tag{3.4}$$

and using the monotonicity of logarithms

$$\Delta^2\geqslant\frac{\sigma_x^2\sigma_w^2}{P^2+\sigma_w^2}. \tag{3.5}$$

If $J_p$ denotes the infimum of $J$ under the constraint $E[u_0^2] = P^2$, i.e.,

$$J_p\triangleq\inf_{\gamma_0,\gamma_1,E[\gamma_0^2(x)]=P^2}J(\gamma_0,\gamma_1),$$

we have the following series of inequalities:

$$J_p\geqslant k_0P^2+\inf_{E[\gamma_0^2]=P^2}\Delta^2+\inf_{E[\gamma_0^2]=P^2}s_{01}E[u_0x]$$

$$\geqslant k_0P^2+\inf_{E[\gamma_0^2]\leqslant P^2}\Delta^2+\inf_{E[\gamma_0^2]\leqslant P^2}s_{01}E[u_0x], \tag{3.6a}$$

$$\geqslant k_0P^2+\frac{\sigma_x^2\sigma_w^2}{P^2+\sigma_w^2}-|s_{01}|P\sigma_x \tag{3.6b}$$

$$\geqslant\min_{P\geqslant 0}\left[k_0P^2+\frac{\sigma_x^2\sigma_w^2}{P^2+\sigma_w^2}-|s_{01}|P\sigma_x\right] \tag{3.6c}$$

$$=k_0P^{*2}+\frac{\sigma_x^2\sigma_w^2}{P^{*2}+\sigma_w^2}-|s_{01}|P^*\sigma_x. \tag{3.6d}$$

Note that in going from (3.6a) to (3.6b) we have utilized the Cauchy-Schwartz inequality. In (3.6d), $P^* > 0$ necessarily exists since at $P = 0$ the function is decreasing, and as $P \to \infty$, $J_p \to \infty$, implying that we can restrict our search to a compact region. A continuous function on a compact set always admits a minimum, and in this case the value of $P = P^*$ which attains this minimum satisfies

$$(2k_0P^*-|s_{01}|\sigma_x)(P^{*2}+\sigma_w^2)^2=2P^*\sigma_x^2\sigma_w^2. \tag{3.7}$$

Now, since the right-hand side of (3.6d) is independent of $P$, we have

$$J_{\text{opt}}\triangleq\inf_{\gamma_0,\gamma_1}J(\gamma_0,\gamma_1)=\inf_{P\geqslant 0}J_p$$

$$\geqslant\frac{\sigma_x^2\sigma_w^2}{P^{*2}+\sigma_w^2}+k_0P^{*2}-|s_{01}|P^*\sigma_x \tag{3.8}$$

which gives us a lower bound for the infimum of $J$. Our next task is to show that this bound is tight, and can be achieved by linear policies $\gamma_0$ and $\gamma_1$.

Towards this end, we first find the minimum value of performance index achievable when $\gamma_0$ is restricted to be linear, i.e., $u_0 = \lambda x$. Here the policy $\gamma_1$ is taken to be a general, possibly nonlinear, Borel measurable mapping. With $u_0$ as above, the optimal $\gamma_1$ is also linear, given by

$$u_1=E[x|u_0+w]=\frac{\lambda\sigma_x^2}{(\lambda^2\sigma_x^2+\sigma_w^2)}(\lambda x+w)\equiv\mu(u_0+w). \tag{3.9}$$

The expected value of cost with the linear policy as above is

$$\bar{J}(\lambda)=k_0\lambda^2\sigma_x^2+s_{01}\lambda\sigma_x^2+\frac{\sigma_x^2\sigma_w^2}{\lambda^2\sigma_x^2+\sigma_w^2}. \tag{3.10}$$

As before, $\bar{J}(\lambda)$ admits a minimum, and the $\lambda$ that attains this minimum is given by $\lambda^*$, where $\lambda^*$ is a solution of (3.11) below:

$$(2k_0\lambda^*\sigma_x^2+s_{01}\sigma_x^2)(\lambda^{*2}\sigma_x^2+\sigma_w^2)^2=\sigma_x^2\sigma_w^2 2\lambda^*\sigma_x^2. \tag{3.11}$$

The minimum value of performance index over the linear class is then found by substituting the solution from (3.11) into (3.10), i.e.,

$$\bar{J}(\lambda^*)=k_0\lambda^{*2}\sigma_x^2+s_{01}\lambda^*\sigma_x^2+\frac{\sigma_x^2\sigma_w^2}{\lambda^{*2}\sigma_x^2+\sigma_w^2}. \tag{3.12}$$

Note that (3.7) is identical to (3.11) with $P^* = -(\text{sgn } s_{01})\lambda^*\sigma_x$. In other words, with $\lambda^* = -(\text{sgn } s_{01})P^*/\sigma_x$, the optimal linear solution achieves the lowest possible value of distortion attainable, with (3.12) and (3.6d) yielding the same value.

Hence, we see that the bound in (3.6) is indeed tight and is attained by linear policies for $u_0$ and $u_1$, implying that the solution to this team problem with nonclassical information pattern is indeed linear.

We now summarize the essentials of this result in the following theorem.

*Theorem 3.1:* i) Problem P1, with $k_{01}' = 0$, admits an optimal solution which is *linear* in the measurements available to the decision makers. The

TABLE I
DEPENDENCE OF OPTIMAL COST ON $k_0$

| S. No. | $k_0$ | $\lambda^*$ | $\mu^*$ | $J_{opt}$ |
|---|---|---|---|---|
| 1 | 0.0125 | −40.0002 | $-2.49973\times10^{-2}$ | −125.999 |
| 2 | 0.125 | −4.02011 | −0.246210 | −17.939 |
| 3 | 0.5 | −1.16663 | −0.763655 | −8.262 |
| 4 | 0.1 | −0.741043 | −1.03525 | −5.754 |
| 5 | 2.0 | −0.494034 | −1.20280 | −3.601 |
| 6 | 4.0 | −0.311618 | −1.18139 | −1.748 |
| 7 | 8.0 | −0.149754 | −0.791960 | −0.537 |
| 8 | 80.0 | $-6.7565\times10^{-3}$ | $-4.05277\times10^{-2}$ | $-2.027\times10^{-2}$ |

corresponding gain coefficients can be obtained by solving for a root of a fifth-order polynomial. The polynomial is given by (3.11) in terms of the parameters of a related Problem P4, and the optimum value for $J$ is (3.12), or equivalently the right-hand side of (3.8), again in terms of the parameters of Problem P4.

ii) If $\lambda^*$ denotes the root of (3.11) rendering a minimum value to (3.12), the optimum solution for Problem P4 is given by

$$u_0=\gamma_0^*(x)=\lambda^*x$$

$$u_1=\gamma_1^*(u_0+w)=[\lambda^*\sigma_x^2/(\lambda^{*2}\sigma_x^2+\sigma_w^2)][u_0+w]. \qquad \square$$

## IV. NUMERICAL RESULTS

In this section we present some numerical results on optimal linear policies, along with numerical evidence that nonlinear policies may indeed outperform the optimal linear policy when a product term between the decision variables is present in the performance index. We shall consider two types of nonlinear policies, $u_0 = \epsilon \operatorname{sgn} x$, as considered by Witsenhausen earlier, and $u_0 = \epsilon \operatorname{sgn} x + \lambda x$, that is, a combination of the two-point policy and a linear policy.

### A. Linear Optimal Policies for P4

Knowing that linear policies are optimal over the class of all strategies for Problem P4 we can find the optimal performance index for various values of the parameters.

Table I gives the optimal strategies and corresponding costs when $k_0$ increases from a small positive value to larger values. Here $s_{01}$, $\sigma_x^2$, and $\sigma_w^2$ are assumed to be fixed at 1.0, 6.0, and 1.0, respectively. An increase in $k_0$ leads to an increase in the signaling cost, and hence as $k_0$ increases the signal energy decreases.

Table II gives the optimal strategies and the corresponding costs when $s_{01}$ increases from a negative value to a positive one. Here $k_0$, $\sigma_x^2$, and $\sigma_w^2$ are assumed to be fixed at 2.0, 6.0, and 1.0, respectively. When the sign of $s_{01}$ is reversed, the signs of both $\lambda^*$ and $\mu^*$ are reversed and the value of the performance index remains unchanged.

Table III gives the optimal strategies and the corresponding costs when $\sigma_x^2$ increases from a small positive value, with $k_0$, $s_{01}$, and $\sigma_w^2$ fixed at 2.0, 1.0, and 1.0, respectively.

Table IV gives the optimal strategies and corresponding costs when $\sigma_w^2$ increases, with $k_0$, $s_{01}$, and $\sigma_x^2$ fixed at 1.0, 1.0, and 6.0, respectively.

### B. Improvement Using Nonlinear Policies

Witsenhausen [6] observed an improvement in the achievable performance index when the linear policy is replaced by a policy of the form $u_0 = \epsilon \operatorname{sgn} x$, and the performance index includes the product term between $u_0$ and $u_1$. In particular, it has been observed that for Problem W2 (Section I), with $k_0\sigma_x^2 = 1$ and $k_0 \to 0$ the nonlinear policy above with $\epsilon = \sigma_x$ performs better than the optimal linear policy. One can actually show that further considerable improvement can be made if $\epsilon$ is chosen as $\sqrt{2/\pi}\cdot\sigma_x$, rather than as $\sigma_x$, the resulting bound on the asymptotic cost: $(1-(2/\pi)) \approx 0.363$ comparing favorably to the one reported in Witsenhausen [6]: $2(1-\sqrt{2/\pi}) \approx 0.404$. We shall next see that such improvements by

TABLE II
DEPENDENCE OF OPTIMAL COST ON $s_{01}$

| S. No. | $s_{01}$ | $\lambda^*$ | $\mu^*$ | $J_{opt}$ |
|---|---|---|---|---|
| 1 | −16.0 | 3.99275 | 0.247863 | −197.938 |
| 2 | −4.0 | 1.05383 | 0.825094 | −17.182 |
| 3 | −1.0 | 0.494034 | 1.20280 | −3.601 |
| 4 | 0.0 | ±0.349297 | ±1.210000 | −1.072 |
| 5 | 0.01 | −0.350774 | −1.21078 | −1.09280 |
| 6 | 1.0 | −0.494034 | −1.20280 | −3.601 |
| 7 | 4.0 | −1.05383 | −0.825094 | −17.182 |
| 8 | 16.0 | −3.99275 | −0.247863 | −197.938 |

TABLE III
DEPENDENCE OF OPTIMAL COST ON $\sigma_x^2$

| S. No. | $\sigma_x^2$ | $\lambda^*$ | $\mu^*$ | $J_{opt}$ |
|---|---|---|---|---|
| 1 | 0.25 | −0.284124 | $-6.96258\times10^{-2}$ | $-3.56\times10^{-2}$ |
| 2 | 1.0 | −0.398357 | −0.343800 | −0.2179 |
| 3 | 6.0 | −0.494034 | −1.20280 | −3.60072 |
| 4 | 12.0 | −0.465487 | −1.55156 | −9.0524 |
| 5 | 24.0 | −0.427155 | −1.90585 | −21.0318 |

TABLE IV
DEPENDENCE OF OPTIMAL COST ON $\sigma_w^2$

| S. No. | $\sigma_w^2$ | $\lambda^*$ | $\mu^*$ | $J_{opt}$ |
|---|---|---|---|---|
| 1 | 0.1 | −0.578220 | −1.64733 | −7.718 |
| 2 | 1.0 | −0.741043 | −1.03525 | −5.757 |
| 3 | 6.0 | −0.797883 | −0.487520 | −3.302 |
| 4 | 12.0 | −0.727443 | −0.287621 | −2.445 |
| 5 | 100.0 | −0.530798 | $-3.13183\times10^{-2}$ | −1.594 |

nonlinear policies of the type given above are parameter-specific, and do not carry over for arbitrary values of the parameters.

Towards this end we have Problem NL1.

*Problem NL1:*

$$\text{minimize } J=\{k_0u_0^2+k_1u_1^2+k_{01}u_0u_1+s_{01}u_0x+s_{11}u_1x\}$$

with

$$u_0=\epsilon \operatorname{sgn} x \text{ and } u_1=\gamma_1(u_0+w)$$

over all Borel measurable $\gamma_1$. $\square$

Using the results from Appendix I, we can compute the optimum value of $J$ for different values of $\epsilon$. For the case where the parameters in NL1 are:

$$k_0=1.01;\ k_1=1;\ k_{01}=-2;\ s_{01}=-0.02;$$

$$s_{11}=0;\ \sigma_x^2=100;\ \sigma_w^2=1$$

the optimum value of $J$ with $\epsilon = 5$ is $-0.5474$ which is much better than the optimal linear policy which yields $J = -0.01$.

However, it is notable that a very small change of the parameters in either direction causes an increase in the performance index achievable by this nonlinear policy to such an extent that it ceases to be better than the linear policy. For instance, if $s_{11}$ is changed to $-0.01$, with all other parameters being the same, the optimal linear policy gives $J = -1.25$, whereas the best policy of the form $\epsilon \operatorname{sgn} x$ gives $J = -1.39$. With $s_{01} = -1$, a linear policy achieves $J = -2625$ while the best policy in the form $\epsilon \operatorname{sgn} x$ achieves $J = -729$. Similarly, if $\sigma_x^2$ is increased to 500, all other parameters being the same, the optimal linear policy achieves $-4.002$, whereas the best policy in the form $\epsilon \operatorname{sgn} x$ achieves $-3.13$.

A similar situation is observed with changes in $s_{11}$, $k_0$, and $k_1$, implying

TABLE V
IMPROVEMENT IN $J$ USING NONLINEAR POLICIES

| $k_0$ | $k_1$ | $k_{01}$ | $s_{01}$ | $s_{11}$ | $\sigma_x^2$ | $\sigma_w^2$ | $\lambda$ | $\epsilon$ | $J$ |
|---|---|---|---|---|---|---|---|---|---|
| 2 | 1 | 0.1 | 2 | 2 | 6 | 1 | $\lambda^l_{opt} = -0.635424$ | 0 | $-.6713$ |
| 2 | 1 | 0.1 | 2 | 2 | 6 | 1 | $-0.635424$ | 0.1 | $-6.7307$ |
| 2 | 1 | 0.1 | 2 | 2 | 6 | 1 | $-0.635424$ | $-0.1$ | $-6.7316$ |
| 2 | 1 | 2 | 2 | 2 | 6 | 1 | $\lambda^l_{opt} = -0.438828$ | 0.0 | $-3.9677$ |
| 2 | 1 | 2 | 2 | 2 | 6 | 1 | $-0.48828$ | 0.1 | $-3.9323$ |
| 2 | 1 | 2 | 2 | 2 | 6 | 1 | $-0.438828$ | $-0.1$ | $-3.9348$ |
| 3 | 2 | 4.5 | 4 | 2 | 6 | 1 | $\lambda^l_{opt} = -1.76136$ | 0 | $-11.4263$ |
| 3 | 2 | 4.5 | 4 | 2 | 6 | 1 | $-1.76136$ | 0.1 | $-11.4229$ |
| 3 | 2 | 4.5 | 4 | 2 | 6 | 1 | $-1.76136$ | $-0.1$ | $-11.4230$ |
| 1.1 | 1 | $-2.0$ | $-0.2$ | 0 | 10 | 1 | $\lambda^l_{opt} = -0.1127$ | 0 | $-0.100$ |
| 1.1 | 1 | $-2.0$ | $-0.2$ | 0 | 10 | 1 | $-0.1127$ | 2 | $-0.4797$ |
| 1.1 | 1 | $-2.0$ | $-0.2$ | 0 | 10 | 1 | $-0.1127$ | $-2$ | 1.4984 |
| 1.01 | 1 | $-2$ | $-0.02$ | 0 | 80 | 1 | $\lambda^l_{opt} = 1.006 \times 10^{-2}$ | 0 | $-7.98 \times 10^{-3}$ |
| 1.01 | 1 | $-2.0$ | $-0.02$ | 0 | 80 | 1 | $1.006 \times 10^{-2}$ | 5 | $-0.4691$ |
| 1.1 | 1 | $-2.0$ | $-0.3$ | 0.1 | 12 | 1 | $\lambda^l_{opt} = -0.880412$ | 0 | $-0.4091$ |
| 1.1 | 1 | $-2.0$ | $-0.3$ | 0.1 | 12 | 1 | 0 | 2 | $-0.4870$ |

that the best policy in the form $\epsilon$ sgn $x$ is better than the optimal linear policy only for very specific parameter values.

We now consider the minimum value of the performance index attainable when we use a policy of the type

$$\gamma_0(x) = \epsilon \text{ sgn } x + \lambda x,$$

that is, we use a combination of the two-point distribution and a linear policy. Such a choice is motivated by the fact that a linear policy is optimal for the case when $k_{01}$ is zero and that in the presence of $k_{01}$ (and for some values of $k_{01}$) a policy of the type $\epsilon$ sgn $x$ can outperform the optimal linear policy.

The problem is now the same as problem NL1 earlier. Using expressions from Bansal [1] the improvement over the optimal linear policy when a nonlinear term is included in $u_0$, either with or without the linear term, may now be observed. The corresponding numerical results are presented in Table V, where $\lambda^l_{opt}$ denotes the gain coefficient of the optimal (pure) linear policy $\gamma_0$. It is observed that in some cases the policy

$$\gamma_0(x) = \lambda^l_{opt} x + \epsilon \text{ sgn } x \qquad \text{for fixed } \epsilon$$

actually performs worse than the policy $\lambda^l_{opt} x$. However, improvements in $J$ are possible in the presence of a nonzero $k_{01}$ (over that achieved by the best linear policy) by including also the $\epsilon$ sgn $x$ term. This improvement has been observed particularly when $|k_{01}|$ is large, close to its limiting value of $\pm\sqrt{4k_0k_1}$. However, it seems that the other parameters $s_{01}$, $s_{11}$, $\sigma_x^2$, etc., also play a significant role and improvement is particularly pronounced when $\sigma_x^2/\sigma_w^2$ is large.

## V. CONCLUSIONS

In this note we have shown that when the performance index does not contain a product term between the decision variables, the LQG team problem with two decision makers indeed admits optimal policies which are affine, even when the information structure is nonclassical. However, when the performance index contains a product term between the two decision variables, the best linear solution ceases to be optimal, with a corrective term $\epsilon$ sgn $x$ added to the first agent's policy leading to significant improvements over the best cost attainable under linear policies. Such a term alone (i.e., without a linear part) does not necessarily outperform the best linear policy, except in a rather restricted region in the parameter space.

The proof given here on the existence of linear policies as solutions of Problem P4 can be carried over to a vector version of P4 where all variables are of the same dimension, by working this time with expressions for the mutual information between two random vectors. There, again, the optimal solution will be linear when a cross term between the decision vectors of the two agents is absent in the cost function. Furthermore, for the case when the first decision maker bases his decision not on the pure observation $x$, but on a noise corrupted version of it, it has been shown in Bansal [1] that the optimal solution is also linear, provided that the product term $u_0u_1$ does not appear in the performance index. The best attainable performance when this cross term is present still remains today as an open problem.

## APPENDIX I

In this Appendix we obtain an expression for the value of Problem NL1 as a function of the parameter $\epsilon$. Toward this end, we first rewrite NL1 in the equivalent form.

*Problem NL2:*

$$\text{minimize } J = \left\{ k_1 \left( u_1 + \frac{k_{01}}{2k_1} u_0 + \frac{s_{11}}{2k_1} x \right)^2 + \left( k_0 - \frac{k_{01}^2}{4k_1} \right) u_0^2 + \left( s_{01} - \frac{s_{11}}{2k_1} k_{01} \right) u_0 x - \frac{s_{11}^2}{4k_1} x^2 \right\}$$

with

$$u_0 = \epsilon \text{ sgn } x \text{ and } u_1 = \gamma_1(u_0 + w)$$

over all Borel measurable $\gamma_1$.

Clearly, the optimal $u_1$ is $E[z|y]$

$$\text{where } y = u_0 + w = \epsilon \text{ sgn } x + w$$

and

$$z = \frac{-k_{01}}{2k_1} u_0 + \frac{-s_{11}}{2k_1} x.$$

Next, we find $E[u_0|y]$ and $E[x|y]$.

The density for $\epsilon$ sgn $x$ = $u_0$ is

$$\frac{1}{2}\delta(u_0 - \epsilon) + \frac{1}{2}\delta(u_0 + \epsilon)$$

and convolving this with the density for $w$ we find the density for $y$ to be

$$p_y(y) = \frac{1}{\sqrt{2\pi}\,\sigma_w} \exp \frac{-(y^2 + \epsilon^2)}{2\sigma_w^2} \cosh \frac{\epsilon y}{\sigma_w^2}.$$

Therefore,

$$E[u_0|y]=\frac{\int_{-\infty}^{\infty} u_0 p(u_0, y)\, du_0}{p_y(y)}=\epsilon \operatorname{Tanh}\frac{\epsilon y}{\sigma_w^2},$$

and

$$E[x|y]=\int x p_{x|y}(x|y)\, dx$$

$$=\sqrt{2/\pi}\,\sigma_x \operatorname{Tanh}\frac{\epsilon y}{\sigma_w^2}.$$

The minimum achievable $J$, with $u_0 = \epsilon \operatorname{sgn} x$, is

$$E\left[k_1(E(z|y)-z)^2+\left(k_0-\frac{k_{01}^2}{4k_1}\right)u_0^2+\left(s_{01}-\frac{s_{11}}{2k_1}k_{01}\right)u_0 x-\frac{s_{11}^2}{4k_1}x^2\right].$$

Since

$$E[u_0^2]=\epsilon^2$$

and

$$E[u_0 x]=\epsilon\sqrt{2/\pi}\,\sigma_x$$

we have

$$E[z^2]=\frac{k_{01}^2}{4k_1^2}\epsilon^2+\frac{s_{11}^2}{4k_1^2}\sigma_x^2+\frac{2k_{01}}{4k_1^2}s_{11}\epsilon\sqrt{2/\pi}\,\sigma_x$$

and

$$(E[z|y])^2=\left(\frac{k_{01}\epsilon}{2k_1}+\frac{s_{11}\sigma_x}{k_1\sqrt{2\pi}}\right)^2\operatorname{Tanh}^2\frac{\epsilon y}{\sigma_w^2}.$$

Therefore, the achieved value of $J$ is

$$=k_0\epsilon^2+s_{01}\epsilon\sqrt{2/\pi}\,\sigma_x$$

$$-\left(\frac{k_{01}^2\epsilon^2}{4k_1}+\frac{s_{11}^2\sigma_x^2}{2\pi k_1}+\frac{k_{01}s_{11}\epsilon}{k_1\sqrt{2\pi}}\sigma_x\right)E\left(\operatorname{Tanh}^2\frac{\epsilon y}{\sigma_w^2}\right)$$

where

$$E\left(\operatorname{Tanh}^2\frac{\epsilon y}{\sigma_w^2}\right)=1-\int_{\infty}^{\infty}\operatorname{sech}^2\frac{\epsilon y}{\sigma_w^2}\frac{1}{\sqrt{2\pi}\,\sigma_w}\exp\left(\frac{-(y^2+\epsilon^2)}{2\sigma_w^2}\right).$$

$$\cosh\frac{\epsilon y}{\sigma_w^2}\,dy=1-\sqrt{2/\pi}\,\frac{1}{\sigma_w}\exp\left(\frac{-\epsilon^2}{2\sigma_w^2}\right)[\tilde{i}]$$

where $\tilde{i}$ is the integral

$$\int_{-\infty}^{\infty}\exp\left(\frac{-y^2}{2\sigma_w^2}\right)\frac{1}{\left(\exp\left(\frac{\epsilon y}{\sigma_w^2}\right)+\exp\left(\frac{-\epsilon y}{\sigma_w^2}\right)\right)}\,dy.$$

The value of this integral can be numerically evaluated and we can, therefore, find the best performance achievable with $u_0 = \epsilon \operatorname{sgn} x$.

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