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**STONE - ČECH COMPACTIFICATION AND REPRESENTATIONS  
OF OPERATOR ALGEBRAS**

**W.I.M. WILS**



# STONE - ČECH COMPACTIFICATION AND REPRESENTATIONS OF OPERATOR ALGEBRAS

**PROMOTOR: PROF. H. FREUDENTHAL**

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OF OPERATOR ALGEBRAS**

**P R O E F S C H R I F T**

**TER VERKRIJGING VAN DE GRAAD VAN DOCTOR  
IN DE WISKUNDE EN NATUURWETENSCHAPPEN  
AAN DE KATHOLIEKE UNIVERSITEIT TE NIJMEGEN,  
OP GEZAG VAN DE RECTOR MAGNIFICUS DR. A.Th.L.M. MERTENS,  
HOOGLERAAR IN DE FACULTEIT DER GENEESKUNDE,  
VOLGENS BESLUIT VAN DE SENAAT  
IN HET OPENBAAR TE VERDEDIGEN  
OP DONDERDAG 13 JUNI DES NAMIDDAGS TE 4 UUR**

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Dit proefschrift is bewerkt mede onder leiding van Dr.R.A. Hirschfeld.

# NOTATIONS

The set of positive integers is denoted by  $\mathbb{N}$ . The symbol  $\mathbb{R}$  stands for the field of real numbers and  $\mathbb{C}$  for the complex numbers. We use  $c$  for the cardinality of  $\mathbb{R}$ .

We consider normed vector spaces  $(X, ||\cdot||)$  over  $\mathbb{C}$ . The dual of a normed space  $(X, ||\cdot||)$  is denoted by  $(X^*, ||\cdot||)$ . Often we just write  $X$  and  $X^*$ . If  $x \in X$  and  $x^* \in X^*$  then the value of  $x^*$  at  $x$  is given by  $\langle x, x^* \rangle$ .

The letters  $S$  and  $T$  denote topological spaces. If  $T \subseteq S$ , then  $T^c = S/T$ . For complex valued functions we use  $\phi$  and  $\psi$ , whereas  $f, g$  and  $h$  stand for vector valued maps. The restriction of a map  $f$ , defined on  $S$ , to a subset  $T \subseteq S$  is  $f|T$ . The space of bounded, complex valued, continuous functions on  $S$  is  $(C(S), ||\cdot||)$ , where  $||\phi|| = \sup \{ |\phi(s)| \mid s \in S \}$  for  $\phi \in C(S)$ .

The following notation was introduced by Professor Dr. H. Freudenthal. Let  $M$  and  $N$  be sets and  $f$  a map from  $M$  into  $N$  such that the value of  $f$  at  $m \in M$  is given by  $r_m$ , where  $r_m \in N$ , then we write  $f = \bigvee_{m \in M} r_m$  or  $f = \bigvee_m r_m$ . For example, let  $\phi \in C(S)$  then  $|\phi| = \bigvee_s |\phi(s)|$  is the element of  $C(S)$  which in  $s \in S$  assumes the value  $|\phi(s)|$ . Similarly if  $\phi, \psi \in C(S)$  then  $\phi \cdot \psi = \bigvee_s \phi(s)\psi(s)$  is the pointwise product of  $\phi$  and  $\psi$ . The numbers 1 and 0 will also denote respectively the maps  $1 = \bigvee_{s \in S} 1$  and  $0 = \bigvee_{s \in S} 0$ .

We list some symbols and abbreviations together with the page on which they occur for the first time.

$CL(X)$	34	$L_X^p(\beta S, \mu)$	24	$\Lambda_T, \Lambda_\omega$	16
$C(S),$	19	$R_{\mu, p}, R_s$	33, 34	$\Lambda_\mu$	18
$C_0(S)$	13	$S^\wedge$	13	$\rho_{\mu, p}$	36
$C(S, X), C_0(S, X)$	21	$X(T), X(\omega)$	16	$\rho_{\mu, p}, \hat{x}$	36
$C_{0, w}(S, X)$	21	$X(\mu)$	18	$\tau_f$	43
$f^\beta$	13	$X_{\mu, p}$	21	$\Pi_\omega, \Pi_\mu$	26, 27
$F$	15	$X(T)^0, X(T)^\wedge$	27	$  \cdot  _T,   \cdot  _\omega$	16
$H_{\mu, 2}^0, H_{\mu, 2}^\wedge, H_{\mu, 2}^+$	41	$X_{\mu, p}^0, X_{\mu, p}^\wedge$	31	$  \cdot  _{\mu, p}$	21
$J$	22	$\beta S$	13		
$K(S, X)$	21	$\Gamma$	15		
$L(X, Y)$	33	$\hat{\mu}, \mu$	14		
$L(H)$	40	$\nu$	15		



## INTRODUCTION

Let  $X$  be a Banach space and  $S$  a completely regular space. We denote by  $(C(S, X), ||\cdot||)$  the Banach space of bounded, continuous maps from  $S$  into  $X$ , with norm  $||f|| = \sup\{||f(s)|| \mid s \in S\}$ ,  $f \in C(S, X)$  and define a map  $v : C(S, X) \rightarrow C(S)$  by  $v(f) = \bigvee_s ||f(s)||$ .

If  $\tau$  is a locally convex topology on  $C(S)$  then  $v^{-1}(\tau)$  is a locally convex topology on  $C(S, X)$ . We denote by  $N_\tau$  the closure of  $0 \in C(S, X)$  with respect to  $v^{-1}(\tau)$ . R.A. Hirschfeld defined in [10] the  $\tau$ -hull of  $C(S, X)$  as the topological quotient space  $C(S, X)/N_\tau$ , where  $C(S, X)$  has the  $v^{-1}(\tau)$  topology. This generalizes work of J.W. Calkin [4] and S.K. Berberian [2].

In 1.3 below, this definition is generalized to so-called continuous families of normed spaces. To each  $s \in S$  corresponds a normed space  $X(s)$ . We consider linear subspaces  $\Gamma \subseteq \prod_{s \in S} X(s)$  and define  $v$  on  $\Gamma$  by  $v(f) = \bigvee_{s \in S} ||f(s)||$ ,  $f \in \Gamma$ . The pair  $F = \{X(s) \mid s \in S\}, \Gamma\}$  is called a continuous family if  $v(\Gamma) \subseteq C(S)$ . (1.3.1). If  $\tau$  is a topology on  $C(S)$  then we define the  $\tau$ -hull of  $F$  in a way which is very similar to the one described above for the constant family with  $\Gamma = C(S, X)$ .

Two classes of examples of hulls are discussed in 1.4. Let  $T$  be a closed subset of the Stone - Čech compactification  $\beta S$  of  $S$ . We define a semi-norm  $||\cdot||_T$  on  $C(S)$  by  $||\phi||_T = \sup\{|\phi(s)| \mid s \in T\}$ ,  $\phi \in C(S)$  and we study the hull of  $F$ , the so-called  $T$ -hull, with respect to the  $||\cdot||_T$  - topology on  $C(S)$ . We denote the projection of  $\Gamma$  into the  $T$ -hull by  $\Lambda_T$  and we put  $X(T) = \Lambda_T \Gamma$ . Special emphasis is put on sets  $T$  which consist of only one point  $\omega \in \beta S/S$  and the corresponding  $\omega$ -hulls  $X(\omega)$ .

Also let  $\hat{\mu}$  be a state on  $C(S)$ , that is,  $\hat{\mu} \in C(S)^*$ ,  $||\hat{\mu}|| = \langle 1, \hat{\mu} \rangle = 1$ , where  $1$  is the identity in  $C(S)$ , and  $1 \leq p < \infty$ . Then one defines  $||\cdot||_{\mu, p}$  on  $C(S)$  by  $||\phi||_{\mu, p} = \langle |\phi|^p, \hat{\mu} \rangle^{1/p}$  with  $|\phi| = \bigvee_s |\phi(s)|$ . It is well-known that  $||\cdot||_{\mu, p}$  defines a semi-norm on  $C(S)$ . The hulls corresponding to the  $||\cdot||_{\mu, p}$  - topology are called  $(\mu, p)$ -hulls. The canonical projection of  $\Gamma$  onto the  $(\mu, p)$ -hull is denoted by  $\Lambda_\mu$  and we put  $\Lambda_\mu \Gamma = X(\mu)$ .

Various properties of  $X(T)$  and  $X(\mu)$  are discussed. We endow  $X(T)$  and  $X(\mu)$  with the norms  $||\Lambda_T f||_T = ||v(f)||_T$  and  $||\Lambda_\mu f||_{\mu, p} = ||v(f)||_{\mu, p}$  respectively, and prove that if  $\Gamma$  is such that for all  $\phi \in C(S)$  and  $f \in \Gamma$  also  $\phi \cdot f \in \Gamma$ , where  $(\phi \cdot f) = \bigvee_s \phi(s) f(s)$  and if  $\Gamma$  is complete with respect to  $||\cdot||$ , then  $(X(T), ||\cdot||_T)$  is complete. (1.4.1.3).

We note that by the Riesz - representation theorem to each state  $\hat{\mu}$  on  $C(S)$  there corresponds a positive Radon-measure  $\mu$  on  $\beta S$ . We show that under the same conditions as for  $T$  - hulls, plus a mild additional condition, the  $(\mu, p)$  - hulls are complete iff  $\text{supp } \mu$  is finite. (1.4.2.1). We denote the completion of  $(X(\mu), ||\cdot||_{\mu, p})$  by  $X_{\mu, p}$ .

It is remarked that if  $F$  is a continuous family of Hilbert spaces then  $X_{\mu, 2}$  is also a Hilbert space. (1.4.2.2).

From 1.5 onwards we consider but constant families, i.e.  $\Gamma = C(S, X)$ . Let  $K(S, X)$  be the subspace of  $C(S, X)$  consisting of those  $f \in C(S, X)$  for which  $f(S)$  is relatively compact. According to the compactification theorem (1.1.1) each  $f \in K(S, X)$  has a continuous extension  $f^\beta$  over  $\beta S$ . The map  $f \rightarrow f^\beta$  is an isomorphism from  $K(S, X)$  onto  $C(\beta S, X)$ . By means of this map we construct an isomorphism  $J$  from the  $\Lambda_\mu$  - image of  $C(\beta S, X)$  in its  $(\mu, p)$  - hull onto the  $\Lambda_\mu$  - image of  $K(S, X)$  in the  $(\mu, p)$  - hull of  $C(S, X)$  as follows :  $J\Lambda_\mu f^\beta = \Lambda_\mu f$ ,  $f \in K(S, X)$ . By definition the completed  $(\mu, p)$  - hull of  $C(\beta S, X)$  is the space  $L_X^p(\beta S, \mu)$ . So we can embed  $L_X^p(\beta S, \mu)$  in the completed  $(\mu, p)$  - hull  $X_{\mu, p}$  of  $C(S, X)$ . (1.5.3). It is shown that  $JL_X^p(\beta S, \mu) \neq X_{\mu, p}$  if  $S = \mathbb{N}$ ,  $X$  infinite-dimensional and  $\hat{\mu}$  a state on  $C(\mathbb{N})$ , which annihilates the functions, which vanish at infinity. (1.5.4).

We note that for  $T \subseteq \beta S$ ,  $T$  closed, the  $\Lambda_T$  image of  $K(S, X)$  in  $X(T)$  can be identified with the space  $C(T, X)$ .

In the next section 1.6 we assume that  $X$  is the dual of a normed space  $Y$ . Every  $f \in C(S, X)$  has a bounded, thus relatively weak\* - compact range. Since all  $f \in C(S, X)$  are also weak\* continuous on  $S$ , there exists a weak\* continuous extension  $f_w^\beta$  of  $f$  over  $\beta S$  by the compactification theorem 1.1.1. We have  $v(f_w^\beta) \leq v(f)^\beta$  (1.6.1) and it can be proved that if  $X$  is reflexive and  $\hat{\mu}$  a state on  $C(S)$  then each  $f_w^\beta$  is  $\mu$  - measurable. Moreover  $||f_w^\beta||_{\mu, p} \leq ||f||_{\mu, p}$  for all  $f \in C(S, X)$  and so  $f_w^\beta \in L_X^p(\beta S, \mu)$ . (1.6.3). We define  $\Pi_\mu$  on  $X(\mu)$  by  $\Pi_\mu \Lambda_\mu f = Jf_w^\beta$ . Thus the continuous extension of  $\Pi_\mu$  is a projection of norm one from  $X_{\mu, p}$  onto  $JL_X^p(\beta S, \mu)$  and  $\Pi_\mu$  is called the Stone - Čech operator. (1.6.4). In 1.7 we deal with the following problem. For  $T \subseteq \beta S$  we put  $X(T)^0 = \{\Lambda_T f | f \in C(S, X), ||f_w^\beta||_T = 0\}$ . Let  $C_{0, w}(S, X)$  be the set  $\{f | f \in C(S, X), \langle y, f \rangle \in C_0(S) \text{ for all } y \in Y\}$  and  $X(T)^\wedge = \Lambda_T C_{0, w}(S, X)$ . What is the relation between  $X(T)^0$  and  $X(T)^\wedge$ ? It is readily seen that  $X(T)^\wedge \subseteq X(T)^0$ .

We give the following characterization. For  $f \in C(S, X)$  we have  $\Lambda_T f \in \Lambda_T(K(S, X) + C_{0, w}(S, X))$  iff  $f_w^\beta|_{\beta S/S}$  is strongly continuous at every point

of T.(1.7.4). Using this characterization we show that if  $T \subseteq \beta S$  contains P - points only, and thus is finite, then  $X(T)^0 = X(T)^\wedge$ . If in addition X is reflexive than this condition is also necessary (1.7.7), (1.7.8).

For chapter II let X and Y be Banach spaces and  $L(X,Y)$  the Banach space of continuous linear transformations from X into Y. If  $\Phi \in C(S, L(X,Y))$  then we can define a transformation  $A\Phi$  from  $C(S,X)$  into  $C(S,Y)$  by

$$(A\Phi)(f) = \bigvee_s \Phi(s)f(s), \quad f \in C(S,X).$$

Now if  $X(\mu)$  is the  $(\mu, p)$  - hull of  $C(S,X)$  with respect to a state  $\hat{\mu}$  on  $C(S)$  and  $Y(\mu)$  similarly for  $C(S,Y)$ , then we can define a map  $R_{\mu,p} \Phi$ ,  $\Phi \in C(S, L(X,Y))$  from  $X(\mu)$  into  $Y(\mu)$  by  $(R_{\mu,p} \Phi)(\bigwedge_\mu f) = \bigwedge_\mu (A\Phi)(f)$ ,  $f \in C(S,X)$ . A similar thing can be done for T - hulls.

If  $X = Y$  then  $C(S, L(X,X))$  is, with point-wise multiplication, a Banach algebra. In this case  $R_{\mu,p}$  is an algebraic homomorphism. (2.1.2). If  $\hat{\mu}$  is a point measure then the  $R_{\mu,p}$  - image of  $C(S, L(X,X))$  in  $L(X_{\mu,p}, X_{\mu,p})$  acts irreducibly and contains even all degenerate operators on  $X_{\mu,p}$ . (2.1.3)

Next we turn our attention to the subspace of  $C(S, L(X,X))$ , which consists of the constant maps and is isomorphic to  $L(X,X)$ . The restriction of  $R_{\mu,p}$  to this subspace induces a representation  $\rho_{\mu,p}$  of  $L(X,X)$  into  $L(X_{\mu,p}, X_{\mu,p})$ .

We prove that if X is separable then every cyclic subrepresentation of a  $\rho_{\mu,p}^S$ , that is  $\rho_{\mu,p}$  and the underlying topological space is S, is equivalent to a cyclic subrepresentation of a  $\rho_{\mu,p}^{\mathbb{N}}$ . (2.2.2). The subspace  $JL_X^p(\beta S, \mu)$  is invariant for  $\rho_{\mu,p}$  and if X is reflexive also the kernel  $X_{\mu,p}^0$  of the Stone-Čech operator is invariant. The restriction  $\rho_{\mu,p}^0$  to  $X_{\mu,p}^0$  of  $\rho_{\mu,p}$  annihilates the compact operators on X. (2.2.4).

In 2.3 finally we assume that X is a Hilbert space, notation H. The representation  $\rho_{\mu,2}$  is a  $*$ representation of  $L(H,H)$ . The principal result is that every cyclic  $*$ representation of  $L(H,H)$ , which annihilates the compact operators is  $*$ isomorphic to a cyclic subrepresentation of a  $\rho_{\mu,2}^0$  with  $\mu$  a point measure on  $\beta \mathbb{N} / \mathbb{N}$ . The cyclic vectors  $\bigwedge_\mu f \in H_{\mu,2}$  can be chosen to be orthogonal to  $\bigwedge_\mu (K(\mathbb{N}, H) + C_{0,w}(\mathbb{N}, H))$ . (2.3.5).



## CHAPTER I

### 1.1 The Stone - Čech compactification.

#### 1.1.1 Compactification theorem. <9, th 6.5>.

Every completely regular topological space  $S$  has a compactification  $\beta S$  with the following equivalent properties :

i. (Stone). Every continuous map  $f : S \rightarrow K$ , where  $K$  is a compact space, has a continuous extension  $f^\beta : \beta S \rightarrow K$ .

ii. (Stone). Every bounded, continuous and real-valued function  $\phi$  on  $S$  has a continuous extension  $\phi^\beta$  to  $\beta S$ .

$\beta S$  is unique in the sense that, if a compactification  $\bar{S}$  of  $S$  satisfies i or ii, then there exists a homeomorphism of  $\beta S$  onto  $\bar{S}$  which leaves  $S$  pointwise invariant.

$\beta S$  is called the Stone - Čech compactification of  $S$ . For a map  $f$  defined on  $S$ , we shall refer to  $f^\beta$  as the Stone - extension of  $f$ . Since  $S$  is by definition dense in  $\beta S$  <9, 3.14>  $f^\beta$  is uniquely determined by  $f$ .

The map  $\bigvee_{\phi \in C(S)} \phi^\beta$  is an isometric, positive, algebraic homomorphism from  $C(S)$  onto  $C(\beta S)$ . A model for  $\beta S$  is the structure space of  $C(S)$  in its hull-kernel topology <9, 7M,N>. Thus the map  $\bigvee_{\phi \in C(S)} \phi^\beta$  turns out to be the Gelfand representation of  $C(S)$ .

The space  $\beta S/S$  will be denoted by  $S^\wedge$ . The symbol  $\omega$  always means a point of  $S^\wedge$ .

The following lemma will be useful <9, 7F>.

1.1.2 If  $C_0(S)$  is the closed subspace of  $C(S)$ , consisting of the functions vanishing at infinity, then  $\phi \in C_0(S)$  iff  $\phi|_{S^\wedge} = 0$ .

proof : Take  $\phi \in C_0(S)$  and  $\epsilon > 0$ . Whence the set  $A = \{s \mid s \in S, |\phi(s)| \geq \epsilon\}$  is compact in  $S$ . As such  $A$  is closed in  $\beta S$ . For  $\omega \in S^\wedge$ , there exists  $\psi \in C(\beta S)$  with  $0 \leq \psi \leq 1$ , such that  $\psi(\omega) = 1$  and  $\psi(A) = 0$ .

Consider  $\psi \cdot \phi$ . Then  $||\psi \cdot \phi|| \leq \epsilon$  and  $|\phi(\omega)| = |\psi(\omega)\phi(\omega)| = |\psi \cdot \phi(\omega)| \leq ||\psi \cdot \phi|| \leq \epsilon$ . Since  $\epsilon > 0$  is arbitrary, we conclude  $\phi(\omega) = 0$ .

Now let  $\phi \in C(\beta S)$  be such that  $\phi|_{S^\wedge} = 0$ . For  $\epsilon > 0$ , let  $A = \{s \mid s \in \beta S, |\phi(s)| \geq \epsilon\}$ . Then  $A$  is a subset of  $S$ , which is compact in  $\beta S$ . Consequently  $A$  is compact in  $S$ .

In the sequel we impose the assumption of complete regularity on all given topological spaces  $S$ .

## 1.2 States.

1.2.1 Definition : A linear functional  $\hat{\mu}$  on  $C(S)$  is called a state if:

- i)  $\phi \in C(S)$ ,  $\phi \geq 0$ , implies  $\langle \phi, \hat{\mu} \rangle \geq 0$ .
- ii)  $\langle 1, \hat{\mu} \rangle = 1$ .

We shall use  $1$  both for the number  $1$  and for the identity in  $C(S)$ .

It can be readily verified that a state is bounded and has norm  $1$ . The Riesz - representation theorem applies to states of  $C(\beta S)$  to the effect that each state  $\hat{\mu}$  determines a unique positive regular Borel measure  $\mu$  of mass  $1$  on  $\beta S$  such that for  $\phi \in C(\beta S)$  we have

$$\langle \phi, \hat{\mu} \rangle = \int \phi \, d\mu.$$

Henceforth we shall denote the functional associated with a measure  $\mu$  by  $\hat{\mu}$ .

We are especially interested in states  $\hat{\mu}$  so that the support of  $\mu$  is contained in  $S^\wedge$ . We shall refer to these states as to free-states. A state is called singular if it annihilates  $C_0(S)$ . The set of free states is contained in the set of singular states. This is a consequence of 1.1.2.

1.2.2 All singular states are free iff  $S$  is locally compact.

proof : Suppose  $S$  is locally compact and  $\hat{\mu}$  a singular state. For every  $s \in S$ , there exists an open neighborhood  $O$  of  $s$  with compact closure. By complete regularity we can find  $\phi \in C(S)$  with  $0 \leq \phi \leq 1$  such that  $\phi(s) = 1$  and  $\phi(\bar{O}) = 0$ . Clearly  $\langle \phi, \hat{\mu} \rangle = 0$  and thus  $s \notin \text{supp } \mu$ .

If  $S$  is not locally compact, there exists a point  $s \in S$  which has no compact neighborhood. Every  $\phi \in C_0(S)$  vanishes at  $s$ . The assignment  $\bigvee \phi \in C(S) \phi(s)$  defines a singular state, which is not free.

An example of a non-locally compact space, is the set  $Q$  of rational numbers, with the topology induced by the ordinary topology of the real numbers. The space  $C_0(Q)$  consists of the zero function only  $\langle 9, 7F \rangle$ .

## 1.3 Continuous families of normed spaces.

Let  $S$  be a completely regular space and  $\{X(s) \mid s \in S\}$  a family of normed vector spaces. We consider elements from  $\prod_{s \in S} X(s)$ , that is maps  $f$

defined on  $S$  such that  $f(s) \in X(s)$  for  $s \in S$ . The set  $\prod_{s \in S} X(s)$  is a linear vector space, where addition and scalar multiplication are defined pointwise. Multiplication with  $\phi \in C(S)$  is made possible by putting  $\phi \cdot f = \prod_{s \in S} \phi(s) f(s)$ , where  $f \in \prod_{s \in S} X(s)$ . Also we define a map  $v$  from  $\prod_{s \in S} X(s)$  into the ring of real valued functions on  $S$  by

$$v(f) = \prod_{s \in S} ||f(s)|| \quad f \in \prod_{s \in S} X(s).$$

**1.3.1 Definition :** A continuous family  $F = \{ \{X(s) \mid s \in S\}, \Gamma \}$  of normed spaces consists of a set  $\{X(s) \mid s \in S\}$  of normed spaces and a linear subspace  $\Gamma \subseteq \prod_{s \in S} X(s)$ , so that  $v(\Gamma) \subseteq C(S)$ .

The space  $\Gamma$  is endowed with the supremum norm  $||\cdot||$ , defined by

$$||f|| = ||v(f)|| = \sup \{ ||f(s)|| \mid s \in S \} \quad f \in \Gamma.$$

The completion  $\bar{\Gamma}$  of  $\Gamma$  with respect to  $||\cdot||$  can be identified with a subspace of  $\prod_{s \in S} \bar{X(s)}$ . Here  $\bar{X(s)}$  is, as usual, the completion of  $X(s)$ .

**1.3.2 Definition :** The pair  $\bar{F} = \{ \{ \bar{X}(s) \mid s \in S \}, \bar{\Gamma} \}$  is called the completion of  $F$ . A continuous family  $F$  is said to be complete if  $F = \bar{F}$ .

In general the inverse image under  $v$  of a locally convex topology  $\tau$  on  $C(S)$  is a locally-convex topology  $v^{-1}(\tau)$  on  $\Gamma$ . We denote the closure of  $0$  in  $\Gamma$  with respect to  $v^{-1}(\tau)$  by  $N_\tau$ .

**1.3.3 Definition :** (R.A. Hirschfeld <10, def 2> ). The hull of a continuous family  $F$  with respect to a topology  $\tau$  on  $C(S)$  is the topological quotient-space  $\Gamma/N_\tau$  of the topological vector space  $\Gamma$  provided with the topology  $v^{-1}(\tau)$ .

We see that  $(\Gamma, ||\cdot||)$  can be considered as the hull of  $F$  with respect to the topology of uniform convergence on  $S$  for  $C(S)$ . A similar definition for hulls may be given for metric spaces. The analogue of  $\Gamma$  should be a set of maps such that the pair distance function  $\prod_{s \in S} d(f(s), g(s))$ ,  $f, g \in \Gamma$ , is contained in  $C(S)$ . Locally convex spaces can be handled by treating each semi-norm separately.

The above definition is a variation on work done by J.W. Calkin <4>, and S.K. Berberian <2> for Hilbert spaces and by R.A. Hirschfeld <10> for Banach spaces. In <2>, <4> and <10> only constant families are considered, that is,  $X(s) = X$  for all  $s \in S$ ,  $S$  being the set  $\mathbb{N}$  of positive

integers in its discrete topology. In <4>  $\Gamma$  is taken to be the set of vector sequences weakly converging to zero, whereas in <2> and <10>  $\Gamma$  is made up of all bounded sequences in  $X$ . Calkin <4> obtained singular representations of the algebra of bounded operators in Hilbert space in terms of hulls.

#### 1.4 Examples and elementary properties of hulls.

1.4.1 Let  $F = \{X(s) \mid s \in S\}, \Gamma$  be a continuous family of normed spaces. Suppose  $T$  is a closed subset of  $\beta S$ . We provide  $C(S)$  with the semi-norm  $||\cdot||_T$  defined by

$$||\phi||_T = \sup\{|\phi(s)| \mid s \in T\}.$$

The corresponding semi-norm, also denoted by  $||\cdot||_T$ , on  $\Gamma$  is then  $||f||_T = ||v(f)||_T = \sup\{||f(s)|| \mid s \in T\}$ . We denote the hull of  $F$  with respect to the  $||\cdot||_T$ -topology, or shortly the  $||\cdot||_T$ -hull of  $F$ , by  $(X(T), ||\cdot||_T)$  and the map from  $\Gamma$  onto  $X(T)$  by  $\Lambda_T$ . Then we have for all  $f \in \Gamma$ ,

$$\Lambda_T f = \{g \mid g \in \Gamma, ||g-f||_T = 0\}.$$

If  $T$  consists of one point  $s \in S$ , then the  $||\cdot||_T$ -hull of  $F$  can be identified with the subspace  $\{f(s) \mid f \in \Gamma\}$  of  $X(s)$ . We may as well suppose that  $\{f(s) \mid f \in \Gamma\} = X(s)$ .

We introduce a special notation for the case where  $T$  consists of a point  $\omega \in S^\wedge$ . Instead of  $||\cdot||_{\{\omega\}}$  we simply write  $||\cdot||_\omega$ .

1.4.1.1 Definition : The  $||\cdot||_\omega$ -hull of  $F$  will be denoted by  $(X(\omega), ||\cdot||_\omega)$ , and is called the  $\omega$ -hull of  $F$ . The canonical projection from  $\Gamma$  onto  $X(\omega)$  will be denoted by  $\Lambda_\omega$ .

Now it is possible to extend, for a continuous family, the maps  $f \in \Gamma$  from  $S$  to  $\beta S$  by putting for  $\omega \in S^\wedge$  :

$$f(\omega) = \Lambda_\omega f \in X(\omega).$$

The set of thus extended maps is denoted by  $\Gamma^\beta$ .

1.4.1.2 Definition : Let  $F = \{X(s) \mid s \in S\}, \Gamma$  be a continuous family of normed spaces. The set of normed spaces  $\{X(s) \mid s \in \beta S\}$  together with  $\Gamma^\beta \subseteq \prod_{s \in \beta S} X(s)$

forms a continuous family  $F^\beta$  of normed spaces on  $\beta S$ . The family  $F^\beta$  is called the hull of  $F$ .

We remark that the hulling-device, as given in definition 1.3.3., set up with the family  $F^\beta$  instead of  $F$  leads to the same spaces. The advantage of working with  $F^\beta$  instead of  $F$  is that sometimes we get a better insight in the hulls of  $F$  with respect to the various topologies on  $C(S)$ .

The following result concerns the completeness of the  $||\cdot||_T$  - hulls for a special class of continuous families.

1.4.1.3 Let  $F = \{ \{X(s) \mid s \in S\}, \Gamma \}$  be a complete continuous family of Banach spaces. We suppose that if  $f \in \Gamma$  and  $\phi \in C(S)$  then  $\phi \cdot f \in \Gamma$ . We have for all  $f \in \Gamma$  and all (closed) subsets  $T \subseteq \beta S$ .

$$||f||_T = \inf \{ ||g|| \mid g \in \Gamma, ||g-f||_T = 0 \}.$$

The  $||\cdot||_T$  - hull of  $F$  is complete.

proof : We put for  $f \in \Gamma$ ,  $|||f|||_T = \inf \{ ||g|| \mid g \in \Gamma, ||g-f||_T = 0 \}$ . If  $g \in \Gamma$  is such that  $||f-g||_T = 0$ , then  $||f||_T = ||g||_T \leq ||g||$ . Upon taking the infimum over all such  $g$ , we get  $||f||_T \leq |||f|||_T$ .

To prove the converse we may suppose  $||f||_T > 0$ . Let  $A = \{s \mid s \in \beta S, v(f)(s) \geq ||f||_T\}$ . The function  $\phi = v(f)^{-1} ||f||_T \chi_A + \chi_{A^c}$ , where  $\chi_A$  is the characteristic function of  $A$  and  $\chi_{A^c} = 1 - \chi_A$ , is contained in  $C(S)$  and  $\phi(T) = 1$ . Clearly we have  $||\phi \cdot f - f||_T = 0$  and  $||\phi \cdot f|| = ||f||_T$ . This implies  $||f||_T \geq |||f|||_T$ .

From the equality  $||f||_T = |||f|||_T$  it follows that the topology of the  $||\cdot||_T$  - hull of  $F$  is the same as the quotient topology of the space  $X(T) = \Gamma / \Lambda_T 0$ , where  $\Gamma$  is taken with the  $||\cdot||$ -topology. It is well-known that a quotient-space of a Banach space with respect to a closed subspace is complete in the quotient topology.

Next we consider an other special case. If each  $X(s)$  is a normed algebra, then  $\prod_{s \in S} X(s)$  can be made into an algebra, where multiplication is defined pointwise.

1.4.1.4 Let  $F = \{ \{X(s) \mid s \in S\}, \Gamma \}$  be a continuous family of normed algebras. If  $\Gamma$  is closed under multiplication, then  $\Gamma$  is a normed algebra with respect

to the supremum norm (1.3.1). The  $||\cdot||_T$  - hulls of  $F$  are normed algebras and  $\Lambda_T : \Gamma \rightarrow X(T)$  is for all  $T \in \beta S$  a continuous algebra homomorphism.

proof : For  $f$  and  $g$  in  $\Gamma$  we find  $\bigvee_{s \in S} ||f(s) g(s)|| \leq \bigvee_{s \in S} ||f(s)|| \bigvee_{s \in S} ||g(s)||$  and consequently  $||fg|| \leq ||f|| \cdot ||g||$  as well as  $||fg||_T \leq ||f||_T ||g||_T$  for all  $T \in \beta S$ .

The first inequality shows that  $(\Gamma, ||\cdot||)$  is a normed algebra. From the second one we infer that the multiplication in  $X(T)$  is well-defined by  $\Lambda_T f \cdot \Lambda_T g = \Lambda_T f \cdot g$ . It is clear that  $(X(T), ||\cdot||_T)$  is a normed algebra with multiplication defined in this way. The last statement is obvious.

1.4.2 A second set of examples, which is of interest, can be got in the following way. Let  $\hat{\mu}$  be a state on  $C(S)$  and  $1 \leq p < \infty$ . We consider  $C(S)$  with the semi-norm  $||\cdot||_{\mu,p}$  defined by :

$$||\phi||_{\mu,p} = (\langle |\phi|^p, \hat{\mu} \rangle)^{1/p} \quad \phi \in C(S).$$

We denote the  $||\cdot||_{\mu,p}$  - hull of a continuous family  $F$  by  $(X(\mu), ||\cdot||_{\mu,p})$ . We remark that  $X(\mu) = X(\text{supp } \mu)$  and that  $X(\mu)$  is independent of  $p$ . The canonical projection from  $\Gamma$  onto  $X(\mu)$  is denoted by  $\Lambda_\mu$ . We have  $\Lambda_\mu = \Lambda_{\text{supp } \mu}$ .

If  $\text{supp } \mu$  consists of only one point, then  $\hat{\mu}$  is multiplicative and  $||\cdot||_{\mu,p}$  is independent of  $p$ . Moreover we get  $||\cdot||_{\mu,p} = ||\cdot||_{\text{supp } \mu}$  in this case. In general the inequality  $||\cdot||_{\mu,p} \leq ||\cdot||_{\text{supp } \mu}$  holds.

As a corollary to 1.4.1.3 we have.

1.4.2.1 Let  $F$  be as in 1.4.1.3. If the set  $\{s \mid s \in \beta S, v(f)^\beta(s) = 0 \text{ for all } f \in \Gamma\}$  is empty, then  $(X(\mu), ||\cdot||_{\mu,p})$  is complete iff  $\text{supp } \mu$  is finite.

proof : For complete  $(X(\mu), ||\cdot||_{\mu,p})$  the inverse-mapping theorem applies to the canonical injection of  $(X(\mu), ||\cdot||_{\text{supp } \mu})$  onto  $(X(\mu), ||\cdot||_{\mu,p})$  to the effect that the norms  $||\cdot||_{\mu,p}$  and  $||\cdot||_{\text{supp } \mu}$  are equivalent in that case. On the other hand the equivalence of the norms and the completeness of  $(X(\mu), ||\cdot||_{\text{supp } \mu})$  ensure the completeness of  $(X(\mu), ||\cdot||_{\mu,p})$ . Therefore  $(X(\mu), ||\cdot||_{\mu,p})$  is complete iff  $||\cdot||_{\mu,p}$  is equivalent to  $||\cdot||_{\text{supp } \mu}$ .

If  $\text{supp } \mu$  is finite the norms  $||\cdot||_{\mu,p}$  and  $||\cdot||_{\text{supp } \mu}$  obviously are equivalent. For  $\text{supp } \mu$  infinite, take  $\varepsilon > 0$ . There exists a point  $t \in \text{supp } \mu$  and an open neighborhood  $0$  of  $t$  such that  $\mu(0) \leq \varepsilon$ . Let  $\phi \in C(\beta S)$  be such that  $0 \leq \phi \leq 1$ ,  $\phi(t) = 1$  and  $\phi(0^c) = 0$ . There is an  $f \in \Gamma$  such that  $v(f)^\beta(t) \neq 0$ .

Because  $\Gamma$  is a module over  $C(S)$  we may suppose  $v(f)^k(t)=1$  and  $\|f\|_{\mu,2} \leq 1$ .

We consider  $\phi \cdot f$ . Then we find  $\|\phi \cdot f\|_{\text{supp } \mu} = 1$  and  $\|\phi \cdot f\|_{\mu,p} \leq \epsilon^{1/p}$ . Since  $\epsilon > 0$  is arbitrary we see that  $\|\cdot\|_{\mu,p}$  and  $\|\cdot\|_{\text{supp } \mu}$  are not equivalent in this case.

We infer from 1.4.2.1 that theorem 4.1 in <4>, which asserts for the complete family  $F$  considered, that  $(X(\mu), \|\cdot\|_{\mu,2})$  is incomplete for every state  $\hat{\mu}$  on  $C(S)$ , is not true in the alleged generality. The proof as it is given hinges on the suggestion made by J. von Neumann, that a Hilbert space of dimension  $c$ , the cardinality of the continuum, should contain  $2^c$  elements. This remark, however, is not valid, as can be seen from the following argument.

Let  $\{e_\alpha \mid \alpha \in \mathbb{R}\}$  be an orthonormal basis for  $H$ . Every element from  $H$  has an expansion  $\sum_{\alpha \in \mathbb{R}} a_\alpha e_\alpha$ , where  $\sum_{\alpha \in \mathbb{R}} |a_\alpha|^2 < \infty$ . This implies that at most countably many of the coefficients  $a_\alpha$  differ from zero. The cardinality of  $H$  is not greater than the product of the cardinality of a separable Hilbert space, equalling  $c$ , and the cardinality of the set  $\mathbb{R}^{\mathbb{N}}$ , consisting of the maps from  $\mathbb{N}$  into  $\mathbb{R}$ . The latter also equals  $c$  and we have  $c \times c = c$ .

1.4.2.1 is a generalization of theorem 1.6.1 in <6> of J.B. Deeds, covering the case  $X(s) = C$ ,  $\Gamma = C(S)$ ,  $S = \mathbb{N}$  and  $\hat{\mu}$  a translation invariant generalized limit. The proof given here is much simpler than the one in <6>.

A property of  $\|\cdot\|_{\mu,2}$  - hulls worth mentioning is the following.

1.4.2.2 Let  $F = \{X(s) \mid s \in S\}, \Gamma$  be a continuous family of normed spaces, such that each  $X(s)$  is a Hilbert space. Suppose  $\hat{\mu}$  is a state on  $C(S)$ . Then the  $\|\cdot\|_{\mu,2}$  - hull of  $F$  is an innerproduct space. The innerproduct is given by :

$$(\Lambda_\mu f, \Lambda_\mu g) = \langle \bigvee_{s \in S} (f(s), g(s)), \hat{\mu} \rangle \quad f, g \in \Gamma.$$

proof : It is readily verified that  $(\cdot, \cdot)$  is an innerproduct. If  $f = g$  we find  $(\Lambda_\mu f, \Lambda_\mu g) = (v(f)^2, \hat{\mu}) = \|f\|_{\mu,2}^2$ , which shows that the norm  $\|\cdot\|_{\mu,2}$  is derived from the innerproduct.

1.4.3 Next we shall give some simple properties of hulls.

Proposition 1.4.2.1 brings to our attention the question of the relation between the various hulls of a continuous family  $F$  and those of its completion.

We consider this problem only for the two examples of hulls mentioned here.

Therefore, let us look at a not necessarily complete, continuous family  $F$  of normed spaces. We consider the map  $\Lambda_T$ ,  $T$  being a closed subset of  $\beta S$ , from  $\bar{\Gamma}$  onto the  $||\cdot||_T$  - hull  $\bar{X}(T)$  of  $\bar{F}$ . The image of  $\Gamma$  under this map can be identified with  $X(T)$ . It is clear from the facts that  $\Gamma$  is dense in  $\bar{\Gamma}$  with respect to  $||\cdot||$  and that  $\Lambda_T$  is  $||\cdot||$  continuous that  $(X(T), ||\cdot||_T)$  is dense in  $(\bar{X}(T), ||\cdot||_T)$  after this identification.

It is not hard to prove that the extension operator for continuous families commutes with the closure operator, i.e.,  $\overline{F^\beta} = \overline{F}^\beta$ .

At first sight it seems rather surprising that, in some cases, it makes no difference, whether we start with  $F$  or  $\bar{F}$ . The resulting  $||\cdot||_T$  - hulls are equal.

1.4.3.1 Let  $X$  be a normed space with completion  $\bar{X}$ . We consider the constant family  $F$  with  $X(s) = X$  for all  $s \in S$ , where  $S = \mathbb{N}$ , and with  $\Gamma$  the set of bounded sequences in  $X$ . If  $T \subseteq \mathbb{N}^\wedge$ , then  $X(T) = \bar{X}(T)$ .

proof : For every  $f \in \bar{\Gamma}$  there exists  $h \in \Gamma$  such that  $v(f - h) \in C_0(\mathbb{N})$ . From 1.1.2 we infer  $v(f - h)^\beta \mid \mathbb{N}^\wedge = 0$  and  $h \in \Lambda_T f$ . Consequently  $\Lambda_T f = \Lambda_T h$  and  $\bar{X}(T) \subseteq X(T)$ . The converse inclusion is known.

It follows from the preceding discussion that the respective completions of the  $||\cdot||_T$  - hulls  $(X(T), ||\cdot||_T)$  and  $(\bar{X}(T), ||\cdot||_T)$  are the same. If  $\hat{\mu}$  is a state on  $C(S)$ , then the same can be said about the completions of  $(X(\mu), ||\cdot||_{\mu,p})$  and  $(\bar{X}(\mu), ||\cdot||_{\mu,p})$ . We denote these completions by  $(X_{\mu,p}, ||\cdot||_{\mu,p})$ . A modification of the usual procedure for  $L_p$  spaces, shows that the elements of  $X_{\mu,p}$  are equivalence classes of elements from  $\prod_{s \in \beta S} X(s)$ .

Often we shall meet situations where a subset  $\Gamma_0 \subseteq \Gamma$  is given. With the image of  $\Gamma_0$  in  $X(T)$ ,  $T \subseteq \beta S$ , we mean the set  $\Lambda_T \Gamma_0$ . If a continuous family  $F' = \{ \{X(s) \mid s \in S\}, \Gamma' \}$  is given and another  $F = \{ \{X(s) \mid s \in S\}, \Gamma \}$  such that  $\Gamma' \subseteq \Gamma$ , then we can identify the hulls of  $F'$  in a canonical way with the image of  $\Gamma'$  in the corresponding hulls of  $F$ .

A slightly more complicated situation is treated in the following proposition, which we record for later use.

1.4.3.2 Let  $F = \{ \{X(s) \mid s \in S\}, \Gamma \}$  be a continuous family of normed spaces on  $S$  and  $F' = \{ \{X'(s) \mid s \in S'\}, \Gamma' \}$  ditto on  $S'$ . Suppose  $\theta : S \rightarrow S'$  is a continuous map such that for  $f \in \Gamma'$  we have  $f \circ \theta \in \Gamma$ . If  $\hat{\mu}$  is a state on

that  $K(S, X)$  is isometrically isomorphic to  $C(\beta S, X)$ . Every  $f \in K(S, X)$  has a continuous Stone-extension  $f^\beta$  to  $\beta S$  by (1.1.1). The map  $\bigvee_{f \in K(S, X)} f^\beta$  is an isometry from  $K(S, X)$  onto  $C(\beta S, X)$ . In fact for  $f \in C(\beta S, X)$  the range  $f(\beta S)$  is compact. The restriction  $f|_S$  is in  $K(S, X)$ . The uniqueness of the Stone - extension implies  $(f|_S)^\beta = f$  on  $\beta S$ .

For  $f \in C(\beta S, X)$  and  $T \subseteq \beta S$ , let  $\bar{\Lambda}_T f$  be the class of elements in  $C(\beta S, X)$  equivalent to  $f$  with respect to  $||\cdot||_T$ , i.e.

$$\bar{\Lambda}_T f = \{g \mid g \in C(\beta S, X), ||f-g||_T = 0\}.$$

1.5.3 The assignment  $J : \Lambda_T f \rightarrow \bar{\Lambda}_T f^\beta$ ,  $f \in K(S, X)$  is well-defined and sets up an isomorphism from  $\Lambda_T K(S, X)$  onto  $\{\bar{\Lambda}_T f \mid f \in C(\beta S, X)\}$ . This isomorphism is isometric in the  $||\cdot||_T$  - norm. If  $\hat{\mu}$  is a state on  $C(S)$  and  $T = \text{supp } \mu$ , then the isomorphism is isometric in every norm  $||\cdot||_{\mu, p}$ ,  $1 \leq p < \infty$ .

proof : For  $g \in K(S, X)$  we have  $v(g)^\beta \in C(\beta S)$  and thus  $v(g)^\beta = v(g^\beta)$ .

We find

$$||\Lambda_T g||_T = \sup\{v(g)^\beta(s) \mid s \in T\} = \sup\{v(g^\beta)(s) \mid s \in T\} = ||\bar{\Lambda}_T g^\beta||_T.$$

The second statement of 1.5.2, concerning the  $||\cdot||_{\mu, p}$  - norms, follows as easily and it is clear that the defined map is onto.

In <3, ch IV, §3, n<sup>o</sup> 4> Bourbaki defines the spaces  $L_X^p(\beta S, \mu)$ , with  $\mu$  a positive measure on  $\beta S$ , as the completion of the set of equivalence classes of maps  $f \in C(\beta S, X)$  with respect to the semi-norm

$$||f||_{\mu, p} = \{\int |f|^p d\mu\}^{1/p} \quad 1 \leq p < \infty$$

In other words  $L_X^p(\beta S, \mu)$  is the completion of the space  $\{\bar{\Lambda}_{\text{supp } \mu} f \mid f \in C(\beta S, X)\}$  with respect to  $||\cdot||_{\mu, p}$ .

By means of the isomorphism in 1.5.3 we can establish an identification  $J$  between  $(\Lambda_{\text{supp } \mu} K(S, X), ||\cdot||_{\mu, p})$  and  $(\{\bar{\Lambda}_{\text{supp } \mu} f \mid f \in C(\beta S, X)\}, ||\cdot||_{\mu, p})$ .

By continuity this identification can be extended to hold between all of  $L_X^p(\beta S, \mu)$  and the closure of  $\Lambda_{\text{supp } \mu} K(S, X)$  in the  $||\cdot||_{\mu, p}$  - hull,  $X_{\mu, p}$ , of the constant family considered here.

- 1.5.4 i) If  $C(S, X) = K(S, X)$  then  $X_{\mu, p} = JL_X^p(\beta S, \mu)$   $1 \leq p < \infty$ .  
 ii) For  $\dim X = \infty$ ,  $\hat{\mu}$  a singular state and  $S = \mathbb{N}$ ,  $X_{\mu, p} \neq JL_X^p(\beta S, \mu)$   $1 \leq p < \infty$ .  
 iii)  $(\bigwedge_{\text{supp } \mu} K(S, X), \|\cdot\|_{\mu, p})$  is complete iff  $\text{supp } \mu$  is finite.

proof : i) If  $C(S, X) = K(S, X)$  then  $(X(\mu), \|\cdot\|_{\mu, p}) \subseteq JL_X^p(\mu)$ , whence the statement.

ii) We infer from a well-known lemma of Riesz's that if  $\dim X = \infty$ , there exists a sequence  $\{x_n \mid n \in \mathbb{N}\}$  in  $X$  with  $\|x_n\| = 1$  and  $\|x_n - x_m\| \geq 1/2$  for  $n \neq m$ . If for  $x \in X$  and some  $n \in \mathbb{N}$ ,  $\|x - x_n\| \leq 1/4$ , then for  $m \neq n$  we find  $\|x - x_m\| \geq 1/4$ . This proves  $\liminf_{n \rightarrow \infty} \|x - x_n\| \geq 1/4$  for all  $x \in X$ .

Consider  $g \in K(\mathbb{N}, X)$  and  $f \in C(\mathbb{N}, X)$  defined by  $f = \bigvee_{n \in \mathbb{N}} x_n$ . Then  $v(f-g)^\beta \mid \mathbb{N}^\wedge = v(f-g^\beta)^\beta \mid \mathbb{N}^\wedge \geq 1/4$  and thus we get  $\|f-g\|_{\mu, p} \geq 1/4$  for  $1 \leq p < \infty$ . Since  $\bigwedge_{\text{supp } \mu} K(S, X)$  is dense in  $JL_X^p(\beta S, \mu)$  this implies  $f \notin JL_X^p(\beta S, \mu)$ .

iii) We apply 1.4.2.1 to the complete family  $X(s) = X$  for  $s \in S$  and  $\Gamma = K(S, X)$ .

Remarks. i) If  $X$  is finite dimensional, the bounded sets in  $X$  are relatively compact, therefore  $C(S, X) = K(S, X)$ .

ii) In case we would admit also locally convex spaces  $X$ ,  $C(S, X)$  would consist of all continuous maps from  $S$  into  $X$  with bounded range. In Montel spaces the closed bounded sets are compact, hence  $C(S, X) = K(S, X)$ . It is readily verified that 1.5.4 i) is also valid for locally convex spaces. For Montel spaces hulling does not give anything new.

iii) It is interesting to find conditions on  $S$ ,  $X$  and  $\omega \in S^\wedge$ , guaranteeing  $X = X(\omega)$ . We shall exhibit an example where  $X = X(\omega)$  for every  $X$  whatsoever.

Let  $W$  denote the space of all countable ordinals in its interval topology. The Stone - Čech compactification  $\beta W$  of  $W$  is nothing but the Alexandroff one-point-compactification of  $W$  <9. 5.12>. For  $f \in C(W, X)$ ,  $f(W)$  is countably compact since  $W$  is countably compact. Because  $f(W)$  is also metric it is compact and it follows that  $f \in K(S, X)$ . From 1.5.2 ii) we infer  $X = X(\omega)$ .

iv) If  $S = \mathbb{N}$  and  $X$  is separable, then the density character of  $X_{\mu, p}$ , with  $\hat{\mu}$  a singular state, equals  $c$ , the cardinality of  $\mathbb{R}$ . The argument runs as follows.

Let  $Y$  be a countable and dense set in  $X$ . The same reasoning as

in 1.4.2.2 shows that the image under  $\Lambda_\mu$  of the set  $C(\mathbb{N}, Y)$  is all of  $X(\mu)$ . The cardinal number of  $C(\mathbb{N}, Y)$  equals  $c$  and the image is dense in  $X_{\mu,p}$ ,  $1 \leq p < \infty$ .

On the other hand consider the map  $f \in C(S, X)$  defined in the proof of 1.5.4 ii). Using well-known techniques [4, 4.2] it can be easily shown that there exists a subset  $\{\phi_\alpha \mid \alpha \in I\}$  of the set of all permutations of  $\mathbb{N}$ , where the cardinality of the index set  $I$  is  $c$ , such that  $v(f \circ \phi_\alpha - f \circ \phi_{\alpha'})^\beta \mid \mathbb{N}^\wedge \geq 1/2$  for  $\alpha \neq \alpha'$ . Then we find for  $1 \leq p < \infty$  that  $\|\bigwedge_{\text{supp } \mu} (f \circ \phi_\alpha - f \circ \phi_{\alpha'})\|_{\mu,p} \geq 1/2$  for  $\alpha \neq \alpha'$ . Thus the density character of  $X_{\mu,p}$  is at least  $c$ , since it is also not bigger than  $c$ , it equals  $c$ .

A similar result holds for the  $\|\cdot\|_T$  - hulls, provided  $T \subseteq \mathbb{N}^\wedge$ .

## 1.6 The Stone - Čech operator

In the sequel of this section we shall assume that the Banach space  $X$  is the dual of some space  $Y$ . The space  $X$  can be endowed with the  $\sigma(X, Y)$  topology, or so called weak\* topology, notation  $X_w$ . Every map  $f \in C(S, X)$  is also continuous with respect to the weak\* topology on  $X$ . It is well known that bounded sets in  $X_w$  are relatively compact. By the Stone - theorem 1.1.1 each  $f \in C(S, X_w)$  has a continuous extension  $f_w^\beta : \beta S \rightarrow X_w$ . The value of  $f_w^\beta$  in  $\omega \in S^\wedge$  is given by

$$f_w^\beta(\omega) = \bigvee_{y \in Y} \langle y, f \rangle^\beta(\omega)$$

where  $\langle y, f \rangle = \bigvee_{s \in S} \langle y, f(s) \rangle$ .

For  $f \in K(S, X)$  we have two extensions ; the strongly continuous one  $f^\beta$  and the weak\* continuous one  $f_w^\beta$ . It is easy to see that they coincide.

1.6.1 Let  $X$  be the (normed) dual of a normed space  $Y$  and  $f \in C(S, X)$ .

The weak\* continuous extension  $f_w^\beta$  of  $f$  to  $\beta S$  satisfies

$$v(f_w^\beta) \leq v(f)^\beta$$

proof : Let  $y \in Y$ ,  $\|y\| \leq 1$ , then  $v(\langle y, f \rangle) \leq v(f)$  and  $v(\langle y, f \rangle)^\beta = v(\langle y, f \rangle^\beta) = v(\langle y, f_w^\beta \rangle) \leq v(f)^\beta$ . Upon taking the supremum over all  $y \in Y$  with  $\|y\| \leq 1$  we get  $v(f_w^\beta) \leq v(f)^\beta$ .

1.6.2 The map  $\Pi_\omega : X(\omega) \rightarrow JX$ , where  $J$  is as in 1.5.3, is defined by  $\Pi_\omega \wedge f = Jf_w^\beta(\omega)$ ,  $f \in C(S, X)$ .  $\Pi_\omega$  is a projection of norm 1 onto  $JX$ .

proof : If  $g \in \wedge_\omega f$  we find  $||f_w^\beta(\omega) - g_w^\beta(\omega)|| \leq v(f-g)^\beta(\omega)$  from 1.6.1. This shows that  $\Pi_\omega$  is well-defined and has norm one. It is obvious that  $\Pi_\omega$  is linear, idempotent and onto  $JX$ .

Following J.B. Deeds <6> we call  $\Pi_\omega$  the  $\omega$ -Stone - Čech operator.

To obtain a corresponding result for states  $\hat{\mu}$  not pertaining to point measures, we have to impose the additional condition of reflexivity on  $X$ .

1.6.3 Suppose  $X$  is a reflexive Banach space and  $\hat{\mu}$  a state on  $C(S)$ .

For  $f \in C(S, X)$  we have :

- i)  $f_w^\beta$  is  $\mu$  - measurable
- ii)  $f_w^\beta \in L_X^p(\beta S, \mu)$  and  $||f_w^\beta||_{\mu, p} \leq ||f||_{\mu, p}$   $1 \leq p < \infty$ .

proof : i) According to <3, ch IV. § 5. n<sup>o</sup> 5. prop 10> to prove that  $f_w^\beta$  is  $\mu$  - measurable it suffices to show that  $f_w^\beta$  sends  $\text{supp } \mu$  into a separable part of  $X$ , because for each  $x^* \in X^*$ , the function  $\langle f_w^\beta, x^* \rangle$ , being continuous, is certainly measurable.

We consider  $X$  with its weak topology and define a state  $\hat{\nu}$  on  $C(X)$  by  $\hat{\nu} = \bigvee_{\phi \in C(X)} \langle \phi \circ f_w^\beta, \hat{\mu} \rangle$ . Then we have  $\text{supp } \nu = \{f_w^\beta(\text{supp } \mu)\}$ . App. 1 applies to  $\nu$ , to the effect that  $f_w^\beta(\text{supp } \mu) \subseteq \text{supp } \nu$  is a separable subset of  $X$ .

ii) Since  $f_w^\beta$  is measurable we need to prove only that  $||f_w^\beta||_{\mu, p} \leq ||f||_{\mu, p}$ . It follows from 1.6.1 that  $v(f_w^\beta) \leq v(f)^\beta$  on  $\beta S$  and consequently

$$||f_w^\beta||_{\mu, p} = \{ \int v(f_w^\beta)^p d\mu \}^{1/p} \leq \{ \int (v(f)^\beta)^p d\mu \}^{1/p} = ||f||_{\mu, p} \quad 1 \leq p < \infty.$$

We define a map  $\Pi_\mu : X_{\mu, p} \rightarrow X_{\mu, p}$ , called the  $\mu$ -Stone - Čech operator, by

$$\Pi_\mu \wedge f = J \tilde{\wedge}_\mu f_w^\beta \quad f \in C(S, X),$$

where  $\tilde{\wedge}_\mu f_w^\beta$  is the equivalence class of  $f_w^\beta$  in  $L_X^p(\beta S, \mu)$ .

1.6.4 The continuous extension of  $\Pi_\mu$  to  $X_{\mu, p}$  is a projection of norm one of  $X_{\mu, p}$  onto  $JL_X^p(\beta S, \mu)$  for  $1 \leq p < \infty$ .

proof : We remark that the assumption that  $X$  is reflexive still holds.

It follows from 1.6.3 that  $\Pi_\mu$  is well-defined, maps  $(X(\mu), ||\cdot||_{\mu,p})$  into  $JL_X^p(\beta S, \mu)$  and has norm not exceeding one. By continuity we get  $\Pi_\mu X_{\mu,p} \subseteq JL_X^p(\beta S, \mu)$ .

If  $f \in K(S, X)$  then  $\Pi_\mu \Lambda_\mu f = J \Lambda_\mu f_w^\beta = \cdot_\mu f$ . Since  $\Lambda_\mu K(S, X)$  is dense in  $JL_X^p(\beta S, \mu)$  we infer that  $\Pi_\mu$  leaves  $JL_X^p(\beta S, \mu)$  invariant.

Remarks. i) If  $f \in C(S, X)$  then  $\Lambda_\mu f_w^\beta \in L_X^\infty(\beta S, \mu)$  as follows readily from 1.6.3. We have no identification of the image  $\{\Lambda_\mu f_w^\beta \mid f \in C(S, X)\}$  with a subspace of  $(X(\mu), ||\cdot||_{\text{supp } \mu})$ .

ii) Proposition 1.6.4 is a generalization of 1.5.8 in <6>. The use of  $K(S, X)$  in the proof is reduced to those points, where it is essential, to wit, the identification of  $L_X^p(\beta S, \mu)$  with a subspace of  $X_{\mu,p}$  and the fact that  $L_X^p(\beta S, \mu)$  is invariant under application of  $\Pi_\mu$ .

## 1.7 The kernel of the Stone - Čech operator.

In this paragraph we are interested in the kernel of the Stone - Čech operator. Suppose  $X$  is the dual of a normed space  $Y$ ,  $S$  a completely regular space,  $T \subseteq \beta S$ , then we consider the set

$$X(T)^0 = \{\Lambda_T f \mid f \in C(S, X), ||f_w^\beta||_T = 0\}.$$

We suppose that  $T \subseteq \hat{S}$  and consider the  $\Lambda_T$  image of  $C_{0,w}(S, X)$  in the  $||\cdot||_T$  - hull of the constant family we study. We put

$$X(T)^\wedge = \{\Lambda_T f \mid f \in C_{0,w}(S, X)\}.$$

For  $f \in C_{0,w}(S, X)$  we have  $f_w^\beta \mid \hat{S} = 0$ , because  $\langle y, f_w^\beta \rangle \mid \hat{S} = 0$  for every  $y \in Y$  and therefore  $||f_w^\beta||_T = 0$ . This implies  $X(T)^\wedge \subseteq X(T)^0$ . Our point will be the relation between  $X(T)^\wedge$  and  $X(T)^0$ .

The following special case of a theorem of Arens's <1> will be used in the sequel.

1.7.1 (R. Arens <1>). Let  $A$  be a closed subset of a compact Hausdorff space  $K$  and let  $h$  be a continuous map from  $A$  into a complete convex metric subset  $M$

of a locally convex topological space. Then  $h$  can be continuously extended over  $K$  with values in  $M$ .

Using the Arens theorem it is easy to characterize the maps  $f \in K(S, X) + C_{0,w}(S, X)$ , if  $S$  is locally compact.

1.7.2 Let  $S$  be locally compact. If  $f \in C(S, X)$ , then  $f \in K(S, X) + C_{0,w}(S, X)$  iff  $f_w^\beta \mid S^\wedge$  is continuous in the strong topology of  $X$ . The subset  $K(S, X) + C_{0,w}(S, X)$  of  $C(S, X)$  determines a complete subfamily.

proof : If  $f = f_1 + f_2$  with  $f_1 \in K(S, X)$  and  $f_2 \in C_{0,w}(S, X)$  then  $f_w^\beta = f_{1,w}^\beta + f_{2,w}^\beta$  and  $f_{2,w}^\beta \mid S^\wedge = 0$ . Moreover  $f_{1,w}^\beta = f_1^\beta$  and thus  $f_w^\beta \mid S^\wedge = f_1^\beta \mid S^\wedge$ , where the last map is strongly continuous.

Now suppose  $f \in C(S, X)$  is such that  $f_w^\beta \mid S^\wedge$  is strongly continuous. Because  $S$  is locally compact,  $S^\wedge$  is a closed subset of  $\beta S$ . The Arens theorem 1.7.1 guarantees the existence of  $g^\beta \in C(\beta S, X)$  such that  $g^\beta \mid S^\wedge = f_w^\beta \mid S^\wedge$ . It is clear that  $(g-f)_w^\beta \mid S^\wedge = 0$  and this implies  $g - f \in C_{0,w}(S, X)$ .

It is easy to see that  $K(S, X) + C_{0,w}(S, X)$  is a subspace of  $C(S, X)$  which is closed with respect to  $\|\cdot\|$  if one uses the characterization, just given.

It follows from 1.7.4 and 1.4.1.3 that the space  $\Lambda_T(K(S, X) + C_{0,w}(S, X))$  is a closed subspace of  $(X(T), \|\cdot\|_T)$ .

To characterize the elements of this subspace we also need the following property.

1.7.3 Let  $S$  be discrete. Suppose one of the following conditions is satisfied :

- i)  $X$  is the dual of a separable Banach space  $Y$ .
- ii)  $X$  is reflexive and  $S = \mathbb{N}$ .

Then for  $f \in C(S, X)$ , such that  $\nu(f_w^\beta) \mid S^\wedge \leq \varepsilon$  there exists a  $g \in C(S, X)$  so that  $\|g\| \leq \varepsilon$  and  $g^\beta \mid S^\wedge = f^\beta \mid S^\wedge$ .

proof : If i) is satisfied, then the weak\* topology of bounded sets in  $X$  is metric.

Now assume ii). For  $f \in C(\mathbb{N}, X)$  the closed linear span  $Z$  of  $f(\mathbb{N})$  is also weakly closed in  $X$  and as a consequence contains  $f^\beta(\beta \mathbb{N})$ . The  $\sigma(Z, Z^*)$  and  $\sigma(Z, X^*)$  topologies on  $Z$  are equal. In addition  $Z$  is reflexive and separable. We infer that  $Z^*$  is separable and that  $\sigma(Z, Z^*)$  is metric on bounded sets.

We note that in both cases the premises of 1.7.1 are satisfied with  $K = \beta S$ ,  $A = S^\wedge$  and  $h = f_w^\beta \mid S^\wedge$ ;  $M$  is the closed ball in  $X$ , respectively  $Z$ , with radius  $\varepsilon$  and center  $O$ . By 1.7.1 there exists a weak\* continuous extension of  $h$  over  $\beta S$  with values in  $M$ . Since  $S$  is discrete, this extension is also strongly continuous on  $S$ .

Now we have

1.7.4 Assume the conditions of 1.7.3. Let  $T \subseteq S^\wedge$  and  $f \in C(S, X)$ . There exists a  $g \in \Lambda_T f$  with  $g \in K(S, X) + C_{0,w}(S, X)$  iff  $f_w^\beta \mid S^\wedge$  is strongly continuous at every point  $T$ .

proof : Suppose such  $g$  exists with  $g = g_1 + g_2$  and  $g_1 \in K(S, X)$ ,  $g_2 \in C_{0,w}(S, X)$ . Then  $0 \leq v(f_w^\beta - g_w^\beta) = v(f_w^\beta - g_1^\beta) \leq v(f - g)^\beta$  on  $S^\wedge$  because  $g_2^\beta \mid S^\wedge = 0$ . Since  $v(f - g)^\beta$  vanishes on  $T$ , is continuous on  $\beta S$  and majorizes  $v(f_w^\beta - g_1^\beta)$  on  $S^\wedge$ , the map  $f_w^\beta - g_1^\beta \mid S^\wedge$  is strongly continuous at every point of  $T$ . Because  $g_1^\beta$  is also strongly continuous, the necessity obtains easily.

Now the sufficiency. By the Arens theorem 1.7.1  $f_w^\beta \mid T$  has a strongly continuous extension  $g_1$  over  $\beta S$ . Consider  $h = f - g_1$ . Then we have  $h_w^\beta \mid T = 0$  and  $h_w^\beta \mid S^\wedge$  is strongly continuous at every point of  $T$ . We next show that  $\Lambda_T h$  can be approximated by  $\Lambda_T g$  with  $g \in C_{0,w}(S, X)$  with respect to  $\|\cdot\|_T$ . Because  $\Lambda_T C_{0,w}(S, X)$  is closed in  $(X(T), \|\cdot\|_T)$  this implies that  $\Lambda_T h \in \Lambda_T C_{0,w}(S, X)$ . We recall that  $C_{0,w}(S, X)$  determines a complete subfamily of  $C(S, X)$  and 1.4.1.3 applies.

Since  $T$  is compact the hypothesis implies that for every  $\varepsilon > 0$  there exists an open neighborhood  $O$  of  $T$  such that  $v(h_w^\beta) \leq \varepsilon$  on  $O$ . There is a  $\phi \in C(\beta S)$  with  $0 \leq \phi \leq 1$ ,  $\phi \mid T = 1$  and  $\phi \mid S^\wedge/O = 0$ . Consider  $\phi \cdot f$ . Then  $\|\phi \cdot f - f\|_T \leq \|\phi - 1\|_T \|f\|_T = 0$  and  $v(\phi \cdot f) \mid S^\wedge \leq \varepsilon$ . By 1.7.2 there exists  $g \in C(S, X)$  such that  $g_w^\beta \mid S^\wedge = \phi \cdot f \mid S^\wedge$  and  $\|g\| \leq \varepsilon$ . We infer  $\|f - (\phi \cdot f - g)\|_T = \|g\|_T \leq \varepsilon$  and  $(\phi \cdot f - g)_w^\beta \mid S^\wedge = 0$ . That is,  $\phi \cdot f - g \in C_{0,w}(S, X)$  and  $\|\Lambda_T f - \Lambda_T(\phi \cdot f - g)\|_T \leq \varepsilon$ .

We remark that the necessity of the condition in 1.7.3 is valid for any space  $S$  and set  $T \subseteq \beta S$ .

Before we can state our next result we need one more definition.

1.7.5 Definition : Let  $T$  be a topological space. A point  $t \in T$  is called a  $P$ -point iff every  $G_\delta$  containing  $t$  is a neighborhood of  $t$  <9, 4L>.

If we assume the continuum hypothesis it is possible to prove that the set of P-points in  $\mathbb{N}^\omega$  is not empty <12>.

1.7.6 Let  $\omega$  be a P-point in  $S^\omega$  with  $S$  locally compact. Under the assumptions of 1.7.2 we have  $X(\omega)^0 = X(\omega)^\omega$ .

proof : Let  $f \in C(S, X)$  and  $f_w^\beta(\omega) = 0$ . In 1.7.3 we showed that  $f_w^\beta(\beta S)$  is contained in a metric space. This implies that the sets of constancy of  $f_w^\beta$  are  $G_\delta$  sets. Consequently  $f_w^\beta$  vanishes on a  $G_\delta$ , say  $A$ , containing  $\omega$ . By the very definition of P-points  $\omega \in A^\circ$ , with  $A^\circ$  the interior of  $A$ .

There exists  $\phi \in C(\beta S)$  with  $0 \leq \phi \leq 1$ ,  $\phi(\omega) = 1$  and  $\phi \upharpoonright S^\omega \setminus A^\circ = 0$ . We consider  $\phi \cdot f$ . Then  $\phi \cdot f \in \bigwedge_\omega f$  and  $(\phi \cdot f)_w^\beta \upharpoonright S^\omega = 0$ . The latter implies  $\phi \cdot f \in C_{0, \omega}(S, X)$ .

For reflexive spaces we have a converse theorem.

1.7.7 Let  $S$  be discrete and suppose  $X$  is a reflexive Banach space.

If  $\omega \in S^\omega$  is a non - P-point then  $X(\omega)^\omega \neq X(\omega)^0$ .

proof : In view of 1.7.4 it is sufficient to construct  $f \in C(S, X)$  such that  $f_w^\beta \upharpoonright S^\omega$  is not strongly continuous at  $\omega$  and  $f_w^\beta(\omega) = 0$ , where  $\omega$  is a given non - P-point.

M.M. Day <5, ch IV, § 3.5> has proved that in every infinite dimensional Banach space  $X$ , there is a closed infinite dimensional subspace  $Z$ , admitting a normalized Schauder basis  $\{b_i \mid i \in \mathbb{N}\}$ . Also, passing to an equivalent norm will not affect our result. Put  $x = \sum_{i=1}^\infty \beta_i(x) b_i$  for  $x \in Z$  and  $U_m x = \sum_{i=1}^m \beta_i(x) b_i$ . We define  $\|x\|' = \sup \{\|U_m x\| \mid m \in \mathbb{N}\}$ , then  $\|x\|' \geq \|x\|$ . It is well known that  $\|\cdot\|'$  is equivalent to  $\|\cdot\|$  <5, ch IV, § 3, th 1>. We may assume  $Z = X$  and  $\|\cdot\|' = \|\cdot\|$ .

Since  $\omega$  is a non - P-point there exists  $\phi \in C(\beta S)$  such that  $0 < \phi < 1$ ,  $\phi(\omega) = 0$  and  $\phi$  is not constant on any neighborhood of  $\omega$  in  $S^\omega$  <9, 4L>.

By induction we shall define for every  $s \in S$  a bounded sequence  $\{f_m(s) \mid m \in \mathbb{N}\}$  in  $X$  such that for  $m \in \mathbb{N}$  and all  $s \in S$ .

- i)  $f_{m+1}(s) - f_m(s)$  is a multiple of  $b_{m+1}$ .
- ii)  $\|f_m(s)\|^m = \phi(s)$ .

We put  $f_1(s) = \phi(s) b_1$  and suppose  $f_1(s), \dots, f_m(s)$  are defined and satisfy i) and ii). We note that  $\phi(s)^{1/m}$  is non - decreasing in  $m$  for fixed  $s$ . There

exists a number  $\beta_{m+1}^s$  such that  $||f_m(s) + \beta_{m+1}^s b_{m+1}|| = \phi(s)^{1/m+1}$ . Let  $f_{m+1}(s) = f_m(s) + \beta_{m+1}^s b_{m+1}$ . This completes the induction. We note that  $U_k f_m(s) = f_k(s)$  for  $m \geq k$ .

For each  $s$ , the sequence  $\{f_m(s) \mid m \in \mathbb{N}\}$  is contained in the unit ball of  $X$ , which is weakly compact, since  $X$  is reflexive. Let  $f(s)$  be a cluster-point, then there exists a subsequence  $\{f_{m_k} \mid k \in \mathbb{N}\}$  weakly converging to  $f(s)$ . Since the projections  $U_m$  are continuous in the weak topology we find  $\lim_{k \rightarrow \infty} U_m f_{m_k}(s) = U_m f(s)$ . For  $m_k \geq m$  we have  $U_m f_{m_k}(s) = f_m(s)$  and therefore  $U_m f(s) = f_m(s)$ . This proves that  $f(s)$  is uniquely determined and is in fact the weak limit of  $\{f_m(s) \mid m \in \mathbb{N}\}$ . Put  $f = \bigvee_{s \in S} f(s)$ , then  $f \in C(S, X)$ . This  $f$  will do the job.

For  $x^* \in X^*$  we see  $\langle U_m f, x^* \rangle^\beta = \langle f, U_m^* x^* \rangle^\beta = \langle f_w^\beta, U_m^* x^* \rangle = \langle U_m f_w^\beta, x^* \rangle$ , therefore  $(U_m f)_w^\beta = U_m f_w^\beta$ . We remark that  $U_m f \in K(S, X)$  and  $v(f^\beta) \geq v(U_m f^\beta) = v(U_m f)^\beta = \phi^{1/m}$ . In every neighborhood of  $\omega$  there are points  $\omega'$  with  $\phi(\omega') \neq 0$  and then  $||f_w^\beta(\omega')|| \geq \sup \{\phi(\omega')^{1/m} \mid m \in \mathbb{N}\} = 1$ . On the other hand  $||U_m f^\beta(\omega)|| = 0$  for all  $m$ , thence  $f_w^\beta(\omega) = 0$ .

An extension of 1.7.5 should deal with sets  $T$  such that  $T$  consists of  $P$  - points only. However, it is well-known that compact  $P$  - spaces are finite <9. 4L>. Thus a trivial generalization only can be obtained.

A direct corollary to 1.7.7 and 1.7.2 is the following,

**1.7.8** Let  $S$  be discrete and  $X$  an (infinite-dimensional) reflexive Banach space.

i) If  $T \subseteq S^\wedge$  is such that there exists a non -  $P$ -point  $\omega \in T$  then  $X(T) \not\subseteq \Lambda_T(K(S, X) + C_{O, \omega}(S, X))$ .

ii) If  $\hat{\mu}$  is a state on  $C(S)$  with  $\text{supp } \mu$  finite and so that  $\text{supp } \mu$  contains a non -  $P$ -points  $\omega$  and if  $X_{\mu, P}^\wedge$  denotes the closure of  $\Lambda_\mu C_{O, \omega}(S, X)$  in  $X_{\mu, P}$ , then  $X_{\mu, P} \not\subseteq J L_X^P(\beta S, \mu) + X_{\mu, P}^\wedge$ .

proof : i) In 1.7.7 the existence was proved of  $f \in C(S, X)$  such that  $f_w^\beta \mid S^\wedge$  is not strongly continuous at the given non -  $P$ -point  $\omega$ . It follows from the remark following 1.7.4 that  $\Lambda_T f \notin \Lambda_T(K(S, X) + C_{O, \omega}(S, X))$ .

ii) We conclude from 1.4.1.3 that  $\Lambda_T(K(S,X) + C_{O,w}(S,X) = JL_X^P(\beta S, \mu) + X_{\mu,p}^\wedge$ . The statement is now readily proved.

If  $S$  is locally compact, then  $S^\wedge$  is closed and every infinite closed subset  $T \subseteq S^\wedge$  contains non -  $P$ -points. Nevertheless the existence of states  $\hat{\mu}$ , with  $\text{supp } \mu$  not finite, can be proved so that  $X_{\mu,p} = JL_X^P(\beta S, \mu) + X_{\mu,p}^\wedge$ ,  $1 \leq p < \infty$ .

Let  $\{\omega_i \mid i \in \mathbb{N}\}$  be an infinite discrete subset of  $\mathbb{N}$  consisting of  $P$ -points only. Give  $\omega_i$  a measure  $1/2^i$  and define  $\langle \phi, \hat{\mu} \rangle = \sum_{i=1}^{\infty} (1/2)^i \phi(\omega_i)$ . It is readily verified that every element of  $X_{\mu,p}$ ,  $1 \leq p < \infty$ , can be approximated by  $\Lambda_T f$  with  $f \in K(\mathbb{N}, X) + C_{O,w}(\mathbb{N}, X)$ .

In <7> the existence is proved of generalized limits  $\hat{\mu}$  on  $C(\mathbb{N})$  such that  $X_{\mu,2}^0 \neq X_{\mu,2}^\wedge$ . In this case  $Y$  is taken to be a Hilbert space. R. Raimi has proved that  $\hat{\mu}$  can be taken to be translation invariant <7>. We note that  $X_{\mu,p}^0$ , the kernel of the  $\mu$  - Stone - Čech operator, denotes the closure of  $X(\text{supp } \mu)^0$  in  $X_{\mu,p}$ .

## CHAPTER II

### 2.1 Representations on hulls.

Let  $X$  and  $Y$  be Banach spaces and  $L(X, Y)$  the Banach space of all continuous linear transformations from  $X$  into  $Y$ . Suppose  $S$  is a completely regular space. We denote the elements of  $C(S, L(X, Y))$ , the space of continuous maps from  $S$  into  $L(X, Y)$ , by  $\phi$  and  $\psi$ . We define a map  $A$  from  $C(S, L(X, Y))$  into the space  $L(C(S, X), C(S, Y))$  of bounded linear maps from  $C(S, X)$  into  $C(S, Y)$  by

$$A\phi = \left\{ f \in C(S, X) \mapsto \bigvee_{s \in S} \phi(s)f(s) \right\} \quad \phi \in C(S, L(X, Y)).$$

For all  $s \in S$ ,  $f \in C(S, X)$  and  $\phi \in C(S, L(X, Y))$  we have

$$\|A\phi f(s)\| \leq \|\phi(s)\| \cdot \|f(s)\|.$$

It is easily verified that  $A$  acts isometrically on  $C(S, L(X, Y))$ .

Now let us look at the constant family  $F^X = \{\{X(s) \mid s \in S\}, \Gamma\}$  with  $X(s) = X$  for all  $s \in S$  and  $\Gamma = C(S, X)$ . Correspondingly we have  $F^Y$ . Suppose  $\hat{\mu}$  is a state on  $C(S)$ . We form the  $\|\cdot\|_{\mu, p}$ -hulls of  $F^X$  and  $F^Y$ , which we denote respectively by  $(X(\mu), \|\cdot\|_{\mu, p})$  and  $(Y(\mu), \|\cdot\|_{\mu, p})$ .

It follows from the above inequality that for  $\phi \in C(S, L(X, Y))$ ,  $A\phi$  leaves  $\Lambda_\mu 0 = \{f \mid f \in C(S, X), \|f\|_{\mu, p} = 0\}$  invariant. Thus we can define a map  $R_{\mu, p}$  from  $C(S, L(X, Y))$  into  $L(X(\mu), Y(\mu))$  by

$$R_{\mu, p} \phi \upharpoonright_{\Lambda_\mu 0} = \upharpoonright_{\Lambda_\mu 0} A\phi$$

where  $\phi \in C(S, L(X, Y))$  and  $f \in C(S, X)$ .

If we provide  $X(\mu)$  and  $Y(\mu)$  with the  $\|\cdot\|_{\mu, p}$ -topologies, then  $R_{\mu, p} \phi$  becomes a bounded operator with norm  $\|R_{\mu, p} \phi\|$ . If the state  $\hat{\mu}$  is such that  $\text{supp } \mu$  consists of one point  $s \in S$ , then  $R_{\mu, p}$  is independent of  $p$  and we simply write  $R_s$ .

2.1.1 i) For all  $\phi \in C(S, L(X, Y))$  we have  $\|R_{\mu, p} \phi\| \leq \|\phi\|_{\text{supp } \mu} = \sup \{\|\phi(s)\| \mid s \in \text{supp } \mu\}.$

ii) The value of  $||R_{\mu,p}\phi||$  is independent of  $p$  and  $||R_{\mu,p}\phi|| = \sup \{ ||R_s\phi|| \mid s \in \text{supp } \mu \}.$

iii) If  $S$  is discrete then  $||R_{\mu,p}\phi|| = ||\phi||_{\text{supp } \mu}.$

proof : i)  $||R_{\mu,p}\phi \wedge_\mu f||_{\mu,p} \leq ||\phi||_{\text{supp } \mu} ||\wedge_\mu f||_{\mu,p}.$

ii) We put  $\alpha = \sup \{ ||R_s\phi|| \mid s \in \text{supp } \mu \}.$  Then we get for  $f \in C(S,X)$  :

$$\begin{aligned} ||R_{\mu,p}\phi \wedge_\mu f||_{\mu,p}^p &= \langle \bigvee_s ||\phi(s)f(s)||^p, \hat{\mu} \rangle \\ &\leq \langle \bigvee_s ||R_s||^p ||f(s)||^p, \hat{\mu} \rangle \leq \alpha^p ||\wedge_\mu f||_{\mu,p}^p \end{aligned}$$

and therefore  $||R_{\mu,p}\phi|| \leq \alpha.$

For every  $\epsilon > 0$ , there exists  $f \in C(S,X)$  with  $v(f) = 1$  and an open set  $C \subseteq \beta S$  such that  $C \cap \text{supp } \mu \neq \emptyset$  and  $v(\phi f)^\beta \geq \alpha(1-\epsilon)$  on  $C$ . By complete regularity there is a  $\phi \in C(\beta S)$  such that  $\phi|_{\beta S/C} = 0$  and  $||\phi||_{\mu,p} = 1.$  We consider

$$\begin{aligned} ||R_{\mu,p}\phi \wedge_\mu \phi \cdot f||_{\mu,p}^p &= \langle \bigvee_s ||\phi(s)\phi(s)f(s)||^p, \hat{\mu} \rangle \\ &\geq \alpha(1-\epsilon) \langle (v(\phi \cdot f)^\beta)^p, \hat{\mu} \rangle = \alpha^p(1-\epsilon)^p ||\wedge_\mu \phi \cdot f||_{\mu,p}^p. \end{aligned}$$

This being true for all  $\epsilon$ , we find  $||R_{\mu,p}\phi|| \geq \alpha.$

iii) For every  $\epsilon > 0$  there exists  $f \in C(S,X)$  with  $v(f) = 1$  and  $v(A\phi f) \geq v(\phi)(1-\epsilon).$  Therefore we have  $||R_s\phi|| \geq v(\phi)^\beta(s)$  for  $s \in \beta S.$

Because the reverse inequality is also valid we get  $||R_s\phi|| = v(\phi)^\beta(s)$  for all  $s \in S.$  In view of ii) the desired equality is easily proved,

If  $X = Y$  then  $C(S, L(X,X))$  is a Banach algebra.

2.1.2 The map  $R_{\mu,p}$  from  $C(S, L(X,X))$  into  $L(X(\mu), X(\mu))$  is multiplicative.

proof : For  $\phi, \psi \in C(S, L(X,X))$  and  $f \in C(S,X)$  we find

$$\begin{aligned} R_{\mu,p}\phi \cdot \psi \wedge_\mu f &= \wedge_\mu A\phi \cdot \psi f = \wedge_\mu A\phi \cdot A\psi f \\ &= R_{\mu,p}\phi \cdot \wedge_\mu A\psi f \\ &= R_{\mu,p}\phi \cdot R_{\mu,p}\psi \wedge_\mu f \end{aligned}$$

and thus  $R_{\mu,p}\phi \cdot \psi = R_{\mu,p}\phi \cdot R_{\mu,p}\psi$  by continuity.

2.1.3 If  $S$  is discrete and  $\omega \in S^\wedge$  then  $R_\omega(C(S, L(X, X)))$  is an algebraically irreducible subalgebra of  $L(X(\omega))$ .

proof : It suffices to prove that  $R_\omega(C(S, L(X, X)))$  acts transitively on the unit sphere of  $X(\omega)$ . For  $\Lambda_\omega f$  and  $\Lambda_\omega g$  from  $X(\omega)$  with  $\|f\|_\omega = \|g\|_\omega = 1$ , there exists  $\bar{f} \in \Lambda_\omega f$  and  $\bar{g} \in \Lambda_\omega g$  such that  $v(\bar{f}) = v(\bar{g}) = 1$  on  $S$ . For every  $s \in S$  there is a  $\phi(s) \in L(X, X)$  so that  $\|\phi(s)\| = 1$  and  $\phi(s)\bar{f}(s) = \bar{g}(s)$ . The map  $\bigvee_{s \in S} \phi(s)$  transforms  $\bar{f}$  into  $\bar{g}$  and  $R_\omega \phi \Lambda_\omega f = \Lambda_\omega g$ .

Each  $\phi(s)$  in the proof of 2.1.3 can be chosen to have one-dimensional range. Let  $FL(X)$  be the algebra of degenerate operators in  $L(X, X)$ . Already  $R_\omega(C(S, FL(X)))$  acts irreducibly on  $X(\omega)$ . In fact it is easy to see that  $R_\omega(C(S, FL(X))) \supseteq FL(X(\omega))$ .

An extension of 2.1.3 to more general states is the next.

2.1.4 For discrete spaces  $S$ , the center of  $R_{\mu, p}(C(S, L(X, X)))$  is isometrically isomorphic with  $C(\text{supp } \mu)$ .

proof : For  $\phi \in C(S)$  we define  $\phi_\phi$  by  $\phi_\phi(s) = \phi(s)E$  where  $E$  is the identity in  $L(X, X)$ . Obviously  $\phi_\phi$  is in the center of  $C(S, L(X, X))$ . For  $f \in C(S, X)$  we have  $R_{\mu, p} \phi_\phi \Lambda_\mu f = \Lambda_\mu \phi f$  and  $\|R_{\mu, p} \phi_\phi\|_{\text{supp } \mu} = \|\phi\|_{\text{supp } \mu}$ . The restriction of  $C(\beta S)$  to  $\text{supp } \mu$  is all of  $C(\text{supp } \mu)$  and thus  $C(\text{supp } \mu)$  is isometrically isomorphic to a subalgebra of the center of  $R_{\mu, p}(C(S, L(X, X)))$ .

Suppose  $R_{\mu, p} \phi$  commutes with  $R_{\mu, p} \psi$ . We infer from 2.1.1 that  $\|R_\omega(\phi\psi - \psi\phi)\| \leq \|R_{\mu, p}(\phi\psi - \psi\phi)\| = 0$  for all  $\omega \in \text{supp } \mu$ . This shows that  $R_\omega \phi$  is in the center of  $R_\omega(C(S, L(X, X)))$  for  $\omega \in \text{supp } \mu$  if  $R_{\mu, p} \phi$  commutes with  $R_{\mu, p}(C(S, L(X, X)))$ . We obtain from 2.1.3 that  $R_\omega \phi$  is a multiple  $\lambda(\omega)E_\omega$  of the identity  $E_\omega$  on  $X(\omega)$ . We must show that  $\lambda = \bigvee_\omega \lambda(\omega)$  is continuous on  $\text{supp } \mu$ .

We put  $\phi_1(s) = E$  for all  $s \in S$  and consider  $\phi_1 = \bigvee_s \phi_1(s)$ . For  $f \in C(S, X)$  and any constant  $\alpha$  we find  $v(A(\phi - \alpha\phi_1)f)^\beta = |\lambda - \alpha|v(f)^\beta$  on  $\text{supp } \mu$ . Since  $|\lambda - \alpha|$  is continuous for all  $\alpha$ , also  $\lambda$  is continuous on  $\text{supp } \mu$ .

Of course it is not necessary for the theorems just proved that to every  $s \in S$ , there corresponds one and the same space  $X$ . One readily obtains corresponding theorems for continuous families of Banach spaces after the necessary modifications.

## 2.2 Representations of $L(X, X)$ .

The subalgebra of  $C(S, L(X, X))$  consisting of the constant maps is isometrically isomorphic to  $L(X, X)$ . The restriction of the representations  $\rho_{\mu, p}$  of  $C(S, L(X, X))$  to the subspace of constant maps induces by means of this isomorphism representations, called  $\rho_{\mu, p}$ , of  $L(X, X)$  on  $X_{\mu, p}$ .

A representation  $\rho$  of  $L(X, X)$  on a Banach space  $Y$  is called cyclic if there exists a vector  $y \in Y$  such that  $\{\rho(A)y \mid A \in L(X, X)\}$  is dense in  $Y$ . We denote the restriction of  $\rho_{\mu, p}$  to the closure of  $\{\rho_{\mu, p}(A)\hat{x} \mid A \in L(X, X)\}$ , where  $\hat{x} \in X_{\mu, p}$  by  $\rho_{\mu, p, \hat{x}}$ . Then  $\rho_{\mu, p, \hat{x}}$  is a cyclic representation of  $L(X, X)$  with cyclic vector  $\hat{x}$ . We shall show that for a separable Banach space the class of cyclic representations  $\rho_{\mu, p, \hat{x}}$ , which can be gotten with  $\mathbb{N}$  as the underlying topological space is complete in a certain sense.

**2.2.1 Definition.** A representation  $\rho$  of  $L(X, X)$  on a Banach space  $Y$  is said to be isometrically equivalent to a representation  $\sigma$  of  $L(X, X)$  on a Banach space  $Z$  iff there exists an invertible isometric operator  $U \in L(Y, Z)$  such that  $\sigma(A)U = U\rho(A)$  for all  $A \in L(X, X)$ .

To indicate in the next theorem the underlying space  $S$  we shall write  $\rho_{\mu, p}^S, X_{\mu, p}^S$  etc. A similar result holds for the  $\|\cdot\|_T$  - hulls.

**2.2.2** Let  $X$  be a separable Banach space,  $S$  a completely regular space and  $\mu$  a state on  $S$ . For every  $\hat{x} \in X_{\mu, p}^S$  there exists a vector  $\hat{z} \in X_{\mu', p}^{\mathbb{N}}$  such that  $\rho_{\mu, p, \hat{x}}^S$  is isometrically equivalent to  $\rho_{\mu', p, \hat{z}}^{\mathbb{N}}$ .

**proof :** First we show that  $S$  can be chosen separable. Let  $\hat{x} \in X_{\mu, p}^S$  be given. There exist  $f_k \in C(S, X)$ ,  $k \in \mathbb{N}$ , such that  $\bigwedge_{\mu} f_k$  converges to  $\hat{x}$  in  $X_{\mu, p}^S$ . We define an equivalence relation  $\sim$  in  $S$  by putting for  $s, t \in S$  :  $s \sim t$  iff  $f_k(s) = f_k(t)$  for every  $k \in \mathbb{N}$ . The equivalence class containing  $s$  is denoted by  $\theta(s)$  and  $T = \{\theta(s) \mid s \in S\}$ . The map  $\bigvee_s \theta(s)$  maps  $S$  onto  $T$ . With each  $f_k$  we associate the unique  $\tilde{f}_k$  from  $T$  into  $X$  such that  $\tilde{f}_k \circ \theta = f_k$ . The space  $T$  is endowed with the topology generated by the semi-norms  $q_k$  defined by  $q_k(\theta(s), \theta(t)) = \|f_k(s) - f_k(t)\|$  for  $k \in \mathbb{N}$  and  $s, t \in S$ . This topology makes  $T$  completely regular. The maps  $\tilde{f}_k$  as well as the map  $\theta$  are continuous in this topology. The following argument shows that  $T$  is separable.

There exists a countable base  $\{O_k \mid k \in \mathbb{N}\}$  for the topology of  $X$ . We put  $O'_{1, k} = \tilde{f}_k^{-1}(O_k)$ , then  $\{O'_{1, k} \mid 1, k \in \mathbb{N}\}$  is a countable subbase for the

topology of  $T$ . Hence  $T$  is separable.

Now we apply 1.4.3.1 to  $\theta : S \rightarrow T$  with  $\Gamma = C(S, X)$  and  $\Gamma' = C(T, X)$  to the effect that  $X_{\mu, p}^T$  is isometrically mapped into  $X_{\mu, p}^S$  by the map  $\theta$  defined in 1.4.3.1. It can be readily verified that for  $A \in L(X, X)$  we have  $\rho_{\mu, p}^S(A)\theta = \theta\rho_{\mu, p}^T(A)$ . Since  $f_k = \hat{f}_k \circ \theta$  we have  $\Lambda_{\mu, p} f_k \in \theta X_{\mu, p}^T$  and because  $\theta X_{\mu, p}^T$  is closed  $\hat{x} \in \theta X_{\mu, p}^T$ . It is now obvious that  $\rho_{\mu, p, \hat{x}}^S$  is isometrically equivalent to  $\rho_{\mu, p, \theta^{-1}\hat{x}}^T$ . This finishes the first part of the proof.

Now suppose  $T$  is separable and consider a representation  $\rho_{\mu, p, \hat{x}}^T$ , where  $\hat{\mu}$  is a state on  $C(T)$  and  $\hat{x} \in X_{\mu, p}^T$ . There exists a dense subset  $\{t_k \mid k \in \mathbb{N}\}$  of  $T$ . We consider  $\theta' = \bigcup_{k \in \mathbb{N}} t_k$ . Again we apply 1.4.3.1, now to  $\theta', \Gamma = C(\mathbb{N}, X)$ , and  $\Gamma' = C(T, X)$ . We find an isometry  $\theta' : X_{\theta', \mu', p}^T \rightarrow X_{\mu, p}^{\mathbb{N}}$ , where  $\hat{\mu}'$  is a state on  $C(\mathbb{N})$  such that  $\theta'\hat{\mu}' = \hat{\mu}$ . We note that  $\theta'$  maps  $\mathbb{N}$  onto a dense subset of  $T$  and according to the remark following 1.4.3.1, then  $\hat{\mu} = \theta'\hat{\mu}'$  for some state  $\hat{\mu}'$  on  $C(\mathbb{N})$ .

The representation  $\rho_{\theta', \mu', p}^T$  is isometrically equivalent to a subrepresentation of  $\rho_{\mu', p}^{\mathbb{N}}$  and clearly  $\rho_{\mu, p, \hat{x}}^T$  is isometrically equivalent to  $\rho_{\mu', p, \theta'\hat{x}}^{\mathbb{N}}$ .

We note that 2.2.2 holds not only for the  $\|\cdot\|_{\mu, p}$ -hulls. It is not hard to see that if we consider the hull of  $C(S, X)$  with respect to a metric topology on  $C(S)$ , such that a continuous representation of  $L(X, X)$  on this hull can be defined, that then a corresponding theorem can be proved.

In <6 Introduction> J.B. Deeds poses the question whether for separable Hilbert spaces the class of cyclic representations  $\rho_{\mu, 2, \hat{x}}^{\mathbb{N}}$  contains, up to equivalence, all  $\rho_{\nu, 2, \hat{y}}^S$ . Theorem 2.2.2 answers this question in the affirmative. In 2.3.5 a more detailed answer even is given.

If  $X$  is not separable,  $\mathbb{N}$  in 2.2.2 must be replaced by a discrete set with a cardinality equal to the density character of  $X$  to obtain a corresponding theorem. The proof of 2.2.2 carries over to this more general situation almost without change.

It is an interesting question whether every cyclic representation of  $L(X, X)$  is isometrically equivalent to a cyclic subrepresentation of a representation of  $L(X, X)$  constructed by means of a hull. A partial result is the following.

**2.2.3** If  $X$  is a not-separable, reflexive Banach space, then there exists a cyclic representation  $\rho$  of  $L(X, X)$  such that  $\rho$  is not isometrically equivalent

to any subrepresentation of a representation of  $L(X, X)$  by means of a hull of  $C(\mathbb{N}, X)$  with respect to a metric topology on  $C(\mathbb{N})$ .

proof : We consider in  $L(X, X)$  the set  $M$  of operators which have a separable range, that is  $M = \{A \mid AX \text{ is separable}\}$ . The set  $M$  is a closed two-sided ideal in  $L(X, X)$ . The identity  $E$  on  $X$  is not contained in  $M$  and so  $M \neq L(X)$ . Let  $\pi$  be the canonical projection of  $L(X)$  onto  $L(X)/M$ . Since  $M$  is an ideal we can define  $\rho(A)$  in  $L(L(X)/M)$  by  $\rho(A)\pi(B) = \pi(AB)$  for  $A, B \in L(X)$ . Then  $\rho$  is a cyclic representation of  $L(X, X)$ , with cyclic vector  $\pi(E)$ , which annihilates  $M$ .

Let  $\sigma$  be a cyclic subrepresentation of a representation of  $L(X, X)$  in a hull  $X_\tau$  of  $C(\mathbb{N}, X)$  with respect to a metric topology  $\tau$ . We suppose  $\hat{x} \in X_\tau$  is the cyclic vector. There is a sequence  $\{f_k \mid k \in \mathbb{N}\}$  such that their images  $\bigwedge_\tau f_k$  in  $X_\tau$  converge to  $\hat{x}$ . The set  $\{f_k \mid (k \in \mathbb{N}) \mid k \in \mathbb{N}\}$  is contained in a separable subspace  $Y$  of  $X$ . According to theorem 1 of J. Lindenstrauss, cited in the appendix, there exists a projection  $P$  in  $L(X, X)$  such that  $PX \supseteq Y$  and  $PX$  is separable. The continuity of  $\sigma$  and the fact that  $\bigwedge_\tau f_k = \bigwedge_\tau P f_k$  for all  $k \in \mathbb{N}$  imply that  $\hat{x}$  is left invariant by  $\sigma(P)$  and that therefore  $\sigma(P) \neq 0$ . Yet, we have  $P \in M$  and thus  $\sigma$  cannot be isometrically equivalent to  $\rho$ .

Now we go back to the full representations  $\rho_{\mu, p}$ .

2.2.4 Let  $S$  be a completely regular space and  $\hat{\mu}$  a state on  $S$ .

- i) The subspace  $JL_X^P(\beta S, \mu)$  of  $X_{\mu, p}$  is invariant with respect to  $\rho_{\mu, p}(L(X, X))$ ,  $1 \leq p < \infty$ .
- ii) If  $X$  is reflexive, also  $X_{\mu, p}^0$  and  $\hat{X}_{\mu, p}$  are invariant subspaces for  $\rho_{\mu, p}$ ,  $1 \leq p < \infty$ .

We remark that similar results hold for the  $\|\cdot\|_\tau$  - hulls.

proof : i) Every  $A \in L(X, X)$  maps  $K(S, X)$  into itself. By continuity of  $\rho_{\mu, p}$  then  $\rho_{\mu, p}(A) L_X^P(\beta S, \mu) \subseteq L_X^P(\beta S, \mu)$ .

ii) The operators in  $L(X, X)$  are continuous with respect to the weak topology. If  $f \in C_{0, w}(S, X)$  then  $Af \in C_{0, w}(S, X)$  for every  $A \in L(X, X)$ . For the Stone - extensions  $f^\beta$  of  $f \in C(S, X)$  we have  $(Af)^\beta = A(f^\beta)$  on  $\beta S$ .

2.2.5 i) If  $A \in L(X, X)$  is compact then  $\rho_{\mu, p}(A)X_{\mu, p} \subseteq L_X^p(\beta S, \mu)$ .

If  $X$  is reflexive and

ii)  $A$  is compact, then  $\rho_{\mu, p}(A)$  annihilates  $X_{\mu, p}^0$ .

iii)  $S = \mathbb{N}$  then, if for  $A \in L(X, X)$  the operator  $\rho_{\mu, p}(A)$  annihilates  $X_{\mu, p}^\wedge$  for some  $p$ ,  $1 \leq p < \infty$  and some state  $\hat{\mu}$ , then  $A$  is compact.

proof : i) A compact operator maps bounded sets into relatively compact sets, therefore,  $A C(S, X) \subseteq K(S, X)$  and  $\rho_{\mu, p}(A) \wedge_\mu C(S, X) \subseteq \wedge_\mu K(S, X) \subseteq L_X^p(\beta S, \mu)$ . By continuity of  $\rho_{\mu, p}(A)$  this inclusion relation can be extended to  $X_{\mu, p}$ .

ii) We know from i)  $\rho_{\mu, p}(A)X_{\mu, p} \subseteq L_X^p(\beta S, \mu)$ . We infer from 2.2.4 ii) that  $\rho_{\mu, p}(A)X_{\mu, p}^0 \subseteq X_{\mu, p}^0$ . The intersection of the spaces  $L_X^p(\beta S, \mu)$  and  $X_{\mu, p}^0$ , being respectively the range and the null space of the projection  $\Pi_\mu$ , consists of the 0 vector only.

iii) We want to show that  $A$  maps sequences which converge weakly to zero into sequences which converge strongly to zero. Since  $X$  is reflexive this implies that  $A$  is compact.

Therefore let  $f \in C_{0, w}(\mathbb{N}, X)$ . For a permutation  $\phi$  of  $\mathbb{N}$  we have  $f \circ \phi \in C_0(\mathbb{N}, X)$ .

Suppose  $\hat{\mu}$  and  $p$  are such that  $\rho_{\mu, p}(A)X_{\mu, p}^\wedge = 0$ . Then we have  $\|\rho_{\mu, p}(A) \wedge_\mu f \circ \phi\|_{\mu, p} = 0$  and  $v(A(f \circ \phi))^\beta \mid \text{supp } \mu = 0$ . Every map  $\phi : \mathbb{N} \rightarrow \mathbb{N}$  has by the Stone - theorem 1.1.1 a continuous extension  $\phi^\beta : \beta \mathbb{N} \rightarrow \beta \mathbb{N}$ . Then we have for every permutation  $\phi$  of  $\mathbb{N}$  that  $v(Af)^\beta \mid \phi^\beta(\text{supp } \mu) = 0$ . It is readily verified that for  $\omega \in \mathbb{N}^\wedge$  the orbit  $\{\phi(\omega) \mid \phi \text{ a permutation of } \mathbb{N}\}$  is dense in  $\mathbb{N}^\wedge$ . We obtain  $v(Af)^\beta \mid \mathbb{N}^\wedge = 0$ . We infer from 1.1.2 that  $Af(k)$  converges strongly to zero as  $k$  increases.

The condition  $S = \mathbb{N}$  in 2.2.5 iii) cannot be replaced by e.g.  $S$  is discrete. We shall give two counter-examples where  $S = \mathbb{R}_\alpha$ , and where  $\mathbb{R}_\alpha$  is the set of real numbers in the discrete topology.

Let  $M$  be the closed ideal in  $C(\mathbb{R}_\alpha)$  consisting of all functions which are zero except for at most countably many points. From the general theory of continuous-function spaces we know that there exists a closed subset  $T \subseteq \beta S$  such that  $\phi \in M$  iff  $\phi^\beta(T) = 0$ .

a) We suppose  $X = Y^*$  with  $Y$  separable. For  $f \in C_{0, w}(\mathbb{R}_\alpha, X)$  we have by definition  $\langle y, f \rangle \in C_0(\mathbb{R}_\alpha)$  for  $y \in Y$ . We note that  $C_0(\mathbb{R}_\alpha)$  is contained in  $M$ . Let  $\{y_k \mid k \in \mathbb{N}\}$  be a dense set in the unit ball of  $Y$ . It is easy to see that  $v(f) = \sup \{|\langle y_k, f \rangle| \mid k \in \mathbb{N}\}$  is contained in  $M$ . Consequently we find

for  $\omega \in T$  that  $v(f)^\beta(\omega) = 0$ . In this case  $X(\omega)^\wedge = 0$ .

b) Let  $X$  be a non-separable reflexive Banach space and  $P$  a projection onto an infinite-dimensional, but separable subspace of  $X$ . Then  $P$  is not compact. If  $f \in C_{0,w}(\mathbb{R}_\alpha, X)$  then also  $Pf \in C_{0,w}(\mathbb{R}_\alpha, X)$ . The same reasoning as in the preceding example shows that for  $\omega \in T$  the operator  $\rho_{\mu,p}^{(P)}$  annihilates  $X(\omega)^\wedge$  with  $\omega \in T$ .

We remark that the set of operators  $A$  such that  $\rho_{\mu,p}^{(A)}$  annihilates  $X_{\mu,p}^\wedge$  for some  $\hat{\mu}$  and  $p$  is always a two-sided ideal in  $L(X, X)$ , if  $X$  is reflexive.

We denote by  $CL(X, X)$  the ideal of compact operators in  $L(X, X)$ . If  $X$  is a reflexive Banach space, then the restriction of  $\rho_{\mu,p}^0$  to  $X_{\mu,p}^0$  induces a representation  $\rho_{\mu,p}^0$  of the quotient algebra  $L(X, X) / CL(X, X)$  in the following way. Let  $\pi$  be the canonical projection of  $L(X, X)$  in  $L(X, X) / CL(X, X)$ . We define  $\rho_{\mu,p}^0(\pi(A)) = \rho_{\mu,p}^0(A) \mid X_{\mu,p}^0$ . It follows from 2.2.5 that this definition is independent of the chosen representative in  $\pi(A)$ .

2.2.6 Let  $X$  be a reflexive Banach space and  $\hat{\mu}$  a singular state on  $C(S)$ .

- i) The representation  $\rho_{\mu,p}^0$  has norm one.
- ii) If  $S = \mathbb{N}$ , then  $\rho_{\mu,p}^0$  is one-to-one.

proof : i) For  $B \in \pi(A)$  we have  $||B|| \geq ||\rho_{\mu,p}^0(B)|| \geq ||\rho_{\mu,p}^0(B) \mid X_{\mu,p}^0|| = ||\rho_{\mu,p}^0(\pi(A))||$ . Upon taking the infimum over all  $B \in \pi(A)$  we find indeed  $||\pi(A)|| \geq ||\rho_{\mu,p}^0(\pi(A))||$ . The  $\rho_{\mu,p}^0$ -image of the identity on  $X$  is the identity on  $X_{\mu,p}^0$ . Thus we find  $||\rho_{\mu,p}^0|| = 1$ .

ii) If  $\rho_{\mu,p}^0(\pi(A)) = 0$ , then  $\rho_{\mu,p}^0(A) \mid X_{\mu,p}^\wedge = 0$ . We obtain from 2.2.5 that  $A \in CL(X, X)$  or  $\pi(A) = 0$ .

### 2.3 \*Representations of $L(H)$ .

We shall apply the foregoing to the representations  $\rho_{\mu,2}$  of  $L(H)$ , where  $L(H) = L(H, H)$  and  $H$  a Hilbert space. First a little attention will be paid to  $H_{\mu,2}$ .

Let  $S$  be a completely regular space and  $\hat{\mu}$  a singular state on  $C(S)$ . We infer from 1.4.2.3 that  $H_{\mu,2}$  is a Hilbert - space. The innerproduct for elements from  $(H(\mu), ||\cdot||_{\mu,2})$  is given by

$$(\Lambda_\mu f, \Lambda_\mu g) = \langle (f, g), \hat{\mu} \rangle$$

$$f, g \in C(S, H).$$

A first result is the following.

2.3.1 The projection  $\pi_\mu : H_{\mu,2} \rightarrow JL_H^2(\beta S, \mu)$  defined in 1.6.4 is self adjoint.

proof : This is a direct consequence of the next identity

$$(\pi_\omega \wedge_\omega f, \wedge_\omega g) = (Jf_\omega^\beta(\omega), \wedge_\omega g) = (f_\omega^\beta(\omega), g_\omega^\beta(\omega)) = (\wedge_\omega f, \pi_\omega \wedge_\omega g),$$

where  $\omega \in \beta S$  and  $f, g \in C(S, H)$ . Integration with respect to  $\mu$  yields

$$(\pi_\mu \wedge_\mu f, \wedge_\mu g) = (\wedge_\mu f, \pi_\mu \wedge_\mu g).$$

Consequently the kernel  $H_{\mu,2}^0$  of  $\pi_\mu$  is orthogonal to  $JL_H^2(\beta S, \mu)$ ; in particular the subspace  $\hat{H}_{\mu,2}^0$  of  $H_{\mu,2}^0$  is orthogonal to  $JL_H^2(\beta S, \mu)$ . Let us denote the orthogonal complement in  $\hat{H}_{\mu,2}^0$  of  $H_{\mu,2}^0$  by  $H_{\mu,2}^+$ .

We note that the spaces  $H(\omega)$  for  $\omega \in S^\wedge$  are Hilbert spaces too and that they are equal to the  $\wedge_\omega$ -image of  $C(S, H)$  (1.4.1.3). The  $\wedge_\omega$ -image of  $C_{0,\omega}(S, H)$  in  $H(\omega)$  is closed and equals  $\hat{H}(\omega)$  (1.4.1.3). We know by 1.7.6 that for  $S = \mathbb{N}$  and P-points  $\omega \in \mathbb{N}^\wedge$  the space  $H(\omega)^+$  consists of the 0 only.

We consider the representations  $\rho_{\mu,2}$  of  $L(H)$  into  $L(H_{\mu,2})$ . These representations were studied in <2>, <4> and <6>. We collect some information concerning  $\rho_{\mu,2}$ .

2.3.2 i) The representation  $\rho_{\mu,2}$  are  $C^*$ -isomorphisms.

ii) The subspaces  $JL_H^2(\beta S, \mu)$ ,  $\hat{H}_{\mu,2}$  and  $H_{\mu,2}^+$  reduce  $\rho_{\mu,2}$ .

iii) The restriction of  $\rho_{\mu,2}$  to  $JL_H^2(\beta S, \mu)$  is normal <8, A 27>.

iv) The restriction of  $\rho_{\mu,2}$  to  $H_{\mu,2}^0$  annihilates the ideal  $CL(H)$  of compact operators in  $L(H)$  and induces a  $C^*$ -homomorphism  $\rho_{\mu,2}^0$  of the quotient algebra  $L(H)/CL(H)$  into  $L(H_{\mu,2}^0)$ .

proof : i) For  $f, g \in C(S, H)$  and  $A \in L(H)$  we have

$$(\rho_{\mu,2}(A) \wedge_\mu f, \wedge_\mu g) = \langle Af, g \rangle_\mu = \langle f, A^*g \rangle_\mu = (\wedge_\mu f, \rho_{\mu,2}(A^*) \wedge_\mu g).$$

By continuity this extends to all of  $H_{\mu,2}$ .

Because  $\rho_{\mu,2}(A) \mid JH = JAJ^*$  we get that  $\rho_{\mu,2}$  is an isomorphism.

- ii) We infer from 2.2.4 that  $JL_H^2(\beta S, \mu)$  and  $H_{\mu,2}^+$  are invariant for  $\rho_{\mu,2}$ . It is well known that then  $JL_H^2(\beta S, \mu)$ ,  $H_{\mu,2}^+$  and their common orthocomplement  $H_{\mu,2}^\perp$  are reducing.
- iii) The restriction of  $\rho_{\mu,2}$  to  $JL_H^2(\beta S, \mu)$  has no singular subrepresentations. Since every  $*$ representation of  $L(H)$  is the direct sum of a normal  $*$ representation and a singular representation the conclusion obtains  $\langle 8, 4.7. 22c \rangle$ .
- iv) This is a direct consequence of 2.2.5 and of the fact that  $L(H) / CL(H)$  is a  $C^*$ - algebra in its quotientnorm.

The restriction of  $\rho_{\mu,2}$  to  $L_H^2(\beta S, \mu)$  is nothing but a direct sum of identity representations  $\langle 8, 4.7. 22c \rangle$ . The interesting part of  $\rho_{\mu,2}$  is therefore the restriction of  $\rho_{\mu,2}$  to  $H_{\mu,2}^0$ . Every  $*$ representation of a  $C^*$ - algebra is the orthogonal sum of cyclic  $*$ subrepresentations. Therefore we are interested mostly in the cyclic  $*$ subrepresentations  $\rho_{\mu,2,\hat{x}}$ , with  $\hat{x} \in H_{\mu,2}^0$  as a cyclic vector, of  $\rho_{\mu,2}$ .

A first question is : How many representations do we get in this way. The sequel of this section will be devoted to this question. We start by making the problem more precise.

**2.3.3 Definition.** A state on the algebra  $L(H)$  is an element  $\hat{a} \in L(H)^*$  such that  $\langle A^*A, \hat{a} \rangle \geq 0$  for every  $A \in L(H)$  and with  $\langle E, \hat{a} \rangle = 1$ , where  $E$  is the identity in  $L(H)$ .

We denote the set of states on  $L(H)$  by  $\sigma(L(H))$ .

States of the form  $\bigvee_{A \in L(H)} (Ax, x)$ , with  $x \in H$ ,  $\|x\| = 1$ , are called vector states.

States which annihilate the compact operators are called singular.

It is clear from the definition that  $\sigma(L(H))$  is a weak\* closed subset of  $L(H)^*$ .

There is a one-to-one correspondence between cyclic  $*$ representations of  $L(H)$  and states on  $L(H)$   $\langle 8, 2.4.1 \rangle$ . The state determined by  $\rho_{\mu,2,\hat{x}}$ , with  $\hat{x} \in H_{\mu,2}^0$ , on  $L(H)$  is given by  $\bigvee_{A \in L(H)} (\rho_{\mu,2}(A)\hat{x}, \hat{x})$ .

Now we may formulate our question on how many representations  $\rho_{\mu,2,\hat{x}}$  are there, as follows. Which states on  $L(H)$  are of the form  $\bigvee_{A \in L(H)} (\rho_{\mu,2}(A)\hat{x}, \hat{x})$ , where  $\hat{x} \in H_{\mu,2}^0$  and  $\hat{\mu}$  is a state on  $C(S)$ ? Of course only singular states can be written in this form with  $\hat{x} \in H_{\mu,2}^0$ .

2.3.4 Let  $f \in C(S, H)$  satisfy  $v(f) = 1$  on  $S$ . We define a map  $\tau_f : \beta S \rightarrow \sigma(L(H))$  by

$$\langle A, \tau_f(s) \rangle = (Af, f)^\beta(s) \quad A \in L(H), s \in \beta S.$$

Then  $\tau_f$  is weak\* continuous on  $\beta S$ .

proof : It is easy to see that  $\tau_f(\beta S) \subseteq \sigma(L(H))$ . The weak\* continuity of  $\tau_f$  is equivalent to the continuity of each of the functions  $\langle A, \tau_f \rangle$  with  $A \in L(H)$ . The latter functions clearly are continuous.

From 2.2.2 we know that we do not find more cyclic representations, and thus more states, by considering topological spaces  $S \neq \mathbb{N}$  as soon as the Hilbert space  $H$  is separable. For Hilbert spaces and cyclic\* representations of  $L(H)$  (2.2.2) can be much improved.

2.3.5 Let  $H$  be a separable Hilbert space. There exists  $f \in C(\mathbb{N}, H)$ ,  $v(f) = 1$  such that every singular state on  $L(H)$  is of the form

$$\bigvee_{A \in L(H)} (Af, f)^\beta(\omega) \quad \omega \in \mathbb{N}^\wedge.$$

It is possible to choose  $\omega \in \mathbb{N}^\wedge$  so that  $\bigwedge_\omega f \in H(\omega)^\perp$ .

proof : A lemma of Dixmier's <8, 11.2.1> states that the set of singular states on  $L(H)$  is contained in the weak\* closure of the set of vector states on  $L(H)$ .

Since  $H$  is separable there exists a sequence  $\{x_n \mid n \in \mathbb{N}\}$  in the unit sphere  $\{x \mid \|x\| = 1\} = H_1$  of  $H$ , which is dense in  $H_1$ . We put  $f = \bigvee_n x_n$ .

Consider the set  $T = \{\omega \mid \omega \in \mathbb{N}^\wedge, \|f_w^\beta(\omega)\| = 1\}$ . For all  $x \in H$  with  $\|x\| = 1$ , there exists a subsequence  $\{n_k \mid k \in \mathbb{N}\}$  of  $\mathbb{N}$  such that  $\lim_{k \rightarrow \infty} f(n_k) = x$ . The intersection of the closure in  $\beta \mathbb{N}$  of the infinite set  $\{n_k \mid k \in \mathbb{N}\}$  with  $\mathbb{N}^\wedge$  is not empty. For  $\omega$  in this intersection we find  $f_w^\beta(\omega) = x$  and  $\langle A, \tau_f(\omega) \rangle = \lim_{k \rightarrow \infty} (Af(n_k), f(n_k)) = (Ax, x)$ . We conclude that  $\omega \in T$  and that  $\tau_f(T)$  contains all vector states.

Since the closure  $\bar{T}$  of  $T$  in  $\mathbb{N}^\wedge$  is compact and because  $\tau_f$  is weak\* continuous, the image  $\tau_f(\bar{T})$  is weak\* compact in  $\sigma(L(H))$ . From the Dixmier lemma we infer that  $\tau_f(\bar{T})$  contains all singular states.

If for  $\omega \in \bar{T}$  the state  $\tau_f(\omega)$  is singular then  $f_w^\beta(\omega) = 0$ . This implies

that  $\omega \notin T$  and that if  $\tau_f(\omega)$  is singular then  $f_w^\beta \mid \mathbb{N}^\wedge$  is not strongly continuous at  $\omega$ . The last statement of 2.3.5 can now be easily proved by invoking 1.7.4.

Remarks : i) The map  $\bigvee_A(Af, f)$ , with  $f \in C(\mathbb{N}, H)$  as defined in 2.3.5 is a bicontinuous positive map of  $L(H)$  into  $C(\mathbb{N})$ . The norm  $\| (Af, f) \|$  of the image of  $A$  is equal to the numerical radius  $r(A) = \sup\{ |(Ax, x)| \mid \|x\| = 1 \}$  of  $A$ . For self-adjoint operators we have  $r(A) = \|A\|$ . Thus  $\bigvee_A(Af, f)$  sets up an isometric, positive map from the real Banach space  $L(H)_h$  of all self-adjoint operators on  $H$  into  $C(\mathbb{N})$ . In this way the study of  $\sigma(L(H))$  can be reduced to the study of the restriction of the set of states  $\sigma(C(\mathbb{N}))$ , to a real subspace of  $C(\mathbb{N})$ .

ii) If  $H$  is not separable, a theorem which corresponds to 2.3.5 can be stated if  $\mathbb{N}$  is replaced by a discrete set with cardinality equal to the Hilbert-dimension of  $H$ .

iii) It follows from the proof of 2.3.5 that  $\{ \langle A, \hat{\mu} \rangle \mid \hat{\mu} \in \sigma(L(H)) \}$  equals the closure of the numerical range  $W(A) = \{ (Ax, x) \mid \|x\| = 1 \}$  of  $A$ . In fact, with  $f$  as in 2.3.5, we find  $\{ \langle A, \hat{\mu} \rangle \mid \hat{\mu} \in \sigma(L(H)) \} = \{ (Af, f)^\beta(\omega) \mid \omega \in \beta\mathbb{N} \} = \overline{W(A)}$ .

We infer from 2.2.3 that if  $H$  is not separable, then we do not get all of  $\sigma(L(H))$  in the form  $\bigvee_A(\rho_{\mu, 2}(A)\hat{x}, \hat{x})$  with  $\hat{x} \in H_{\mu, 2}$ , if we allow  $S = \mathbb{N}$  only. Yet we can ask: Which states do we get in the above form and with  $S = \mathbb{N}$ ?

2.3.6 Let  $H$  be an infinite - dimensional Hilbert space.

- i) The singular states on  $L(H)$ , which are of the form  $\tau_f(\omega)$  with  $\omega \in \mathbb{N}^\wedge$  and  $f \in C(\mathbb{N}, H)$  are precisely those singular states  $\hat{a}$ , for which there exists a projection  $P \in L(H)$  with separable range and such that  $\langle P, \hat{a} \rangle = 1$ .
- ii) Those are the states which induce a representation of  $L(H) / CL(H)$ , any subrepresentation of which is isometric.

proof : i) For any state  $\nu_f(\omega)$  with  $f \in C(\mathbb{N}, H)$  and  $\omega \in \mathbb{N}^\wedge$ , let  $P$  be the projection on the closed linear space of  $f(\mathbb{N})$  in  $H$ . Then  $\langle P, \tau_f(\omega) \rangle = 1$ .

If there exists a separable projection  $P$  such that for the singular state  $\hat{a}$  under consideration we get  $\langle P, \hat{a} \rangle = 1$ , then  $\hat{a}$  induces a singular state on  $L(PH)$ . Since  $PH$  is separable, there is a  $f \in C(\mathbb{N}, PH)$ ,  $\nu(f) = 1$  and a  $\omega \in \mathbb{N}^\wedge$  such that  $\tau_f(\omega)$  equals this state. We have  $\langle A, \hat{a} \rangle = \langle PAP, \hat{a} \rangle$  for  $A \in L(H)$  and therefore  $\langle A, \hat{a} \rangle = \langle A, \tau_f(\omega) \rangle$ .

ii) We infer from i) that if a singular state  $\hat{\alpha}$  lives only on a separable subspace then  $\hat{\alpha} = \tau_f(\omega)$  for some  $f \in C(\mathbb{N}, H)$  and  $\omega \in \mathbb{N}^*$ . If we put  $\hat{\mu} = \bigvee_{\phi} \phi^{\beta}(\omega)$ , then the corresponding representation is  $\rho_{\mu, 2, \wedge_{\mu} f}$ . It can be readily verified that every subrepresentation is isometric on  $L(H) / CL(H)$ .

If a state  $\hat{\alpha}$  induces a representation which is isometric on  $L(H) / CL(H)$ , then it follows from the construction of this representation <cf. 8. 2.4.1> that there exists a projection  $P \in L(H)$  with separable range such that  $\langle P, \hat{\alpha} \rangle \neq 0$ . We consider the family, partially ordered by inclusion, of all sets  $\{P_i \mid i \in I\}$  of mutually orthogonal, separable projections such that  $\langle P_i, \hat{\alpha} \rangle \neq 0$ . By Zorn's lemma there exists a maximal set  $\{P_i \mid i \in I_m\}$ . A simple argument shows that  $I_m$  is at most countable. We put  $P = \sum_{i \in I_m} P_i$  and  $Q = E - P$ . Then  $P$  is separable and  $\langle Q, \hat{\alpha} \rangle = 0$ .

The last assertion can be proved as follows. If  $x$  is the cyclic vector for the representation  $\rho$  constructed with  $\hat{\alpha}$ , then it is well known that  $\|\rho(Q)x\|^2 = \langle Q, \hat{\alpha} \rangle$ . Since for every separable projection  $P' \leq Q$  we have  $\langle P', \hat{\alpha} \rangle = 0$ , the cyclic \* subrepresentation of  $\rho$  with cyclic vector  $\rho(Q)x$  annihilates all operators with separable range. This subrepresentation is, consequently, not isometric on  $L(H) / CL(H)$  and must vanish. This implies  $\|\rho(Q)x\|^2 = \langle Q, \hat{\alpha} \rangle = 0$  and  $\langle P, \hat{\alpha} \rangle = 1$ .

We go back to the separable case. It follows from 2.3.5 that every singular state can be given the form  $\tau_f(\omega)$  with  $f \in C(\mathbb{N}, H)$ ,  $v(f) = 1$  and  $\wedge_{\omega} f \in H(\omega)^+$ . It is an open question whether we can replace  $H(\omega)^+$  by  $H(\omega)^{\wedge}$  or more generally whether all singular states can be put in the form  $\bigvee_A (\rho_{\mu, 2}(A)\hat{x}, \hat{x})$  with  $\hat{x} \in H_{\mu, 2}^{\wedge}$  for some state  $\hat{\mu}$  on  $C(\mathbb{N})$ .

Let us make some remarks concerning this problem.

The fact that we showed in 2.3.5 that it was sufficient to consider only one special  $f \in C(\mathbb{N}, H)$  does not help us very much. In fact for  $g \in C(\mathbb{N}, H)$  with  $v(g) = 1$ , there is a subsequence  $\{n_k \mid k \in \mathbb{N}\}$  of  $\mathbb{N}$  such that  $\|f(n_k) - g(k)\| \leq k^{-1}$  for  $k \in \mathbb{N}$  and  $f$  as defined in 2.3.5. The closure of  $\{n_k \mid k \in \mathbb{N}\}$  in  $\beta\mathbb{N}$  is homeomorphic with  $\beta\mathbb{N}$ . If  $\omega'$  is the point, which corresponds to  $\omega \in \mathbb{N}^{\wedge}$  then  $\tau_f(\omega') = \tau_g(\omega)$ .

By considering a continuous map  $\tau : \mathbb{N} \rightarrow \mathbb{N}$  and forming  $f \circ \tau$  for  $f \in C_{0, \omega}(\mathbb{N}, H)$  it was shown in <7> that not always  $H_{\mu, 2}^{\wedge} = H_{\mu, 2}^0$ . If  $\tau$  as well as  $\hat{\mu}$  are properly chosen then  $\wedge_{\mu}(f \circ \tau) \in H_{\mu, 2}^+$ . The following identity shows that one gets in this way always functionals which can be described by vectors from  $H_{\mu, 2}^{\wedge}$ . For  $A \in L(H)$  we find

$$(\rho_{\mu, 2}(A)\wedge_{\mu} f \circ \tau, \wedge_{\mu} f \circ \tau) = \langle A f, f \rangle, \tau \hat{\mu} \rangle = (\rho_{\tau \mu, 2}(A)\wedge_{\tau \mu} f, \wedge_{\tau \mu} f).$$

We note that  $\tau\hat{\mu}$  is defined by  $\langle \phi \circ \tau, \hat{\mu} \rangle = \langle \phi, \tau\mu \rangle$  for  $\phi \in C(\mathbb{N})$ . Although it might be possible that  $\Lambda_{\mu} f \circ \tau \in H_{\mu,2}^+$  we yet have  $\Lambda_{\tau\mu} f \in H_{\tau\mu,2}^+$ .

There is a variant on this procedure. Let  $f \in C_{0,w}(\mathbb{N}, H)$  and  $\{e_n \mid n \in \mathbb{N}\}$  an orthonormal basis for  $H$ . We put  $K_i = \{x \mid |(x, e_n)| = |(f(i), e_n)| \text{ for all } n \in \mathbb{N}\}$ . Every  $K_i$  is compact and  $K = \bigcup_{i=1}^{\infty} K_i$  is weakly compact. We consider maps  $g : \mathbb{N} \rightarrow K$ . By means of such  $g$  many singular functionals can be constructed. It is again an open problem whether we get thus all singular states and whether these states can be described by vectors from  $H_{\mu,2}^+$  for some  $\hat{\mu}$ . It can be shown that the maps  $f$ , defined in 1.7.7, such that  $\Lambda_{\omega} f \in H(\omega)^+$ , are of the form  $f = f_1 + f_2$  with  $f_1 \in K(\mathbb{N}, H)$  and  $f_2$  of the type just introduced, such that  $v(f) = v(f_2)$  on  $\beta\mathbb{N}$ .

## APPENDIX

App. 1. Let  $X$  be a reflexive Banach space. We consider  $X$  with its weak topology. If  $\nu$  is a positive Radon measure on  $X$  then  $\text{supp } \nu$  is contained in a separable subspace of  $X$ .

proof : In the course of the proof we will invoke the following recent results of J. Lindenstrauss < 11 >.

L.1 < 11, Prop. 1 >. Let  $X$  be a reflexive Banach space and let  $Y$  be a separable subspace of  $X$ . Then there is a linear projection  $P$  from  $X$  into itself so that  $\|P\| = 1$ ,  $PX \supseteq Y$  and  $PX$  is separable.

L.2 < 11, corr 1 >. Let  $X$  be a reflexive Banach space. Then  $X$  has an equivalent strictly convex norm.

It follows from L 2 that we may assume that  $X$  has a strictly convex norm, that is, if  $x$  and  $y$  are in  $X$  and  $\|x+y\| = \|x\| + \|y\|$ , then  $x = \lambda y$  with  $\lambda \in \mathbb{C}$ .

It is sufficient to prove App 1 for the case where  $\text{supp } \nu$  is a bounded subset of  $X$ .

We denote the unit ball of  $X^*$  by  $U$  and the set of finite subsets of  $U$  by  $F(U)$ . If  $V$  is a subset of  $U$  and  $A \in L(X)$  we put

$$|A|_V = \bigvee_{x^* \in X} \sup\{|\langle Ax, x^* \rangle| \mid x^* \in V\}.$$

and  $|A| = |A|_U$ . The identity in  $L(X)$  is denoted by  $I$ .

We remark that the functions  $|A|_V$  are lower semi-continuous. We apply < 3, ch IV, §1, n° 1, th 1 > to the effect that

$$\int |I| d\nu = \int \sup\{|I|_K \mid K \in F(U)\} d\nu = \sup\{\int |I|_K d\nu \mid K \in F(U)\}.$$

The first integral exists because  $|I|$  is bounded on  $\text{supp } \mu$  and  $|I|$  is certainly measurable. There exists an increasing sequence  $\{K_n \mid n \in \mathbb{N}\}$  in  $F(U)$  such that

$$\int |I| d\nu = \sup\{\int |I|_{K_n} d\nu \mid n \in \mathbb{N}\} = \int \sup\{|I|_{K_n} \mid n \in \mathbb{N}\} d\nu \quad (1).$$

We put  $K = \bigcup_{n=1}^{\infty} K_n$  and let  $Y$  be the closure of the linear span of  $K$  in  $X^*$ .

The subspace  $Y$  is separable. We infer from L 1, that there is a projection  $P^*$  on  $X^*$  of norm one, which leaves  $Y$  invariant and which has separable range. Suppose  $P^*$  is the adjoint of  $P$  then  $P$  has norm one and separable range too.

$$\begin{aligned} |I| &\geq |P| = \sup \left\{ \bigvee_{x \in X} |\langle x, P^* x \rangle| \mid x^* \in U \right\} \\ &\geq \sup \{ |I|_{K_n} \mid n \in \mathbb{N} \}. \end{aligned}$$

Upon integration and using (1) we find

$$\int |I| d\nu \geq \int |P| d\nu \geq \int \sup \{ |I|_{K_n} \mid n \in \mathbb{N} \} d\nu = \int |I| d\nu.$$

Consequently we get  $\int |I| d\nu = \int |P| d\nu$  and since  $|I| \geq |P|$  we have  $|I| = |P|$   $\nu$ -almost everywhere. Because  $\|P\| = 1$  we find for  $0 < \rho < 1$ .

$$|I| \geq |\rho I + (1-\rho)P| \geq |\rho P + (1-\rho)P^2| = |P| = |I|$$

$\nu$ -almost everywhere. The strict convexity of the norm gives us then  $Px = x$   $\nu$ -almost everywhere. The operator  $P$  is continuous for the weak topology and we readily get that  $\text{supp } \mu$  is contained in  $PX$ , which is separable.

## BIBLIOGRAPHY

- <1> ARENS, R.

Extension of functions on fully normal spaces. Pac.J.Math. 2(1952), 11-22.
- <2> BERBERIAN, S.K.

Approximate proper vectors.  
Proc.Amer.Math.Soc. 13(1962), 111-114.
- <3> BOURBAKI, N.

Integration. Ch. I,II,III,IV.  
Hermann, Paris, 1952.
- <4> CALKIN, J.W.

Two sided ideals and congruences in the ring of bounded operators on Hilbert space. Ann.of Math. 42(1941), 839-873.
- <5> DAY, M.M.

Normed linear spaces. Springer, Berlin 1962.
- <6> DEEDS, J.B.

The Stone-Ćech operators and its associated functionals. Thesis. Univ.of Michigan 1966.
- <7> DEEDS, J.B.

The Stone-Ćech operator and its associated functionals. Preprint.
- <8> DIXMIER, J.

Les  $C^*$  algebras et leurs representations.  
Gauthier-Villars. Paris, 1964.
- <9> GILLMAN, L.; JERISON, M.

Rings of continuous functions.  
Princeton, 1960.
- <10> HIRSCHFELD, R.A.

On hulls of linear operators.  
Math.Zeitschrift 96(1967), 216-222.
- <11> LINDENSTRAUSS, J.

On non-separable reflexive Banach spaces.  
Bull.Amer.Math.Soc. 72(1966), 967-970.
- <12> RUDIN, W.

Homogeneity problems in the theory of the Ćech - compactification.  
Duke Math.J. 23(1956), 409-420.



## SAMENVATTING

Zij  $S$  een volledig reguliere topologische ruimte. Als  $(X, ||\cdot||)$  een genormeerde ruimte is, dan geven we met  $(C(S, X), ||\cdot||)$  de ruimte aan van alle continue begrensde afbeeldingen van  $S$  in  $X$ . De norm  $||\cdot||$  in  $C(S, X)$  is de supremum norm :  $||f|| = \sup \{ ||f(s)|| \mid s \in S \}$ ,  $f \in C(S, X)$ . We definiëren  $v(f) = \bigvee_{s \in S} ||f(s)||$ . De afbeelding  $v = \bigvee_f v(f)$  is een isometrische afbeelding van  $C(S, X)$  in  $C(S, \mathbb{R}) = C(S)$ . Indien  $\tau$  een lokaal-convexe topologie is voor  $C(S)$ , dan is  $v^{-1}(\tau)$  een lokaal-convexe topologie voor  $C(S, X)$ . De afsluiting van  $0$  ten opzichte van  $v^{-1}(\tau)$  is  $N_\tau$ . R.A. Hirschfeld definiëerde in [10] de  $\tau$ -hull van  $C(S, X)$  als de topologische quotientruimte  $C(S, X)/N_\tau$  van  $(C(S, X), v^{-1}(\tau))$ .

We bekijken twee klassen van voorbeelden van hulls. Zij  $T$  een gesloten deelverzameling van de Stone - Čech compactificatie  $\beta S$  van  $S$  en  $||\phi||_T = \sup \{ |\phi^\beta(s)| \mid s \in T \}$ , waar  $\phi^\beta$  de Stone-uitbreiding is tot  $\beta S$  van  $\phi \in C(S)$ . De hull  $X(T)$  van  $C(S, X)$  ten opzichte van de  $||\cdot||_T$ -topologie wordt de  $||\cdot||_T$ -hull van  $C(S, X)$  genoemd. De canonieke afbeelding van  $C(S, X)$  op  $X(T)$  wordt aangegeven met  $\Lambda_T$ . De  $||\cdot||_T$ -hulls van  $C(S, X)$  zijn volledig als  $X$  volledig is.

Vervolgens nemen we een toestand  $\hat{\mu}$  van  $C(S)$ , d.w.z.  $\hat{\mu} \in C(S)^*$ ,  $\langle 1, \hat{\mu} \rangle = 1$  en  $||\hat{\mu}|| = 1$ . Iedere toestand  $\hat{\mu}$  correspondeert met een waarschijnlijkheidsmaat  $\mu$  op  $\beta S$  via  $\langle \phi, \hat{\mu} \rangle = \int \phi^\beta d\mu$  voor  $\phi \in C(S)$ . Zij voor  $\phi \in C(S)$ ,  $1 \leq p < \infty$ ,  $||\phi||_{\mu, p} = \langle |\phi|^p, \hat{\mu} \rangle^{1/p}$ , dan is  $||\cdot||_{\mu, p}$  een seminorm op  $C(S)$ . De overeenkomstige hull van  $C(S, X)$  heet de  $\mu, p$ -hull van  $C(S, X)$ . De notatie voor de canonieke projectie van  $C(S, X)$  op de  $\mu, p$ -hull is  $\Lambda_\mu$ . De completering van de  $\mu, p$ -hull wordt aangegeven met  $(X_{\mu, p}, ||\cdot||_{\mu, p})$ . Het is eenvoudig in te zien dat als  $X$  een Hilbertruimte is dat dan  $X_{\mu, 2}$  ook een Hilbertruimte is.

We bestuderen de structuur van de ruimten  $X_{\mu, p}$ . Als  $f \in C(S, X)$  een relatief compacte waardenvoorraad heeft, schrijven we  $f \in K(S, X)$ . Iedere  $f \in K(S, X)$  heeft een continue Stone - uitbreiding  $f^\beta$  tot  $\beta S$ . De afbeelding  $\bigvee_{f \in K(S, X)} f^\beta$  is een isomorfie van  $K(S, X)$  op  $C(\beta S, X)$ . Gebruikmakend van dit gegeven tonen we aan dat de afsluiting in  $X_{\mu, p}$  van  $\bigwedge_\mu K(S, X)$  isomorf is, via een isomorfie  $J$ , met de wel bekende ruimte  $L_X^p(\beta S, \mu)$ . Indien  $X$  eindig dimensionaal is, bijvoorbeeld, geldt  $X_{\mu, p} = J L_X^p(\beta S, \mu)$ . Als  $X$  oneindig dimensionaal is daarentegen,  $S = \mathbb{N}$  en  $\text{supp } \mu \subseteq \beta \mathbb{N} / \mathbb{N}$  dan is  $J L_X^p(\beta S, \mu) \neq X_{\mu, p}$ .

Voor reflexieve Banachruimten  $X$  heeft  $JL_X^P(\beta S, \mu)$  een directe summand in  $X_{\mu, p}$ . We construeren als volgt een projectie op  $JL_X^P(\beta S, \mu)$ . Indien  $X$  reflexief is, zijn gesloten en begrensde verzamelingen in  $X$  zwak compact en heeft iedere  $f \in C(S, X)$  dus relatief zwak compacte-waarden-voorraad. Zodoende heeft elke  $f \in C(S, X)$  een zwak continue Stone - uitbreiding  $f_w^\beta$  tot  $\beta S$ . We bewijzen dat  $f_w^\beta$   $\mu$ -meetbaar is in de norm topologie van  $X$  en een element  $\bigwedge_\mu f_w^\beta$  in  $L_X^P(\beta S, \mu)$  bepaalt. De projectie  $\Pi_\mu$  van  $X_{\mu, p}$  of  $JL_X^P(\beta S, \mu)$  wordt gedefinieerd door  $\Pi_\mu \bigwedge_\mu f = J \bigwedge_\mu f_w^\beta$  voor  $f \in C(S, X)$  en verder met behulp van continuïteit.

In de laatste paragraaf van het eerste hoofdstuk tenslotte bestuderen we de relatie tussen de nulruimte  $X_{\mu, p}^0$  van  $\Pi_\mu$  en de afsluiting  $\hat{X}_{\mu, p}$  van het  $\bigwedge_\mu$ -beeld van de verzameling  $C_{0, w}(S, X)$ , die bestaat uit die  $f \in C(S, X)$  waarvoor  $\bigvee_s \langle f(s), x^* \rangle \in C_0(S)$  voor alle  $x^* \in X^*$ . We laten zien dat in sommige gevallen  $X_{\mu, p}^0 = \hat{X}_{\mu, p}$ , maar geven ook voorbeelden met  $X_{\mu, p}^0 \neq \hat{X}_{\mu, p}$ .

In het tweede hoofdstuk bekijken we voornamelijk representaties van de algebra  $L(X, X)$  van alle continue operatoren op  $X$ . Als  $A \in L(X, X)$  en  $f \in C(S, X)$  dan is  $Af = \bigvee_s Af(s) \in C(S, X)$ . We definiëren nu een representatie  $\rho_{\mu, p}$  van  $L(X)$  in  $L(X_{\mu, p}, X_{\mu, p})$  door  $\rho_{\mu, p}(A) \bigwedge_\mu f = \bigwedge_\mu Af$  voor  $f \in C(S, X), A \in L(X, X)$ . De beperking  $\rho_{\mu, p}^0$  van  $\rho_{\mu, p}$  tot  $X_{\mu, p}^0$  annihileert de verzameling  $CL(X, X)$  van compacte operatoren op  $X$ . Indien  $p = 2$  en  $X$  een Hilbertruimte is, is  $\rho_{\mu, 2}$  een  $*$ representatie van  $L(X, X)$ . Het belangrijkste resultaat is dat iedere cyclische  $*$ representatie van  $L(X, X)$ ,  $X$  een separabele Hilbertruimte, die  $CL(X, X)$  annihileert,  $*$ isomorf is met een cyclische subrepresentatie van een  $\rho_{\mu, 2}$ , waar  $\mu$  een puntmaat is op  $\beta \mathbb{N}/\mathbb{N}$ .

## STELLINGEN

### I

Laat  $S$  een volledig reguliere topologische ruimte zijn en  $X$  een gelijkmatig gladde of gelijkmatig convexe Banachruimte. Dan zijn de gecompleteerde  $\mu, p$ -hulls,  $X_{\mu, p}$ , van  $C(S, X)$  (cf. opmerking na 1.4.3.1 in dit proefschrift) voor  $1 < p < \infty$  reflexief.

### II

De klassificatie van alle twee-zijdige idealen in de ring  $L(H)$  van continue operatoren op een separabele Hilbertruimte  $H$ , zoals gegeven door J.W. Calkin, kan generaliseerd worden tot  $W^*$  algebras.

J.W.Calkin : Ann.of Math.42(1941)839-873.

B.Gramsch : Journ.f.d.R.und Ang.Mat.225(1967)  
97-115.

W.Wils : Aarhus preprint series.1968.

### III

Als  $\pi$  een singuliere\*representatie is van  $L(H)$ ,  $H$  een Hilbertruimte, op een Hilbertruimte  $H'$  en  $A \in L(H)$  is normaal, dan heeft  $\pi(A)$  een zuiver punkspectrum. In het algemeen echter zullen de eigenvectoren van  $\pi(A)$  niet de hele  $H'$  opspannen.

J.W.Calkin : Ann.of Math.42(1941)839-873.

### IV

Bij de beantwoording van de vraag van R.V.Kadison en I.Singer of de extremale toestanden op de maximaal abelse deelalgebra  $\mathcal{A}_d$  van  $L(H)$ ,  $H$  een separabele Hilbertruimte, die bestaat uit alle operatoren die diagonaliseerbaar zijn t.o.v. van een gegeven orthogonale basis in  $H$ , een unieke uitbreiding hebben tot  $L(H)$ , is het relevant zich af te vragen of het beeld van  $\mathcal{A}_d$  onder de kanonieke projectie van  $L(H)$  in  $L(H)/CL(H)$ , waar  $CL(H)$  de algebra van compacte operatoren is, maximaal abels is.

Bovenstaande vraag kan positief beantwoord worden voor Hilbertruimten en meer algemeen voor reflexieve Banachruimten met een onvoorwaardelijke basis.

R.V.Kadison, I.Singer : Amer.J. of Math.81(1959)  
383-400.



## V

Het is mogelijk een uitbreiding tot Banachruimten te geven van het begrip  $\sigma$ -additieve  $*$ representatie van  $L(H)$ ,  $H$  een Hilbertruimte. Een structuurstelling voor zulke zgn. reguliere representaties kan bewezen worden.

## VI

Een bevredigende definitie van meetbare families van Hilbertruimten kan gegeven worden zonder de gebruikelijke aftelbaarheidsvoorwaarden op te leggen. Met behulp van een structuurstelling voor deze gegeneraliseerde meetbare families is het mogelijk een eenvoudigere benadering te geven voor de multipliciteiten-theorie van spectrale maten.

J.Dixmier : Algèbres d'Opérateurs. Ch.III,  
Paris, 1957.

R.Halmos : An Introduction to Hilbertspace. Ch.III,  
New York, 1957.

W.Wils : Aarhus preprint series. 1968.

## VII

Een aantal belangrijke stellingen uit de theorie van de ontbinding van  $W^*$  algebras kan gegeneraliseerd worden tot het niet-separabele geval.

J.Dixmier : Algèbres d'Opérateurs. Ch.II,  
Paris, 1957.

J.Vesterstrøm, W.Wils : Aarhus preprint series. 1968.

## VIII

Zij  $(S, \mu)$  een eindige maatruimte,  $M$  een  $W^*$  algebra of  $M = L(X, X)$ , waar  $X$  een reflexieve Banachruimte is. In  $L^\infty(M, \Omega, \mu)$  kan op natuurlijke wijze een product gedefinieerd worden, zodat  $L^\infty(M, \Omega, \mu)$  een Banachalgebra wordt. Dit product hoeft niet samen te vallen met het puntsgewijs gedefinieerde product, zo dit al gedefinieerd is.

S.Sakai : Bull.Amer.Math.Soc. 70(1964)393-398.

## IX

Het door Braun en Koecher gegeven bewijs van Satz 7.4, Kap.III, bevat een cirkelredenering.

H.Braun und M.Koecher : Jordan - Algebren,  
Springer, Berlin, 1968.



## X

Het Heisenberg model voor de quantummechanica laat zich beter generaliseren tot de quantumveldentheorie dan het Schrödingermodel.

## XI

Het verdient aanbeveling na te gaan of het volgen van een anti-slipcursus voor automobilisten, zoals die gegeven wordt b.v. door de K.N.A.C., de kans op het krijgen van een ongeluk verkleint. Indien de uitslag van een dergelijk onderzoek positief is, zou het volgen van anti-slipcursussen gestimuleerd moeten worden door het geven van extra reducties op auto-verzekeringspremies en wegenbelasting voor hen die zo'n cursus met succes gevolgd hebben.

## XII

Het is een goed idee om de allerbeste studenten in hun tweede studiejaar in te schakelen bij het tweedejaars practicumwerk.

## XIII

Inleidende colleges voor jongere jaars studenten dienen door de beste docenten gegeven te worden.

## XIV

Unesco cursussen in de wiskunde zoals die het afgelopen jaar gegeven werden in Denemarken en Polen zijn gebaat met een uitgebreide propaganda en een goede selectie-procedure.





