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STRAIN ENERGY BOUNDS IN FINITE ELEMENT ANALYSIS BY SLAB ANALOGY

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I. INTRODUCTION.

In formulating approximate solutions by the finite element method it is possible to obtain "bounding" values on the true strain energy content.
If the approximation involves the use of a compatible displacement field then it will represent in general the lower bound. Alternatively, if an equilibrating stress field is used, then an upper bound on the stress energy will be obtained.(1)
Energy bounds can be translated into bounds for the structural deflections, providing a direct measure of convergence of the analysis.

Generation of compatible solutions, while not always easy, presents nevertheless fewer difficulties than the derivation of equilibrating solutions. In fact, many efficient solutions to both the plane elasticity and plate bending problems have been derived (2)(3) using compatible displacement formulations. The object of this paper is to show how the slab analogy can help by utilising such solutions to generate equilibrating solutions and obtain reciprocal bounds.

In addition the slab analogy will yield always an alternative formulation of the problem which at times may be more efficient from the computational point of view.

Before proceeding further it will be convenient to recapitulate some facts about the general slab analogy.

2. SLAB ANALOGY.

The recognition of an analogy between the stress functions in plane problems and the lateral displacements of plates was evident early (4) via the identical bi-harmonic relationships valid for homogeneous and isotropic situations.
An extension of this to multiply connected regions and to non-homogeneous situations came later (5)(6). Southwell (7) extended the analogy concept to a direct relationship between displacements in the plane problem and two new stress functions introduced for plate bending. Fung (8) derived Southwell's equations by the complementary energy principle, extending them to plates of variable thickness and mixed boundary conditions.

In the notes given below it will be seen that a "one to one" analogy is evident for all steps of the formulation of the two problem, irrespective of material properties assumed. The variables entering each problem will first be listed and then the analogy stated.

A. Plane elasticity.

The stress dependant part of strain can be defined in terms of displacements u and v in direction of the x and y axes :

Thus

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{Bmatrix} = \begin{Bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \frac{1}{2}(\partial u/\partial y + \partial v/\partial x) \end{Bmatrix} - \begin{Bmatrix} C\alpha T \\ C\alpha T \\ o \end{Bmatrix} \qquad \text{(IA)}$$

The last vector stands for thermal strains due to a temperature rise T and an expansion coefficient α. (The coefficient C is equal to unity for plane stress or $(1+\nu)$ for plane strain).
In the absence of dislocations the displacements u and v are single-valued. With the help of (IA) they can be determined by the variational principle of displacements

$$\int W(\varepsilon_x, \varepsilon_y, \varepsilon_{xy})dxdy - \int (\bar{X}u + \bar{Y}v)dxdy - \int (\bar{\sigma}_n u_n + \bar{\tau}_{tn} u_t)ds \quad \text{minimum} \qquad \text{(2A)}$$

In (2A), the strain energy density W is a quadratic form containing the appropriate elastic constants to produce the linear stress-strain relations

$$\sigma_x = \frac{\partial W}{\partial \varepsilon_x} \qquad \sigma_y = \frac{\partial W}{\partial \varepsilon_y} \qquad \tau_{xy} = \frac{1}{2}\frac{\partial W}{\partial \varepsilon_{xy}} \qquad \text{(3A)}$$

$\bar{X}$ and $\bar{Y}$ are specified body forces; $\bar{\sigma}_n$ and $\bar{\tau}_{tn}$ normal and tangential stresses specified along parts of the boundary. Along complementary parts of the boundary there are specified normal and tangential displacements

$$u_n = \bar{u}_n \qquad u_t = \bar{u}_t \qquad \text{(4A)}$$

The variational derivatives of the principle (2A), produce a pair of partial differential equations for the unknowns u and v, together with boundary conditions supplementing (4A). They are respectively statements of equilibrium with the body forces

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \bar{X} = o \qquad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \bar{Y} = o \qquad \text{(5A)}$$

and equilibrium at the boundary with the specified stresses.

$$\sigma_n = \bar{\sigma}_n \qquad \tau_{nt} = \bar{\tau}_{nt} \tag{6A}$$

This defines uniquely the problem and permits its solution.
The finite element method of approximation, relevant to this formulation of the problem, makes use of displacement models for the finite elements. Within each element the displacements are single-valued and differentiable; the whole field being then piecewise differentiable. The elements are said to be "conforming" if the displacements are furthermore single-valued at the interfaces. In such a case the variational principle (2A) remains applicable to the structure as a whole and predicts a lower strain energy bound if $\bar{u}_n = \bar{u}_s = o$, an upper bound if $\bar{X} = \bar{Y} = o$ and $\bar{\sigma}_n = \bar{\tau}_{tn} = o$.

An alternate approach to the problem is through the use of a general solution to the equilibrium equations. Such a solution is provided by setting

$$\{ \sigma \} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{Bmatrix} \partial^2\phi/\partial y^2 + F \\ \partial^2\phi/\partial x^2 + F \\ - \partial^2\phi/\partial x\partial y \end{Bmatrix} \tag{7A}$$

in which F is a body force potential such that

$$\bar{X} = - \partial F/\partial x \qquad \bar{Y} = - \partial F/\partial y$$

Along those parts of the boundary where stresses are specified (fig. I)

$$\begin{aligned} \bar{\sigma}_n &= \frac{\partial^2\phi}{\partial t^2} = \frac{\partial^2\phi}{\partial s^2} + \frac{\partial\phi}{sn}\,\frac{d\psi}{ds} + F \\ - \bar{\tau}_{nt} &= \frac{\partial^2\phi}{\partial t\partial n} = \frac{\partial^2\phi}{\partial s\partial n} - \frac{\partial\phi}{\partial s}\,\frac{d\psi}{ds} \end{aligned} \tag{8A}$$

where $d\psi/ds$ is the curvature of the boundary. The stress function $\phi(x,y)$ can be determined by the complementary energy principle

$$\int \left(\phi(\sigma_x, \sigma_y, \tau_{xy}) + \alpha CT(\sigma_x + \sigma_y)\right) dxdy$$

$$- \int \left(\overline{u}_n \left(\frac{\partial^2\phi}{\partial s^2} + \frac{\partial\phi}{\partial n}\,\frac{d\psi}{ds} + F\right) - \overline{u}_s \left(\frac{\partial^2\phi}{\partial n\partial s} - \frac{\partial\phi}{\partial s}\,\frac{d\psi}{ds}\right)\right) ds \quad \text{minimum} \tag{9A}$$

The complementary energy density ϕ (stress energy) provides the same stress-strain relations (3A), but solved for the strains

$$\varepsilon_x = \frac{\partial\Phi}{\partial\sigma_x} \qquad \varepsilon_y = \frac{\partial\Phi}{\partial\sigma_y} \qquad \varepsilon_{xy} = \frac{1}{2}\,\frac{\partial\Phi}{\partial\tau_{xy}} \tag{IOA}$$

The variational derivative of this principle for ϕ, yields

$$\frac{\partial^2\varepsilon_x}{\partial y^2} + \frac{\partial^2\varepsilon_y}{\partial x^2} - 2\,\frac{\partial^2\varepsilon_{xy}}{\partial x\partial y} = -\,\nabla^2(\alpha CT) \tag{IIA}$$

It is a necessary condition for the integrability of single-valued displacements u and v, defined according to (1A). After substitution of (IOA) and (7A), it is also the partial differential equation governing the stress function, which reduces to the non-homogeneous bi-harmonic equation

$$\nabla^4\phi = -\frac{1}{mE}\,\nabla^2\ (\alpha CT)$$

in the isotropic case. (The constant m is equal to unity for plane stress or $(1-\nu^2)$ for plane strain).
The variational boundary conditions, which supplement (8A), are :

$$\text{from}\quad \delta\phi_n \qquad \frac{\partial\overline{u}_t}{\partial s} + \overline{u}_n\,\frac{d\psi}{ds} = \varepsilon_t + \alpha CT$$

$$\tag{I2A}$$

$$\text{from}\quad \delta\phi \qquad \frac{\partial^2\overline{u}_n}{\partial s^2} - \frac{\partial}{\partial s}\left(\overline{u}_t\,\frac{d\psi}{ds}\right) = 2\,\frac{\partial\varepsilon_{nt}}{\partial s} + \frac{d\psi}{ds}\,(\varepsilon_n - \varepsilon_t) - \frac{\partial}{\partial n}\,(\varepsilon_t + \alpha CT)$$

This permits the problem to be solved, provided the domain is simply connected. For simplicity we use straight barriers to reduce the multiply-connected case to the simply connected one (fig. I). Further statements of equilibrium are necessary : across each barrier stresses should be continuously transmitted

$$\sigma_n^+ = \frac{\partial^2\phi^+}{\partial t^2} = \sigma_n^- = \frac{\partial^2\phi^-}{\partial t^2}$$

$$\tau_{nt}^+ = -\frac{\partial^2\phi^+}{\partial n\partial t} = \tau_{nt}^- = -\frac{\partial^2\phi^-}{\partial n\partial t} \tag{I3A}$$

Equivalent statements are

$$\phi^+ = \phi^- + t\,N_n + M_B$$

$$\phi_n^+ = \phi_n^- - N_t \tag{I4A}$$

The significance of the constants (N_n, N_t, M_B) appears in Appendix 1 .
The variational transition equations based on (I4A) are obtained directly from (I2A). Remembering that there are no specified displacements on a barrier, the variation on ϕ_n^- gives

$$(\varepsilon_t + \alpha CT)^+ = (\varepsilon_t + \alpha CT)^-$$

and the variation on ϕ^-

$$2\frac{\partial\varepsilon_{nt}}{\partial t} - \frac{\partial}{\partial n}(\varepsilon_t + \alpha CT)\Big|^+ = 2\frac{\partial\varepsilon_{nt}}{\partial t} - \frac{\partial}{\partial n}(\varepsilon_t + \alpha CT)\Big|^-$$

In view of definitions (1A), these results are equivalent to

$$\left(\frac{\partial u_t}{\partial t}\right)^+ = \left(\frac{\partial u_t}{\partial t}\right)^- \qquad \left(\frac{\partial^2 u_n}{\partial t^2}\right)^+ = \left(\frac{\partial^2 u_n}{\partial t^2}\right)^-$$

and yield the classical property discovered by Weingarten (9) and V. Volterra (10) that both sides of the barrier can undergo a kinematical relative displacement :

$$u_t^+ = u_t^- + p$$

$$u_n^+ = u_n^- + \omega\,t + q \tag{I5A}$$

Hence (IIA) is not sufficient for single-valuedness of the displacements. Parameters in the solution for ϕ must be further adjusted to implement for each barrier the single-valuedness conditions "in the large"

$$p = o \qquad q = o \qquad \omega = o \tag{I6A}$$

Those parameters are, for instance, the "Michell" constants of the expressions $\alpha x + \beta y + \gamma$ that can be added to ϕ along each internal boundary of a cavity without disturbing satisfaction of stress boundary conditions.
The finite element approximation, relevant to this alternate formulation, makes use of equilibrium models for the finite elements. An equilibrium model can be generated by a single-valued, twice differentiable stress function, defined in its interior. The elements will be "stress diffusing" if the stresses defined at the interfaces are continuously transmitted between adjacent elements. In view of (8A), this property can be insured if F, ϕ and its normal slope $\partial\phi/\partial n$ remain single-valued at the interfaces. This condition is however not quite necessary, since the surface $z = \phi(x,y)$ of any element can be moved bodily (a vertical translation and two rotations about the x and y axes) without disturbing the stress field. This freedom, used along barriers, allows the treatment of multiply connected cases.
The variational principle (9A) remains then applicable to the whole structure and predicts upper stress energy bounds if $\bar{u}_n = \bar{u}_t = o$, lower bounds if $\bar{X} = \bar{Y} = o$ and $\bar{\sigma}_n = \bar{\tau}_{nt} = o$.
Because, in the general case, the approximations will not allow ϕ to satisfy the partial differential equation issued from (IIA), the displacement field within an element will not be integrable. The only knowledge provided about displacements will be in the form of weighted averages [1][11].

B. Plate flexure.

In the Kirchhoff-Love theory of plate flexure, the curvatures produced by the internal moments are related to the lateral displacement, w, by

$$\{\kappa\} = \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} = \begin{Bmatrix} \partial^2 w/\partial x^2 - \kappa_o \\ \partial^2 w/\partial y^2 - \kappa_o \\ \partial^2 w/\partial x\partial y \end{Bmatrix} \tag{IB}$$

in which κ_o is an initial, isotropic, curvature of the type resulting from a temperature change. The internal moments can be related to the curvatures through an energy $W(\kappa_x, \kappa_y, \kappa_{xy})$ per unit area containing appropriate elastic constants

$$M_x = -\frac{\partial W}{\partial \kappa_x} \qquad M_y = -\frac{\partial W}{\partial \kappa_y} \qquad M_{xy} = -\frac{1}{2}\frac{\partial W}{\partial \kappa_{xy}} \tag{2B}$$

The problem for the lateral displacement is governed by the variational principle

$$\int W\, dxdy - \int \bar{q}\, w\, dxdy + \int (\bar{M}_n \frac{\partial w}{\partial n} - \bar{K}_n w) ds - \sum_i \bar{Z}_i w_i \quad \text{minimum} \tag{3B}$$

The lateral displacement can be specified along parts of the boundary

$$w = \bar{w} \tag{4B}$$

Then, along complementary part the shear distribution $\bar{K}_n$ is given, with, possibly, some concentrated loads $\bar{Z}_i$.
In the same manner the normal bending moment $\bar{M}_n$ is given along parts of the boundary complementary to those where the normal slope is known

$$\frac{\partial w}{\partial n} = \overline{\left(\frac{\partial w}{\partial n}\right)} \tag{5B}$$

Finally, a known transverse pressure $\bar{q}(x,y)$ is applied.
The variational derivative of the minimum total energy principle, using (2B) and (1B), is

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = -\bar{q} \tag{6B}$$

It is convenient, for later use, to introduce as auxiliary quantity the shear vector (Q_x, Q_y)

$$Q_x = \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} \qquad Q_y = \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} \tag{7B}$$

Then, the variational derivative is more clearly a statement of vertical equilibrium

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = -\bar{q} \tag{8B}$$

Expressed through (2B) and (1B) in terms of w, it becomes the partial differential equation governing the lateral displacement. For the isotropic case it reduces to

$$\nabla^4 w = \frac{\bar{q}}{D} \qquad \left(D = \frac{Et^3}{12(1-\nu^2)} \text{ the bending rigidity}\right)$$

The variational boundary conditions, supplementing (4B) and (5B) are :

$$\text{from } \delta w \qquad K_n = Q_n + \frac{\partial M_{nt}}{\partial s} = \bar{K}_n \tag{9B}$$

$$\text{from } \delta\left(\frac{\partial w}{\partial n}\right) \qquad M_n = \bar{M}_n \tag{10B}$$

$$\text{from } \delta w_i \qquad M_{nt}\,(s_i + o) - M_{nt}\,(s_i - o) = \bar{Z}_i \tag{11B}$$

Equation (9B) introduces the general definition of the Kirchhoff equivalent shear distribution along a curved boundary.
A displacement model for a finite plate flexure element will be defined by a parametric, single-valued, twice-differentiable lateral displacement field.
The elements will be conforming if w and $\partial w/\partial n$ are single-valued at the interfaces. The variational principle is applicable to a gridwork of conforming elements and predicts a lower bound to the strain energy if $\bar{w} = o$, $\left(\frac{\partial \bar{w}}{\partial n}\right) = o$; an upper bound if $\bar{q} = o$, $\bar{K}_n = o$, $\bar{M}_n = o$, $\bar{Z}_i = o$.
The stress-diffusing properties of the elements, represented here by the single-valuedness of K_n and M_n at the interfaces, will only be averaged in the approximation, because the variations on w and $\partial w/\partial n$ at the interfaces are constrained by the finite number of degrees of freedom.

The alternate approach to plate flexure is again through a general solution to the equilibrium problem. If a stress-function vector (U, V) is introduced such that

$$\begin{pmatrix} M_y \\ M_x \\ -M_{xy} \end{pmatrix} = \begin{pmatrix} \partial U/\partial x - P_o \\ \partial V/\partial y - P_o \\ \frac{1}{2}(\partial U/\partial y + \partial V/\partial x) \end{pmatrix} \tag{12B}$$

Then, we have for the shear load vector (7B)

$$Q_x = \frac{\partial \Omega}{\partial y} - \frac{\partial P_o}{\partial x} \qquad Q_y = - \frac{\partial \Omega}{\partial x} - \frac{\partial P_o}{\partial y} \tag{I3B}$$

with

$$\Omega = \frac{1}{2} \left(\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}\right) \tag{I4B}$$

and the equilibrium equation (8B) is satisfied, provided

$$\nabla^2 P_o = \bar{q} \tag{I5B}$$

The stress-function vector can be determined from the variational principle

$$\int \left\{ \phi(M_x, M_y, M_{xy}) - \kappa_o (M_x + M_y) \right\} dxdy - \int \left(\bar{w} K_n - \left(\frac{\partial \bar{w}}{\partial n}\right) M_n\right) ds \quad \text{minimum} \tag{I6B}$$

The complementary energy ϕ , per unit area, yields the inner moments-curvatures relations (2B) in inverted form

$$\kappa_x = - \frac{\partial \phi}{\partial M_x} \qquad \kappa_y = - \frac{\partial \phi}{\partial M_y} \qquad \kappa_{xy} = - \frac{1}{2} \frac{\partial \phi}{\partial M_{xy}} \tag{I7B}$$

To apply the principle by varying the quantities U and V we can rely on definitions (I2B) but we must also express the boundary loads in terms of the stress-functions. In natural boundary coordinates, let U_n , U_t be the components of the stress-function vector. Then

$$\Omega = \frac{1}{2} \left(\frac{\partial U_t}{\partial n} - \frac{\partial U_n}{\partial t}\right) \tag{I8B}$$

and, using a set of transformation rules detailed in Appendix 2,

$$M_n = \frac{\partial U_t}{\partial t} - P_o = \frac{\partial U_t}{\partial s} + U_n \frac{d\phi}{ds} - P_o \tag{I9B}$$

$$K_n = Q_n + \frac{\partial M_{nt}}{\partial s} = \frac{\partial \Omega}{\partial t} - \frac{\partial P_o}{\partial n} - \frac{1}{2} \frac{\partial}{\partial s} \left(\frac{\partial U_t}{\partial n} + \frac{\partial U_n}{\partial t}\right)$$

$$= - \frac{\partial^2 U_n}{\partial s \partial t} - \frac{\partial P_o}{\partial n}$$

Or, finally

$$K_n = -\frac{\partial^2 U_n}{\partial s^2} + \frac{\partial}{\partial s}\left(U_t \frac{d\psi}{ds}\right) - \frac{\partial P_o}{\partial n} \qquad (20B)$$

M_n and K_n as given by (I9B) and (20B) should be substituted into the boundary term of the principle. The variational derivatives for U and V are respectively

$$\frac{\partial \kappa_y}{\partial x} - \frac{\partial \kappa_{xy}}{\partial y} = -\frac{\partial \kappa_o}{\partial x}$$

$$(2IB)$$

$$-\frac{\partial \kappa_{xy}}{\partial x} + \frac{\partial \kappa_x}{\partial y} = -\frac{\partial \kappa_o}{\partial y}$$

They represent necessary integrability conditions for a single-valued lateral displacement w, as defined by (1B). When modified through (I7B) and (I2B), they become the pair of partial differential equations governing the stress functions. The boundary values associated to them are (I9B) and (20B), where K_n and M_n are specified and the complementary set, where deflections and slopes are specified :

$$\frac{\partial^2 \overline{w}}{\partial s^2} + \left(\frac{\partial \overline{w}}{\partial n}\right)\frac{d\psi}{ds} = \kappa_t + \kappa_o = \frac{\partial^2 w}{\partial t^2} \qquad (22B)$$

stemming from the boundary variations on U_n, and

$$\frac{\partial}{\partial s}\left(\frac{\partial \overline{w}}{\partial n}\right) - \frac{\partial \overline{w}}{\partial s}\frac{d\psi}{ds} = \kappa_{nt} = \frac{\partial^2 w}{\partial n \partial t} \qquad (23B)$$

stemming from the boundary variations on U_t.

This, in principle, formulates the problem to be solved for a simply connected domain. The additional statements of equilibrium required for a multiply-connected domain are the continuity of K_n and M_n across the barriers. Once again, taking straight barriers for simplicity, (I9B) and (20B) show that those equilibrium conditions are equivalent to

$$\left(\frac{\partial U_t}{\partial t}\right)^+ = \left(\frac{\partial U_t}{\partial t}\right)^-$$

$$\left(\frac{\partial^2 U_n}{\partial t^2}\right)^+ = \left(\frac{\partial^2 U_n}{\partial t^2}\right)^-$$

($\dot{P}_o$ and $\partial P_o/\partial n$ are assumed single-valued throughout).

Hence

$$U_t^+ = U_t^- + \mu_t$$

$$U_n^+ = U_n^- + \mu_n - T\,t \tag{24B}$$

The significance of the constants μ_t , μ_n and T appears in Appendix I . The variational transition equations based on (24B) can be written directly from (22B) and (23B). Because there are no specified displacements on the barrier, the variation on U_t^- yields

$$(\kappa_{nt})^+ = (\kappa_{nt})^-$$

and the variation on U_n^-

$$(\kappa_t + \kappa_o)^+ = (\kappa_t + \kappa_o)^-$$

In view of definitions (1B), these results are equivalent to

$$\left(\frac{\partial^2 w}{\partial n \partial t}\right)^+ = \left(\frac{\partial^2 w}{\partial n \partial t}\right)^- \qquad \text{and} \qquad \left(\frac{\partial^2 w}{\partial t^2}\right)^+ = \left(\frac{\partial^2 w}{\partial t^2}\right)^-$$

or, finally

$$w^+ = w^- + h_B + \omega_n\,t$$

$$\left(\frac{\partial w}{\partial n}\right)^+ = \left(\frac{\partial w}{\partial n}\right)^- - \omega_t \tag{25B}$$

As in the case of plane elasticity, this expresses that the two sides of the barrier can undergo a kinematical relative displacement : a relative vertical translation h_B and two relative rotations about the local axes n and t in B .

Again conditions (21B) are not sufficient for single-valuedness of the lateral displacement and its slopes. To avoid dislocations, parameters in U and V must be adjusted to obtain

$$h_B = o \qquad \omega_n = o \qquad \omega_t = o \tag{26B}$$

on each barrier. Those parameters can be the arbitrary constants in the expressions $(\alpha - \omega y)$ and $(\beta + \omega x)$ that can be added respectively to U and V along each internal boundary of cavity without disturbing stress boundary conditions. The equilibrium models of finite plate flexure elements can be generated by a parametric, single-valued, differentiable stress-function vector coupled with a parametric loading function P_o, if transverse loading modes are desired. The elements will be stress-diffusing if the stress-function vector remains single-valued at the interfaces, together with the loading function and its normal slope. This follows immediatly on inspection of equations (I9B) and (20B). In the multiply-connected case U and V can be taken single-valued in the domain cut by a set of barriers. Across those, U and V will eventually suffer a rigid body type discontinuity (distributed like $u_o - \omega y$, $v_o + \omega x$ along the barrier). The variational principle (I6B) will remain applicable to the gridwork of elements and produce upper stress energy bounds if $\bar{w} = o$, $\left(\frac{\partial \bar{w}}{\partial n}\right) = o$, lower bounds if $\bar{q} = o$, $\bar{K}_n = o$, $\bar{M}_n = o$.
Again, since in the approximation, the functions U and V are not required to and will generally not satisfy the partial differential equations issued from (2IB), the lateral displacement w will not be integrable. Numerical information on lateral displacements and slopes is only provided in the form of weighted averages.

C. The analogies.

The comparison of relations A and B for both systems show clearly the mathematical analogies which exist on a one to one basis between various quantities and equations. Table I shows this in detail.

The formal elegance of the analogies could be improved by the adoption of the following unusual notation for plate flexure theory

M_y	and	κ_y	instead of	M_x	and	κ_x
M_x	and	κ_x	" "	M_y	and	κ_y
$-M_{xy}$	and	$-\kappa_{xy}$	" "	M_{xy}	and	κ_{xy}

It will be observed that the role of stresses and strains, stress functions and displacements, equilibrium conditions and integrability conditions become at all times reversed.
This includes the analogy between physical dislocations in multiply connected domains and multi-valuedness of stress functions.

TABLE I.

(IA)	$\longleftrightarrow$	(I2B)
$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_{xy} \end{bmatrix}$	$\longleftrightarrow$	$\begin{bmatrix} M_y \\ M_x \\ -M_{xy} \end{bmatrix}$
αCT	$\longleftrightarrow$	P_o
$\begin{bmatrix} u \\ v \end{bmatrix}$	$\longleftrightarrow$	$\begin{bmatrix} U \\ V \end{bmatrix}$
(IIA)	$\longleftrightarrow$	(6B)
(I6A)	$\longleftrightarrow$	(24B)
$\begin{bmatrix} p \\ q \\ \omega \end{bmatrix}$	$\longleftrightarrow$	$\begin{bmatrix} \mu_t \\ \mu_n \\ -T \end{bmatrix}$
(7A)	$\longleftrightarrow$	(IB)
$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}$	$\longleftrightarrow$	$\begin{bmatrix} \kappa_y \\ \kappa_x \\ -\kappa_{xy} \end{bmatrix}$
F	$\longleftrightarrow$	$-\kappa_o$
ϕ	$\longleftrightarrow$	w
(5A)	$\longleftrightarrow$	(2IB)
(I4A)	$\longleftrightarrow$	(25B)
$\begin{bmatrix} N_n \\ N_t \\ M_B \end{bmatrix}$	$\longleftrightarrow$	$\begin{bmatrix} \omega_n \\ \omega_t \\ h_B \end{bmatrix}$
(3A)	$\longleftrightarrow$	(I7B)
(IOA)	$\longleftrightarrow$	(2B)

3. APPLICATIONS TO FINITE ELEMENT TECHNIQUES.

Conforming displacement models, analyzed by matrix methods, have discrete elastic characteristics described in terms of a set of generalized displacements $\{ q \}_e$ and a corresponding set of generalized loads $\{ g \}_e$

$$\{ g \}_e = K_e \{ q \}_e \qquad (I)$$

where K_e is the stiffness matrix of the element (I) (2) (3) (II).
Stress diffusing equilibrium models can be described in the same way; the stiffness matrix in this case is of the form

$$K_e = C_e F_e^{-1} C_e' \qquad (2)$$

where F_e is the flexibility matrix of the element (the matrix of the stress energy in terms of stress parameters) and C_e a load connection matrix, relating generalized loads and the set $\{ b \}_e$ of stress parameters.

$$\{ g \}_e = C_e \{ b \}_e \qquad (3)$$

This procedure can lead to elements with spurious kinematical degrees of freedom, which then require special handling in the treatment of the problem at the structural level (1) (11) (12).
The use of the analogies opens new possibilities in the construction of stress-diffusing equilibrium models and in their handling at the structural level.
From the analysis of the analogies it becomes clear that each conforming displacement model of a finite element produces an analogue stress-diffusing equilibrium model by identification of the parametric displacement field of the former to a parametric stress-function field of the latter. However a plate flexure element generates a plane stress (or strain) element and vice-versa.
A good example is the conforming plate flexure quadrilateral which goes over into a plate extension equilibrium element with linear stress variations by analogy between w and ϕ (I3).
However, even the treatment of the problem at the structural level can benefit from the analogies.
The analogue matrix relation to (I) is

$$\{ \varepsilon \}_e = F_e \{ c \}_e \qquad (4)$$

where $\{ c \}_e$ is the set of local values of (ϕ and $\partial\phi/\partial n$) or (U and V) analogous to the local displacements (w and $\partial w/\partial n$) or (u and v) .
F_e is now an "extended" flexibility matrix.

While the flexibility matrix in (2) is non-singular, the extended one is singular because the equation

$$F_e \; \{ z \} = o \tag{5}$$

has three independant solutions, representing rigid modes of the stress functions (the analogues of kinematical displacement modes) which generate no stresses and consequently no strains. The column matrix $\{ \varepsilon \}_e$ is precisely in the nature of generalized strains as can be seen as follows.
From (4) and Clapeyron's theorem, the strain energy of the element is

$$\frac{1}{2} \; \{ c \}_e' \; \{ \varepsilon \}_e \tag{6}$$

Generalized loads can now be defined as in (3) through an "extended" load connection matrix C_e

$$\{ g \}_e = C_e \; \{ c \}_e \tag{7}$$

There are exactly as many generalized loads as there are local values of the stress function so that, in contrast to (3) the load connection matrix is <u>square</u>.
The associated generalized displacements are such that the energy of the element is also

$$\frac{1}{2} \; \{ g \}_e' \; \{ q \}_e = \frac{1}{2} \; \{ c \}_e' \; C_e' \; \{ q \}_e \tag{8}$$

Comparing (8) with (6) and noting that the equality must hold for any $\{ c \}_e$

$$\{ \varepsilon \}_e = C_e' \; \{ q \}_e \tag{9}$$

This shows how the generalized strains are deduced from the generalized displacements.
The extended load connection matrix is singular for it is obvious that, since stresses vanish for $\{ c \}_e = \{ z \}$, so must the generalized loads in (7)

$$C_e \{ z \} = o \tag{I0}$$

From a classical theorem of algebra, if the homogeneous system (I0) has three independant non trivial solutions, so has the system

$$C'_e \{ y \} = o \tag{II}$$

In view of (9) those solutions are to be identified with the 3 rigid displacement modes of the element (which generate no strains).
Since there is no other solution to (II) there are also no spurious kinematical freedoms in the element. Hence the fact the stress-diffusing equilibrium model derives from an analogy with a conforming displacement model is sufficient to guarantee the absence of spurious kinematical freedoms.
In the displacement models the laws for assembling elements are those of stiffness addition or load addition [1] [2] [3] producing for the whole structure a relation between all the generalized external loads $\{ g \}$ and the nodal displacements $\{ q \}$

$$\{ g \} = K \{ q \}$$

with a master stiffness matrix

$$K = \sum_e L'_e K_e L_e$$

generated by the "localizing" matrices L_e defined by

$$\{ q \}_e = L_e \{ q \}$$

In the analogous equilibrium models the laws of assembly are those of flexibility addition or strain addition, producing for the whole structure a relation

$$\{ e \} = F \{ c \}$$

between the generalized strains for the whole structure and the complete array of nodal stress-function values. The master flexibility matrix

$$F = \sum_e L'_e F_e L_e$$

is obtained by the same localizing matrices as in the analogue structure.
The procedure is particularly well adapted to equilibrium models because the number of self-stressing states, represented by each element of $\{ c \}$, is markedly lower than the number of generalized displacements. The equations to be solved are both well conditioned and fewer in number.

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Appendix I .

Interpretations of the stress functions.

A. The Airy stress function $\phi(x,y)$.

Let (N_x, N_y) denote the resultants of stresses generated by ϕ along a path from a reference point A to (x,y) leaving matter on the left-hand side.

$$dN_x = - \tau_{xy}\, dx + \sigma_x\, dy = d\left(\frac{\partial\phi}{\partial y}\right)$$

$$dN_y = - \sigma_y\, dx + \tau_{xy}\, dy = - d\left(\frac{\partial\phi}{\partial x}\right)$$

Hence
$$N_x = \frac{\partial\phi}{\partial y}(x,y) - \left(\frac{\partial\phi}{\partial y}\right)_A$$

$$N_y = - \frac{\partial\phi}{\partial x}(x,y) + \left(\frac{\partial\phi}{\partial x}\right)_A$$

Similarly for the moment M about the point (x,y)

$$dM = N_x\, dy - N_y\, dx = d\phi - \left(\frac{\partial\phi}{\partial x}\right)_A dx - \left(\frac{\partial\phi}{\partial y}\right)_A dy$$

and
$$M = \phi(x,y) - \phi_A - (x - x_A)\left(\frac{\partial\phi}{\partial x}\right)_A - (y - y_A)\left(\frac{\partial\phi}{\partial y}\right)_A$$

Since the stress field determines ϕ except for the addition of an arbitrary linear form $a_o + a_1 x + a_2 y$, it is always possible to make ϕ and its first derivatives vanish at the chosen reference point.
In this case we have simply

$$\phi(x,y) = M \qquad - \frac{\partial\phi}{\partial x}(x,y) = N_y \qquad \frac{\partial\phi}{\partial y}(x,y) = N_x$$

The result is the same for all reconcileable paths from A to (x,y), because the stress field generated by ϕ does not involve body loads. Applying this to the closed path from A to B around a cavity (fig. 2), we conclude that in I4A, N_n, N_t and M_B are respectively the resultants in the directions of n and t and the moment about B of the loads applied to the boundary of the cavity. Should the cavity be loaded by a system statically equivalent to zero, the stress function would remain single-valued;

otherwise not. In particular, considering the application of a concentrated load or couple at an interior point as the limit of a loading distribution inside a circular cavity of vanishing radius, we conclude that such loads destroy the single-valuedness of the stress function.

B. The Southwell stress-functions U and V.

The total transverse load generated by the stress-functions along the path from A to (x,y) (matter on the left-hand side) is denoted by T.

$$dT = - Q_y \, dx + Q_x \, dy = d\Omega$$

Hence
$$T = \Omega(x,y) - \Omega_A$$

The moments of the same stresses about axes parallel to x and y through the final point (x,y) are denoted by μ_x and μ_y. Then

$$d\mu_x = M_y \, dx - M_{xy} \, dy - T \, dy = dU + \Omega_A \, dy$$

$$d\mu_y = - M_{xy} \, dx + M_x \, dy + T \, dx = dV - \Omega_A \, dx$$

and
$$\mu_x = U(x,y) - U_A + (y - y_A) \, \Omega_A$$

$$\mu_y = V(x,y) - V_A - (x - x_A) \, \Omega_A$$

Since U and V are determined except for the addition of arbitrary terms $(u_o - \omega y \, , \, v_o + \omega x)$, it is always possible to let U, V and Ω vanish at the reference point A. Then we have simple interpretations

$$T = \Omega(x,y) \qquad \mu_x = U(x,y) \qquad \mu_y = V(x,y)$$

The result is the same for all reconcileable paths because U and V alone imply zero transverse pressure on the plate.
Applying it to the closed path of fig. 2, we conclude that in (24B), T, μ_n and μ_t are respectively the total transverse load and the bending moments about the axes n and t of the loads applied to the cavity boundary.
Unless the loading system is statically equivalent to zero, the functions U and V are not single-valued. In particular, considering the application of a concentrated transverse load or bending couple at an interior point of the

plate as a limiting case of cavity loading, we conclude that such loads destroy the single-valuedness of the stress functions [6].

This raises an interesting question in the finite element approach, when an interior point becomes a vertex common to several plate elements. Because in each element a corner load

$$Z = \Delta M_{nt}$$

will appear, due to the jump in M_{nt} as we turn around the corner, we should infer from the foregoing considerations that if U and V are single-valued, the sum of all the corner loads must vanish.

The slab analogy provides an elegant proof of this. The analog to M_{nt} being $-\varepsilon_{nt}$, the corner load is, except for a constant factor, measured by the change in wedge angle in the analogue state of plane stress described by single-valued displacement functions $u = U$ and $v = V$. Since, at an interior point, the sum of all wedge angles remains equal to 2π, the sum of all wedge angle alterations must vanish and the corner loads consequently add up to zero.

Appendix 2 .

Boundary transformation rules.

As the local axes (n,t) of fig. I are moved along the boundary, the unit vectors on them undergo the following changes

$$\frac{d\vec{t}}{ds} = -\vec{n}\,\frac{d\psi}{ds} \qquad\qquad \frac{d\vec{n}}{ds} = \vec{t}\,\frac{d\psi}{ds} \tag{I}$$

which involve the radius of curvature of the boundary $(ds = R\,d\psi)$.
The notations $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial t}$ will indicate that we take or do not take into account the change in orientation of the local axes.

a) For a scalar $\phi(x,y)$ there is no difference

$$\frac{\partial\phi}{\partial t} \equiv \frac{\partial\phi}{\partial s} \tag{2}$$

b) For a vector $\vec{u} = u_n\,\vec{n} + u_t\,\vec{t}$, the identity

$$\frac{\partial\vec{u}}{\partial t} \equiv \frac{\partial\vec{u}}{\partial s}$$

produces in view of (I) the formulas for components

$$\frac{\partial u_n}{\partial t} = \frac{\partial u_n}{\partial s} - u_t\,\frac{d\psi}{ds} \qquad\qquad \frac{\partial u_t}{\partial t} = \frac{\partial u_t}{\partial s} + u_n\,\frac{d\psi}{ds} \tag{3}$$

This, applied to the gradient vector

$$\text{grad}\,\phi = \frac{\partial\phi}{\partial n}\,\vec{n} + \frac{\partial\phi}{\partial t}\,\vec{t} \;, \qquad \text{yields}$$

$$\frac{\partial^2\phi}{\partial t\partial n} = \frac{\partial^2\phi}{\partial n\partial s} - \frac{\partial\phi}{\partial s}\,\frac{d\psi}{ds} \qquad\qquad \frac{\partial^2\phi}{\partial t^2} = \frac{\partial^2\phi}{\partial s^2} + \frac{\partial\phi}{\partial n}\,\frac{d\psi}{ds} \tag{4}$$

c) For a symmetrical cartesian tensor, we have first the rules of component transformation

$$M_n = \ell^2 M_x + m^2 M_y + 2 \ell m M_{xy}$$

$$M_t = m^2 M_x + \ell^2 M_y - 2 \ell m M_{xy} \qquad (5)$$

$$M_{nt} = \ell m (M_y - M_x) + (\ell^2 - m^2) M_{xy}$$

where ℓ and m are the components of the unit vector $\vec{n}$, so that the following formulas hold

$$\frac{\partial \ell}{\partial n} = o \qquad \frac{\partial \ell}{\partial s} = - m \frac{d\psi}{ds} \qquad \frac{\partial m}{\partial n} = o \qquad \frac{\partial m}{\partial s} = \ell \frac{d\psi}{ds} \qquad (6)$$

It follows then that

$$\frac{\partial M_n}{\partial s} = \frac{\partial M_n}{\partial t} + 2 \left(\ell \frac{\partial \ell}{\partial s} M_x + m \frac{\partial m}{\partial s} M_y + \left(\ell \frac{\partial m}{\partial s} + m \frac{\partial \ell}{\partial s}\right) M_{xy} \right)$$

or, using (6) and interpreting

$$\frac{\partial M_n}{\partial s} = \frac{\partial M_n}{\partial t} + 2 \frac{d\psi}{ds} M_{nt}$$

Similarly

$$\frac{\partial M_t}{\partial s} = \frac{\partial M_t}{\partial t} - 2 \frac{d\psi}{ds} M_{nt} \qquad (7)$$

$$\frac{\partial M_{nt}}{\partial s} = \frac{\partial M_{nt}}{\partial t} + \frac{d\psi}{ds} (M_t - M_n)$$

We note that

$$\begin{pmatrix} \frac{\partial U_n}{\partial n} & \frac{\partial U_n}{\partial t} \\ \\ \frac{\partial U_t}{\partial n} & \frac{\partial U_t}{\partial t} \end{pmatrix}$$

is a cartesian tensor. Hence by a similar procedure we can establish the following

$$\frac{\partial^2 U_n}{\partial s \partial n} = \frac{\partial^2 U_n}{\partial t \partial n} + \frac{d\psi}{ds}\left(\frac{\partial U_n}{\partial t} + \frac{\partial U_t}{\partial n}\right) \tag{8}$$

$$\frac{\partial^2 U_n}{\partial t^2} = \frac{\partial}{\partial s}\left(\frac{\partial U_t}{\partial t}\right) + \frac{d\psi}{ds}\left(\frac{\partial U_t}{\partial n} + \frac{\partial U_n}{\partial t}\right) \tag{9}$$

$$\frac{\partial^2 U_n}{\partial t^2} = \frac{\partial}{\partial s}\left(\frac{\partial U_n}{\partial t}\right) + \frac{d\psi}{ds}\left(\frac{\partial U_n}{\partial n} - \frac{\partial U_t}{\partial t}\right) \tag{IO}$$

$$\frac{\partial^2 U_t}{\partial s \partial n} = \frac{\partial^2 U_t}{\partial t \partial n} + \frac{d\psi}{ds}\left(\frac{\partial U_t}{\partial t} - \frac{\partial U_n}{\partial n}\right) \tag{II}$$

Further transformations of those results by (3) allow to express the right hand sides entirely in terms of s and n derivatives.