STRAIN MEASURES AND COMPATIBILITY EQUATIONS IN THE LINEAR HIGH-ORDER SHELL THEORIES*

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1. Introduction The development of consistent refined theories of shells (and plates) based upon discarding the Love-Kirchhoff (L.K.) assumptions has attracted increasing attention during recent years. Outside the purely heuristic interest, there is the advent of the new exotic materials, such as composites, which has constituted the greatest stimulus for these refined theories. From the multitude of works dealing with this problem we shall refer only to two groups of works which involve similar features in their mathematical treatment.

The first group concerns the so-called Timoshenko shell (or plate) theory (T.S.T). The displacement field appropriate to model its kinematic behavior is represented as:

$$\bar{V}_\alpha(x^\omega, x^3) = V_\alpha^{(0)}(x^\omega) + x^3 V_\alpha^{(1)}(x^\omega), \qquad \bar{V}_3(x^\omega, x^3) = V_3^{(0)}(x^\omega). \tag{1}$$

The second group of works concerns the so-called higher-order shell theory (H.O.S.T). It constitutes in fact an extension of the former model, in the sense that the appropriate displacement field may be represented as:

$$\bar{V}_\alpha(x^\omega, x^3) = \sum_{r=0}^{R} (x^3)^r V_\alpha^{(r)}(x^\omega); \qquad \bar{V}_3(x^\omega, x^3) = \sum_{s=0}^{S} (x^3)^s V_3^{(s)}(x^\omega). \tag{2}$$

In Eq. (2), R and S ($R \gtrless S$) are two natural numbers defining the level of truncation in the series expansion (across the shell wall thickness).

In this connection it should be stressed that the large diversity of high-order shell (or plate) theories relies upon the various selection of R and S. The extant literature on the problem may be relevant in this regard (see, e.g., the works devoted to high-order shell [1-8] and plate [9–12] theories). In contrast to the T.S.T., a framework in which a great number of basic results have been obtained (see, e.g., [13]–[18]), the H.O.S.T. still needs improvements in order to become a complete and consistent theory. Nevertheless, it is worth reporting that:

(i) in the monograph [19], the general theory of shells as substantiated in the framework of the Cosserat continuum concept furnishes valuable results for the high-order shell (and plate) theories, as well, and
(ii) a large part of the monograph [20] deals with the substantiation of the high-order shell (and plate) theories, treated both in linear and nonlinear formulations.

However, as far as the authors of the present paper are aware, no results concerning the

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compatibility equations (C.E.) appropriate to H.O.S.T. are available in the literature. And it may be inferred from the classical and Timoshenko shell theories that the mere existence of such results has constituted the necessary premise for further remarkable contributions in the field.

The derivation of compatibility equations appropriate to the high-order linear shell theory is the basic object of the following sections.

2. Preliminaries. Let the points of the 3D space of the shell be referred to a set of curvilinear normal coordinates $x^i(x^\alpha, x^3)$ where $x^3 = 0$ defines the shell middle surface. Accordingly, the following relations for the spatial metric tensor hold valid:

$$g_{\alpha\beta} = \mu_\alpha^\lambda \mu_\beta^\omega a_{\lambda\omega}; \qquad g_{\alpha 3} = g^{\alpha 3} = 0; \qquad g_{33} = g^{33} = 1; \tag{3}$$

here $a_{\lambda\omega}$ denotes the metric tensor of the middle surface, while μ_β^α is defined by:

$$\mu_\beta^\alpha = \delta_\beta^\alpha - x^3 b_\beta^\alpha \tag{4}$$

δ_β^α being Kronecker's symbol; $b_{\alpha\beta}$ is the second fundamental form of the middle surface. As shown in [21], μ_β^α is nonsingular. Its inverse, denoted by $(\mu^{-1})_\beta^\alpha$ and satisfying

$$\mu_\beta^\alpha (\mu^{-1})_\lambda^\beta = \delta_\lambda^\alpha, \tag{5}$$

may be expressed as (see [21])

$$(\mu^{-1})_\beta^\alpha = \sum_{n=0}^{\infty} (x^3)^n (b^n)_\beta^\alpha. \tag{6}$$

In (5), $(b^n)_\beta^\alpha$ is defined by

$$(b^n)_\beta^\alpha = b_\beta^\lambda (b^{n-1})_\lambda^\alpha = b_\lambda^\alpha (b^{n-1})_\beta^\lambda \tag{7}$$

where, in addition,

$$\begin{aligned} (b^n)_\beta^\alpha &\equiv \delta_\beta^\alpha \quad \text{for} \quad n = 0 \\ &\equiv 0 \quad \text{for} \quad n < 0 \end{aligned} \tag{8}$$

As was emphasized in [21], μ_β^α and $(\mu^{-1})_\beta^\alpha$ (called shifters) play an important role in establishing the relationships between space tensor components and their surface (shifted) counterparts.

Concerning the relationships between covariant differentiation of space and surface tensors (see [21]), the following relations will prove useful in what follows:

$$T_{\alpha\|\beta} = \mu_\alpha^\nu (\bar{T}_{\nu|\beta} - b_{\nu\beta} \bar{T}_3); \qquad T_{\alpha\|3} = \mu_\alpha^\nu \bar{T}_{\nu,3}; \qquad T_{3\|\alpha} = \bar{T}_{3,\alpha} + b_\alpha^\varepsilon \bar{T}_\varepsilon; \qquad T_{3\|3} = \bar{T}_{3,3}, \tag{9}$$

where the shifted components are denoted by an upper bar; the double and single strokes are used to identify the covariant differentiation with respect to the space and surface metrics, respectively, while a comma denotes partial differentiation. Throughout this paper, Latin indices run over the range 1, 2, 3, and Greek indices over the range 1, 2.

It is emphasized that the shifted components can be functions of x^3.

3. Displacement field in the shell: definition of the nth order strain measures. Let us consider the displacement vector $\mathbf{V}(x^\omega, x^3)$ of the 3D points of the shell, expressed in terms of the spatial and their shifted components as

$$\mathbf{V} = \bar{V}_\alpha \mathbf{a}^\alpha + \bar{V}_3 \mathbf{a}^3 = V_\alpha \mathbf{g}^\alpha + \bar{V}_3 \mathbf{g}^3 \tag{10}$$

where $\mathbf{g}_\alpha$ and $\mathbf{a}_\alpha$ denote the space and surface base vectors (related by $\mathbf{g}_\alpha = \mu_\alpha^\lambda\, \mathbf{a}_\lambda$) while $\mathbf{g}_3$ and $\mathbf{a}_3$ (where $\mathbf{g}_3 \equiv \mathbf{a}_3$) denote the unit normal vector to the mid-surface. It is easily seen from (10) that:

$$V_\alpha = \mu_\alpha^\beta\, \bar{V}_\beta; \qquad V_3 = \bar{V}_3 \tag{11}$$

where

$$\bar{V}_\alpha \equiv \bar{V}_\alpha(x^\omega, x^3); \qquad \bar{V}_3 \equiv \bar{V}_3(x^\omega, x^3).$$

Let us adopt for the shifted displacements $\bar{V}_\alpha$, $\bar{V}_3$ the general representation (2), in which, for the sake of simplicity, we shall assume that $R = S \equiv N$. As will be shown later, the results so obtained may easily be modified when such an equality is not invoked *a priori*.

Consideration of (9) and (2) in the strain-displacement linear equations

$$2e_{ij} = V_{i\|j} + V_{j\|i} \tag{12}$$

yields the following representation of the strain components e_{ij} in the 3D medium of the shell:

$$\begin{aligned} 2e_{\alpha\beta} &= \mu_\alpha^\mu \sum_{n=0}^{N} \gamma_{\mu\beta}^{(n)}\, (x^3)^n + \mu_\beta^\mu \sum_{n=0}^{N} \gamma_{\mu\alpha}^{(n)}\, (x_3)^n, \\ e_{33} &= \sum_{n=0}^{N} \gamma_{33}^{(n)}\, (x^3)^n, \qquad 2e_{\alpha 3} = \sum_{n=0}^{N} \gamma_{\alpha 3}^{(n)}\, (x^3)^n. \end{aligned} \tag{13}$$

In the following development we choose $\gamma_{\mu\beta}^{(n)} \equiv \gamma_{\mu\beta}^{(n)}(x^\omega)$; $\gamma_{\alpha 3}^{(n)} \equiv \gamma_{\alpha 3}^{(n)}(x^\omega)$ and $\gamma_{33}^{(n)}(x^\omega)$ as the nth-order shell-strain measures. Consistent with (2) and the restriction $R = S \equiv N$, these nth-order strain measures may be written in the form:

$$\begin{aligned} \gamma_{\alpha\beta}^{(n)} &= V_{\alpha|\beta}^{(n)} - b_{\alpha\beta} V_3^{(n)} && \text{for} \quad n = \overline{0, N-1}, \\ &= V_{\alpha|\beta}^{(N)} - b_{\alpha\beta}\, V_3^{(N)} && \text{for} \quad n = N, \\ \gamma_{\alpha 3}^{(n)} &= (n+1)\, V_\alpha^{(n+1)} - (n-1)\, b_\alpha^\nu\, V_\nu^{(n)} + V_{3,\alpha}^{(n)} && \text{for} \quad n = \overline{0, N-1}, \\ &= -(N-1)\, b_\alpha^\nu V_\nu^{(N)} + V_{3,\alpha}^{(N)} && \text{for} \quad n = N, \\ \gamma_{33}^{(n)} &= (n+1)\, V_3^{(n+1)} && \text{for}\ n = \overline{0, N-1}, \\ &= 0 && \text{for} \quad n = N. \end{aligned} \tag{14}$$

We now consider some special cases of the above results.

For the L.K. model, consistent with its geometrical content stipulating the vanishing of transverse shear and transverse normal strains (i.e. with $\gamma_{\alpha 3}^{(n)} = \gamma_{33}^{(n)} = 0$ for all $n \geq 0$), one obtains from (14) the following restrictions on the coefficients in (2)

$$\begin{aligned} V_3^{(n)} &= V_3^{(0)} && \text{for} \quad n = 0, \\ &= 0 && \text{for} \quad \forall n \geq 1; \\ V_\alpha^{(n)} &= V_\alpha^{(0)} && \text{for}\ n = 0, \\ &= -(b_\alpha^\nu\, V_\nu^{(0)} + V_{3,\alpha}^{(0)}) && \text{for} \quad n = 1, \\ &= 0 && \text{for} \quad \forall n \geq 2. \end{aligned} \tag{15}$$

In light of the above, there results that in the framework of the L.K. theory the non-vanishing strain measures are:

$$\begin{aligned} \gamma^{(n)}_{\alpha\beta} &= \gamma^{(0)}_{\alpha\beta} = V^{(0)}_{\alpha\,|\,\beta} - b_{\alpha\beta}\, V^{(0)}_3 \equiv \gamma_{\alpha\beta} \quad \text{for} \quad \text{n} = 0, \\ &= \gamma^{(1)}_{\alpha\beta} = V^{(1)}_{\alpha|\beta} \equiv \kappa_{\alpha\beta} \qquad\qquad \text{for} \quad n = 1 \end{aligned} \tag{16}$$

Consistent with (15), the representation of shifted displacements is:

$$\bar{V}_\alpha(x^\omega,x^3) = V^{(0)}_\alpha - x^3(b^\nu_\alpha\, V^{(0)}_\nu + V^{(0)}_{3,\alpha}), \qquad \bar{V}_3\,(x^\omega, x^3) = V^{(0)}_3(x^\omega). \tag{17}$$

In the case of T.S.T. the pertinent representation of the shifted displacements is given by (1), or in a more extended form as:

$$\bar{V}_\alpha = V^{(0)}_\alpha + x^3 V^{(1)}_\alpha; \qquad \bar{V}_3 = V^{(0)}_3 + x^3 V^{(1)}_3. \tag{18}$$

Consistent with (18), the appropriate shell-strain measures as derived from (14), are:

$$\begin{aligned} \gamma^{(n)}_{\alpha\beta} &= \gamma^{(0)}_{\alpha\beta} = V^{(0)}_{\alpha\,|\,\beta} - b_{\alpha\beta}\, V^{(0)}_3 \equiv \gamma_{\alpha\beta} && \text{for} \quad n = 0, \\ &= \gamma^{(1)}_{\alpha\beta} = V^{(1)}_{\alpha\,|\,\beta} - b_{\alpha\beta}\, V^{(1)}_3 \equiv \kappa_{\alpha\beta} && \text{for} \quad n = 1, \\ &= 0 && n \geq 2; \\ \gamma^{(n)}_{\alpha 3} &= \gamma^{(0)}_{\alpha 3} = V^{(1)}_\alpha + b^\nu_\alpha V^{(0)}_\nu + V^{(0)}_{3,\alpha} \equiv \gamma_{\alpha 3} && \text{for} \quad n = 0, \\ &= \gamma^{(1)}_{\alpha 3} = V^{(1)}_{3,\alpha} \equiv \kappa_{\alpha 3} && \text{for} \quad n = 1, \\ &= 0 && \text{for} \quad n \geq 2; \\ \gamma^{(n)}_{33} &= \gamma^{(0)}_{33} = V^{(1)}_3 \equiv \gamma_{33} && \text{for} \quad n = 0, \\ &= 0 && \text{for} \quad \forall n \geq 1. \end{aligned} \tag{19$_1$}$$

Consistent with (1), the appropriate strain-measures are obtained from (19) by modifying only $\gamma^{(1)}_{\alpha\beta}$, $\gamma^{(1)}_{\alpha 3}$, $\gamma^{(0)}_{33}$, which in this instance become:

$$\gamma^{(1)}_{\alpha\beta} = V^{(1)}_{\alpha\,|\,\beta}; \qquad \gamma^{(1)}_{\alpha 3} = 0; \qquad \gamma^{(0)}_{33} = 0. \tag{19$_2$}$$

For other representations of shifted displacements encountered in the field literature (see e.g. [1]–[12]), the specialization of the general results (14) is a simple matter. Therefore we shall not proceed to further specializations of the above results.

4. Other representations of shell-strain measures. At this point a few remarks about some other possible representations of the nth-order strain measures are in order. In this connection it must be emphasized that in addition to $\gamma^{(n)}_{\alpha\beta}$ (intervening in the kinematical set $\{\Gamma\} \equiv \{\gamma^{(n)}_{\alpha\beta},\ \gamma^{(n)}_{\alpha 3},\ \gamma^{(n)}_{33}\}(n = \overline{0, N})$), still other variants of them could be defined. The coefficients $e^{(n)}_{\alpha\beta}$ associated with the various powers of x^3 in the expansion of $e_{\alpha\beta}$ as under:

$$e_{\alpha\beta} = \sum_{n=0}^{N} (x^3)^n e^{(n)}_{\alpha\beta}, \tag{20}$$

play the role of a such new variant. By identifying the coefficients of the same powers in (13) and (20), the following relationship between $e_{\alpha\beta}^{(n)}$ and $\gamma_{\alpha\beta}^{(n)}$ results:

$$\begin{aligned}
2e_{\alpha\beta}^{(n)} &= \gamma_{\alpha\beta}^{(0)} + \gamma_{\beta\alpha}^{(0)} && (n = 0),\\
&= \gamma_{\alpha\beta}^{(1)} + \gamma_{\beta\alpha}^{(1)} - b_\alpha^\mu \gamma_{\mu\beta}^{(0)} - b_\beta^\mu \gamma_{\mu\alpha}^{(0)} && (n = 1),\\
&= \gamma_{\alpha\beta}^{(2)} + \gamma_{\beta\alpha}^{(2)} - b_\alpha^\mu \gamma_{\mu\beta}^{(1)} - b_\beta^\mu \gamma_{\mu\alpha}^{(1)} && (n = 2),\\
&\equiv (\gamma_{\alpha\beta}^{(2)} + \gamma_{\beta\alpha}^{(2)} - b_\alpha^\nu b_\beta^\varepsilon e_{\nu\varepsilon}^{(0)} - \tfrac{1}{2}(b_\alpha^\nu e_{\nu\beta}^{(1)} + b_\beta^\nu e_{\nu\alpha}^{(1)})), \ldots\\
&\leq \gamma_{\alpha\beta}^{(N)} + \gamma_{\beta\alpha}^{(N)} - \mu_\alpha^\mu \gamma_{\mu\beta}^{(N-1)} - \mu_\beta^\mu \gamma_{\mu\alpha}^{(N-1)} && (n = N),
\end{aligned} \tag{21}$$

$2e_{\alpha 3}^{(n)} = \gamma_{\alpha 3}^{(n)}$; $e_{33}^{(n)} = \gamma_{33}^{(n)}$ $(n = \overline{0, N})$. $\{E\} \equiv \{e_{\alpha\beta}^{(n)}, e_{\alpha 3}^{(n)}, e_{33}^{(n)}\}$ $(n = \overline{0, N})$ could constitute another variant of strain measures in the linear H.O.S.T., as in [22]. It is easily seen that in contrast to $\{\Gamma\}$, where the components of $\gamma_{\alpha\beta}^{(n)}$ are asymmetric, their counterparts $e_{\alpha\beta}^{(n)}$ in $\{E\}$ are symmetric. It is worth mentioning that the strain measure set $\{E\}$ may result also through linearization of the pertinent kinematical measures as obtained either in [19], with the use of the Cosserat continuum concept, or in [20], on the basis of $3D$ elasticity theory. In the framework of Timoshenko shell theory, consistent with (18), the strain measure set $\{E\}$ reduces to $\{\tilde{E}\}$, i.e. $\{E\} \to \{\tilde{E}\} \equiv \{e_{\alpha\beta}^{(n)}, e_{\alpha 3}^{(n)}, e_{33}^{(0)}\}$ $(n = 0, 1)$. These strain measures have been obtained in [19] through the appropriate specialization of their nth-order counterparts. In the framework of L.K.T., the symmetrical counterpart of $\{\tilde{\tilde{\Gamma}}\} \equiv \{\gamma_{\alpha\beta}^{(n)}\}$ is $\{\tilde{\tilde{E}}\} \equiv \{e_{\alpha\beta}^{(n)}\}$ $(n = 0, 1)$. These last symmetric kinematic measures coincide with those derived in another way in [23] (and referred to as Naghdi's strain measures). It is to be pointed out that in both $\{\tilde{E}\}$ and $\{\tilde{\tilde{E}}\}$, the strain measures $e_{\alpha\beta}^{(2)}$ are not intervening, although they are different from zero quantities. This is because, in the framework of T.S.T. and L.K.T., $e_{\alpha\beta}^{(2)}$ are not independent quantities, being expressible exactly and entirely in terms of $e_{\alpha\beta}^{(0)}$ and $e_{\alpha\beta}^{(1)}$, as it may result from (21). In addition to $\{\Gamma\}$ and $\{E\}$, still other possible strain measure representations could be defined in H.O.S.T. However, it is most desirable that such new representations should be obtained in close connection with the requirement of the reduction in he number of compatibility equations, as it was done in the classical framework in [23, 24]. This problem will be considered at a later stage, after the deduction of compatibility equations appropriate to H.O.S.T.

5. The compatibility equations. The relations expressing the nth-order shell-strain measures in terms of displacement components may be regarded as a set of partial differential equations (in number $(7n - 1)$ or $(6n - 1)$ according as $\{\Gamma\}$ or $\{E\}$ are used as strain measure variants, respectively) for only $3n$ unknown functions $V_\alpha^{(n)}$, $V_3^{(n)}$ $(n = \overline{0, N})$.

As in $3D$ elasticity theory (see e.g. [25]), it may be argued that the differential equation system will assume the existence of a single-valued solution (within a rigid-body motion) if the selected nth-order strain measures satisfy certain conditions referred to as compatibility equations (C.E.). These will be derived by using the condition ensuring the continuity of the deformed shell space (assumed to be a simply connected one). The condition is

$$\varepsilon^{\alpha\beta} \mathbf{V}_{,\alpha\beta} = 0. \tag{22_1}$$

From the purely mathematical point of view, (22) expresses the integrability condition of partial differential equations correlating $\mathbf{V}_{,\alpha}$ with the linearized nth-order strain measures. Special forms of (22) have been used in [16, 20]; for a comprehensive discussion of this condition see [19]. In (22) and in what now follows, $\varepsilon^{\alpha\beta}$ and $\varepsilon_{\alpha\beta}$ denote the ε-system of the middle surface.

Consistent with the representation (2) of shifted displacements and the definition (14) of nth-order strain measures, the condition (22_1) used in conjunction with (10) and the well-known Gauss-Weingarten equations (see e.g. [21] or [20, Appendix C]), there results:

$$\varepsilon^{\beta\gamma}\{[\gamma_{\alpha\beta\,|\,\gamma}^{(n)} - b_{\gamma\alpha}(\gamma_{\beta 3}^{(n)} - (n+1)V_{\beta}^{(n+1)} + n\, b_{\beta}^{\nu}\, V_{\nu}^{(n)})]\mathbf{a}^{\alpha} + [\gamma_{\alpha\beta}^{(n)} b_{\gamma}^{\alpha} + \gamma_{\beta 3,\gamma}^{(n)} - (n+1)V_{\beta,\gamma}^{(n+1)} + n(b_{\beta}^{\nu}\, V_{\nu}^{(n)})_{,\gamma}]\mathbf{a}^{3}\} = 0 \qquad (n = \overline{0, N}). \quad (22_2)$$

From (22_2) it is clear that the coefficients of $\mathbf{a}^{\alpha}$ and $\mathbf{a}^{3}$ must vanish separately, yielding:

$$\varepsilon^{\beta\gamma}[\gamma_{\alpha\beta\,|\,\gamma}^{(n)} - b_{\gamma\alpha}(\gamma_{\beta 3}^{(n)} - (n+1)V_{\beta}^{(n+1)} + n\, b_{\beta}^{\nu}\, V_{\nu}^{(n)})] = 0;$$
$$\varepsilon^{\beta\gamma}[\gamma_{\alpha\beta}^{(n)} b_{\gamma}^{\alpha} + \gamma_{\beta 3,\gamma}^{(n)} - (n+1)V_{\beta,\gamma}^{(n+1)} + n(b_{\beta}^{\nu}\, V_{\nu}^{(n)})_{,\gamma}] = 0 \qquad (n = \overline{0, N}). \quad (23)$$

The equations in (23) are basic in deducing the compatibility equations. In order to express them explicitly in terms of the selected strains, we differentiate covariantly (23_1) with respect to x^{δ}, multiply the result by $\varepsilon^{\alpha\delta}$ and use Mainardi-Codazzi equations and also $(14)_3$. This yields

$$\varepsilon^{\alpha\delta}\varepsilon^{\beta\gamma}[\gamma_{\alpha\beta\,|\,\gamma\delta}^{(n)} - b_{\gamma\alpha}\,\gamma_{\beta 3\,|\,\delta}^{(n)} + (n+1)b_{\gamma\alpha}\,\gamma_{\beta\delta}^{(n+1)} + b_{\gamma\alpha}\, b_{\beta\delta}\,\gamma_{33}^{(n)} - (n b_{\gamma\alpha}\, b_{\beta}^{\nu}\, V_{\nu}^{(n)})_{\,|\,\delta}] = 0 \qquad (n = \overline{0, N}), \quad (24)$$

which again contains the displacement $V_{\nu}^{(n)}$. It is easily seen that for $n = 0$, (24) may be expressed entirely in terms of strain measures as:

$$\varepsilon^{\alpha\delta}\varepsilon^{\beta\gamma}[\gamma_{\alpha\beta\,|\,\gamma\delta}^{(0)} - b_{\alpha\gamma}\,\gamma_{\beta 3\,|\,\delta}^{(0)} + b_{\gamma\alpha}\,\gamma_{\beta\delta}^{(1)} + b_{\gamma\alpha}\, b_{\beta\delta}\,\gamma_{33}^{(0)}] = 0. \quad (25)$$

(25) constitutes a first C.E. In order to obtain the remaining compatibility equations, (23_1) will be used again. For $n = 1$ it yields:

$$\varepsilon^{\beta\gamma}[\gamma_{\alpha\beta\,|\,\gamma}^{(1)} - b_{\gamma\alpha}(\gamma_{\beta 3}^{(1)} - 2V_{\beta}^{(2)} + b_{\beta}^{\nu}\, V_{\nu}^{(1)})] = 0. \quad (26)$$

In order to express (26) in terms of the selected strains, we shall use (23_1) specialized for $n = 0$ which gives:

$$-\varepsilon^{\beta\gamma} b_{\gamma\alpha}\, V_{\beta}^{(1)} = -\varepsilon^{\beta\gamma} b_{\gamma\alpha}\,\gamma_{\beta 3}^{(0)} + \varepsilon^{\beta\gamma}\gamma_{\alpha\beta\,|\,\gamma}^{(0)}. \quad (27)$$

Insertion of (27) into (26) followed by covariant differentiation with respect to x^{δ} and subsequent multiplication by $\varepsilon^{\alpha\delta}$ yields a second C. E.:

$$\varepsilon^{\alpha\delta}\varepsilon^{\beta\gamma}[\gamma_{\alpha\beta\,|\,\gamma\delta}^{(1)} + (b_{\beta}^{\nu}\,\gamma_{\alpha\nu\,|\,\gamma}^{(0)})]_{\,|\,\delta} - b_{\gamma\alpha}(\gamma_{\beta 3}^{(1)} + b_{\beta}^{\nu}\,\gamma_{\nu 3}^{(0)})_{\,|\,\delta} + 2b_{\gamma\alpha}(\gamma_{\beta\delta}^{(2)} + \tfrac{1}{2} b_{\beta\delta}\,\gamma_{33}^{(1)})] = 0. \quad (28)$$

A similar procedure can be followed step by step so as to obtain from (23_1) succesively for $n = 2, \ldots N-1$, the pertinent C. E. However, it may be shown that we may write:

$$-p\,\varepsilon^{\beta\gamma} b_{\gamma\alpha}\, V_{\beta}^{(p)} = \varepsilon^{\beta\gamma} \sum_{n=0}^{p-1} (b^{n})_{\beta}^{\nu}[\gamma_{\alpha\nu\,|\,\gamma}^{(p-n-1)} - b_{\gamma\alpha}\,\gamma_{\nu 3}^{(p-n-1)}] \quad (p = \overline{0, N}) \quad (29)$$

and consequently the remaining C. E. following from (23_1) may be written compactly as:

$$\varepsilon^{\alpha\delta}\varepsilon^{\beta\gamma}\left\{\sum_{n=0}^{p} [(b^{n})_{\beta}^{\nu}(\gamma_{\alpha\nu\,|\,\gamma}^{(p-n)} - b_{\gamma\alpha}\,\gamma_{\nu 3}^{(p-n)})]_{\,|\,\delta} + (p+1)b_{\gamma\alpha}[\gamma_{\beta\delta}^{(p+1)} + (p+1)^{-1} b_{\beta\delta}\,\gamma_{33}^{(p)}]\right\} = 0 \qquad (p = \overline{0, N-1}). \quad (30)$$

It is easily seen that the simple specialization of (30) for $p = 0$ and $p = 1$ allow us to recover (25) and (28).

The C. E. following from (23_1) for $n = N$ requires special attention. For this case, the term $V_\beta^{(N+1)}$ is obviously zero and the resulting equation becomes:

$$\varepsilon^{\beta\gamma}[\gamma_{\alpha\beta}^{(N)}|_\gamma - b_{\gamma\alpha}\gamma_{\beta 3}^{(N)} - N\, b_{\gamma\alpha} b_\beta^\nu V_\nu^{(N)}] = 0. \tag{31}$$

The problem is ow to express the last displacement term in (31) in terms of shell-strain measures. For this purpose (29) specialized for $p = N$ will be substituted into (31). Multiplying the result by $\varepsilon_{\alpha\omega}$ and using the identity

$$\delta_{\alpha\omega}^{\gamma\beta} b_\gamma^\alpha b_\beta^\nu = \delta_{\alpha\beta}^{\gamma\nu} b_\gamma^\alpha b_\omega^\beta \qquad (\delta_{\alpha\omega}^{\gamma\beta} \equiv \varepsilon^{\gamma\beta}\varepsilon_{\alpha\omega}) \tag{32}$$

yields another C. E. The result is:

$$\varepsilon^{\beta\gamma}\left\{\varepsilon^{\pi\alpha}(\gamma_{\alpha\beta}^{(N)}|_\gamma - b_{\gamma\alpha}\gamma_{\beta 3}^{(N)}) + \varepsilon^{\rho\alpha} b_{\rho\pi} \sum_{n=0}^{N-1} [(b^n)_\beta^\nu [\gamma_{\alpha\nu}^{(N-n-1)}|_\gamma - b_{\gamma\alpha}\gamma_{\nu 3}^{(N-n-1)}]\right\} = 0. \tag{33}$$

We now consider the remaining equation, (23_2). This may easily be expressed in terms of the nth-order linearized shell-strain measures as:

$$\varepsilon^{\beta\gamma}[(n+1)\gamma_{\gamma\beta}^{(n+1)} + (1-n) b_\gamma^\alpha \gamma_{\alpha\beta}^{(n)} + \gamma_{\beta 3}^{(n)}|_\gamma] = 0 \qquad (n = \overline{0, N-1}), \tag{34}$$

while for $n = N$ it reduces to:

$$\varepsilon^{\beta\gamma}[(1-N) b_\gamma^\alpha \gamma_{\alpha\beta}^{(N)} + \gamma_{\beta 3}^{(N)}|_\gamma] = 0. \tag{35}$$

It is easily seen that for $N = 1$ (i.e. in the framework of both L. K. and T. S. models), (35) reduces to trivial identities. Consequently, in these cases (35) will be suppressed. Eq. (30), (32)–(35) (in number $(2N + 3)$), expressed in terms of nth-order strain measures, are the exact C. E. pertinent to the high-order linear theory of shells.

Some specifications emerging from the non-fulfilment of the equality relation $R = S$, on which the C. E. derived above are based, are now in order. In this respect it is worth stressing that the C. E. derived for $R = S$ also subsist when $R \neq S$. However, in this last instance some precautions are to be taken. Thus, when $R > S$, then $R = N$ will be considered and in the summation process in the C. E. only the appropriate non-vanishing strain terms are retained. Conversely, when $S > R$, then $S = N$ will be considered and further the procedure mentioned above will be introduced.

6. Special cases. The results derived in the previous sections will now be specialized for two known cases, i.e. for the L. K. and T. S. T. models. In the first case, making use of (16), the general derived C. E. reduce to:

$$\begin{aligned} &\varepsilon^{\alpha\delta}\varepsilon^{\beta\gamma}[\gamma_{\alpha\beta}^{(0)}|_{\gamma\delta} + b_{\gamma\alpha}\gamma_{\beta\delta}^{(1)}] = 0, \\ &\varepsilon^{\beta\gamma}[\varepsilon^{\pi\alpha}\gamma_{\alpha\beta}^{(1)}|_\gamma + \varepsilon^{\rho\alpha} b_\rho^\pi \gamma_{\alpha\beta}^{(0)}|_\gamma] = 0, \\ &\varepsilon^{\beta\gamma}[\gamma_{\gamma\beta}^{(1)} + b_\gamma^\alpha \gamma_{\alpha\beta}^{(0)}] = 0. \end{aligned} \tag{36}$$

These equations are identical to those obtained in [21] and are equivalent—see e.g. [20, Chapter II] - to those derived in [26].

Specializing the general C. E. for the T. S. T. model, consistent with (18) and (19) one obtains:

$$\begin{aligned}&\varepsilon^{\alpha\delta}\varepsilon^{\beta\gamma}[\gamma^{(0)}_{\alpha\beta\,|\,\gamma\delta} - \gamma^{(0)}_{\beta 3\,|\,\delta} b_{\gamma\alpha} + b_{\gamma\alpha}(\gamma^{(1)}_{\beta\delta} + b_{\beta\delta}\gamma^{(0)}_{33})] = 0,\\ &\varepsilon^{\beta\gamma}\{\varepsilon^{\pi\alpha}(\gamma^{(1)}_{\alpha\beta\,|\,\gamma} - b_{\gamma\alpha}\gamma^{(1)}_{\beta 3}) + \varepsilon^{\rho\alpha} b^{\pi}_{\rho}(\gamma^{(0)}_{\alpha\beta\,|\,\gamma} - b_{\gamma\alpha}\gamma^{(0)}_{\beta 3})\} = 0,\\ &\varepsilon^{\beta\gamma}[\gamma^{(0)}_{\beta 3\,|\,\gamma} + \gamma^{(1)}_{\gamma\beta} + \gamma^{(0)}_{\alpha\beta} b^{\alpha}_{\gamma}] = 0.\end{aligned} \tag{37}$$

The above C.E. coincide with the ones derived in an *ad hoc* manner in [16, Part 1]. Later on, consistent with the representation (1) of shifted displacements, another variant of C.E. (also belonging to the T.S.T. model) may be derived. This may result either from (37) (which are to be modified by $(19)_2$) or from the general set of C.E., where the precautions already mentioned (arising from the fact that in this instance $R = 1$, $S = 0$) are to be applied. All these lead to the C.E. as derived in lines of curvature in [27] and in invariant form in [28].

7. Additional remarks. In the foregoing sections of the paper some variants of the nth-order strain measures have been defined. In the same context, the C.E. appropriate to linear, high-order shell theories have been derived. These involve the strain measure set as identified by $\{\Gamma\}$, i.e. the asymmetric strain measures $\gamma^{(p)}_{\alpha\beta}$, as well as the transverse shear $\gamma^{(p)}_{\alpha 3}$ and transverse-normal $\gamma^{(p)}_{33}$ strain measures ($p = \overline{0, N}$). The results include as special cases the corresponding C.E. pertinent to L.K. and T.S.T. theories.

A problem worthy of further study is the appropriate representations of symmetric strain measures which could lead to a reduction in the number of C.E. In the framework of T.S.T. such representations may easily be obtained. A first such new representation may be obtained by modifying $\gamma^{(1)}_{\alpha\beta}$ appearing in the set $\{\tilde{\Gamma}\} \equiv \{\gamma^{(0)}_{\alpha\beta}, \gamma^{(1)}_{\alpha\beta}, \gamma^{(n)}_{\alpha 3}, \gamma^{(0)}_{33}\}$ $(n = 0, 1)$. In this sense, by defining the modified counterpart of $\gamma^{(1)}_{\alpha\beta}$ as:

$$\boldsymbol{\gamma}^{(1)}_{\gamma\beta} = \gamma^{(1)}_{\gamma\beta} - b^{\alpha}_{\beta}\gamma^{(0)}_{\alpha\gamma} + \gamma^{(0)}_{\beta 3\,|\,\gamma}, \tag{38}$$

it results from (37_3) that the antisymmetric part of (38), i.e. $\boldsymbol{\gamma}^{(1)}_{[\alpha\beta]}$, is identically zero, from which the symmetry $\boldsymbol{\gamma}^{(1)}_{\beta\gamma} = \boldsymbol{\gamma}^{(1)}_{\gamma\beta}$ simply follows. As a result, a symmetric representation corresponding to $\gamma^{(1)}_{\gamma\beta}$ may be chosen as:

$$\boldsymbol{\gamma}^{(1)}_{\gamma\beta} = \boldsymbol{\gamma}^{(1)}_{\beta\gamma} = \tfrac{1}{2}(\gamma^{(1)}_{\beta\gamma} + \gamma^{(1)}_{\gamma\beta}) - \tfrac{1}{2}(b^{\alpha}_{\beta}\gamma^{(0)}_{\alpha\gamma} + b^{\alpha}_{\gamma}\gamma^{(0)}_{\alpha\beta}) + \tfrac{1}{2}(\gamma^{(0)}_{\beta 3\,|\,\gamma} + \gamma^{(0)}_{\gamma 3\,|\,\beta}). \tag{39}$$

$\boldsymbol{\gamma}^{(1)}_{\alpha\beta}$ considered in conjunction with $\gamma^{(0)}_{(\alpha\beta)}$, $\gamma^{(0)}_{\alpha 3}$, $\gamma^{(1)}_{\alpha 3}$, $\gamma^{(0)}_{33}$ may constitute a first new set of shell-strain measures appropriate to T.S.T. model. It appears evident that Eq. (37_3) expressed in terms of the strain measures as modified above becomes a trivial identity which may consequently be suppressed.

Let us consider now the two symmetric strain measure sets $\{\tilde{\Gamma}\}_{\text{Mod}} \equiv \{\gamma^{(0)}_{(\alpha\beta)}, \boldsymbol{\gamma}^{(1)}_{\alpha\beta},; \boldsymbol{\gamma}^{(n)}_{\alpha 3}, \gamma^{(0)}_{33}\}$ and $\{\tilde{E}\} \equiv \{e^{(n)}_{\alpha\beta}, e^{(n)}_{\alpha 3}, e^{(0)}_{33}\}$ $(n = 0, 1)$, both of which are appropriate to T.S.T. From their comparison it simply emerges that: (i) the difference between them occurs in the expressions of $\boldsymbol{\gamma}^{(1)}_{\alpha\beta}$, and $e^{(1)}_{\alpha\beta}$ only; (ii) in terms of the former representation only, Eq. $(37)_3$ becomes a trivial identity which is to be suppressed, and (iii) in the framework of the classical L.K. theory the two variants reduce to a single one, i.e. to the set $\{\tilde{\tilde{E}}\} \equiv \{e^{(0)}_{\alpha\beta}, e^{(1)}_{\alpha\beta}\}$, in terms of which the property of the reduction in the number of C.E. still maintains. As a matter of fact, it is worth remarking that $\{\tilde{E}\}$ has been obtained first in [23], just by requiring the identical fulfillment of (36).

Using a development similar to that performed in the L.K. theory (see [23, 24]), another representation of the strain measures allowing the reduction in the number of C.E. may be obtained. For this purpose and according to this procedure, $\gamma_{\alpha\beta}^{(0)}$ appearing in (39) is decomposed into its unique symmetric and antisymmetric parts according to $\gamma_{\alpha\beta}^{(0)} = \gamma_{(\alpha\beta)}^{(0)} + \gamma_{[\alpha\beta]}^{(0)}$, in which

$$\gamma_{(\alpha\beta)}^{(0)} = \frac{1}{2}(\gamma_{\alpha\beta}^{(0)} + \gamma_{\beta\alpha}^{(0)}); \qquad \gamma_{[\alpha\beta]}^{(0)} = \frac{1}{2}\,\gamma_{\alpha\beta}^{(0)} - \gamma_{\beta\alpha}^{(0)}). \tag{40}$$

This yields the symmetric representation

$$\hat{\tilde{\gamma}}_{\beta\gamma}^{(1)} = \hat{\tilde{\gamma}}_{\gamma\beta}^{(1)} = \boldsymbol{\gamma}_{\beta\gamma}^{(1)} + \frac{1}{2}(b_\gamma^\alpha \gamma_{\alpha\beta}^{(0)} + b_\beta^\alpha \gamma_{(\alpha\gamma)}^{(0)}) - \frac{1}{2}(\gamma_{\beta 3\,|\,\gamma}^{(0)} + \gamma_{\gamma 3\,|\,\beta}^{(0)})$$

$$= \gamma_{(\beta\gamma)}^{(1)} - \frac{1}{2}(b_\gamma^\alpha \gamma_{[\alpha\beta]}^{(0)} + b_\beta^\alpha \gamma_{[\alpha\gamma]}^{(0)}).$$

$\{\hat{\tilde{\Gamma}}\}_{\text{Mod}} \equiv \{\hat{\tilde{\gamma}}_{\alpha\beta}^{(1)}, \gamma_{(\alpha\beta)}^{(0)}, \gamma_{\alpha 3}^{(0)}, \gamma_{\alpha 3}^{(1)}, \gamma_{33}^{(0)}\}$ may be selected as a new set of strain measures in the T.S.T. A comparison of the two sets of strain measures $\{\tilde{\Gamma}\}_{\text{Mod}}$ and $\{\hat{\tilde{\Gamma}}\}_{\text{Mod}}$ reveals that (i) the difference occurs in the expressions of $\boldsymbol{\gamma}_{\alpha\beta}^{(1)}$ and $\hat{\tilde{\gamma}}_{\alpha\beta}^{(1)}$ only; (ii) both of them fulfill identically Eq. (37_3) (which is to be suppressed when such strain representations are used); (iii) as per L.K.T. the modified strain variants $\{\tilde{\Gamma}\}_{\text{Mod}}$ and $\{\hat{\tilde{\Gamma}}\}_{\text{Mod}}$ reduce to those referred to as Nagdhi's [23, 24] and Koiter-Sanders-Budiansky's (see [29, 31]) strain representations, respectively.

It is to be mentioned, in addition, that the strain set $\{\hat{\tilde{\Gamma}}\}_{\text{Mod}}$ agrees with the one derived in a different manner in [32] (for the case when $\gamma_{33}^{(0)} = 0$. The formulation of similar symmetric strain representations in the more general framework of H.O.S.T. still remains an open problem, which nevertheless merits further work.

It is worth remarking, also, that the results previously derived are founded upon the concept of the expansion of all the field variables in *power series* across the shell thickness. However, another expansion procedure in terms of *Legendre polynomials* (L.P.) has been also successfully employed in the theory of shells (see e.g. [33, 34]). It is instructive to compare the expressions of the nth-order strain measures in H.O.S.T. as resulting from the two above-mentioned approaches. To do this, these kinematical relations will be briefly deduced by using the latter expansion approach. Let us represent the shifted components $\bar{V}_i(x^\omega, x^3)$ of the displacement vector as:

$$\bar{V}_i(x^\omega, x^3) = \sum_{n=0}^{N} h^n v_i^{(n)}(x^\omega) P_n(\bar{x}_3) \tag{41}$$

where $P_n(\bar{x}_3)$ are the Legendre polynomials, $\bar{x}_3 \equiv x_3/h$ is the reduced transverse coordinate, $2h$ denotes the shell thickness, and N is a natural number denoting the level of truncation of the Legendre series.

It is easily inferred that, consistent with (41), the strain tensor e_{ij} may be expressed as:

$$e_{ij}(x^\omega, x^3) = \sum_{n=0}^{N} h^n e_{ij}^{(n)}(x^\omega)\, P_n(\bar{x}_3). \tag{42}$$

Making use of the orthoganality property of Legendre polynomials as expressed by $\int_{-1}^{1} P_m(\bar{x}_3)\, P_n(\bar{x}_3)\, d\bar{x}_3 = 2\delta_{mn}/(2n+1)$, (42) may be inverted to give:

$$e_{ij}^{(m)} = (m + 1/2)h^{m+1} \int_{-1}^{1} e_{ij} P_m(\bar{x}_3) d\bar{x}_3 . \tag{43}$$

Employment in (43) of (12), (11), (9), (4), (41) and of the recurrence formulae (see e.g. [35]):

$$P'_K(\bar{x}_3) = (2K-1)P_{K-1}(\bar{x}_3) + (2K-5)P_{K-3}(\bar{x}_3) + \dots,$$

$$\bar{x}_3 P'_K(\bar{x}_3) = KP_K(\bar{x}_3) + (2K-3)P_{K-2}(\bar{x}_3) + (2K-7)P_{K-4}(\bar{x}_3) + \dots,$$

$$\bar{x}_3 P_K(\bar{x}_3) = (2K+1)^{-1}((K+1)P_{K+1}(\bar{x}_3) + KP_{K-1}(\bar{x}_3)), \qquad (P'_K \equiv dP_K/d\bar{x}_3)$$

yields the kinematical equations correlating the nth-order shell-strain measures $e_{ij}^{(n)}$ to the *displacement components* $v_i^{(n)}$. These are given by:

$$2e_{\alpha\beta}^{(n)} = h^{2n+1}\left[\gamma_{\alpha\beta}^{(n)} + \gamma_{\beta\alpha}^{(n)} - \frac{n}{2n-1}(b_\alpha^\mu \gamma_{\mu\beta}^{(n-1)} + b_\beta^\mu \gamma_{\mu\alpha}^{(n-1)}) \underline{- h^2 \frac{n+1}{2n+3}(b_\alpha^\mu \gamma_{\mu\beta}^{(n+1)} + b_\beta^\mu \gamma_{\mu\alpha}^{(n+1)})}\right], \tag{44}$$

$$2e_{\alpha 3}^{(n)} = h^{2n+1}\left[(v_{3,\alpha}^{(n)} + b_\alpha^\lambda v_\lambda^{(n)} - nb_\alpha^\lambda v_\lambda^{(n)} + (2n+1)v_\alpha^{(n+1)} \underline{+ (2n+1)\Sigma_{K=1,2,\dots} v_\alpha^{(n+1+2K)} h^{2K}} - (2n+1)b_\alpha^\lambda \sum_{K=1,2} v_\lambda^{(n+2K)} h^{2K}\right] \quad (= \gamma_{\alpha 3}^{(n)}),$$

$$e_{33}^{(n)} = (2n+1)h^{2n+1}\left[v_3^{(n+1)} + \sum_{K=1,2} h^{2K} v_3^{(n+2K+1)}\right] \quad (= \gamma_{33}^{(n)},$$

where

$$\gamma_{\alpha\beta}^{(n)} = v_{\alpha|\beta}^{(n)} - b_{\alpha\beta} v^{(n)}{}_3 \qquad (n = \overline{0, N}).$$

Comparison of (44) with (21) and (14) allows us to conclude that, with the exception in (44) of some scale factors and of the underlined terms, the corresponding expressions for the nth-order strain measures are formally similar one another. Moreover, this formal similarity entails the conclusion that the general form of compatibility equations as deduced in Sec. 5, in the framework of the former approach, will be preserved in the latter instance too. Nevertheless, the effective derivation of C.E. in this last context may be performed without any difficulty by paralleling the developments in Sec. 5.

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