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Abstract

We consider a mirror symmetry between invertible weighted homogeneous polynomials in three variables. We define Dolgachev and Gabrielov numbers for them and show that we get a duality between these polynomials generalizing Arnold's strange duality between the 14 exceptional unimodal singularities.

Introduction

Mirror symmetry is now understood as a categorical duality between algebraic geometry and symplectic geometry. One of our motivations is to apply some ideas of mirror symmetry to singularity theory in order to understand various mysterious correspondences among isolated singularities, root systems, Weyl groups, Lie algebras, discrete groups, finite-dimensional algebras and so on. In this paper, we shall generalize Arnold's strange duality for the 14 exceptional unimodal singularities to a specific class of weighted homogeneous polynomials in three variables called invertible polynomials.

Let $f(x, y, z)$ be a polynomial which has an isolated singularity only at the origin $0 \in \mathbb{C}^3$. A distinguished basis of vanishing (graded) Lagrangian submanifolds in the Milnor fiber of f can be categorified to an A_∞-category $\mathrm{Fuk}^{\rightarrow}(f)$ called the directed Fukaya category whose derived category $D^b\mathrm{Fuk}^{\rightarrow}(f)$ is, as a triangulated category, an invariant of the polynomial f. Note that the triangulated category $D^b\mathrm{Fuk}^{\rightarrow}(f)$ has a full exceptional collection.

If $f(x, y, z)$ is a weighted homogeneous polynomial, then one can consider another interesting triangulated category, the category of a maximally graded singularity $D^{L_f}_{\mathrm{Sg}}(R_f)$:

$$D^{L_f}_{\mathrm{Sg}}(R_f) := D^b(\mathrm{gr}^{L_f}\text{-}R_f)/D^b(\mathrm{proj}^{L_f}\text{-}R_f), \tag{0.1}$$

where $R_f := \mathbb{C}[x, y, z]/(f)$ and L_f is the maximal grading of f (see § 1). This category $D^{L_f}_{\mathrm{Sg}}(R_f)$ is considered as an analogue of the bounded derived category of coherent sheaves on a smooth proper algebraic variety.

It is known that the Berglund–Hübsch transpose [BH93] for some polynomials with nice properties induces the topological mirror symmetry which gives the systematic construction of mirror pairs of Calabi–Yau manifolds. Therefore, we may expect that the topological mirror symmetry can also be categorified to the following.

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Conjecture [Tak09, Tak10]. Let $f(x, y, z)$ be an invertible polynomial (see § 1 for the definition).

(i) There should exist a quiver with relations (Q, I) and triangulated equivalences

$$D^{L_f}_{\mathrm{Sg}}(R_f) \simeq D^b(\text{mod-}\mathbb{C}Q/I) \simeq D^b\mathrm{Fuk}^{\to}(f^t). \tag{0.2}$$

(ii) There should exist a quiver with relations (Q', I') and triangulated equivalences

$$D^b\mathrm{coh}(\mathcal{C}_{G_f}) \simeq D^b(\text{mod-}\mathbb{C}Q'/I') \simeq D^b\mathrm{Fuk}^{\to}(T_{\gamma_1,\gamma_2,\gamma_3}), \tag{0.3}$$

which should be compatible with the triangulated equivalence (0.2), where $\mathcal{C}_{G_f}$ is the weighted projective line associated to the maximal abelian symmetry group G_f of f and $T_{\gamma_1,\gamma_2,\gamma_3}$ is a 'cusp singularity' (see § 3).

There is much evidence of the above conjectures, which follows from related results by several authors. Among them, the most important one in this paper is that one should be able to choose as (Q', I') the quiver obtained by the graph $T(\gamma_1, \gamma_2, \gamma_3)$ in § 3 with a suitable orientation together with two relations along the dotted edges. This leads us to our main theorem, the strange duality for invertible polynomials.

Theorem 13. *Let $f(x, y, z)$ be an invertible polynomial. The Dolgachev numbers $(\alpha_1, \alpha_2, \alpha_3)$ for G_f, the orders of isotropy of the weighted projective line $\mathcal{C}_{G_f}$, coincide with the Gabrielov numbers $(\gamma_1, \gamma_2, \gamma_3)$ for f^t, the index of the 'cusp singularity' $T_{\gamma_1,\gamma_2,\gamma_3}$ associated to $f^t(x, y, z) - xyz$.*

We give here an outline of the paper. Section 1 introduces the definition of invertible polynomials and their maximal abelian symmetry groups. We also recall the Berglund–Hübsch transpose of invertible polynomials, which plays an essential role in this paper. In § 2, we first give the classification of invertible polynomials in three variables. Most of the results in this paper rely on this classification of invertible polynomials and several data given explicitly by them. The main purpose of this section is to define the Dolgachev numbers. We associate to each pair of an invertible polynomial f and its maximal abelian symmetry group G_f a quotient stack $\mathcal{C}_{G_f}$. We show in both a categorical way and a geometrical way that this quotient stack is a weighted projective line with three isotropic points of orders $\alpha_1, \alpha_2, \alpha_3$. The numbers $A_{G_f} := (\alpha_1, \alpha_2, \alpha_3)$ are our Dolgachev numbers. In § 3, we associate to an invertible polynomial $f(x, y, z)$ the Gabrielov numbers $\Gamma_f := (\gamma_1, \gamma_2, \gamma_3)$ by the 'cusp singularity' $T_{\gamma_1,\gamma_2,\gamma_3}$ obtained as the deformation of the polynomial $f(x, y, z) - xyz$. Note that the triple of Gabrielov numbers is not an invariant of the singularity defined by f but an invariant of the polynomial f.

Section 4 gives the main theorem of this paper, a generalization of Arnold's strange duality between the 14 exceptional unimodal singularities. We show that $A_{G_f} = \Gamma_{f^t}$ and $A_{G_{f^t}} = \Gamma_f$ for all invertible polynomials $f(x, y, z)$. This means that strange duality is one aspect of a mirror symmetry among the isolated hypersurface singularities with good group actions. Therefore, it is now a 'charm' duality and no more a 'strange' duality.

In § 5, we collect some additional features of the duality. We show the coincidence of certain invariants for dual invertible polynomials. An additional feature of Arnold's strange duality is a duality between the characteristic polynomials of the Milnor monodromy discovered by Saito. Moreover, the first author observed a relation of these polynomials with the Poincaré series of the coordinate rings. We discuss to which extent these facts continue to hold for our duality.

In § 6, we show how our results fit into the classification of singularities. We recover Arnold's strange duality between the 14 exceptional unimodal singularities. We obtain a new

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strange duality embracing the 14 exceptional bimodal singularities and some other ones. Note that this duality depends on the chosen invertible polynomials for the 14 exceptional bimodal singularities. In [KPABR10], a slightly different version of a duality for these singularities is considered. Finally, we discuss how our Gabrielov numbers are related to Coxeter–Dynkin diagrams of the singularities.

1. Invertible polynomials

Let $f(x_1, \ldots, x_n)$ be a weighted homogeneous complex polynomial. This means that there are positive integers $w_1, \ldots, w_n$ and d such that $f(\lambda^{w_1} x_1, \ldots, \lambda^{w_n} x_n) = \lambda^d f(x_1, \ldots, x_n)$ for $\lambda \in \mathbb{C}^*$. We call $(w_1, \ldots, w_n; d)$ a system of *weights*. If $\gcd(w_1, \ldots, w_n, d) = 1$, then a system of weights is called *reduced*. A system of weights which is not reduced is called *non-reduced*. We shall also consider non-reduced systems of weights in this paper.

Definition. A weighted homogeneous polynomial $f(x_1, \ldots, x_n)$ is called *invertible* if the following conditions are satisfied:

(i) the number of variables ($=n$) coincides with the number of monomials in the polynomial $f(x_1, \ldots, x_n)$, namely,

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} a_i \prod_{j=1}^{n} x_j^{E_{ij}}$$

for some coefficients $a_i \in \mathbb{C}^*$ and non-negative integers E_{ij} for $i, j = 1, \ldots, n$;

(ii) a system of weights $(w_1, \ldots, w_n; d)$ can be uniquely determined by the polynomial $f(x_1, \ldots, x_n)$ up to a constant factor $\gcd(w_1, \ldots, w_n; d)$, namely, the matrix $E := (E_{ij})$ is invertible over $\mathbb{Q}$;

(iii) $f(x_1, \ldots, x_n)$ and the *Berglund–Hübsch transpose* $f^t(x_1, \ldots, x_n)$ of $f(x_1, \ldots, x_n)$ defined by

$$f^t(x_1, \ldots, x_n) := \sum_{i=1}^{n} a_i \prod_{j=1}^{n} x_j^{E_{ji}}$$

have singularities only at the origin $0 \in \mathbb{C}^n$ which are isolated. Equivalently, the *Jacobian rings* $\mathrm{Jac}(f)$ of f and $\mathrm{Jac}(f^t)$ of f^t defined by

$$\mathrm{Jac}(f) := \mathbb{C}[x_1, \ldots, x_n] \Big/ \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \right),$$
$$\mathrm{Jac}(f^t) := \mathbb{C}[x_1, \ldots, x_n] \Big/ \left(\frac{\partial f^t}{\partial x_1}, \ldots, \frac{\partial f^t}{\partial x_n} \right)$$

are both finite-dimensional algebras over $\mathbb{C}$ and $\dim_{\mathbb{C}} \mathrm{Jac}(f), \dim_{\mathbb{C}} \mathrm{Jac}(f^t) \geqslant 1$.

Polynomials with these properties have extensively been studied for a long time since the early stage of the mirror symmetry. They are applied to give a lot of topological mirror pairs of Calabi–Yau manifolds. The name *invertible polynomial* was introduced by Kreuzer [Kre94].

Definition. Let $f(x_1, \ldots, x_n) = \sum_{i=1}^{n} a_i \prod_{j=1}^{n} x_j^{E_{ij}}$ be an invertible polynomial. The *canonical system of weights* W_f is the system of weights $(w_1, \ldots, w_n; d)$ given by the unique solution of

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the equation

$$E\begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} = \det(E)\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \quad d := \det(E).$$

Remark 1. It follows from Cramer's rule that $w_1, \ldots, w_n$ are positive integers. Note that the canonical system of weights is in general non-reduced.

Definition. Let $f(x_1, \ldots, x_n)$ be an invertible polynomial and $W_f = (w_1, \ldots, w_n; d)$ the canonical system of weights attached to f. Define

$$c_f := \gcd(w_1, \ldots, w_n, d).$$

Definition. Let $f(x_1, \ldots, x_n) = \sum_{i=1}^n a_i \prod_{j=1}^n x_j^{E_{ij}}$ be an invertible polynomial. Consider the free abelian group $\bigoplus_{i=1}^n \mathbb{Z}\vec{x}_i \oplus \mathbb{Z}\vec{f}$ generated by the symbols $\vec{x}_i$ for the variables x_i for $i = 1, \ldots, n$ and the symbol $\vec{f}$ for the polynomial f. The *maximal grading* L_f of the invertible polynomial f is the abelian group defined by the quotient

$$L_f := \bigoplus_{i=1}^n \mathbb{Z}\vec{x}_i \oplus \mathbb{Z}\vec{f}/I_f,$$

where I_f is the subgroup generated by the elements

$$\vec{f} - \sum_{j=1}^n E_{ij}\vec{x}_j, \quad i = 1, \ldots, n.$$

Note that L_f is an abelian group of rank one which is not necessarily free.

Definition. Let $f(x_1, \ldots, x_n)$ be an invertible polynomial and L_f the maximal grading of f. The *maximal abelian symmetry group* G_f of f is the abelian group defined by

$$G_f := \mathrm{Spec}(\mathbb{C}L_f),$$

where $\mathbb{C}L_f$ denotes the group ring of L_f. Equivalently,

$$G_f = \left\{ (\lambda_1, \ldots, \lambda_n) \in (\mathbb{C}^*)^n \,\middle|\, \prod_{j=1}^n \lambda_j^{E_{1j}} = \cdots = \prod_{j=1}^n \lambda_j^{E_{nj}} \right\}.$$

Note that the polynomial f is homogeneous with respect to the natural action of G_f on the variables. Namely, we have

$$f(\lambda_1 x_1, \ldots, \lambda_n x_n) = \lambda f(x_1, \ldots, x_n)$$

for $(\lambda_1, \ldots, \lambda_n) \in G_f$, where $\lambda := \prod_{j=1}^n \lambda_j^{E_{1j}} = \cdots = \prod_{j=1}^n \lambda_j^{E_{nj}}$.

2. Dolgachev numbers for pairs (f, G_f)

In this paper, we shall only consider invertible polynomials in three variables. We have the following classification result (see [AGV85, 13.2]).

Proposition 2. *Let $f(x, y, z)$ be an invertible polynomial in three variables. Then, by a suitable rescaling of variables, f becomes one of the five types in Table 1.*

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TABLE 1. Invertible polynomials in three variables.

Type	Class	f	f^t
I	I	$x^{p_1}+y^{p_2}+z^{p_3}$ $(p_1,p_2,p_3\in\mathbb{Z}_{\geqslant 2})$	$x^{p_1}+y^{p_2}+z^{p_3}$ $(p_1,p_2,p_3\in\mathbb{Z}_{\geqslant 2})$
II	II	$x^{p_1}+y^{p_2}+yz^{p_3/p_2}$ $\left(p_1,p_2,\dfrac{p_3}{p_2}\in\mathbb{Z}_{\geqslant 2}\right)$	$x^{p_1}+y^{p_2}z+z^{p_3/p_2}$ $\left(p_1,p_2,\dfrac{p_3}{p_2}\in\mathbb{Z}_{\geqslant 2}\right)$
III	IV	$x^{p_1}+zy^{q_2+1}+yz^{q_3+1}$ $(p_1\in\mathbb{Z}_{\geqslant 2},\ q_2,q_3\in\mathbb{Z}_{\geqslant 1})$	$x^{p_1}+zy^{q_2+1}+yz^{q_3+1}$ $(p_1\in\mathbb{Z}_{\geqslant 2},\ q_2,q_3\in\mathbb{Z}_{\geqslant 1})$
IV	V	$x^{p_1}+xy^{p_2/p_1}+yz^{p_3/p_2}$ $\left(p_1,\dfrac{p_3}{p_2}\in\mathbb{Z}_{\geqslant 2},\dfrac{p_2}{p_1}\in\mathbb{Z}_{\geqslant 1}\right)$	$x^{p_1}y+y^{p_2/p_1}z+z^{p_3/p_2}$ $\left(p_1,\dfrac{p_3}{p_2}\in\mathbb{Z}_{\geqslant 2},\dfrac{p_2}{p_1}\in\mathbb{Z}_{\geqslant 1}\right)$
V	VII	$x^{q_1}y+y^{q_2}z+z^{q_3}x$ $(q_1,q_2,q_3\in\mathbb{Z}_{\geqslant 1})$	$zx^{q_1}+xy^{q_2}+yz^{q_3}$ $(q_1,q_2,q_3\in\mathbb{Z}_{\geqslant 1})$

TABLE 2. Dolgachev numbers for pairs (f,G_f).

Type	$f(x,y,z)$	$(\alpha_1,\alpha_2,\alpha_3)$
I	$x^{p_1}+y^{p_2}+z^{p_3}$	(p_1,p_2,p_3)
II	$x^{p_1}+y^{p_2}+yz^{p_3/p_2}$	$\left(p_1,\dfrac{p_3}{p_2},(p_2-1)p_1\right)$
III	$x^{p_1}+zy^{q_2+1}+yz^{q_3+1}$	(p_1,p_1q_2,p_1q_3)
IV	$x^{p_1}+xy^{p_2/p_1}+yz^{p_3/p_2}$	$\left(\dfrac{p_3}{p_2},(p_1-1)\dfrac{p_3}{p_2},p_2-p_1+1\right)$
V	$x^{q_1}y+y^{q_2}z+z^{q_3}x$	$(q_2q_3-q_3+1,q_3q_1-q_1+1,q_1q_2-q_2+1)$

In Table 1, we follow the notation in [Sai98]. Note that the classes in [AGV85] differ from our types; the equivalence is given in Table 1.

We can naturally associate to an invertible polynomial $f(x,y,z)$ the following quotient stack:

$$\mathcal{C}_{G_f}:=[f^{-1}(0)\backslash\{0\}/G_f]. \tag{2.1}$$

Since f has an isolated singularity only at the origin $0\in\mathbb{C}^3$ and G_f is an extension of a one-dimensional torus $\mathbb{C}^*$ by a finite abelian group, the stack $\mathcal{C}_{G_f}$ is a Deligne–Mumford stack and may be regarded as a smooth projective curve with a finite number of isotropic points on it. Moreover, we have the following.

THEOREM 3. *Let $f(x,y,z)$ be an invertible polynomial. The quotient stack $\mathcal{C}_{G_f}$ is a smooth projective line $\mathbb{P}^1$ with at most three isotropic points of orders $\alpha_1,\alpha_2,\alpha_3$ given in Table 2, where the number of isotropic points is the number of i with $\alpha_i\geqslant 2$.*

DEFINITION. The numbers $(\alpha_1,\alpha_2,\alpha_3)$ in Theorem 3 are called the *Dolgachev numbers* of the pair (f,G_f) and the tuple is denoted by A_{G_f}.

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Table 3. The images of the generators in $R_{A_{G_f}}$.

Type	$f(x,y,z)$	x	y	z
I	$x^{p_1}+y^{p_2}+z^{p_3}$	X_1	X_2	X_3
II	$x^{p_1}+y^{p_2}+yz^{p_3/p_2}$	X_1X_3	$X_3^{p_1}$	X_2
III	$x^{p_1}+zy^{q_2+1}+yz^{q_3+1}$	$X_1X_2X_3$	$X_2^{p_1}$	$X_3^{p_1}$
IV	$x^{p_1}+xy^{p_2/p_1}+yz^{p_3/p_2}$	$X_2^{p_3/p_2}X_3$	$X_3^{p_1}$	X_1X_2
V	$x^{q_1}y+y^{q_2}z+z^{q_3}x$	$X_2X_3^{q_2}$	$X_3X_1^{q_3}$	$X_1X_2^{q_1}$

Proof of Theorem 3. There are two ways to prove the statement. One is categorical and the other is geometrical.

First, we give a proof by the use of abelian categories of coherent sheaves, which is already announced in some places (see [Tak09, Proposition 30] for example). It is almost the same proof given in [Tak10, Theorem 5.1], where the case $L_f \simeq \mathbb{Z}$ is considered. Following Geigle–Lenzing [GL87], to a tuple of numbers $A_{G_f}=(\alpha_1,\alpha_2,\alpha_3)$ one can associate the ring

$$R_{A_{G_f}} := \mathbb{C}[X_1,X_2,X_3]/(X_1^{\alpha_1}+X_2^{\alpha_2}+X_3^{\alpha_3}).$$

Since $R_{A_{G_f}}$ is graded with respect to an abelian group

$$L_{A_{G_f}} := \bigoplus_{i=1}^{3}\mathbb{Z}\vec{X}_i/(\alpha_i\vec{X}_i-\alpha_j\vec{X}_j; 1\leqslant i<j\leqslant 3),$$

one can consider the quotient stack

$$\mathcal{C}_{A_{G_f}} := [\operatorname{Spec}(R_{A_{G_f}})\backslash\{0\}/\operatorname{Spec}(\mathbb{C}L_{A_{G_f}})].$$

The quotient stack $\mathcal{C}_{A_{G_f}}$ is a Deligne–Mumford stack which may also be regarded as a smooth projective line $\mathbb{P}^1$ with at most three isotropic points of orders $\alpha_1,\alpha_2,\alpha_3$. It is easy to see this since $R_{A_{G_f}}$ contains the ring $\mathbb{C}[X_1^{\alpha_1},X_2^{\alpha_2}]$ as a subring,

Now, the statement of Theorem 3 follows from the following (see also [Tak10, Theorem 5.1]).

Proposition 4. *The L_f-graded ring $R_f := \mathbb{C}[x,y,z]/(f)$ can be naturally embedded into the $L_{A_{G_f}}$-graded ring $R_{A_{G_f}}$. This embedding induces an equivalence of abelian categories:*

$$\operatorname{mod}^{L_f}\text{-}R_f/\operatorname{tor}^{L_f}\text{-}R_f \simeq \operatorname{mod}^{L_{A_{G_f}}}\text{-}R_{A_{G_f}}/\operatorname{tor}^{L_{A_{G_f}}}\text{-}R_{A_{G_f}}. \tag{2.2}$$

In other words, there is an equivalence of abelian categories:

$$\operatorname{coh}(\mathcal{C}_{G_f}) \simeq \operatorname{coh}(\mathcal{C}_{A_{G_f}}). \tag{2.3}$$

Proof. The proof is the same as the one in [GL87, Tak10] except for the definition of the map $R_f \hookrightarrow R_{A_{G_f}}$ given by Table 3. □

Next, we give another proof which is more geometric.

Lemma 5. *The genus of the underlying smooth projective curve C_{G_f} of the stack $\mathcal{C}_{G_f}$ is zero.*

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Proof. We shall calculate the dimension of the space of holomorphic 1-forms on the curve C_{G_f}. Recall that any holomorphic 1-form on the curve C_{G_f} is of the following form:

$$\omega(g) := g(x, y, z)\frac{w_1 x\, dy \wedge dz - w_2 y\, dx \wedge dz + w_3 z\, dx \wedge dy}{df},$$

where $g(x, y, z)$ is a weighted homogeneous representative of an element in the Jacobian ring $\mathrm{Jac}(f)$. Note also that $\omega(g)$ must be G_f-invariant. By a case by case study based on Table 1, we can show that $g(x, y, z) = 0$ in $\mathrm{Jac}(f)$. □

Remark 6. The above proof is a generalization of the one in [Sai87, Theorem 3].

Lemma 7. *On the underlying smooth projective curve C_{G_f} of the stack $\mathcal{C}_{G_f}$, there exist at most three isotropic points of orders $\alpha_1, \alpha_2, \alpha_3$ given in Table 2.*

Proof. Since each element $(\lambda_1, \lambda_2, \lambda_3) \in G_f$ acts on $\mathbb{C}^3$ diagonally:

$$(x, y, z) \mapsto (\lambda_1 x, \lambda_2 y, \lambda_3 z),$$

any isotropic point must be contained in the subvariety $\{xyz = 0\} \subset C_{G_f}$. By a case by case study based on Table 1, we first see that there are at most three isotropic points. Then, by considering the equation

$$\prod_{j=1}^{3} \lambda_j^{E_{1j}} = \prod_{j=1}^{3} \lambda_j^{E_{2j}} = \prod_{j=1}^{3} \lambda_j^{E_{3j}}$$

at each isotropic point, we can show that the isotropy group is a cyclic group. The triple of orders of these isotropy groups coincides with the one in Table 2. □

One sees that Theorem 3 now follows from the above Lemmas 5 and 7. □

3. Gabrielov numbers for $f(x, y, z)$

Definition. The polynomial

$$x^{\gamma_1} + y^{\gamma_2} + z^{\gamma_3} + axyz \quad \text{for some } a \neq 0$$

is called a polynomial of type $T_{\gamma_1,\gamma_2,\gamma_3}$.

Definition. For a triple (a, b, c) of positive integers, we define

$$\Delta(a, b, c) := abc - bc - ac - ab.$$

Remark 8. If

$$\Delta(\gamma_1, \gamma_2, \gamma_3) > 0,$$

then a polynomial of type $T_{\gamma_1,\gamma_2,\gamma_3}$ defines a cusp singularity of this type. We do not assume this condition here.

Remark 9. A Coxeter–Dynkin diagram of a polynomial of type $T_{\gamma_1,\gamma_2,\gamma_3}$ is the numbered graph encoding an intersection matrix of a distinguished basis of vanishing cycles. For $\Delta(\gamma_1, \gamma_2, \gamma_3) \geqslant 0$, this is obtained from a morsification of the germ at 0. For $\Delta(\gamma_1, \gamma_2, \gamma_3) < 0$, we consider a morsification of the polynomial, i.e. we also take the other singularities outside of 0 into account.

A polynomial of type $T_{\gamma_1,\gamma_2,\gamma_3}$ has a Coxeter–Dynkin diagram of type $T(\gamma_1, \gamma_2, \gamma_3)$ (see Figure 1) (for $\Delta(\gamma_1, \gamma_2, \gamma_3) < 0$, see [Tak09]). It corresponds to the matrix $A = (a_{ij})$ defined

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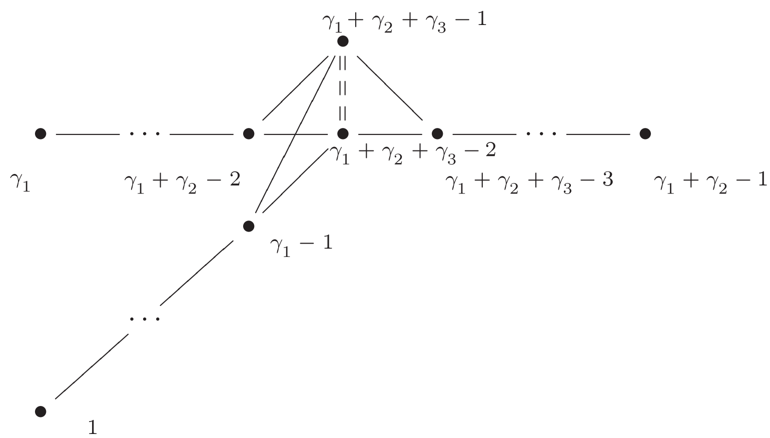

FIGURE 1. The graph $T(\gamma_1, \gamma_2, \gamma_3)$.

by $a_{ii} = -2$, $a_{ij} = 0$ if the vertices $\bullet_i$ and $\bullet_j$ are not connected, and

$$a_{ij} = 1 \Leftrightarrow \bullet_i \text{ —— } \bullet_j, \quad a_{ij} = -2 \Leftrightarrow \bullet_i = = = \bullet_j.$$

The number $(-1)^{\gamma_1+\gamma_2+\gamma_3-2}\Delta(\gamma_1, \gamma_2, \gamma_3)$ is the discriminant of the symmetric bilinear form defined by the matrix A' (i.e. the determinant of A'), where A' is the intersection matrix of the diagram obtained from $T(\gamma_1, \gamma_2, \gamma_3)$ by deleting the last $(\gamma_1 + \gamma_2 + \gamma_3 - 1)$th vertex.

THEOREM 10. *Let $f(x, y, z)$ be an invertible polynomial. We associate to f the numbers $\gamma_1, \gamma_2, \gamma_3$ according to Table 4.*

(i) *If $\Delta(\gamma_1, \gamma_2, \gamma_3) < 0$, then, by a polynomial change of coordinates, the polynomial $f(x, y, z) - xyz$ becomes a deformation of a polynomial of type $T_{\gamma_1,\gamma_2,\gamma_3}$ of the following form:*

$$x^{\gamma_1} + y^{\gamma_2} + z^{\gamma_3} - xyz + \sum_{i=1}^{\gamma_1-1} a_i x^i + \sum_{j=1}^{\gamma_2-1} b_j y^j + \sum_{k=1}^{\gamma_3-1} c_k z^k + c, \quad a_i, b_j, c_k, c \in \mathbb{C}.$$

(ii) *If $\Delta(\gamma_1, \gamma_2, \gamma_3) = 0$, then the polynomial $f(x, y, z) + axyz$ for some $a \in \mathbb{C}^*$ becomes a polynomial of type $T_{\gamma_1,\gamma_2,\gamma_3}$ by a suitable holomorphic change of coordinates at $0 \in \mathbb{C}^3$.*

(iii) *If $\Delta(\gamma_1, \gamma_2, \gamma_3) > 0$, then the polynomial $f(x, y, z) - xyz$ becomes a polynomial of type $T_{\gamma_1,\gamma_2,\gamma_3}$ by a suitable holomorphic change of coordinates at $0 \in \mathbb{C}^3$.*

DEFINITION. The numbers $(\gamma_1, \gamma_2, \gamma_3)$ in Theorem 10 are called the *Gabrielov numbers* of f and the tuple is denoted by Γ_f.

Proof of Theorem 10. (i) The classification of invertible polynomials with $\Delta(\Gamma_f) < 0$ is given in Table 5.

Let $\widetilde{f}(x, y, z) := f(x, y, z) - xyz$. If a monomial of type $x_i x_j^r((x_1, x_2, x_3) := (x, y, z)$, $i \neq j, r \geqslant 1)$ occurs in $\widetilde{f}(x, y, z)$, then, after the polynomial coordinate change $x_k \mapsto x_k + x_j^{r-1}$ $(\{i, j, k\} = \{1, 2, 3\})$, this is eliminated in $\widetilde{f}(x, y, z)$, but new monomials of mixed type (i.e. involving at least two variables) might be introduced. In each of the cases of Table 5, one can find a sequence of such transformations such that all mixed monomials are eliminated and $\widetilde{f}(x, y, z)$ will be of the desired form.

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Table 4. Gabrielov numbers for f.

Type	$f(x, y, z)$	$(\gamma_1, \gamma_2, \gamma_3)$
I	$x^{p_1} + y^{p_2} + z^{p_3}$	(p_1, p_2, p_3)
II	$x^{p_1} + y^{p_2} + yz^{p_3/p_2}$	$\left(p_1, p_2, \left(\frac{p_3}{p_2} - 1\right)p_1\right)$
III	$x^{p_1} + zy^{q_2+1} + yz^{q_3+1}$	(p_1, p_1q_2, p_1q_3)
IV	$x^{p_1} + xy^{p_2/p_1} + yz^{p_3/p_2}$	$\left(p_1, \left(\frac{p_3}{p_2} - 1\right)p_1, \frac{p_3}{p_1} - \frac{p_3}{p_2} + 1\right)$
V	$x^{q_1}y + y^{q_2}z + z^{q_3}x$	$(q_2q_3 - q_2 + 1, q_3q_1 - q_3 + 1, q_1q_2 - q_1 + 1)$

Table 5. The cases $\Delta(\gamma_1, \gamma_2, \gamma_3) < 0$.

Type	$f(x, y, z)$	$\alpha_1, \alpha_2, \alpha_3$	$\gamma_1, \gamma_2, \gamma_3$	Singularity
I	$x^2 + y^2 + z^k$, $k \geqslant 2$	$2, 2, k$	$2, 2, k$	A_{k-1}
	$x^2 + y^3 + z^3$	$2, 3, 3$	$2, 3, 3$	D_4
	$x^2 + y^3 + z^4$	$2, 3, 4$	$2, 3, 4$	E_6
	$x^2 + y^3 + z^5$	$2, 3, 5$	$2, 3, 5$	E_8
II	$x^2 + y^2 + yz^k$, $k \geqslant 2$	$2, k, 2$	$2, 2, 2(k-1)$	A_{2k-1}
	$x^2 + y^k + yz^2$, $k \geqslant 2$	$2, 2, 2(k-1)$	$2, k, 2$	D_{k+1} $(D_3 = A_3)$
	$x^3 + y^2 + yz^2$	$3, 2, 3$	$3, 2, 3$	E_6
	$x^2 + y^3 + yz^3$	$2, 3, 4$	$2, 3, 4$	E_7
III	$x^2 + zy^2 + yz^{k+1}$, $k \geqslant 1$	$2, 2, 2k$	$2, 2, 2k$	D_{2k+2}
IV	$x^l + xy + yz^k$, $k, l \geqslant 2$	$k, (l-1)k, 1$	$l, (k-1)l, 1$	A_{kl-1}
	$x^2 + xy^k + yz^2$, $k \geqslant 2$	$2, 2, 2k-1$	$2, 2, 2k-1$	D_{2k+1}
V	$xy + y^kz + z^lx$, $k, l \geqslant 1$	$kl - l + 1, l, 1$	$kl - k + 1, 1, k$	A_{kl}

We indicate an example of such a sequence of coordinate transformations. Let $f(x, y, z) = x^2 + y^3 + yz^3$, $\widetilde{f}(x, y, z) = x^2 + y^3 + yz^3 - xyz$. By the coordinate transformation

$$(x, y, z) \mapsto (x + z^2, y, z),$$

$\widetilde{f}(x, y, z)$ is transformed to

$$\widetilde{f}_1(x, y, z) = x^2 + 2xz^2 + z^4 + y^3 - xyz.$$

The coordinate change

$$(x, y, z) \mapsto (x + 6y + 12z, y + 2z, z)$$

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Table 6. The cases $\Delta(\gamma_1, \gamma_2, \gamma_3) = 0$.

Type	$f(x,y,z)$	$\alpha_1,\alpha_2,\alpha_3$	$\gamma_1,\gamma_2,\gamma_3$	Singularity
I	$x^2+y^3+z^6$	2, 3, 6	2, 3, 6	$\widetilde{E}_8$
	$x^2+y^4+z^4$	2, 4, 4	2, 4, 4	$\widetilde{E}_7$
	$x^3+y^3+z^3$	3, 3, 3	3, 3, 3	$\widetilde{E}_6$
II	$x^2+y^3+yz^4$	2, 4, 4	2, 3, 6	$\widetilde{E}_8$
	$x^2+y^4+yz^3$	2, 3, 6	2, 4, 4	$\widetilde{E}_7$
	$x^4+y^2+yz^2$	4, 2, 4	4, 2, 4	$\widetilde{E}_7$
	$x^3+y^2+yz^3$	3, 3, 3	3, 2, 6	$\widetilde{E}_8$
	$x^3+y^3+yz^2$	3, 2, 6	3, 3, 3	$\widetilde{E}_6$
III	$x^2+zy^3+yz^3$	2, 4, 4	2, 4, 4	$\widetilde{E}_7$
	$x^3+zy^2+yz^2$	3, 3, 3	3, 3, 3	$\widetilde{E}_6$
IV	$x^2+xy^2+yz^3$	3, 3, 3	2, 4, 4	$\widetilde{E}_7$
	$x^3+xy^2+yz^2$	2, 4, 4	3, 3, 3	$\widetilde{E}_6$
V	$x^2y+y^2z+z^2x$	3, 3, 3	3, 3, 3	$\widetilde{E}_6$

yields

$$\widetilde{f}_2(x,y,z) = x^2 + y^3 + z^4 + 8z^3 + 144z^2 + 36y^2 + 12xy + 24xz + 144yz - xyz.$$

Finally, the transformation

$$(x,y,z) \mapsto (x+144, y+24, z+12)$$

gives the final form

$$\widetilde{f}_3(x,y,z) = x^2 + y^3 + z^4 - xyz + 576x + 108y^2 + 5184y + 56z^3 + 1296z^2 + 17280z + 193536.$$

(ii) The cases with $\Delta(\gamma_1, \gamma_2, \gamma_3) = 0$ are listed in Table 6. The statement follows from Arnold's determinator of singularities [AGV85, 16.2].

(iii) First, suppose that

$$f(x,y,z) = x^2 + f'(y,z).$$

The substitution $x \mapsto x + \frac{1}{2}yz$ transforms the polynomial $f(x,y,z) - xyz$ to

$$x^2 + f'(y,z) - \tfrac{1}{4}y^2z^2.$$

Since $\Delta(\gamma_1, \gamma_2, \gamma_3) > 0$, [AGV85, 16.2, Theorems 3–5] imply that the 3-jet of f' is either:

(1) $j^3 f'(y,z) = y^3$; or

(2) $j^3 f'(y,z) = 0$.

In case (1), [AGV85, 16.2, Theorem 12_2] implies that the polynomial $f(x,y,z) - xyz$ becomes a polynomial of type $T_{2,3,\gamma_3}$ ($\gamma_3 \geqslant 7$) after a suitable holomorphic change of coordinates at $0 \in \mathbb{C}^3$.

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In case (2), let f_0 be the 4-jet of $f'(y,z) - \frac{1}{4}y^2z^2$. According to [AGV85, 16.2, Theorem 13], by a suitable holomorphic coordinate change, one can assume that we have the following possibilities for f_0:

(2a) $f_0(y,z) = y^4 - y^2z^2$;

(2b) $f_0(y,z) = -y^2z^2$.

Now let f be of corank 3 and let f_0 be the 3-jet of $f(x,y,z) - xyz$. According to [AGV85, 16.2, Theorem 50], by a suitable holomorphic coordinate change, one can assume that f_0 is one of the following three polynomials

(3a) $f_0(x,y,z) = x^3 + y^3 - xyz$;

(3b) $f_0(x,y,z) = x^3 - xyz$;

(3c) $f_0(x,y,z) = -xyz$.

Now by the assumption $\Delta(\gamma_1,\gamma_2,\gamma_3) > 0$, in the cases (2) and (3), $f(x,y,z) - xyz$ is of the form $f_0 + f_1$, where f_1 has terms of degree greater than the degree of the homogeneous polynomial f_0. According to [Arn74, Lemma 7.3], by a transformation similar to the transformations in (i), one can get rid of the monomials of lowest degree involving more than one variable by possibly introducing new such monomials, but of higher degree. The monomials which are powers of a variable of lowest degree are not changed. The statement now follows by an iterated application of [Arn74, Lemma 7.3]. In case (2), one has to go back from $x^2 - \frac{1}{4}y^2z^2 + \cdots$ to $x^2 - xyz + \cdots$ via the coordinate change $x \mapsto x - \frac{1}{2}yz$. $\square$

Corollary 11. *Let $f(x,y,z)$ be an invertible polynomial with Gabrielov numbers $\Gamma_f = (\gamma_1,\gamma_2,\gamma_3)$.*

(i) *If $\Delta(\Gamma_f) < 0$, then the Milnor fiber of f (i.e. the level set $f(x,y,z) = 1$) can be deformed to an open submanifold of the Milnor fiber of a polynomial of type $T_{\gamma_1,\gamma_2,\gamma_3}$.*

(ii) *If $\Delta(\Gamma_f) > 0$, then the singularity given by $f(x,y,z) = 0$ deforms to a cusp singularity of type $T_{\gamma_1,\gamma_2,\gamma_3}$.*

If a singularity f deforms to a singularity g, then a Coxeter–Dynkin diagram of g can be extended to a Coxeter–Dynkin diagram of f. Therefore, we obtain the following corollary.

Corollary 12. *Let $f(x,y,z)$ be an invertible polynomial with Gabrielov numbers $\Gamma_f = (\gamma_1,\gamma_2,\gamma_3)$.*

(i) *If $\Delta(\Gamma_f) < 0$, then a Coxeter–Dynkin diagram of f is contained in the graph of type $T(\gamma_1,\gamma_2,\gamma_3)$. More precisely, a Coxeter–Dynkin diagram of f is one of the standard Coxeter–Dynkin diagrams of types A_μ, D_μ, E_6, E_7 or E_8 and the graph $T(\gamma_1,\gamma_2,\gamma_3)$ is an extension of this diagram. The precise relation is given in Table 5.*

(ii) *If $\Delta(\Gamma_f) = 0$, then the graph $T(\gamma_1,\gamma_2,\gamma_3)$ is a Coxeter–Dynkin diagram of f.*

(iii) *If $\Delta(\Gamma_f) > 0$, then the graph of type $T(\gamma_1,\gamma_2,\gamma_3)$ can be extended to a Coxeter–Dynkin diagram of f.*

Moreover, Corollary 11 gives us a relation (a semi-orthogonal decomposition theorem) between the Fukaya categories $D^b\mathrm{Fuk}^{\rightarrow}(f)$ and $D^b\mathrm{Fuk}^{\rightarrow}(T_{\gamma_1,\gamma_2,\gamma_3})$, which is mirror dual to the one proven by Orlov [Orl09].

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4. Strange duality

Now we are ready to state our main theorem in this paper.

Theorem 13. *Let $f(x,y,z)$ be an invertible polynomial. Then we have*

$$A_{G_f} = \Gamma_{f^t}, \quad A_{G_{f^t}} = \Gamma_f. \tag{4.1}$$

Namely, the Dolgachev numbers A_{G_f} for the pair (f, G_f) coincide with the Gabrielov numbers Γ_{f^t} for the Berglund–Hübsch transpose f^t of f, and the Dolgachev numbers $A_{G_{f^t}}$ for the pair (f^t, G_{f^t}) coincide with the Gabrielov numbers Γ_f for f.

Proof. This can be easily checked by Tables 1, 2 and 4. See Table 7. □

It is convenient in the next section to introduce the following.

Definition. Let X and Y be weighted homogeneous isolated hypersurface singularities of dimension two. If there exists an invertible polynomial $f(x,y,z)$ such that f represents the singularity X and f^t represents the singularity Y, then Y is called *f-dual* to X.

Note that Y is f-dual to X if and only if X is f^t-dual to Y.

Definition. Let X be a weighted homogeneous isolated hypersurface singularity of dimension two. The singularity X is called *f-self dual* if X is f-dual to X.

If the invertible polynomial f is clear from the context, we shall often say 'dual' instead of 'f-dual' for simplicity.

5. Additional features of the duality

Theorem 14. *Let $f(x,y,z)$ be an invertible polynomial. Then we have*

$$\Delta(A_{G_f}) = \Delta(A_{G_{f^t}}). \tag{5.1}$$

Proof. This can be easily shown by direct calculation based on Table 7. □

Remark 15. The rational number

$$\chi_{G_f} := 2 + \sum_{i=1}^{3}\left(\frac{1}{\alpha_i} - 1\right) = \sum_{i=1}^{3}\frac{1}{\alpha_i} - 1$$

is called the *orbifold Euler number* of the stack $\mathcal{C}_{A_{G_f}} = \mathcal{C}_{G_f}$. Note that $\Delta(A_{G_f}) = -\alpha_1\alpha_2\alpha_3 \cdot \chi_{G_f}$.

Proposition 16. *Let $f(x,y,z)$ be an invertible polynomial. The canonical system of weights $W_f = (w_1, w_2, w_3; d)$ is as given in Table 8.*

Proof. One can easily show this by direct calculation. □

Proposition 17. *If the canonical system of weights $(w_1, w_2, w_3; d)$ is reduced, then $G_f \simeq \mathbb{C}^*$, where $\mathbb{C}^*$ is the group acting on $\mathbb{C}^3$ by the weights w_1, w_2, w_3.*

Proof. We shall prove the statement by showing for a canonical system of weights which may not be reduced that L_f is an extension of $\mathbb{Z}$ by a finite abelian group of order c_f. It is easy

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TABLE 7. Strange duality.

Type	$A_{G_f} = (\alpha_1, \alpha_2, \alpha_3) = \Gamma_{f^t}$	$\Gamma_f = (\gamma_1, \gamma_2, \gamma_3) = A_{G_{f^t}}$
I	(p_1, p_2, p_3)	(p_1, p_2, p_3)
II	$\left(p_1, \frac{p_3}{p_2}, (p_2 - 1)p_1\right)$	$\left(p_1, p_2, \left(\frac{p_3}{p_2} - 1\right)p_1\right)$
III	(p_1, p_1q_2, p_1q_3)	(p_1, p_1q_2, p_1q_3)
IV	$\left(\frac{p_3}{p_2}, (p_1 - 1)\frac{p_3}{p_2}, p_2 - p_1 + 1\right)$	$\left(p_1, \left(\frac{p_3}{p_2} - 1\right)p_1, \frac{p_3}{p_1} - \frac{p_3}{p_2} + 1\right)$
V	$(q_2q_3 - q_3 + 1, q_3q_1 - q_1 + 1, q_1q_2 - q_2 + 1)$	$(q_2q_3 - q_2 + 1, q_3q_1 - q_3 + 1, q_1q_2 - q_1 + 1)$

TABLE 8. Canonical system of weights attached to f.

Type	$W_f = (w_1, w_2, w_3; d)$
I	$(p_2p_3, p_3p_1, p_1p_2; p_1p_2p_3)$
II	$\left(p_3, \frac{p_1p_3}{p_2}, (p_2 - 1)p_1; p_1p_3\right)$
III	$(p_2, p_1q_3, p_1q_2; p_1p_2)$ $(p_2 + 1 = (q_2 + 1)(q_3 + 1))$
IV	$\left(\frac{p_3}{p_1}, (p_1 - 1)\frac{p_3}{p_2}, p_2 - p_1 + 1; p_3\right)$
V	$(q_2q_3 - q_3 + 1, q_3q_1 - q_1 + 1, q_1q_2 - q_2 + 1; q_1q_2q_3 + 1)$

to see that the homomorphism $\deg : L_f \longrightarrow \mathbb{Z}$ of abelian groups defined by sending $\vec{x}_i \mapsto w_i/c_f$, $i = 1, 2, 3$, is surjective. By the following commutative diagram of abelian groups:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \mathrm{Ker}(\deg) \arrow[r, equal] \arrow[d] & \mathrm{Ker}(\deg) \arrow[d] & \\
0 \arrow[r] & \mathbb{Z}\vec{f} \arrow[r] \arrow[d, equal] & L_f \arrow[r] \arrow[d, "\deg"] & L_f/\mathbb{Z}\vec{f} \arrow[r] \arrow[d, "\deg"] & 0 \\
0 \arrow[r] & \frac{d}{c_f}\mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z}\Big/\frac{d}{c_f}\mathbb{Z} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

one sees that the kernel of the map deg is a finite abelian group of order c_f since the abelian group $L_f/\mathbb{Z}\vec{f}$ is a finite group of order d. $\square$

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Table 9. Canonical system of weights attached to f^t.

Type	$W_{f^t} = (w_1, w_2, w_3; d)$
I	$(p_2p_3, p_3p_1, p_1p_2; p_1p_2p_3)$
II	$\left(p_3, \left(\frac{p_3}{p_2} - 1\right)p_1, p_1p_2; p_1p_3\right)$
III	$(p_2, p_1q_3, p_1q_2; p_1p_2)$ $(p_2 + 1 = (q_2 + 1)(q_3 + 1))$
IV	$\left(\frac{p_3}{p_1} - \frac{p_3}{p_2} + 1, \left(\frac{p_3}{p_2} - 1\right)p_1, p_2; p_3\right)$
V	$(q_2q_3 - q_2 + 1, q_3q_1 - q_3 + 1, q_1q_2 - q_1 + 1; q_1q_2q_3 + 1)$

Definition. Let $f(x, y, z)$ be a weighted homogeneous polynomial and $W := (w_1, w_2, w_3; d)$ a system of weights attached to f. The integer

$$a_W := d - w_1 - w_2 - w_3$$

is called the *Gorenstein parameter* of W.

Remark 18. The Gorenstein parameter a_W is denoted by $-\epsilon_W$ in [Sai98].

Remark 19. Note that the integer $\Delta(A_{G_f})$ can also be regarded as the Gorenstein parameter of the $\mathbb{Z}$-graded ring $R_{A_{G_f}} = \mathbb{C}[X_1, X_2, X_3]/(X_1^{\alpha_1} + X_2^{\alpha_2} + X_3^{\alpha_3})$ with respect to the system of weights $(\alpha_2\alpha_3, \alpha_3\alpha_1, \alpha_1\alpha_2; \alpha_1\alpha_2\alpha_3)$ attached to the polynomial $X_1^{\alpha_1} + X_2^{\alpha_2} + X_3^{\alpha_3}$.

Theorem 20. *Let $f(x, y, z)$ be an invertible polynomial. Let W_f and W_{f^t} be the canonical systems of weights attached to f and f^t. Then we have*

$$a_{W_f} = a_{W_{f^t}}. \tag{5.2}$$

Proof. One can easily show this by direct calculation based on Tables 8 and 9. □

For an invertible polynomial $f(x, y, z)$, the ring $R_f := \mathbb{C}[x, y, z]/(f)$ is a $\mathbb{Z}$-graded ring with respect to the canonical system of weights $(w_1, w_2, w_3; d)$ attached to f. Therefore, we can consider the decomposition of R_f as a $\mathbb{Z}$-graded $\mathbb{C}$-vector space:

$$R_f := \bigoplus_{k \in \mathbb{Z}_{\geqslant 0}} R_{f,k}, \quad R_{f,k} := \left\{ g \in R_f \;\middle|\; w_1 x \frac{\partial g}{\partial x} + w_2 y \frac{\partial g}{\partial y} + w_3 z \frac{\partial g}{\partial z} = kg \right\}.$$

Definition. Let $f(x, y, z)$ be an invertible polynomial. The formal power series

$$p_f(t) := \sum_{k \geqslant 0} (\dim_{\mathbb{C}} R_{f,k}) t^k \tag{5.3}$$

is called the *Poincaré series* of the $\mathbb{Z}$-graded coordinate ring R_f with respect to the canonical system of weights $(w_1, w_2, w_3; d)$ attached to f.

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TABLE 10. Characteristic function $\phi_{G_f}(t)$.

Type	$\phi_{G_f}(t)$
I	$p_1 \cdot p_2 \cdot p_3 \cdot p_1p_2p_3/1 \cdot p_2p_3 \cdot p_3p_1 \cdot p_1p_2$
II	$p_1 \cdot \dfrac{p_3}{p_2} \cdot p_1p_3/1 \cdot p_3 \cdot \dfrac{p_1p_3}{p_2}$
III	$p_1 \cdot p_1p_2/1 \cdot p_2$
IV	$\dfrac{p_3}{p_2} \cdot p_3/1 \cdot \dfrac{p_3}{p_1}$
V	$q_1q_2q_3 + 1/1$

It is easy to see that for an invertible polynomial $f(x,y,z)$ with canonical system of weights $(w_1, w_2, w_3; d)$, the Poincaré series $p_f(t)$ is given by

$$p_f(t) = \frac{(1-t^d)}{(1-t^{w_1})(1-t^{w_2})(1-t^{w_3})}$$

and defines a rational function.

In order to simplify some notation, we shall denote the rational function of the form

$$\frac{\prod_{l=1}^{L}(1-t^{i_l})}{\prod_{m=1}^{M}(1-t^{j_m})}$$

by

$$i_1 \cdot i_2 \cdot \ \cdots \ \cdot i_L/j_1 \cdot j_2 \cdot \ \cdots \ \cdot j_M.$$

For example, the Poincaré series $p_f(t)$ is denoted by $d/w_1 \cdot w_2 \cdot w_3$.

DEFINITION. Let $f(x,y,z)$ be an invertible polynomial. Let $p_f(t)$ be the Poincaré series of the $\mathbb{Z}$-graded coordinate ring R_f with respect to the canonical system of weights attached to f and $A_{G_f} = (\alpha_1, \alpha_2, \alpha_3)$ the Dolgachev numbers of the pair (f, G_f). The rational function

$$\phi_{G_f}(t) := p_f(t)(1-t)^2 \prod_{i=1}^{3} \frac{1-t^{\alpha_i}}{1-t}$$

is called the *characteristic function* of f. The functions $\phi_{G_f}(t)$ are listed in Table 10.

Here we recall the notion of Saito's $*$-duality (see [Sai98]).

DEFINITION. Let d be a positive integer and $\phi(t)$ a rational function of the form

$$\phi(t) = \prod_{i|d}(1-t^i)^{e(i)}, \quad e(i) \in \mathbb{Z}.$$

The *Saito dual* $\phi^*(t)$ of $\phi(t)$ is the rational function given by

$$\phi^*(t) = \prod_{i|d}(1-t^{d/i})^{-e(i)}.$$

One easily sees that $(\phi^*)^*(t) = \phi(t)$.

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The characteristic function $\phi_{G_f}(t)$ may not be a polynomial in general for an invertible polynomial f of type I and type II; however, by [Ebe02, Theorem 1], we have the following.

THEOREM 21. *Let $f(x,y,z)$ be an invertible polynomial whose canonical system of weights $(w_1,w_2,w_3;d)$ is reduced. Then the characteristic function $\phi_{G_f}(t)$ is a polynomial. Moreover, its Saito dual $\phi^*_{G_f}(t)$ is the characteristic polynomial of the Milnor monodromy of the singularity f.*

Here we use a normalized version of the characteristic polynomial. If τ denotes the Milnor monodromy of the singularity f, then its characteristic polynomial is

$$\Phi_f(t) := \det(1-\tau^{-1}t).$$

Even if $f(x,y,z)$ is an invertible polynomial whose canonical system of weights $(w_1,w_2,w_3;d)$ is reduced, the canonical system of weights of its transpose f^t may not be reduced.

We have the following property of our characteristic functions.

THEOREM 22. *Let $f(x,y,z)$ be an invertible polynomial. Then we have the Saito duality*

$$\phi^*_{G_f}(t) = \phi_{G_{f^t}}(t).$$

In particular, if the canonical system of weights attached to f is reduced, then the polynomial $\phi_{G_f}(t)$ is the characteristic polynomial of an operator τ such that $\tau^{c_{f^t}}$ is the monodromy of the singularity f^t.

Proof. One can easily check the first statement by direct calculations.

The characteristic polynomials of the monodromy of f^t can be computed using Varchenko's method [Var76]. They are listed in Table 11. In order to prove the second statement, assume that the canonical system of weights attached to f is reduced. If the canonical system of weights attached to f^t is also reduced, then the second statement follows from Theorem 21. Otherwise, let $c=c_{f^t}$. The case that $c>1$ can only occur for types II (with $c_1=c$ and $c_2=1$), IV or V. Let $\zeta := e^{2\pi i/c}$ be a primitive cth root of unity. There is the following relation between the characteristic polynomial of an operator τ and that of the operator τ^c:

$$\det(1-\tau^{-c}t^c) = \prod_{i=0}^{c-1} \det(1-\tau^{-1}\zeta^i t).$$

Using this relation and Table 11, one can easily show the second statement for the remaining cases. □

Remark 23. Theorem 22 is already shown in [Tak99] for the special case when both the canonical systems of weights for f and f^t are reduced.

6. Examples: Coxeter–Dynkin diagrams

We now show how the weighted homogeneous singularities of Arnold's classification of singularities fit into our scheme.

DEFINITION. Let $f(x,y,x)$ be an invertible polynomial whose canonical system of weights is reduced, $\alpha_1,\alpha_2,\alpha_3$ be the Dolgachev numbers of f and a_{W_f} be the Gorenstein parameter of W_f. We define positive integers β_i, $0<\beta_i<\alpha_i$, by

$$\beta_i a_{W_f} \equiv 1 \bmod \alpha_i, \quad i=1,2,3.$$

The numbers $(\alpha_1,\beta_1),(\alpha_2,\beta_2),(\alpha_3,\beta_3)$ are called the *orbit invariants* of f.

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Table 11. Characteristic polynomial of the monodromy of f^t.

Type	$\Phi_{f^t}(t)$
I	$p_1 \cdot p_2 \cdot p_3 \cdot \left(\frac{p_1p_2p_3}{c}\right)^c \Big/ 1 \cdot \left(\frac{p_2p_3}{c_1}\right)^{c_1} \cdot \left(\frac{p_3p_1}{c_2}\right)^{c_2} \cdot \left(\frac{p_1p_2}{c_3}\right)^{c_3}$ $c_1 = \gcd(p_2, p_3), c_2 = \gcd(p_1, p_3), c_3 = \gcd(p_1, p_2)$
II	$p_1 \cdot \frac{p_3}{p_2} \cdot \left(\frac{p_1p_3}{c}\right)^c \Big/ 1 \cdot \left(\frac{p_3}{c_1}\right)^{c_1} \cdot \left(\frac{p_1p_3}{p_2c_2}\right)^{c_2}$ $c_1 = \gcd\left(p_2, \frac{p_3}{p_2} - 1\right), c_2 = \gcd\left(p_1, \frac{p_3}{p_2}\right)$
III	$p_1 \cdot \left(\frac{p_1p_2}{c}\right)^c \Big/ 1 \cdot \left(\frac{p_2}{c_1}\right)^{c_1}$ $c_1 = \gcd(q_2, q_3)$
IV	$\frac{p_3}{p_2} \cdot \left(\frac{p_3}{c}\right)^c \Big/ 1 \cdot \left(\frac{p_3}{p_1c_1}\right)^{c_1}$ $c_1 = \gcd\left(\frac{p_2}{p_1}, \frac{p_3}{p_2} - 1\right)$
V	$\left(\frac{q_1q_2q_3+1}{c}\right)^c \Big/ 1$

Remark 24. Since the weight system is assumed to be reduced, $G_f \cong \mathbb{C}^*$ and we have the usual $\mathbb{C}^*$-action. The numbers $(\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3)$ are just the usual orbit invariants of the $\mathbb{C}^*$-action (see [Dol83]).

We now consider the classification of the singularities defined by invertible polynomials according to the Gorenstein parameter a_{W_f}.

The invertible polynomials $f(x, y, z)$ with $a_{W_f} < 0$ define the simple singularities. These invertible polynomials together with the corresponding Dolgachev and Gabrielov numbers have already been given in Table 5.

The invertible polynomials $f(x, y, z)$ with $a_{W_f} = 0$ define the simply elliptic singularities. They have already been exhibited in Table 6. The corresponding canonical weight systems are not reduced.

Now we consider invertible polynomials $f(x, y, z)$ with $a_{W_f} > 0$. In Table 12, we have chosen some invertible polynomials for the exceptional unimodal singularities. We obtain Arnold's strange duality. Here the weight systems are all reduced and we have $a_{W_f} = 1$. We obtain a Coxeter–Dynkin diagram for f by adding one new vertex to the graph $T(\gamma_1, \gamma_2, \gamma_3)$ (see Figure 1) and connecting it to the upper central vertex (with index $\gamma_1 + \gamma_2 + \gamma_3 - 1$) by a solid edge. Therefore, our Gabrielov numbers coincide with the numbers defined by Gabrielov in this case.

Now we consider the exceptional bimodal singularities (Table 13). They can all be given by invertible polynomials with a reduced weight system. We see that we also obtain a strange

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Table 12. Arnold's strange duality.

Name	$\alpha_1, \alpha_2, \alpha_3$	f	f^t	$\gamma_1, \gamma_2, \gamma_3$	Dual
E_{12}	$2, 3, 7$	$x^2 + y^3 + z^7$	$x^2 + y^3 + z^7$	$2, 3, 7$	E_{12}
E_{13}	$2, 4, 5$	$x^2 + y^3 + yz^5$	$x^2 + zy^3 + z^5$	$2, 3, 8$	Z_{11}
E_{14}	$3, 3, 4$	$x^3 + y^2 + yz^4$	$x^3 + zy^2 + z^4$	$2, 3, 9$	Q_{10}
Z_{12}	$2, 4, 6$	$x^2 + zy^3 + yz^4$	$x^2 + zy^3 + yz^4$	$2, 4, 6$	Z_{12}
Z_{13}	$3, 3, 5$	$x^2 + xy^3 + yz^3$	$x^2y + y^3z + z^3$	$2, 4, 7$	Q_{11}
Q_{12}	$3, 3, 6$	$x^3 + zy^2 + yz^3$	$x^3 + zy^2 + yz^3$	$3, 3, 6$	Q_{12}
W_{12}	$2, 5, 5$	$x^5 + y^2 + yz^2$	$x^5 + y^2z + z^2$	$2, 5, 5$	W_{12}
W_{13}	$3, 4, 4$	$x^2 + xy^2 + yz^4$	$x^2y + y^2z + z^4$	$2, 5, 6$	S_{11}
S_{12}	$3, 4, 5$	$x^3y + y^2z + z^2x$	$zx^3 + xy^2 + yz^2$	$3, 4, 5$	S_{12}
U_{12}	$4, 4, 4$	$x^4 + zy^2 + yz^2$	$x^4 + zy^2 + yz^2$	$4, 4, 4$	U_{12}

Table 13. Strange duality of the exceptional bimodal singularities.

Name	$\alpha_1, \alpha_2, \alpha_3$	f	f^t	$\gamma_1, \gamma_2, \gamma_3$	Dual
E_{18}	$3, 3, 5$	$x^3 + y^2 + yz^5$	$x^3 + y^2z + z^5$	$2, 3, 12$	Q_{12}
E_{19}	$2, 4, 7$	$x^2 + y^3 + yz^7$	$x^2 + y^3z + z^7$	$2, 3, 12$	$Z_{1,0}$
E_{20}	$2, 3, 11$	$x^2 + y^3 + z^{11}$	$x^2 + y^3 + z^{11}$	$2, 3, 11$	E_{20}
Z_{17}	$3, 3, 7$	$x^2 + xy^4 + yz^3$	$x^2y + y^4z + z^3$	$2, 4, 10$	$Q_{2,0}$
Z_{18}	$2, 4, 10$	$x^2 + zy^3 + yz^6$	$x^2 + zy^3 + yz^6$	$2, 4, 10$	Z_{18}
Z_{19}	$2, 3, 16$	$x^2 + y^9 + yz^3$	$x^2 + y^9z + z^3$	$2, 4, 9$	E_{25}
Q_{16}	$3, 3, 9$	$x^3 + zy^2 + yz^4$	$x^3 + zy^2 + yz^4$	$3, 3, 9$	Q_{16}
Q_{17}	$2, 4, 13$	$x^3 + xy^5 + yz^2$	$x^3y + y^5z + z^2$	$3, 3, 9$	$Z_{2,0}$
Q_{18}	$2, 3, 21$	$x^3 + y^8 + yz^2$	$x^3 + y^8z + z^2$	$3, 3, 8$	E_{30}
W_{17}	$3, 5, 5$	$x^2 + xy^2 + yz^5$	$x^2y + y^2z + z^5$	$2, 6, 8$	$S_{1,0}$
W_{18}	$2, 7, 7$	$x^7 + y^2 + yz^2$	$x^7 + y^2z + z^2$	$2, 7, 7$	W_{18}
S_{16}	$3, 5, 7$	$x^4y + y^2z + z^2x$	$zx^4 + xy^2 + yz^2$	$3, 5, 7$	S_{16}
S_{17}	$2, 7, 10$	$x^6 + xy^2 + yz^2$	$x^6y + y^2z + z^2$	$3, 6, 6$	$X_{2,0}$
U_{16}	$5, 5, 5$	$x^5 + zy^2 + yz^2$	$x^5 + zy^2 + yz^2$	$5, 5, 5$	U_{16}

duality involving the exceptional bimodal singularities and some other singularities which are given by invertible polynomials with in general non-reduced weight systems (see Table 14).

We list the invariants $(\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3)$ in Table 14.

We now indicate Coxeter–Dynkin diagrams for the bimodal exceptional singularities. Coxeter–Dynkin diagrams for these singularities were obtained in [Ebe83]. Let f be an invertible polynomial defining an exceptional bimodal singularity with orbit invariants $(\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3)$ and Gabrielov numbers $\gamma_1, \gamma_2, \gamma_3$. We define numbers $\delta_1, \delta_2, \delta_3$ by Table 14. One can show that there exists a Coxeter–Dynkin diagram which is obtained by an extension of a $T(\gamma_1, \gamma_2, \gamma_3)$-diagram by a_{W_f} vertices in the following way:

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Table 14. Invariants of the singularities involved.

Name	$(\alpha_i, \beta_i), i=1,2,3$	a_{W_f}	$(\gamma_i, \delta_i), i=1,2,3$	c_{f^t}	μ_{f^t}	Dual
E_{18}	$(3,2),(3,2),(5,3)$	2	$(2,1),(3,2),(12,8)$	2	12	Q_{12}
E_{19}	$(2,1),(4,3),(7,5)$	3	$(2,1),(3,2),(12,9)$	3	15	$Z_{1,0}$
E_{20}	$(2,1),(3,2),(11,9)$	5	$(2,1),(3,2),(11,9)$	1	20	E_{20}
Z_{17}	$(3,2),(3,2),(7,4)$	2	$(2,1),(4,3),(10,6)$	2	14	$Q_{2,0}$
Z_{18}	$(2,1),(4,3),(10,7)$	3	$(2,1),(4,3),(10,7)$	1	18	Z_{18}
Z_{19}	$(2,1),(3,2),(16,13)$	5	$(2,1),(4,3),(9,7)$	1	25	E_{25}
Q_{16}	$(3,2),(3,2),(9,5)$	2	$(3,2),(3,2),(9,5)$	1	16	Q_{16}
Q_{17}	$(2,1),(4,3),(13,9)$	3	$(3,2),(3,2),(9,6)$	3	21	$Z_{2,0}$
Q_{18}	$(2,1),(3,2),(21,17)$	5	$(3,2),(3,2),(8,6)$	1	30	E_{30}
W_{17}	$(3,2),(5,3),(5,3)$	2	$(2,1),(6,4),(8,5)$	2	14	$S_{1,0}$
W_{18}	$(2,1),(7,5),(7,5)$	3	$(2,1),(7,5),(7,5)$	1	18	W_{18}
S_{16}	$(3,2),(5,3),(7,4)$	2	$(3,2),(5,3),(7,4)$	1	16	S_{16}
S_{17}	$(2,1),(7,5),(10,7)$	3	$(3,2),(6,4),(6,4)$	3	21	$X_{2,0}$
U_{16}	$(5,3),(5,3),(5,3)$	2	$(5,3),(5,3),(5,3)$	1	16	U_{16}

- if $a_{W_f}=2$, then the diagram $T(\gamma_1,\gamma_2,\gamma_3)$ is extended by $\bullet_1$—$\bullet_2$, where $\bullet_1$ is connected to the upper central vertex and $\bullet_2$ to the $(\gamma_i-\delta_i-1)$th vertex from the outside of the ith arm, unless $\delta_i=\gamma_i-1$ $(i=1,2,3)$;
- if $a_{W_f}=3$, then the diagram $T(\gamma_1,\gamma_2,\gamma_3)$ is extended by $\bullet_1$— $\bullet_2$ —$\bullet_3$, where $\bullet_1$ is connected to the upper central vertex and $\bullet_3$ to the $(\gamma_i-\delta_i-1)$th vertex from the outside of the ith arm, unless $\delta_i=\gamma_i-1$ $(i=1,2,3)$;
- if $a_{W_f}=5$, then the diagram $T(\gamma_1,\gamma_2,\gamma_3)$ is extended by $\bullet_1$— $\bullet_2$ — $\bullet_3$ — $\bullet_4$ —$\bullet_5$, where $\bullet_1$ is connected to the upper central vertex and $\bullet_3$ to the $(\gamma_i-\delta_i-1)$th vertex from the outside of the ith arm, unless $\delta_i=\gamma_i-1$ $(i=1,2,3)$.

We now consider the characteristic functions $\phi_{G_f}(t)$ and $\phi_{G_{f^t}}(t)$ in this case. They are listed in Table 15. By Theorems 21 and 22, the characteristic function $\phi_{G_{f^t}}(t)$ is the characteristic polynomial of the Milnor monodromy of the singularity f. Moreover, the characteristic function ϕ_{G_f} is the characteristic polynomial of an operator τ such that $\tau^{c_{f^t}}$ is the monodromy of the singularity f^t.

In Table 13, pairs (f, f^t) with $c_f=1$, $c_{f^t}>1$ only occur for types II and IV. As already noted in the proof of Theorem 22, such pairs can only exist for types II, IV or V. Here is an example of such a pair for type V.

Example 25. Let $f(x,y,z)=x^2y+y^3z+z^4x$ with canonical system of weights $W_f=(9,7,4;25)$. Then $f^t(x,y,z)=zx^2+xy^3+yz^4$, $W_{f^t}=(10,5,5;25)$. Here $c_f=1$ but $c_{f^t}=5$.

In Table 13, three of the six weighted homogeneous singularities of the bimodal series appear. For completeness, we also indicate invertible polynomials for the remaining three in Table 16. Here both canonical systems of weights are non-reduced. In the case $J_{3,0}$, $\phi_{G_f}(t)$ is a polynomial, but not the characteristic polynomial of an operator τ such that $\tau^{c_{f^t}}$ is the monodromy of Z_{13}. In this case, $\phi_{G_{f^t}}(t)$ is not a polynomial. In the remaining cases, $\phi_{G_f}(t)=\phi^*_{G_f}(t)$ is not a polynomial.

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Table 15. Polynomials ϕ_{G_f} and $\phi_{G_{f^t}}$.

Name	ϕ_{G_f}	$\phi_{G_{f^t}}$	Dual
E_{18}	$3 \cdot 5 \cdot 30/1 \cdot 10 \cdot 15$	$2 \cdot 3 \cdot 30/1 \cdot 10 \cdot 6$	Q_{12}
E_{19}	$2 \cdot 7 \cdot 42/1 \cdot 14 \cdot 21$	$2 \cdot 3 \cdot 42/1 \cdot 6 \cdot 21$	$Z_{1,0}$
E_{20}	$2 \cdot 3 \cdot 11 \cdot 66/1 \cdot 6 \cdot 22 \cdot 33$	$2 \cdot 3 \cdot 11 \cdot 66/1 \cdot 6 \cdot 22 \cdot 33$	E_{20}
Z_{17}	$3 \cdot 24/1 \cdot 12$	$2 \cdot 24/1 \cdot 8$	$Q_{2,0}$
Z_{18}	$2 \cdot 34/1 \cdot 17$	$2 \cdot 34/1 \cdot 17$	Z_{18}
Z_{19}	$2 \cdot 3 \cdot 54/1 \cdot 6 \cdot 27$	$2 \cdot 3 \cdot 54/1 \cdot 6 \cdot 27$	E_{25}
Q_{16}	$3 \cdot 21/1 \cdot 7$	$3 \cdot 21/1 \cdot 7$	Q_{16}
Q_{17}	$2 \cdot 30/1 \cdot 10$	$3 \cdot 30/1 \cdot 15$	$Z_{2,0}$
Q_{18}	$2 \cdot 3 \cdot 48/1 \cdot 6 \cdot 16$	$2 \cdot 3 \cdot 48/1 \cdot 6 \cdot 16$	E_{30}
W_{17}	$5 \cdot 20/1 \cdot 10$	$2 \cdot 20/1 \cdot 4$	$S_{1,0}$
W_{18}	$2 \cdot 7 \cdot 28/1 \cdot 4 \cdot 14$	$2 \cdot 7 \cdot 28/1 \cdot 4 \cdot 14$	W_{18}
S_{16}	$17/1$	$17/1$	S_{16}
S_{17}	$2 \cdot 24/1 \cdot 4$	$6 \cdot 24/1 \cdot 12$	$X_{2,0}$
U_{16}	$5 \cdot 15/1 \cdot 3$	$5 \cdot 15/1 \cdot 3$	U_{16}

Table 16. Bimodal series.

Name	$\alpha_1, \alpha_2, \alpha_3$	f	f^t	$\gamma_1, \gamma_2, \gamma_3$	Dual
$J_{3,0}$	2, 4, 6	$x^2 + y^3 + yz^6$	$x^2 + y^3 z + z^6$	2, 3, 10	Z_{13}
$W_{1,0}$	2, 6, 6	$x^6 + y^2 + yz^2$	$x^6 + y^2 z + z^2$	2, 6, 6	$W_{1,0}$
$U_{1,0}$	3, 3, 6	$x^3 + y^3 + yz^3$	$x^3 + y^3 z + z^3$	3, 3, 6	$U_{1,0}$

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References

Arn74 V. I. Arnold, *Normal forms of functions in neighbourhoods of degenerate critical points*, Uspekhi Mat. Nauk **29** (1974), 11–49; Engl. transl. Russian Math. Surveys **29** (1974), 19–48.

AGV85 V. I. Arnold, S. M. Gusein-Zade and A. N. Varchenko, *Singularities of differentiable maps, Vol. I* (Birkhäuser, Boston–Basel–Berlin, 1985).

BH93 P. Berglund and T. Hübsch, *A generalized construction of mirror manifolds*, Nuclear Phys. B **393** (1993), 377–391.

Dol83 I. V. Dolgachev, *On the link space of a Gorenstein quasihomogeneous surface singularity*, Math. Ann. **265** (1983), 529–540.

Ebe83 W. Ebeling, *Milnor lattices and geometric bases of some special singularities*, in *Nœuds, tresses et singularités*, Monographies de L'Enseignement Mathématique, vol. 31, ed. C. Weber (Genève, 1983), 129–146; Enseign. Math. **29** (1983), 263–280.

Ebe02 W. Ebeling, *Poincaré series and monodromy of a two-dimensional quasihomogeneous hypersurface singularity*, Manuscripta Math. **107** (2002), 271–282.

https://doi.org/10.1112/S0010437X11005288 Published online by Cambridge University Press

GL87 W. Geigle and H. Lenzing, *A class of weighted projective curves arising in representation theory of finite-dimensional algebras*, in *Singularities, representation of algebras, and vector bundles* (Lambrecht, 1985), Lecture Notes in Mathematics, vol. 1273 (Springer, Berlin, 1987), 265–297.

KPABR10 M. Krawitz, N. Priddis, P. Acosta, N. Bergin and H. Rathnakumara, *FJRW-rings and mirror symmetry*, Comm. Math. Phys. **296** (2010), 145–174.

Kre94 M. Kreuzer, *The mirror map for invertible LG models*, Phys. Lett. B **328** (1994), 312–318.

Orl09 D. Orlov, *Derived categories of coherent sheaves and triangulated categories of singularities*, in *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin, Vol. II*, Progress in Mathematics, vol. 270 (Birkhäuser, Boston, MA, 2009), 503–531.

Sai87 K. Saito, *Regular system of weights and associated singularities*, in *Complex analytic singularities*, Adv. Stud. Pure Math. **8** (1987), 479–526.

Sai98 K. Saito, *Duality for regular systems of weights*, Asian J. Math. **2** (1998), 983–1047.

Tak99 A. Takahashi, *K. Saito's duality for regular weight systems and duality for orbifoldized Poincaré polynomials*, Comm. Math. Phys. **205** (1999), 571–586.

Tak09 A. Takahashi, *HMS for isolated hypersurface singularities, Talk at the 'Workshop on Homological Mirror Symmetry and Related Topics'*, 19–24 January 2009, University of Miami, PDF file available from http://www-math.mit.edu/∼auroux/frg/miami09-notes/.

Tak10 A. Takahashi, *Weighted projective lines associated to regular systems of weights of dual type*, Adv. Stud. Pure Math. **59** (2010), 371–388.

Var76 A. N. Varchenko, *Zeta-function of monodromy and Newton's diagram*, Invent. Math. **37** (1976), 253–262.

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