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Strategic and Tactical Asset Allocation and the Effect of Long-Run Equilibrium Relations

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Strategic and Tactical Asset Allocation and the Effect of Long-Run Equilibrium Relations

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This article focuses on the relevance of long-term equilibrium relations for financial decision making. Special attention is devoted to optimal asset allocation in the presence of possibly cointegrated time-series, e.g., asset prices. Using a stylized asset allocation problem, the link is established between the number of cointegrating relations and their precise form on the one hand, and the optimal asset allocation on the other hand. The paper disentangles the different effects of long-term relations on optimal asset allocation with different planning horizons: error-correction mainly affects tactical asset allocation, while cointegration affects strategic asset allocation. The paper also presents results on the effects of incorporating an incorrect number of error-correction mechanisms in financial decision models. Mis-specifying the number of cointegrating relations in a scenario generator can induce either inefficient or overly risky financial management decisions. The findings are illustrated using a stylized empirical example from currency management.
(Cointegration; Error-Correction; Tactical Asset Allocation; *Strategic Asset Allocation; Mean Reversion; Non-Stationarity; Mis-specified Equilibrium Relationships;* Currency *Management)*

1. Introduction

The uncertainty associated with possible future developments of economic and social circumstances constitutes one of the main complications in decision making in general and financial decision making in particular. Banks, for example, have to assess the future prospects of alternative investment opportunities in order to design a solid investment policy. Also, e.g., pension funds have to cope with several sources of uncertainty in strategic policy

making. In addition to predicting the developments of returns on different investment opportunities, pension funds have to forecast the long-term movements in their liabilities. This involves making forecasts of variables like inflation (because of price indexed pension payments) and ageing (for the design of sustainable contribution schemes).

In order to formally include the future uncertainty in a financial decision process, the decision maker needs a compact description of future possible developments, i.e., of possible scenarios. On easy and obvious way to achieve such a description is by using quantitative models for the economic (and social) environment. There are numerous papers in which quantitative models are used for financial decision making, see, e.g., Boender and Romeijn (1991), Cariño et al. (1994), Mulvey (1995), Consigli and Dempster (1996), Boender (1997), Boender et al. (1997), Cariño and Ziemba (1997a,b), Dert (1997), and Mulvey and Thorlacius (1997). In the present paper, the focus is on multivariate time-series models to describe the range of possible scenarios. Time-series models have the attractive property that few a *priori* knowledge about the working of the economic system is needed. By explaining the behavior of a time-series by its own past and by the past of related time-series, one obtains a set of scenarios that is consistent with the observed, past behavior of those time-series.

An important issue over the last decade in time-series analysis concerns the long-term or (non-)stationarity properties of time-series models. Loosely speaking, a time-series is called stationary in the present paper if the effect of present circumstances on developments in the distant future diminishes and eventually dies out. Since the seminal paper of Nelson and Plosser (1982), there has been an abundance of empirical literature demonstrating that the non-stationarity of most economic time-series is very difficult to reject: the effect of present economic developments is very persistent. Good starting references are Ooms (1994) and Hendry (1995). For financial time-series, non-stationarity often emerges quite naturally as a result of the assumption of efficient markets and the absence of arbitrage, see, e.g., de Vries (1994) for foreign exchange markets.

The long-term behavior of non-stationary time-series differs markedly from that of their stationary counterparts, see, e.g., Section 2. Therefore, when using time-series models to generate scenarios for financial decision making, it is important to make the correct choice concerning the (non-)stationarity properties of the time-series under study. This is even more important in multivariate settings, i.e., in situations where we consider several economic variables simultaneously. It may well be the case that two series are individually non-stationary, while a transformation of the two series is stationary. For example, interest rates might be individually non-stationary,

while the term-structure of interest rates is stationary. Similarly, in the context of a pension fund, real estate prices and the consumer price level might each be non-stationary, while some combination of the two series might be stationary due to a partial hedging function of real estate investments for inflation. Engle and Granger (1987) introduced the concept of cointegration as a formalization of the above issue. If there exists a stationary linear combination of two non-stationary time-series, the two series are said to be cointegrated. A lucid explanation of the concept of cointegration can be found in the story of a drunk and here dog, see Murray (1994). Similar to the choice concerning the stationarity properties of univariate time-series, the cointegrating properties of multivariate time-series models can have important consequences for financial decisions based on scenarios that are generated by these models.

The goal of the present paper is to obtain' insight into the *interaction* between the cointegrating properties of econometric time-series models on the one hand, and financial decision models on the other hand. In particular, two questions will be addressed.

1. Do the cointegrating properties of multivariate time-series models, i.e., the long-term time-series properties of scenario generators, have an effect on the (optimal) policies emerging from financial planning models, and if answered affirmatively, what can be said about the relative importance of this effect for different planning horizons?

2. If the cointegrating properties of the time-series model used for financial decision making are incorrect, what are the consequences for the feasibility and efficiency of the adopted policy?

Although the literature on cointegrated time-series is enormous and still expanding rapidly, no effort has yet been made to formalize the effect of the cointegrating properties of time-series and time-series models on (financial) decision models. Most theoretical cointegration papers, on the one hand, focus on new methods for testing for the presence of cointegrating relationships and on distributional properties of proposed estimation and inference procedures. Most empirical cointegration papers, on the other hand, restrict attention to determining the number (and nature) of cointegrating relationships in systems of variables. Clements and Hendry (1995) discuss the accuracy of the predictions obtained with cointegrated and non-cointegrated time-series models. It is the author's opinion that for decision purposes and policy advise the analysis should be taken a step further. For a financial decision maker, it is important to develop a general understanding of the *link*

between the long-run time-series properties of scenario generators and the resulting optimal financial policy. Moreover, the decision maker must be able to assess the sensitivity of his adopted financial policy to mis-specification of the long-term time-series properties of the scenario generating model used for decision making. These issues have (to the author's knowledge) not yet been addressed systematically in the existing literature.

In this paper, I focus on the effects of cointegration on asset allocation. Using a stylized asset allocation problem with a risk averse investment manager, the effect of cointegration on the optimal asset allocation is characterized analytically. The framework is in continuous time. It turns out that for long-term or strategic asset allocation, the long-run cointegrating properties of the time-series are of prime importance. Cointegrating combinations of time-series reveal less long-term variability, and, therefore, less long-term risk. By contrast, for short-term or tactical asset allocation, the most important mechanism concerns the time-series' reaction to temporary states of disequilibrium. If series are cointegrated, they show error-correcting behavior. The error-correction mechanisms allow the financial decision maker to anticipate future developments of key (financial) economic variables, thus affecting tactical management decisions. Concerning the mis-specification of the cointegrating properties of time-series, we obtain a mixed conclusion. For long-term decision problems, it seems advisable not to over-estimate the number of cointegrating relations. By contrast, for short-term forecasting it appears better not to under-estimate the number of relations.

The set-up of the paper is as follows. In Section 2, I introduce the concept of cointegration and explain the main consequences of cointegration and error-correction for forecasting and scenario analysis. In Section 3, I introduce a stylized asset allocation model and characterize its solution using stochastic dynamic programming. In Section 4, I illustrate the findings from Section 3 using a stylized empirical application on currency management. The effect of model mis-specification is considered in Section 5, where I treat the effect of imposing an incorrect number of error-correction mechanisms in the model describing the economy. In Section 6, I present some brief conclusions and suggestions for future research.

2. Cointegration and scenarios

In this section I explain the notions of cointegration and error-correction, especially in their relation to forecasting and scenario generation. Given the dichotomy between the effects of cointegration on short-term versus long-term financial decision making in Section 3, it appears useful to split the

present exposition in two parts. Subsection 2.1 deals with long-term characteristics of cointegrated models. Subsection 2.2 considers the short-term behavior of such models.

2.1. Cointegration

In order to explain the notions of cointegration and error-correction and their relevance for generating scenarios, I start by introducing the class of Vector AutoRegressive (VAR) time-series models. Consider the VAR model of order p,

$$\text{A}y_t = \Pi y_{t-1} + \Phi_1 \Delta y_{t-1} + \ldots + \Phi_{p-1} \Delta y_{t-p+1} + \mu + \text{E}t, \tag{1}$$

with $y_t \in \mathbb{R}^k$, A denoting the first difference operator $\Delta y_t = y_t - y_{t-1}$, Π and $\Phi_1, \ldots, \Phi_{p-1}$ denoting k x k parameter matrices, $\mu \in \mathbb{R}^k$ denoting a vector of constants, and ε_t denoting an independently and identically distributed (i.i.d.) error term. The VAR model in (1) uses a linear projection of the time-series y_t on its own past to obtain a prediction. VAR models have the advantage that the modeler has to rely on very little (possibly incorrect) a *priori* information stemming from, e.g., economic theory. The VAR approach is data oriented and, as such, able to produce scenarios that are compatible with the past behavior of economic processes. For a fuller exposition of the merits of VAR models as opposed to, e.g., structural econometric models, see Sims (1980).

Now for illustrative purposes, consider a VAR model of order one,

$$\Delta y_t = \mu + \Pi y_{t-1} + \varepsilon_t \Leftrightarrow \text{y}t = \mu + (\text{I} + \Pi) y_{t-1} + \text{E}t. \tag{2}$$

By repeated substitution, we obtain

$$Yt = (I + \Pi)^t y_0 + \sum_{i=0}^{t-1} (I + \Pi)^i (\mu + \varepsilon_{t-i}). \tag{3}$$

Note that in order to exclude explosive behavior, the eigenvalues of $(I + \Pi)$ must lie inside or on the unit circle. If the eigenvalues lie strictly inside the unit circle, (3) shows that the past has an ever diminishing influence on the future, i.e., the impact of ε_t (or y_t) on y_{t+k} decreases if k grows to infinity. The classical example of this case is $\Pi = -I$, in which case (2) reduces to a pure white noise model: the past has no influence on the present. If all eigenvalues lie inside the unit circle, the time-series y_t is weakly stationary, see, e.g., Hamilton (1994). By contrast, if some of the roots of $(I + \Pi)$ lie on the unit circle, past observations have an everlasting impact on future observations. A simple example is the case $\Pi = 0$, in which case

all eigenvalues of $(I + \Pi)$ lie on the unit circle. Model (2) then reduces to a pure random walk (with drift) model.

Following Engle and Granger (1987) and Johansen (1988,1991), we restrict our attention to the case where the eigenvalues of $(I + \Pi)$ lie either inside the unit circle, or at $+1$. This is the most interesting case for most economic (non-seasonal) time-series. The number of non-unit eigenvalues of $(I + \Pi)$ coincides with the rank of the matrix Π. Assume that Π has rank $0 < r < k$, then Π can be decomposed as the product of two $(k \times r)$ matrices α and β, $\Pi = \alpha\beta^\top$, where $^\top$ denotes transposition. Consequently, (2) can be rewritten as

$$\Delta y_t = \alpha\beta^\top y_{t-1} + \mu + \varepsilon_t. \tag{4}$$

Now under the assumption that the time-series Δy_t is stationary, it follows directly from (4) that $\beta^\top y_{t-1}$ must be stationary, as both Δy_t and ε_t are stationary processes, see also Johansen (1991). So the r linear combinations $\beta^\top y_t$ of the k possibly non-stationary time-series y_t, are stationary. Put in the terms of Engle and Granger (1987), we say that there are r cointegrating relationships between the elements of y_t. The cointegrating relations $\beta^\top y_t$ are also called equilibrium relations.

The fact whether or not a time-series is cointegrated has important consequences for the scenarios that are generated. To illustrate the main differences between scenarios with different cointegrating properties, I generate 1 through 20-step-ahead predictions and corresponding (pointwise) 95% confidence intervals using a univariate version of (3) with standard normal disturbance terms. The starting value is taken to be $y_0 = 3$, while $\mu = 0$. Three different values of $\Pi = \pi$ are considered. The case $\pi = 0$ corresponds to the absence of cointegration, while $\pi = -0.1$ and $\pi = -0.2$ results in a "cointegrated" or stationary time-series. The results of the experiment are presented in Figure 1.

It is clearly seen in Figure 1 that the confidence bands for $\pi = 0$ are much wider than for $\pi = -0.1$ or $\pi = -0.2$, especially at long horizons. Moreover, the conditional mean of y_t gradually adjusts to the long-term equilibrium value (i.e., the unconditional mean) 0 for $-2 < \pi < 0$, while it remains constant for $\pi = 0$.

Although the above example in Figure 1 is univariate and, strictly speaking, concerns stationarity rather than cointegration, the extension to the multivariate setting with genuine cointegration is straightforward. If a vector time-series exhibits cointegration, certain linear combinations of the series will display behavior as for $\pi = 0$ in Figure 1, while other linear combinations $(\beta^\top y_t)$ will behave like the plots for $-2 < \pi < 0$. So in the multivariate setting, both the level, the range, and the coherency properties of scenarios
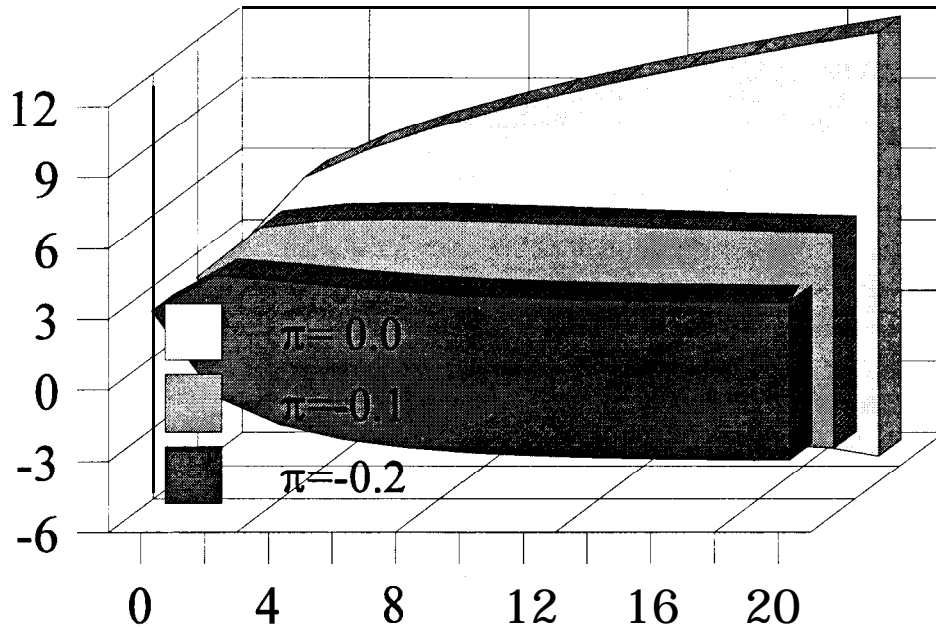
Figure 1: *Forecast Intervals of an $AR(1)$ Model (3) with $y_0 = 3$ and Forecast Horizon 0 through 20.*

are affected by the presence or absence of cointegration. It is intuitively clear that such differences must also have their impact on the decisions that are based on these scenarios. For example, if y_t denotes the price of a specific asset category, this asset would be riskier in the long term and thus potentially less interesting, for $\pi = 0$ than for $-2 < \pi < 0$.

To conclude this subsection, note that if $\mu \neq 0$, the time-series may display trending behavior. Again, this is most easily seen in the univariate case. For $\pi = 0$, the constant μ in (2) then becomes the drift of a random walk process, resulting in a trend term μt in the levels y_t. For $-2 < \pi < 0$, by contrast, μ is directly related to the unconditional mean of y_t, so that the deterministic trending part in y_t eventually levels off and becomes constant. Both effects can be seen directly by-considering the deterministic function of time in (3), $\sum_{i=0}^{t-1}(I + \Pi)^i\mu$. Again, in the multivariate setting the effects are similar. Certain linear combinations of the constant term cause linear trending, while other combinations have a bounded impact on the levels of the series. It is clear, therefore, that the cointegrating properties of the system also have important effects on the trending behavior of scenarios through the specification and interpretation of deterministic regressors in (2), like the

constant   term.

2.2. Error-correction

In the previous subsection, the focus was on the long-run properties of cointegrated systems. Cointegrating combinations of a time-series show much less long-term variability than non-cointegrating combinations. This has direct implications for decision making over long planning horizons. In order to give an assessment of the effect of cointegration on short-term financial decision making, we have to take a slightly different point of view using the concept of error-correction.

As mentioned in Subsection 2.1, the cointegrating relations are also called equilibrium relations. In fact, the elements of $\beta^\top y_t$ give the deviations from the long-run equilibrium relations, i.e., the equilibrium errors. An alternative way to interpret the cointegrated VAR on the right-hand side in (2) is therefore obtained using the error-correction model parameterization on the left-hand side in (2),

$$\Delta y_t = \alpha(EE)_t + \mu + \varepsilon_t, \tag{5}$$

where $EE_t = \beta^\top y_{t-1}$ is the equilibrium error.   (5) clearly shows that if the time-series variables y_t are in a temporary state of disequilibrium, i.e., $EE_t \neq 0$, then y_t will adapt towards the equilibrium. It is illustrative to present a graphic example of this behavior. Figure 2 presents some simulated time-series based on the bivariate VAR model

$$\begin{pmatrix} \Delta r_t \\ \Delta r_t^* \end{pmatrix} = \begin{pmatrix} 0 \\ -\alpha \end{pmatrix} (r_t^* - r_t) + \varepsilon_t, \tag{6}$$

with ε_t i.i.d. standard normally distributed, $r_0 = r_0^* - 0.1 = 4$, and r_t and r_t^* denoting, e.g., a local and foreign short-term interest rate, respectively. For $0 < \alpha < 2$, (6) is error-correcting. This is clearly seen in the Figure 2. For $\alpha = 0.01$, and even clearer for $\alpha = 0.05$, we see that equilibrium errors $(r_t^* - r_t)$ affect the series in such a way that future equilibrium errors become smaller. As a result, we see that the series for larger values of α stay closer together, i.e., the series are cointegrated. By contrast, for $\alpha = 0$ the equilibrium errors do not seem to affect the pattern of the series, such that r_t and r_t^* may drift arbitrarily far apart. Engle and Granger (1987) and Johansen (1991) formally prove the direct relation between cointegrated models and error-correction models.

The relevance of error-correction mechanisms for short-term financial policy making is evident. If y_t is temporarily in a state of disequilibrium, i.e., $\beta^\top y_t \neq 0$, we can predict part of the near future developments of y_t due to
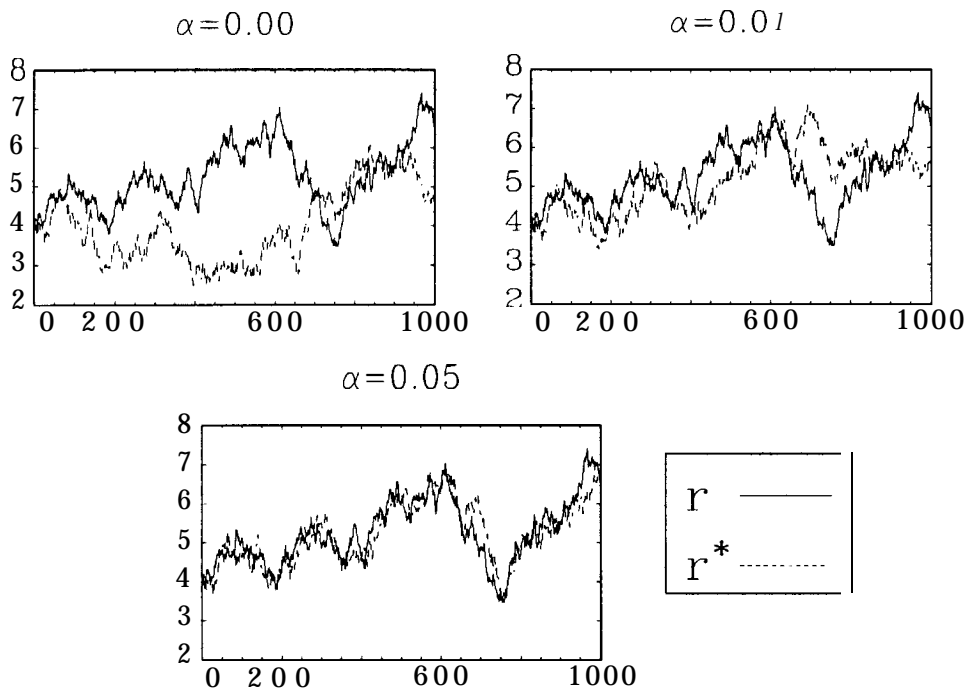
Figure *2: Realization of the Bivariate VAR Model in (6).*

the working of the error-correction mechanisms. The partial predictability of y_t may clearly have an impact on present decisions through the possibility to anticipate future developments in the (economic) state variables.

Concluding, cointegration entails the existence of long-run equilibrium relations between variables. Error-correction, on the other hand, describes the reaction of series to temporary deviations from the long-run equilibrium relations. The presence of long-run equilibrium relations has important consequences for:

1. the short-term predictability of the time-series;

2. the level of forecasts or scenarios;

3. the interpretation of deterministic terms in the regression model and their effect on scenarios;

4. the coherency displayed by the simulated series over time;

5. the confidence bands for forecasts, or put differently, the range of possible scenarios.

The first four consequences mainly affect the level or return dimension of a financial decision process. The fifth consequence, and to some extent also the fourth one, mainly affects the risk dimension of the problem. In all cases, however, it is intuitively clear that the presence or absence of cointegrating relations is relevant for decision making. In the next sections, we formalize these intuitive ideas and disentangle the effects of cointegration on long-term and short-term planning problems.

3. Analytical results

In this section I develop a stylized asset allocation model to qualify the effect of the number of cointegrating relations in a scenario generating model on optimal asset allocations. I first present the model in Subsection 3.1. In Subsection 3.2, I characterize the effect of cointegration on long-term or strategic asset allocation. In Subsection 3.3, I discuss the impact of error-correction on short-term or tactical asset allocation.

3.1. The model

Consider an investment manager that is faced with formulating a dynamic investment policy over a planning period $[0, T]$. The manager has initial funds F(O), while his cumulative funds at time t are equal to F(t). At each point in time, the manager has to decide what amount to invest in each of n + 1 asset categories. Asset category 0 is a money market account giving a risk-free return. The returns on the remaining asset categories are stochastic, such that investing in one of these categories entails a risk. Assume that the performance of the investment manager is measured by his cumulative earnings F(t) , and that the manager, therefore, gets his utility from the total amounts of funds he has to manage. We assume that the utility function of the manager is time-separable and that the manager uses a constant discount factor over the planning period to maximize his expected utility. Formally, the manager tries to maximize

$$E_0 \left[\int_0^T e^{-\rho t} U(F(t)) dt \right] , \qquad (7)$$

where ρ is the discount factor, T denotes the planning horizon, $E_0[\cdot]$ is the expectations operator given the information available at time 0, and $U(\cdot)$ is an increasing, concave utility function. Throughout this section, we will work with a constant relative risk aversion (CRRA) utility function, although the

results in Subsection 3.3 continue to hold for more general utility functions. We have

$$U(F) = \begin{cases} F^{1-\gamma}/(1-\gamma) & \text{for } \gamma \neq 1, \\ \ln(F) & \text{for } \gamma = 1, \end{cases} \tag{8}$$

where $\gamma > 0$ is Pratt's (1964) measure of relative risk aversion. It follows easily that $-U''(F)F/U'(F) = \gamma$, such that the investor is more risk averse for higher values of γ. The framework so far is slightly non-standard, in that we consider an investment manager that has to pick an asset allocation only and derives his utility from the funds under management. In a more standard framework, we would consider an agent who derives his utility from consumption and has to decide on an asset allocation and a consumption plan simultaneously, see, e.g., Merton (1990). In the present paper we abstract from the consumption decision for expository purposes. Incorporating the consumption decision in the model would not alter the qualitative results on the interaction between cointegration and (financial) decision making.

Assume that the return on asset category i at time t over an infinitesimally small holding period dt is given by $\exp(r_i(t)) - 1$. Then the state variable F(t) evolves as follows:

$$dF(t) = F(t)x(t)^{\top}\{r(t) + \tfrac{1}{2}\text{diag}(r(t)r(t)^{\top})\}, \tag{9}$$

with $x(t)^{\top} = (x_0(t), \ldots, x_n(t))$, $r(t)^{\top} = (r_0(t), \ldots, r,(t))$, $F(t)x_i(t)$ denoting the amount invested in asset category $i = 0, \ldots, n$ at time t, and $\sum_{i=0}^{n} x_i(t) \equiv 1$. Equation (9) states that the funds under management grow due to the returns earned by investing the funds available at time t in each of the n + 1 asset categories. The second order term in (9) is due to the possible stochastic nature of the returns.

Given the utility of the manager in (7) and the evolution of the funds that are available for investment, we now describe the nature of the (stochastic) returns on each of the different asset categories. We assume that the return vector $r(t)$ is a function of dt, $q(t)$, and $dq(t)$, where $q(t)$ is an m-dimensional diffusion process

$$dq(t) = \mu dt + \Pi(q(t) - q(0) - \mu t)dt + L^{\top}dW, \tag{10}$$

with IV(t) a standard multivariate Wiener process, and $L^{\top}L = \Omega$. It follows that *q(t)* follows an Ornstein-Uhlenbeck process around the deterministic growth path $q(0)+\mu t$. The definition $r(t) = r(t; dt, q(t), dq(t))$ combined with the evolution of *q(t)* described in (10) comprises a wide variety of stochastic processes for the return vector r(t). For example, if *q(t)* is the logarithm of asset prices at time t and if II = 0, we have $r(t) = dq(t)$ while (10) reduces to the familiar log-Brownian motion model for asset prices. Alternatively, (10)

could be used to describe the return process directly by defining q(t) = r(t). Some special care is needed for the treatment of the risk-free rate of interest. In all models, we assume that $r_0(t)$ does not depend on $dq(t)$, such that the risk-free rate of interest from t to $t + dt$ is known at time t.

The presence of the matrix Πdt in (10) is of central interest in the present paper. In most theoretical papers on asset allocation, we find $\Pi = 0$. By considering non-zero values of Π, we are able to study the effect of cointegration and error-correction. The specification in (10) closely follows the cointegrated VAR model presented in Section 2. Multiplication by the length of the time interval dt follows the approach taken by, e.g., Phillips (1988) and Johansen (1989) for studying the power of cointegration tests under local alternatives. In fact, (10) presents a model exhibiting *local* cointegration. Local cointegration has the advantage that we can employ continuous-time methods as in Merton (1990) to obtain a direct characterization of the optimal asset allocation. Given the discussion in the previous paragraph, we can allow for cointegration in the returns as well as for cointegration in prices. Moreover, by separating the time-series model for $q(t)$ from the asset returns $r(t)$, we allow for situations where the time-series model contains more variables than asset prices only, i.e., m > n+ 1. Such situations can be of interest if the risk of a specific asset category can be split naturally into several risk factors, which can be modeled separately. An example based on empirical data is presented in Section 4.

To complete the model, we abstract from market imperfections, e.g., we assume that short-selling is allowed and that the investment manager can borrow and lend freely on the capital market against the risk-free rate of interest $r_0 (\cdot)$. Incorporating capital market imperfections at the present stage into the model would unnecessarily complicate the exposition.

3.2. Cointegration and strategic asset allocation

In this subsection, I consider the effect of $\Pi \neq 0$ on the "long-term" asset allocation resulting from the financial decision problem described in the previous subsection. For expository purposes, I make the following simplifying assumptions.

First, I assume that the investment manager may decide on the composition of his portfolio only once, namely at time $t = 0$. The chosen portfolio has to be kept until time T, at which time the investment manager is evaluated. So we have $x(t) \equiv$ x. By making this assumption we can abstract from all "short-term" effects of cointegration caused by the ability of the manager to adapt his asset allocation to present economic circumstances. Such short-term effects are considered in detail in Subsection 3.3.

Second, I assume that there is only one (possible) cointegrating relation-ship, i.e., I assume that the rank of Π is equal to one. More specifically, assume that

$$\Pi = \alpha\lambda\beta^{\top}, \tag{11}$$

with α and β column vectors in $\mathbb{R}^m$, $\lambda \in \mathbb{R}$, and $\alpha^{\top}\alpha = \beta^{\top}\beta = 1$. Restricting attention to one cointegrating vector does not affect the generality of the results derived below. Define $\lambda^* = \lambda\alpha^{\top}\beta$, then we can distinguish three situations. First, if $\lambda^* = 0$, there is no local cointegration as $\lambda = 0$ and so $\Pi = 0$. Second, if $\lambda^* > 0$, (10) is locally explosive, as the eigenvalues of $I + \Pi$ lie on or outside the unit circle. Finally, if $\lambda^* < 0$, m $-$ 1 eigenvalues of $I + \Pi$ are equal to one, while the mth eigenvalue is smaller than one, such that $q(t)$ is locally cointegrated. Given the focus of the present paper, I concentrate on the third case, $\lambda^* < 0$. It is interesting to note that in the limiting case $\lambda^* \to -\infty$, the process $\beta^{\top}q(t)$ is stationary, see the comments in Phillips (1988).

The third simplifying assumption made in this subsection concerns the specification of the returns. I assume

$$r(t) = r(t; dt, q, dq) \equiv r(dt, dq). \tag{12}$$

So neither the exact instant in time t nor the level of q(t) affects the returns on the different asset categories. An obvious example in which (12) holds is when q(t) is the vector of log-asset prices. The simplification in (12) is made in order to allow for an explicit solution of the stochastic differential equation (9). This explicit solution can then be used in maximizing the investment manager's utility at time T.

The following theorem can now be proved (see Appendix A).

Theorem 1 *Given the model of Subsection 3.1 and the* three *simplifying assumptions mentioned in the present subsection,*

$$E_0(e^{-\rho T}U(F(T)) = U(F(0)) \cdot \exp\left[h(x, t) + \tfrac{1}{2}(1 - \gamma)^2 x^{\top}r_qQ(t)r_q^{\top}x\right], \tag{13}$$

with $h(\cdot, \cdot)$ *a quadratic* function *in x that does not depend on* α, β, *or* λ, $r_q = \partial r(0, 0)/\partial(dq)$ *a matrix of constants, and*

$$Q(t) = \int_0^t \left(I + (e^{s\lambda^*} - 1)\frac{\alpha\beta^{\top}}{\alpha^{\top}\beta}\right)\Omega\left(I + (e^{s\lambda^*} - 1)\frac{\alpha\beta^{\top}}{\alpha^{\top}\beta}\right)^{\top} ds. \tag{14}$$

The matrix $Q(t)$ in Theorem 1 is the covariance matrix of $q(t)$ given q(0). Theorem 1 implies that if the investment manager is allowed to take

only one strategic decision at the start of the planning period, cointegration only affects this decision through the covariance matrix of the diffusion that determines the asset returns. The fact that $q(t)$ may temporarily be in a state of disequilibrium, $\Pi q(0) \neq 0$, does not affect the long-term decision process. By contrast, the relative long-term riskyness of different asset combinations has a direct effect on the optimal asset allocation.

In order to obtain some more insight into the effect of cointegration, consider the case $|r_q| \neq 0$ with dominant cointegration $\lambda^* \rightarrow$ -co. Then by choosing $x = (r_q^\top)^{-1}\beta$, $x^\top r_q Q(T) r_q^\top x$ reduces to $\beta^\top \Omega \beta (\exp(2\lambda^* T) - 1)/(2\lambda^*) \rightarrow$ 0. Taking into account the sign of $U(F(0))$, the second term in the exponential in (13) has a positive effect on utility if $0 < \gamma < 1$, and a negative effect if $\gamma > 1$. So by choosing $x = (r_q^\top)^{-1}\beta$, we increase utility through this second term for $\gamma > 1$, while a decrease is established for $0 < \gamma < 1$. Naturally, choosing $x = (r_q^\top)^{-1}\beta$ also affects the first term in the exponential in (13), such that the composite effect on expected utility is ambiguous. The formulas show, however, that dominant cointegrating relations will be exploited by the investors who are relatively more risk averse. For extremely risk averse investors $\gamma \rightarrow \infty$, the second term in the exponential in (13) becomes dominant and the optimal asset allocation becomes a direct linear transformation of the cointegrating vector β.

So for long-term decision making, cointegration matters as it reduces the long-term variability of the asset allocations corresponding to the cointegrating relations. 'Though the results for $\lambda^* > -\infty$ and $\gamma < \infty$ will be less clear-cut than the results above, the qualitative conclusions remain similar. Investors that are relatively risk-averse ($\gamma > 1$) will invest in a portfolio that lies closer to the cointegrating space. Investors with a risk aversion parameter $0 < \gamma < 1$, by contrast, will invest in a portfolio further away from the cointegrating space. This last conclusion is intuitively clear once we realize that the variable F(t) is log-normally distributed in the present context. As a result, decreasing the variance of $x^\top r_q q(T)$ also decreases the expected value of F(T). If $\gamma > 1$, this decrease in expected value does not outweigh the variance reduction in the utility trade-off. The opposite holds for $0 < \gamma < 1$.

3.3. Error-correction and tactical asset allocation

After having studied the effect of cointegration on long-term financial planning, we now turn to the short-term decision process. The model is the same as in Subsection 3.1. Note that we drop the three assumptions made in the beginning of Subsection 3.2. Although we stick to the CRRA utility function in the present subsection, it should be kept in mind that the qualitative results obtained here remain valid for more general types of utility functions.

In order to concentrate on the short-term decision process, we assume that the investment manager can revise his portfolio any time in the future. The manager is thus faced with a dynamic decision problem, in which he has to construct an asset allocation rule that produces the optimal allocation given present economic circumstances. So the optimal allocation rule will generally be a function of the economic state variables (q(t)), the funds under management (F(t)), and time (t).

Following Merton (1990, Chapter 4), we start solving the dynamic optimization problem by introducing the value function

$$V(F(t), q(t), t) = \max_{x(t)} E_t \left\{ V(F(t^*), q(t^*), t^*) + \int_t^{t^*} e^{-\rho \cdot s} U(F(s)) ds \right\}, \quad (15)$$

where $t^* > t$, and $E_t(\cdot)$ denotes the expectations operator conditional on the information available at time t. We obtain the following result.

Theorem 2 *A necessary condition for x^* to be an optimal allocation rule for the maximization problem with objective function (7) is*

$$x_i^*(t) = \frac{-V_F}{F \cdot V_{FF}} \tilde{\mu}_i(\Pi(q(t) - q(0) - \mu t), q(t)), \quad (16)$$

for $i = 1, \ldots, n$, and $x_0^*(t) = 1 - \sum_{i=1}^n x_i^*(t)$, with x^* the optimal investment policy, V_F and V_{FF} the first-order and second-order partial derivatives of $V(\cdot)$ with respect to F, and $\tilde{\mu}_i(\cdot, q(t))$ denoting a linear function for every value of $q(t)$.

Theorem 2 states that the optimal short-term asset mix generally depends on time, the amount of funds, and the diffusion process *q(t)* (all through V_F and V_{FF}). If the value function is separable in *t, q(t),* and $F(t)$, however, the optimal asset allocation only depends on $F(t)$ (through $V_F/(FV_{FF})$) and on q(t) (through $\tilde{\mu}_i(\cdot, \cdot)$), see for example Section 4 and Merton (1990). The vector *q(t)* thus affects the optimal mix in two different ways. First, *q(t)* enters directly into $\tilde{\mu}_i$ due to the fact that the returns may be a non-linear function of *dq(t)* and/or *(dt,q(t))*. For example, if *r(t) = dq(t)* with *q(t)* the vector of logarithmic asset prices, $\tilde{\mu}_i$ only depends on $\Pi q(t)$ and not on *q(t)* itself. The effect of cointegration on the optimal asset mix through the second argument of $\tilde{\mu}_i$ is very limited. (Local) cointegration for *q(t)* implies restrictions on the behavior of *q(t)* through time, see also Subsection 3.2. These restrictions, however, are taken as given for the short-term decision process and are not exploited in determining the optimal asset allocation. For example, if *q(t)* contains a long-term and a short-term interest rate and if the

spread between these interest rates is (locally) stationary, then the optimal asset mix takes the levels of the interest rates as given. It does not exploit the fact that the term structure may have forecasting power for the growth rates of interest rates due to the cointegrating properties of the two series. The possible explanatory power of cointegrating relations for future values of $q(t)$ is taken care of by the second argument of $\tilde{\mu}_i$, namely $\Pi(q(t) - q(0) - \mu t)$.

In order to consider the effect of cointegration on the optimal short-term asset allocation in more detail, we again decompose Π into $\alpha\beta^\top$, with α and β being two k x r matrices of full column rank. Furthermore, for simplicity we set $q(0) = \mu = 0$. Using the linearity of $\tilde{\mu}$ in $\Pi q(t)$, (16) shows that it is not the cointegrating relations or equilibrium errors $\beta^\top q(t)$ per se that directly affect the optimal asset mix. What is more important is the way in which the process $q(t)$ reacts to equilibrium errors, i.e., the relative strength of the error-correction mechanism. This mechanism is given by $\alpha\beta^\top q(t) = \Pi q(t)$. The intuition behind this finding is clear. If one of the elements of $q(t)$ is out of equilibrium, we anticipate a price movement towards the equilibrium during the next period, i.e., an error correction, see Section 2. Under the optimal strategy, the investment manager tries to exploit these anticipated price movements. Without loss of generality we can assume that the cointegrating vectors β are normalized such that $\beta^\top\beta = I$. The formal derivations in Appendix A then reveal that the effect of cointegration and error-correction is strongest if the magnitude of the error-correction parameters α is large compared to the variance of diffusion process $\Omega = L^\top L$.

The theoretical analysis in this section has produced two clear-cut results. First, for long-term decision making based on a multivariate time-series model, cointegrating relationships matter for the long-term variability of portfolios. Temporary deviations from equilibrium relations appear to have no influence on strategic decisions. By contrast, for short-term decision making, the temporary deviations from long-term equilibria are of prime importance. In fact, these deviations and the error-correction mechanisms in the time-series model drive the optimal allocation rule. For the short-term, it appears that "error-correction" instead of "cointegration" seems the most important characteristic of the time-series model. The next section applies the findings of the present section to an empirical data set using a stylized investment problem.

4. Empirical illustration: FOREX management

In this section we study a simple international investment problem with one risky, and one risk-free asset. The risk-free asset is a one month deposit,
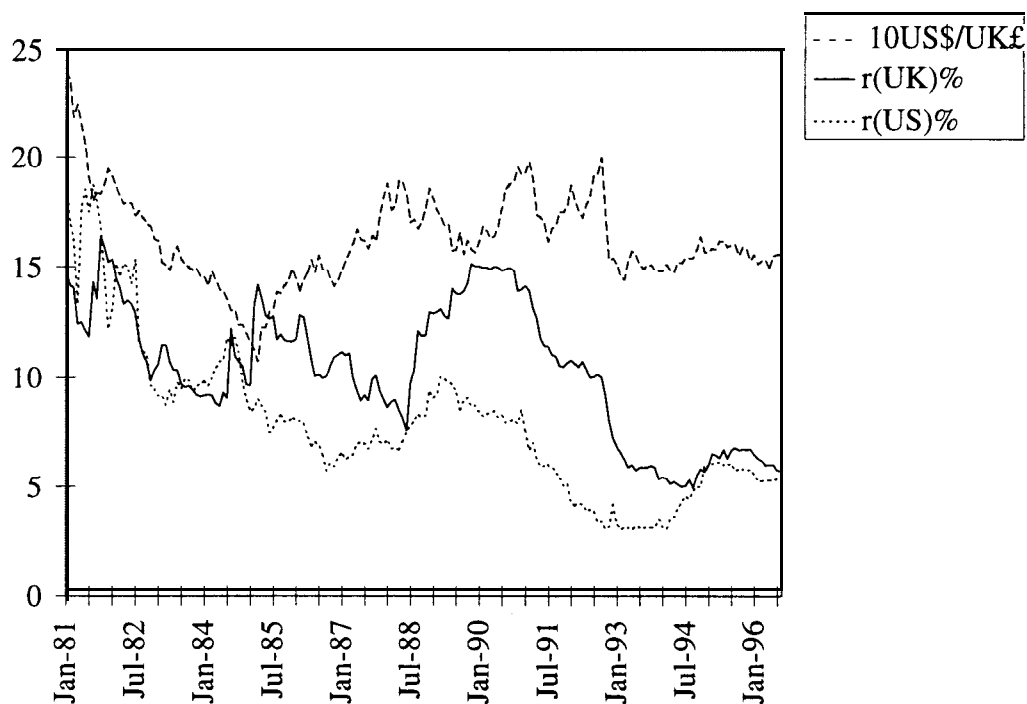
Figure *3: UK and US 1 Month Interest Rates and 10US$/UK£ Spot Exchange Rate.*

giving a certain continuously compounded annualized return of $r_0(t)$ per cent. The risky asset is a one month deposit in a foreign country, giving a certain continuously compounded annualized return of $r_0^*(t)$ in the foreign currency. The exchange (spot) rate at time t is given by *S(t)* and denotes the amount of foreign currency to be received for one unit of the local currency. We use a real data set to illustrate the main effects of cointegration and error-correction on asset allocation. The data are taken from Datastream and contain monthly observations over the period January 1981 until August 1996. We take the UK as the home country, and the US as the foreign country. The data are visualized in Figure 3.

We construct the vector

$$q(t) = (r_0(t), r_0(t) - r_0^*(t), \ln(S(t))^\top.$$

Next, we model $q(t)$ as a VAR process using the empirical data at the monthly frequency. This implies that we take $dt \approx 1/12$. The first element of the vector $q(t)$ is the UK interest rate, the second element is the international UK/US interest rate spread, and the third element is the logarithm of the spot exchange rate (in US$/UK£), which approximately equals the exchange

rate returns. Using standard order selection criteria like Akaike's or Schwarz' information criterion, we select a first order VAR model for *q(t)*. Using the cointegration tests as proposed in Lucas (1996), a cointegrating rank of $r = 1$ is found. In order to investigate the effect of cointegration, we also consider a fully stationary model $(r = 3)$ and a fully non-stationary model $(r = 0)$. In the cointegrated model $(r = 1)$, there appears to be a significant long-run equilibrium relation between the spot exchange rate and the international interest rate spread. For example, if domestic (i.e., UK) interest rates are high compared to the long-run equilibrium, the exchange rate rises, in this case implying a depreciation of the US dollar vis à vis the UK pound. All estimated' models are given in Appendix B. It remains to be mentioned here that we use the specification of the VAR model as explained in Section 2. This is the form of the model that is mostly used in the empirical cointegration literature. As there is a slight difference between the discrete time model introduced in Section 2 and the continuous time model used in Section 3, we have to re-specify $\bar{\mu}(t)$ defined below (A9) as $\bar{\mu}(t) = \mu + \Pi q(t)$.

In discussing the effect of cointegration and error-correction on optimal asset allocation for the above problem, we focus on tactical or short-term asset allocation. This is not uncommon in the present context of currency management. Moreover, in order to implement the (long-term) results from Subsection 3.2, we would need more ad hoc assumptions about, e.g., the length of the planning period and the magnitude of the discount factor. An implementation of the short-term results from Subsection 3.3 requires much less arbitrary choices in this respect.

Given the definition of *q(t)*, we have

$$r_0(t; dt, q(t), dq(t)) \;\; = \;\; q_0(t)dt \tag{17}$$

$$
\begin{aligned}
r_1(t, t + h; \textbf{\textit{q(t)}}, q(t + \textbf{\textit{h}})) \;\; &= \;\; \ln\left(\frac{S(t)\exp(r_0^*(t)dt)}{S(t+h)}\right) \\
&= \;\; (q_0(t) - q_1(t))dt \quad dq_2(t),
\end{aligned}
\tag{18}
$$

where the differential operator must now be interpreted as the difference operator over periods of one month, e.g., $dq_2(t)$ is the one month (logarithmic) exchange rate return. Using the definitions below (Al) in Appendix A, we

---

[1] The models are estimated using usual discrete time methods instead of continuous time estimation methods. The continuous time parameter estimates are obtained using a first order approximation, i.e., dividing the discrete time parameter estimates by $\textbf{\textit{dt}} = 1/12$. This first order approximation turns out to be adequate in the present context to illustrate the main points involved.

obtain

$$r_{0,t} = q_0, \quad r_{1,t} = \mathbf{40} - \mathbf{41}, \quad r_{0,q} = 0, \quad r_{1,q} = (0, 0, -1)^\top, \quad r_{0,qq} = r_{1,qq} = \mathbf{0}.$$

Using these equalities, (A9) reduces to

$$0 = \max_{x_1} \left(e^{-\rho t} U(F) + V_t + V_q \cdot \bar{\mu} + \tfrac{1}{2}\mathrm{trace}(V_{qq} \cdot \Omega) + \tfrac{1}{2}V_{FF} \cdot F^2 \cdot \omega_{33}x_1^2 + \right.$$

$$\left. V_F \cdot F \cdot \left\{ r_{0,t} + x_1 \cdot (r_{1,t} - r_{0,t} - \bar{\mu}_3 + \tfrac{1}{2}\omega_{33}) \right\} \right), \tag{19}$$

with $\bar{\mu}_3$ and ω_{33} the 3rd and (3,3)-element of $\bar{\mu}$ and Ω, respectively. (19) determines the optimal tactical asset allocation for infinitesimally small holding periods. In the present experiment, however, the investment manager is forced to hold his portfolio for the minimum period of one month. As a result, the optimal portfolio according to (19) might prove far too risky for the investment manager. This indeed turns out to be the case. Some preliminary experimentation with (19) revealed that allocations that are optimal for infinitesimally small holding periods can lead to unrealistic allocations for a one month holding period. For example, in some cases more than 40 times[2] the amount of available funds is invested in one of the asset categories, resulting in an investment policy that is both far too risky and difficult to implement in practice. In order to avoid this unrealistic behavior, I abstain from the possibility of short-sales by requiring $x_1 \in [0, 1]$. As a result, using Theorem 2 the optimal policy following from (19) is

$$x_1^*(t) = \mathrm{median}(0, \; x_1^{**}(t), 1), \tag{20}$$

with

$$x_1^{**}(t) = \frac{-V_F}{V_{FF} \cdot F \cdot \omega_{33}} \, (-\mathbf{41} - (0, 0, 1)(\Pi q(t) + \mu) + \tfrac{1}{2}\omega_{33}) \cdot \tag{21}$$

Consider the following trial solution for the value function based on commonly found separability results (see, e.g., Merton, 1990),

$$V(t, F, q) = e^{-\rho t} U(F) g(q), \tag{22}$$

for some function *g(q)*. Substituting this candidate solution in (21) and (19) and dividing by $\exp(-\rho t)U(F)$, (19) reduces for $\gamma \neq 1$ to a second order partial differential equation for *g(q)* with varying coefficients. Assuming a

---

[2] It should be noted that these unrealistic asset allocations are also partly due to the crude first order approximation used to estimate the continuous time parameters, see footnote 1.

function $g(\cdot)$ exists that solves this partial differential equation, x_1^{**} in (21) reduces to

$$x_1^{**}(t) = \frac{-q_1 - (0,0,1)(\Pi q(t) + \mu) + \frac{1}{2}\omega_{33}}{\gamma \omega_{33}};$$ (23)

thus simplifying the optimal asset allocation in (20). Equation (23) clearly shows that the optimal asset allocation is stabler for investors that are more risk averse. Moreover, if exchange rate risk $\left(\omega_{33}\right)$ is high, investments in the risky asset will be smaller in magnitude.

Figure 4 presents the optimal values of xi and the corresponding values for the funds under management if the optimal dynamic strategy is applied monthly over the observation period. Three values of the risk aversion parameter γ are considered.

The first thing to note in Figure 4 is that it can make a considerable difference whether or not a cointegrated or non-cointegrated model is used to forecast the returns. The differences become really apparent for $\gamma > 1$. For $\gamma = 1$, we have the logarithmic utility function U(F) $\sim \ln(F)$. In this case, the differences between the optimal asset allocations for different values of the cointegrating rank r are relatively weak, which appears in line with the results of Subsection 3.2. For $\gamma = 1$, therefore, the amount of funds under management F is approximately equal over the whole sample period for all three models considered. If follows directly from intuition and from (23) that if we increase the risk aversion parameter γ, the optimal fraction invested in the risky US deposit should lie closer to zero if all remaining model parameters remain constant. This is clearly seen by looking at the left-hand side plots in Figure 4. It is striking to note that the variation over γ in the optimal value of x_1 is much smaller for $r = 1$ than for $r = 0$ (and $r = 3$). As a result, the total amount of funds under management for $r = 0$ decreases as we increase γ, while F(t) remains relatively constant for the optimal allocations based on $r = 1$. The smaller amount of funds for $r = 0$, caused by lower average returns on the managed portfolio, is a natural consequence of the reduction in risk. By contrast, the stability of F(t) for $r = 1$ illustrates that the cointegrated model does quite well in this case in forecasting future excess returns of the risky asset category. This can be expected given the test results discussed earlier. Using cointegration tests, the cointegrated VAR with one cointegrating relation seems the best model. Therefore, the difference stationary model with $r = 0$ is mis-specified, as a significant regressor is omitted from the model: the equilibrium error

$$s(t) - 6.23\left(r_0(t) - r;(t)\right) + 1.17r_0(t),$$

see Appendix B. Further testing reveals that the UK interest rate is insignificant in this equilibrium relation, and that we only obtain a relation
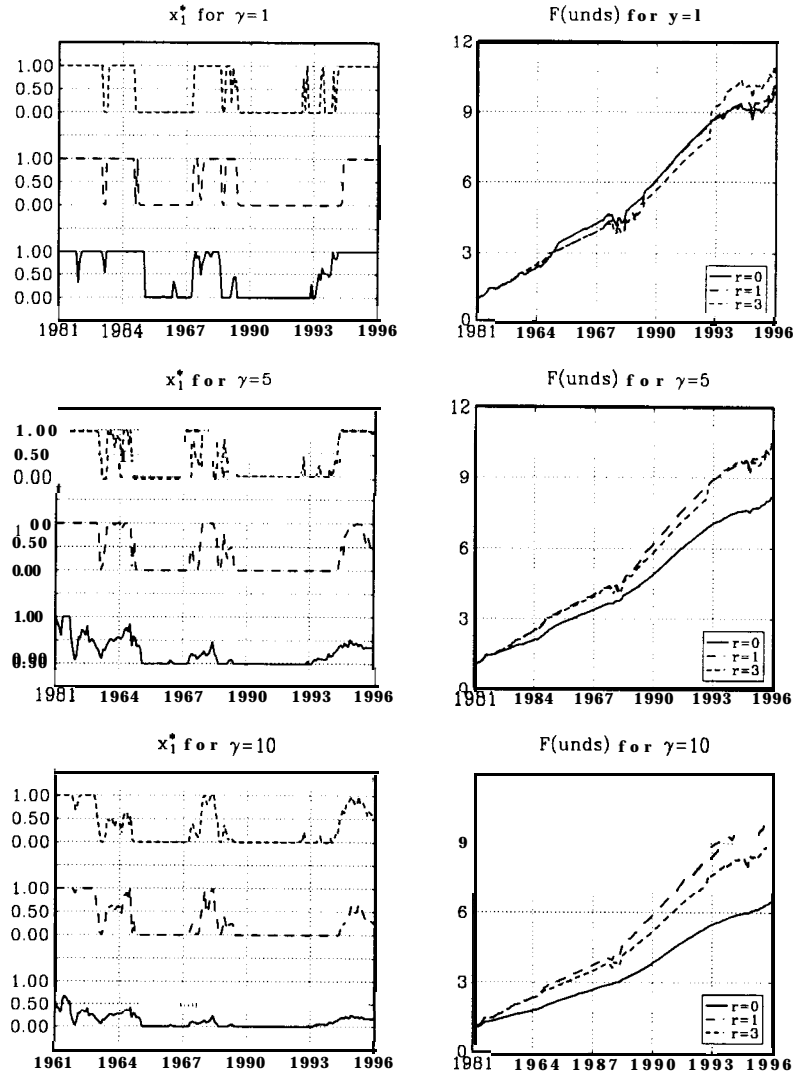
Figure *4: Optimal Fraction* x_1^* *Invested in the US by a UK Investor and the Cumulative Funds* under *Management, Starting from F(0) = 1.*

between the US/UK exchange rate and the UK/US interest rate spread. In particular, a high UK/US interest rate spread causes an upward pressure on the US/UK exchange rate, which appears a plausible result. Omitting this relation as an explanatory variable in the VAR in differences (as for $r = 0$) results in a lack of forecasting power of the corresponding time-series model.

As a final note to Figure 4, it is interesting to see that the results for $r = 1$ and $r = 3$ are quite similar. Only for high values of the risk aversion parameter γ, the cointegrated model seems to do a better job at forecasting future returns than the fully stationary model. These issues are investigated in more depth in Section 5.

5. Model Mis-specification

The findings in Sections 3 and 4 indicate that the presence or absence of cointegrating relations in a time-series model used for financial decision making may have important consequences for financial policy making. Therefore, it is interesting to know the effect of using a mis-specified cointegrating rank in the time-series model. In particular, it is important to find out whether the effects of fixing the cointegrating rank r too high or too low are symmetric. The results in Figure 4 suggest that it is more important not to choose r too low than it is to set r too high. This indeed turns out to be the case for short-term management decisions. For long-term policy making, much more care is needed. The present section investigates these issues in more detail.

In order to allow for mis-specification of the cointegrating rank of the diffusion $q(t)$ in (10), we have to introduce some additional notation. Let μ^*, Ω^*, Π^*, and r^* denote the true drift term, covariance matrix, impact matrix, and cointegrating rank of the process $q(t)$. Note that these values are not necessarily used by the manager in his decision model. By contrast, the manager uses μ^m, Ω^m, Π^m, and r^m to determine his optimal dynamic asset allocation. For example, assume that $r^* = 1$ and that the investment manager is unaware of the concepts of cointegration and non-stationarity. The manager will in that case use either a stationary $(r^m = 3)$ or a difference stationary $(r^m = 0)$ model to base his financial decisions on. Consequently, there will be a discrepancy between the true model generating the $q(t)$ and the model used by the manager. We consider two different cases. First, Subsection 5.1 considers the case $\Pi^m = 0$ $(r^m = 0)$ and $r^* > 0$, such that the investment manager underspecifies the true cointegrating rank. Second, Subsection 5.2 looks at the case $r^m > 0$ and $r^* = 0$, such that we get an idea of the effect of underestimating the cointegrating rank of $q(t)$. Both in Subsection 5.1 and 5.2 the focus is on short-term policy making. Subsection 5.3 contains some comments on the effect of mis-specified cointegrating ranks for long-term policy making.

5.1. Too much non-stationarity: $r^m < r^*$

Assume that the manager uses a difference stationary model $r^m = 0$ and $\Pi^m = 0$, while the true data generating process is either stationary or cointegrated, r* > 0. Equation (4) then clearly shows that the manager uses a mis-specified scenario generator to base his decisions on, as relevant error-correction mechanisms are omitted from the model by restricting r^m to zero.

Apart from the fact that the manager mis-specifies the dynamics of *q(t)*, setting r^m too low compared to r^* also has consequences for the parameter estimates obtained by the manager. Assume that in line with the specification used in Sections 2 and 4, the true data generating process is given by

$$dq(t) = \mu^* dt + \Pi^* q(t) dt + (L^*)^\top dW(t), \tag{24}$$

where $(L^*)^\top L^* = \Omega^*$, and that the manager uses the model

$$dq(t) = \mu^m dt + (L^m)^\top dW(t), \tag{25}$$

where $(L^m)^\top L^m = \Omega^m$. Assume that we observe the process *q(t)* generated by (24) from time *-K* to 0, with *K* > 0. If standard maximum likelihood procedures (or ordinary least-squares) procedures are used by the manager to fix the values of μ^m and Ω^m based on observed data, we get

$$
\begin{aligned}
\mu^m &= \frac{\int_{-K}^0 dq(t)}{K} \\
&= \mu^* + \frac{\Pi^*}{K} \int_{-K}^0 q(t) dt + \frac{(L^*)^\top}{K} W(-K) \\
&= \mu^* + \frac{\Pi^*}{K} \int_{-K}^0 q(t) dt + O_p(K^{-1/2}),
\end{aligned}
\tag{26}
$$

and

$$
\begin{aligned}
\Omega^m &= \frac{\int_{-K}^0 (dq(t) - \mu^m dt)(dq(t) - \mu^m dt)}{K} \\
&= \Omega^*.
\end{aligned}
\tag{27}
$$

So although the variance of the process *q(t)* is estimated correctly, the estimate of the drift term is inconsistent, i.e., $\mu^* \neq \mu^m$. This even holds for long observation periods $K \to \infty$. Together the mis-specification of μ^* and r* generally lead to incorrect predictions of future returns and, consequently, to suboptimal asset allocation strategies. This clearly emerges in the case of tactical asset allocation from the application in Section 4. As mentioned before, formal cointegration tests indicate that r* = 1 for the application

discussed in Section 4. The right panels in Figure 4 illustrate that underestimating the true cointegrating rank indeed leads to suboptimal performance. The amount of funds under management is generally lower, indicating that the portfolio based on the dynamic allocation strategy for $r^m = 0$ generally underperforms with respect to the portfolio obtained with $r^m = r^* = 1$. The effect is more pronounced for investors that are more risk averse, i.e., investors with a higher value of γ. Appendix B also corroborates the above findings. The estimates of Ω are similar for $r = 0, 1, 3$. By contrast, the estimates of the constant term for $r = 0$ are an order of magnitude smaller than the estimates for $r = 1$.

5.2. Too much stationarity: $r^* < r^m$

We now turn to the case $r^m > r^*$, such that the investment manager overestimates the cointegrating rank of $q(t)$. Overestimating the cointegrating rank results in a number of redundant parameters in the VAR model describing the behavior of $q(t)$. So the VAR model is over-specified in this case. Estimates of the model parameters will thus be consistent, but generally inefficient. If the number of observations used for the estimation process is large enough, we therefore obtain $\mu^* \approx \mu^m$, $\Pi^* \approx \Pi^m$, and $\Omega^* \approx \Omega^m$ for $r^* < r^m$. Consequently, the dynamic optimal asset allocation of the manager generally coincides with the true optimal policy. This stands in sharp contrast with the results obtained in Subsection 5.1. The findings are corroborated by the results in Appendix B and Figure 4. The right panels in Figure 4 indicate that the performance of the optimal policy based on $r^m = 3$ is approximately equal to that based on $r^m = r^* = 1$. Both policies outperform the allocation strategy based on $r^m = 0$.

5.3. Cautionary remarks

Although it might seem advisable given the results in Section 4 and Subsections 5.1 and 5.2 to overestimate rather than to underestimate r^*, one should remember that the results in this section are based on continuous time and asymptotic   approximations. Therefore, some cautionary remarks are in order if the results of this section are to be applied in practice. These remarks are especially relevant if long-term decision problems are involved.

The first remark concerns the application of statistical theory for testing in models that overestimate the cointegrating rank of a time-series system. Phillips (1991) d early shows that the statistical theory for determining whether particular parameters in the VAR model for $q(t)$ are significant or not, is a mixture of standard and non-standard distribution theory if one

does not account for the correct cointegrating rank. This should be kept in mind when performing further model selection for the scenario generator if $r^m > r^*$. Furthermore, although the redundant regressors for $r^m > r^*$ do not effect the consistency of the relevant parameter estimates, the implied efficiency loss can substantially affect the estimates and thus the financial decisions in finite samples.

The second remark is even more important and concerns the finite sample bias in the estimate of the parameter matrix II in (1). As is well known, see, e.g., Andrews (1993) and Abadir et al. (1994), the estimate of the first order autoregressive parameter $(I + \mathrm{II})$ is generally biased towards zero in finite samples. This bias becomes worse in higher dimensions, see Abadir et al. (1994). The bias in the autoregressive parameter results in 'too much stationarity' for certain linear combinations of $q(t)$. As was explained in Figure 1 in Section 2, the stationarity properties of a multivariate time-series model have a large impact on the long-term characteristics of scenarios that are generated with these models. Therefore, for long-term decision making, the finite sample bias might prove quite important. By contrast, this bias is less important for short-term policy making, as only one-step-ahead or few-steps-ahead predictions are needed in that case. As illustrated in Figure 1, the differences between stationary models and non-stationary models only become clear at longer horizons.

The above remarks are in line with recent findings of Christoffersen and Diebold (1997). They indicate that from a generalized mean squared error-perspective, it is important for long-run forecasting not to under-estimate the integrating rank of a time-series. Put differently, they claim that one should not over-estimate the cointegrating rank of a system if the time-series model is primarily used for long-run forecasting. Concluding, we can say that for short-term forecasting the results of Subsection 5.1 and 5.2 indicate that one should not impose too many unit roots in order not to miss any advantageous investment opportunities caused by short-term disequilibrium situations. For long-term planning problems, the cautionary remarks in the present subsection apply, indicating that one should be careful in over-estimating the cointegrating rank.

6. Conclusions

In this paper I investigated the effect of the long-term properties of multivariate time-series models on optimal short-term and long-term asset allocation. The prime focus was on the interaction between contemporary econometric time-series methods and financial decision making. The absence or presence

of long-run equilibrium relationships, i.e., cointegrating relations, turned out to be important for financial planning. For long-term decision making, it proved important that cointegrating combinations of time-series generally show less variability than non-cointegrating combinations. This has consequences for the risk characteristics of long-term policies. For short-term decision making, it proved important that cointegrating models are error-correcting: short term deviations from long-run equilibria are annihilated over subsequent periods. The error-correction property affects the optimal financial decision through the partial predictability of short-term economic developments in states of temporary disequilibrium. The analytical results in the paper were corroborated using a small stylized empirical illustration.

Given the importance of the cointegrating properties of a time-series model for financial decision making, the paper also addressed the effect of using a mis-specified cointegrating rank. For short-term policy making, it turned out that is better to over-estimate rather than to under-estimate the cointegrating rank. If the cointegrating rank is set too low, profitable short-term investment possibilities are missed because one does not exploit the error-correcting behavior of the time-series process. Over-estimating the cointegrating rank has not got a substantial adverse effect for short-term policy making. Short-term decisions require only one-step-ahead or few-steps-ahead predictions. Because the differences between predictions of time-series models with unit roots estimated and unit roots imposed are generally fairly close if the prediction horizon is short, it is intuitively clear that too large a cointegrating rank should have little impact on short-term decision making. By contrast, for long-term decision making predictions from models with unit roots imposed can be substantially different from the forecasts based on models with estimated unit roots. Therefore, based on findings of Christoffersen and Diebold (1997), it seems preferable for long-term planning problems not to over-estimate the cointegrating rank.

A. Proofs

This Appendix contains the proofs of the main theorems of this paper.

Proof of Theorem 1.   Using the fact that r(t) = $r(t; dt, q, dq)$ and the evolution of *q(t)* as given by (10), we obtain

$$r_i(t) = r_{i,t}(t; 0, q, 0)dt + r_{i,q}(t; 0, q, 0)^\top dq + \tfrac{1}{2}(dq)^\top r_{i,qq}(t; 0, q, 0)dq + o_p(dt), \quad \text{(A1)}$$

with $r_{i,t} = \partial r_i/\partial(dt)$, $r_{i,q} = \partial r_i/\partial(dq)$, and $r_{i,qq} = \partial r_{i,q}/\partial(dq)^\top$. This follows easily by noting that $r_i(t; 0, q, 0) = 0$, i.e., if the holding period is zero and asset prices are constant, returns are zero. Using the assumption in (12), we have that $r_{i,t} = r_{i,t}(t; 0, q, 0)$, $r_{i,q} = r_{i,q}(t; 0, q, 0)$, and $r_{i,qq} = r_{i,qq}(t; 0, q, 0)$ are constant. Combining (9) and (A1), we obtain

$$\frac{dF}{F} = x^\top(r_t dt + r_q dq) + \sum_{i=0}^{n} \tfrac{1}{2} x_i \text{trace}\{(r_{i,qq} + r_{i,q} r_{i,q}^\top)\Omega\} dt \quad \text{(A2)}$$

and

$$\left(\frac{dF}{F}\right)^2 = x^\top r_q \Omega r_q^\top x \, dt, \quad \text{(A3)}$$

with r_q a matrix with ith row equal to $r_{i,q}^\top$. Moreover,

$$d\ln(F) = \frac{dF}{F} - \tfrac{1}{2}\left(\frac{dF}{F}\right)^2 = x^\top r_q dq + h^*(x)dt, \quad \text{(A4)}$$

with

$$h^*(x) = x^\top r_t + \tfrac{1}{2} \sum_{i=1}^{n} x_i \text{trace}\{(r_{i,qq} + r_{i,q} r_{i,q}^\top)\Omega\} - \tfrac{1}{2} x^\top r_q \Omega r_q^\top x. \quad \text{(A5)}$$

Prom (A4) it follows that

$$F(t) = F(0) \cdot \exp[x^\top r_q(q(t) - q(0)) + h^*(x)t]. \quad \text{(A6)}$$

Note that *q(t)* is normally distributed with mean *q(0)* + μt and variance-covariance matrix *Q(t)*, with

$$Q(t) = \int_0^t \exp(s\Pi)\Omega \exp(s\Pi^\top)ds, \quad \text{(A7)}$$

see, e.g., Phillips (1988). Using the normality of *q(t)* and the CRRA utility function, we obtain

$$E_0(e^{-\rho t}U(F)) = U(F(0)) \cdot \exp\left[\left((1-\gamma)(h^*(x) + x^\top r_q \mu) - \rho\right)t +\right.$$

$$\left. \tfrac{1}{2}(1-\gamma)^2 x^\top r_q Q(t) r_q^\top x\right| \quad \text{(A8)}$$

The result now follows by noting that

$$\exp(s\Pi) = I + (e^{s\lambda^*} - 1)\frac{\alpha\beta^\top}{\alpha^\top\beta}.$$

$$\mathcal{Q.E.D.}$$

Proof of Theorem 2. Following Chapter 4 of Merton (1990, Chapter 4) and using (Al), we obtain the Bellman-Dreyfus fundamental equation for optimality (compare (A2) and (A3))

$$0 = \max_{x_0+\ldots+x_n=1}\left(e^{-\rho t}U(F) + V_t + \mathrm{vq}\cdot\bar{\mu} + \tfrac{1}{2}\mathrm{trace}\left(V_{qq}\cdot\Omega\right) + \right.$$

$$V_F\cdot F\cdot\sum_{i=0}^{n} x_i\left[r_{i,t} + r_{i,q}^\top\bar{\mu} + \tfrac{1}{2}\mathrm{trace}\{ (r_{i,qq} + r_{i,q}r_{i,q}^\top)\Omega\} + \right] +$$

$$\left. \tfrac{1}{2}V_{FF}\cdot F^2\cdot\sum_{i=0}^{n}\sum_{j=0}^{n} x_i x_j\left\{ \mathrm{trace}(r_{i,q}r_{j,q}^\top\Omega)\right\}\right), \tag{A9}$$

with subsripts of the value function $V(\cdot)$ indicating the variable of differentiation (e.g., V_t indicating the partial derivative of $V(\cdot)$ with respect to t), $\bar{\mu} = \bar{\mu}(t) = \mu + \Pi(q(t) - q(0) - \mu t)$, and $r_{i,t}$, $r_{i,q}$, and $r_{i,qq}$ as defined below (Al). It is immediately seen that the objective function in (A9) is quadratic in x (with negative semi-definite Hessian). Therefore, differentiating (A9) with respect to x and solving the first order conditions, we obtain the desired result.

$$\mathcal{Q.E.D.}$$

B. Estimates

This appendix presents the estimates of three models for continuously compounded annualized returns on one-month deposit accounts in the UK $(100\cdot r_0$ per cent) and the US $(100\cdot r_0^*$ per cent) and the logarithm of the spot exchange rate between the US and the UK (s). All models are vector autoregressive of order 1 and differ in the cointegrating rank that is imposed. Using the cointegration testing methodology of Lucas (1996), we find a cointegrating' rank r of 1. In all models, the vector $q(t)$ denotes

$$q(t) = (r_0(t), r_0(t) - r_0^*(t), s(t))^\top. \tag{B10}$$

As we use monthly data to estimate the model's parameters, we let $q(n)$ denote the value of $q(t)$ in month n. Note that the estimates below are multiplied by 12 $(= 1/dt)$ before the numbers are substituted in the formulas of Sections 3 and 4. This amounts to taking a first order approximation to the continuous time estimates. The models are now presented in (B11) through (B16).

B.1. Difference stationary model: $r = 0$

$$q(n+1) - q(n) = \begin{pmatrix} -0.0005 \\ 0.0002 \\ -0.0023 \end{pmatrix} + \varepsilon(n), \qquad \text{(B11)}$$

with

$$\text{Var}(\varepsilon(n)) = 10^{-5} \begin{pmatrix} 4.488 & 3.664 & -3.665 \\ 3.664 & 7.960 & -0.096 \\ -3.665 & -0.096 & 121.500 \end{pmatrix}. \qquad \text{(B12)}$$

B.2. One cointegrating relation: $r = 1$

$$q(n+1) - q(n) = \begin{pmatrix} 0.0014 \\ 0.0053 \\ -0.0456 \end{pmatrix} (\ 1.1665\ \ -6.2266\ \ 1.0000\ )\ q(n) +$$

$$\begin{pmatrix} -0.0011 \\ -0.0021 \\ 0.0180 \end{pmatrix} + \varepsilon(n), \qquad \text{(B13)}$$

with

$$\text{Var}(\varepsilon(n)) = 10^{-5} \begin{pmatrix} 4.478 & 3.627 & -3.342 \\ 3.627 & 7.821 & 1.103 \\ -3.342 & 1.103 & 111.100 \end{pmatrix}. \qquad \text{(B14)}$$

B.3. Fully stationary model: $r = 3$

$$q(n+1) - q(n) = \begin{pmatrix} -0.0065 & -0.0387 & -0.0060 \\ 0.0495 & -0.0795 & -0.0086 \\ 0.0793 & 0.2255 & -0.0659 \end{pmatrix} q(n) + \begin{pmatrix} 0.0040 \\ 0.0012 \\ 0.0155 \end{pmatrix} + \varepsilon(n),$$

$$\text{(B15)}$$

with

$$\text{Var}(\varepsilon(n)) = 10^{-5} \begin{pmatrix} 4.292 & 3.439 & -3.430 \\ 3.439 & 7.425 & 0.456 \\ -3.430 & 0.456 & 109.600 \end{pmatrix}. \qquad \text{(B16)}$$

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