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Strategy-Proof Package Assignment

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Abstract

We examine the strategy-proof allocation of multiple divisible and indivisible resources; an application is the assignment of packages of tasks, workloads, and compensations among the members of an organization. We find that any allocation mechanism obtained by maximizing a separably concave function over a polyhedral extension of the set of Pareto-efficient allocations is strategy-proof. Moreover, these are the only strategy-proof and unanimous mechanisms satisfying a coherence property and responding well to changes in the availability of resources. These mechanisms generalize the parametric rationing mechanisms (Young, 1987), some of which date back to the Babylonian Talmud.

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1 Introduction

This paper introduces incentive compatible mechanisms to allocate multiple divisible and indivisible resources. Applications include the assignment of packages of tasks, workloads, support personnel, and compensations among a hospital's medical staff or among an academic department's faculty. These allocation problems differ from combinatorial auctions and package exchanges (as surveyed by Milgrom, 2007) in that cash transfers are constrained or impossible. Despite their relevance, these "package assignment problems" have not been studied systematically.

As in most economic design problems, the relevant information to evaluate the welfare impact of choosing a mechanism, the preferences of the agents involved, is privately held. Successful real-life mechanisms overcome this difficulty, and the resulting incentives for manipulation, by making truthful preference revelation a dominant strategy. These mechanisms are known as *strategy-proof* and examples include the matching mechanisms in school choice (Abdulkadiroğlu and Sönmez, 2003; Pathak and Sönmez, 2008; Abdulkadiroğlu et al., 2009), kidney exchange (Roth et al., 2004, 2005), and entry level labor markets (as surveyed by Roth, 2002). The focus on dominant strategy incentive compatibility is due to its minimal assumptions about agents' knowledge and behavior. Since reporting preferences truthfully is a dominant strategy, equilibrium behavior does not depend on beliefs, common knowledge of rationality and the information structure, etc. This gives a predictive power and robustness that is important for practical mechanism design (Wilson, 1987; Bergemann and Morris, 2005).

Unfortunately, in package assignment problems, sequential dictatorship is essentially the only strategy-proof and efficient mechanism.[1] This mechanism is neither individually rational nor equitable. Often these distributional objectives will override efficiency and thus exclude this mechanism. In other words, the designer faces a tradeoff between efficiency and any other objective she may want to implement. This paper describes the class of strategy-proof mechanisms that avoid a number of drawbacks once efficiency is relaxed.

First, we exclude the most inefficient mechanisms. Every mechanism in the class is *unanimous*: if an allocation yielding each agent her ideal assignment is feasible, then the mechanism delivers this allocation. Though sequential dictatorship is the only efficient mechanism in the class, strongly egalitarian mechanisms are also members.

Second, we exclude mechanisms that recommend allocations contradicting each

[1] Sequential dictatorship is the mechanism whereby agents are arranged sequentially and resources are allocated accordingly. The first agent in the sequence is assigned her best possible package. Conditional on this, the second agent is assigned her best possible package, and so forth.

other. A mechanisms is *consistent* if its recommendations in problems involving different groups of agents and resources are coherent.[2]

Third, we exclude mechanisms that don't respond well to changes in the availability of resources. A mechanism is *resource-monotonic* if, when the supply of each resource increases, no agent's assignment of any resource decreases. This excludes mechanisms that are discontinuous in changes in the availability of resources. The joint implication of our properties is to exclude mechanisms that depend on information that may be regarded as irrelevant: preferences over unavailable resources, how an agent compares the packages received by agents other than herself, etc.

Every strategy-proof, unanimous, consistent, and resource monotonic mechanism is specified by a list of concave functions (Theorems 1, 2, and 3). These functions determine how heavily an agent's welfare is weighed against another's. According to the scarcity of resources, a function is drawn from this list for each agent and each resource. The sum of these functions is then maximized subject to efficiency constraints. The unique maximizer is the allocation recommended by the mechanism. We call the mechanisms defined in this way *separably concave*. Appropriately specifying the list of concave functions defines mechanisms satisfying additional design objectives: when resources are privately owned so that each agent starts off with an endowment of the resources, individual rationality with respect to these endowments has precise implications on the functional forms of the functions. Fairness properties like "no-envy" (Foley, 1967) or "fair net trades" (Schmeidler and Vind, 1972) can also be achieved specifying a list of functions (see Section 5).

The rest of this paper is organized as follows. Section 2 overviews the most relevant literature. Section 3 introduces the package assignment problem. Section 4 introduces the strategic and normative properties of mechanisms. Section 5 introduces the separably concave mechanisms and contains the main results. Section 6 illustrates the flexibility these mechanisms have to accommodate additional design criteria. All proofs are collected in the Appendix.

2 Related literature

To the best of our knowledge, package assignment problems, involving the allocation of multiple divisible and indivisible resources, have not been studied, especially in connection with strategy-proofness. The framework however does overlap and

[2]Consistency is one the most thoroughly studied principles in resource allocation. See Thomson (2011) for an overview. Balinski (2005) and Thomson (2012) discuss the normative content of consistency which Balinski calls "coherence."

include, as special cases, well known resource allocation problems.

There are two strands of research on allocation problems where agents are assigned or exchange packages of indivisible objects. The most mature strand focuses on price equilibria (Gul and Stacchetti, 1999; Bikhchandani and Ostroy, 2002; Milgrom, 2007; Milgrom and Strulovici, 2009). The analysis relies heavily on the analytical power of transferable utility and necessarily on the assumption that agents are not budget-constrained: each agent has enough cash to pay the dollar value of her preferred package. Preferences are also assumed to be non-decreasing in the amounts received. Moreover, these models only deal with the allocation of packages of indivisible objects and cash transfers whereas our model captures the allocation of multiple divisible and indivisible resources. These features limit the relevance of this strand of research to the problem of allocating resources or tasks among the members of an organization where budgets and institutional constraints restrict what cash transfers are possible.

The second strand of research on the assignment of packages of indivisible objects focuses on situations where institutional constraints bar the use of cash transfers or any other divisible resource altogether. An application is course allocation in business schools (Budish, 2011). Unfortunately, the strategy-proof and efficient mechanisms are somewhat limited to those based on sequential dictatorships (Pápai, 2000; Klaus and Miyagawa, 2001).

Perhaps the simplest package assignment situation is an Edgeworth box economy. Already here, Hurwicz (1972) established that no individually rational allocation mechanism is strategy-proof and efficient. In fact, a strategy-proof and efficient mechanism is dictatorial (Zhou, 1991; Schummer, 1997; Goswami et al., 2013).[3]

These impossibility results depend critically on the multidimensionality of assignments. If a single divisible resource is to be allocated among agents with single-peaked preferences over their assignments, there is a strategy-proof and efficient mechanism satisfying various equity properties, the "uniform rule" (Sprumont, 1991). Moreover, extensive classes of strategy-proof and efficient mechanisms satisfying other desirable properties are known (Barberà et al., 1997; Moulin, 1999; Massó and Neme, 2007). These properties include consistency (Thomson, 1994a; Dagan, 1996) and resource-monotonicity notions (Thomson, 1994b). Most relevantly, Moulin (1999) describes all the strategy-proof, efficient, consistent, and resource-monotonic mechanisms, albeit with no reference to separably concave maximization. Since, in the single resource case, our properties imply efficiency (Lemma 3), Theorems 2 and 3 both strengthen this characterization, proving unanimity is sufficient, and

[3]Essentially, the conclusions are as grim when more than two agents are involved. See Goswami et al. (2013), Serizawa (2002), and the references therein.

give an intuitive description of Moulin's mechanisms in terms of separably concave maximization.

Finally, the separably concave mechanisms contribute to a recent literature that extends Sprumont's uniform rule to allocation problems involving multiple divisible commodities. Here, preferences satisfy a generalized form of single-peakedness, "multi-dimensional single-peakedness."[4] This extension of the uniform rule, a separably concave mechanism (Section 6), is the only strategy-proof mechanism satisfying a weak efficiency notion and no-envy (Amóros, 2002; Adachi, 2010). Weakening efficiency to unanimity and specifying that agents with the same preferences receive welfare-equivalent assignments essentially singles out the extended uniform rule among strategy-proof mechanisms (Morimoto et al., 2013). A systematic study of the joint consequences of strategy-proofness, a weak efficiency notion, and no-envy in broader preference domains is available (Cho and Thomson, 2012); these results establish that the domain of multidimensional single-peaked preference is "maximal" for the existence of non-trivial strategy-proof and envy-free mechanisms.

3 The package assignment problem

We now describe the elements of a package assignment problem.[5] Agents drawn from a finite set $\boldsymbol{A}$ are assigned packages consisting of amounts of one or more different resource kinds. Let $\boldsymbol{\mathcal{N}}$ denote the subsets of A. The finite set $\boldsymbol{K}$ indexes the different resource kinds that may be available. The resource kinds that are available in indivisible units are indexed by $\boldsymbol{I}$ while those available in divisible units are indexed by $\boldsymbol{D}$. Thus, K is partitioned into I and D.

Each agent has a maximum capacity to receive each of the resources. For example, a worker may be assigned a workload ranging from zero hours to at most ten hours per day so her assignment of the workload lies in $[0, 10]$ if the workload is divisible and in $\{0, 1, 2, \ldots, 10\}$ if the workload can only be assigned in one hour shifts. Introducing

[4] Unlike in the allocation of a single commodity, where strict convexity delivers single-peakedness, the vastness of the class of possibly satiated preferences over a multidimensional commodity space requires a stand to be taken on what the relevant preference domain is. Multi-dimensional single-peakedness was introduced by Barberà, Gul, and Stacchetti (1993) who studied the implications strategy-proofness in multidimensional public choice problems. Earlier work by Border and Jordan (1983) studied closely related domain restrictions in spatial public choice problems.

[5] The basic mathematical notation is as follows: Let $\{Y_i\}_{i\in I}$ be a family of sets Y_i indexed by I. Let $Y^I \equiv \times_{i\in I} Y_i$. For each $y \in Y^I$ and each $J \subseteq I$, we denote by y_J the projection of y onto Y^J. If $x, y \in \mathbb{R}^I$, then $x \geq y$ means that, for each $i \in I$, $x_i \geq y_i$. For each $i \in I$, $\mathbf{e}_i \in \mathbb{R}^I$ denotes the ith standard basis vector, the vector with a one in the ith coordinate and zeros elsewhere. We boldface notation when first introduced.

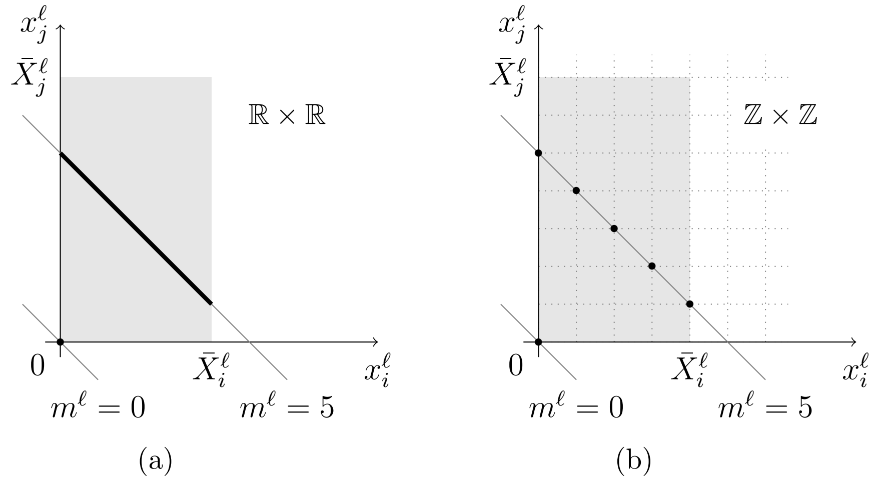


Figure 1: (a) Feasible allocations of a divisible resource of kind ℓ among agents i and j. The allocations corresponding to an an amount $m^\ell = 0$ and $m^\ell = 5$ are represented by the the dot and the thick line, respectively. (b) Feasible allocations of a resource of kind ℓ, available in indivisible units, among agents i and j. The allocations corresponding to available units $m^\ell = 0$ and $m^\ell = 5$ are represented by the the dot and the five aligned dots, respectively.

notation, for each agent $i \in A$, her assignment of resource kind $\ell \in K$ is bounded above by $\boldsymbol{\bar{X}_i^\ell}$; the range of assignments of resource kind ℓ that she may receive, denoted $\boldsymbol{X_i^\ell}$, is thus $[0, \bar{X}_i^\ell]$ if ℓ is in D and $\{0, 1, \dots, \bar{X}_i^\ell\}$ if ℓ is in I. Thus, agent i's assignment lies in $\boldsymbol{X_i} \equiv \times_{\ell \in K} X_i^\ell$ which we refer to as the agent's **assignment space**. We allow for the possibility that agent i is "not qualified" to receive a share of resource ℓ, that is $\bar{X}_i^\ell = 0$.

Each agent, $i \in A$, is equipped with a complete and transitive preference relation $\boldsymbol{R_i}$ over her assignment space, X_i. As usual, $\boldsymbol{P_i}$ denotes the asymmetric part of R_i. The maximizers of R_i over X_i are denoted $\boldsymbol{p(R_i)}$ and are referred to as the **peak of $\boldsymbol{R_i}$**. Whenever $p(R_i)$ is a singleton, say $\{p\}$, we will abuse notation, letting $p(R_i)$ stand for p. The preference relation R_i is **multidimensional single-peaked** if $p(R_i)$ is a singleton and, for each pair of distinct $x, y \in X_i$, $[p(R_i) \geq y \geq x$ or $p(R_i) \leq y \leq x]$ implies $y \; P_i \; x$.[6] Let $\boldsymbol{\mathcal{R}_i}$ denote the class of multidimensional single-peaked preferences over X_i. For each $N \in \mathcal{N}$, let $\boldsymbol{R} \equiv (R_i)_{i \in N}$ and $\boldsymbol{p(R)} \equiv (p(R_i))_{i \in N}$.

For each group of agents $N \in \mathcal{N}$ and each resource kind $\ell \in K$, there is a range of amounts of the resource that can be allocated among the agents in N. This range, denoted $\boldsymbol{M^\ell(N)}$, is $[0, \sum_{i \in N} \bar{X}_i^\ell]$ if ℓ is in D and $\{0, 1, \dots, \sum_{i \in N} \bar{X}_i^\ell\}$ if ℓ is in I.

[6] Barberà, Gul, and Stacchetti (1993) only consider strict preferences over a discrete space. We do not exclude indifferences.

A generic amount in $M^\ell(N)$ is denoted $\boldsymbol{m^\ell}$. Thus, the possible resource profiles available to be allocated among the agents in N is $\boldsymbol{M(N)} \equiv \times_{\ell \in K} M^\ell(N)$. For each $N \in \mathcal{N}$ and each $m \in M(N)$, a feasible **allocation** specifies a distribution of m among the agents in N, a $z \in X^N$ such that $\sum_{i \in N} z_i = m$. For each $N \in \mathcal{N}$ and each $m \in M(N)$, let $\boldsymbol{Z(N,m)}$ denote the collection of feasible allocations, the **feasible set**. Figures 1a and 1b illustrate allocations of a divisible and an indivisible resource, respectively, among two agents.

A package assignment problem or **economy** involving the agents in $N \in \mathcal{N}$ consists of a profile of preferences and a profile of resources to be allocated among the agents in N, $\boldsymbol{(R,m)} \in \mathcal{R}^N \times M(N)$. For each $N \in \mathcal{N}$, let $\boldsymbol{\mathcal{E}^N}$ denote the collection of possible economies involving the agents in N.

4 Allocation mechanisms and their properties

A **mechanism** is a mapping φ that recommends, for each economy (R,m), a unique feasible allocation denoted $\boldsymbol{\varphi(R,m)}$. We now introduce the strategic and normative properties of mechanisms. Unless otherwise specified, we state definitions with respect to a generic group of agents $N \in \mathcal{N}$ and a generic mechanism φ.

We start by recalling the classical efficiency notion. An allocation $x \in Z(N,m)$ is (Pareto) **efficient at $\boldsymbol{(R,m)}$** $\in \mathcal{E}^N$ if there is no $y \in Z(N,m)$ such that, for each $i \in N$, $y_i \; R_i \; x_i$ and, for at least one $i \in N$, $y_i \; P_i \; x_i$. For each $(R,m) \in \mathcal{E}^N$, let $\boldsymbol{P(R,m)}$ denote the set of efficient allocations.

Efficiency: For each $(R,m) \in \mathcal{E}^N$, $\varphi(R,m) \in P(R,m)$.

A minimal efficiency requirement is that the unanimously best allocation is chosen whenever feasible.

Unanimity: For each $(R,m) \in \mathcal{E}^N$, if $p(R) \cap Z(N,m) \neq \varnothing$, then $\varphi(R,m) \in p(R)$.

We turn to strategic issues. As discussed before, strategy-proofness is the most compelling incentive compatibility criterion. This has led to its central place in market design. Examples include second-price auctions, deferred acceptance and top-trading cycles mechanisms for school-choice problems (Abdulkadiroğlu and Sönmez, 2003), and the matching mechanisms proposed for kidney exchange (Roth et al., 2004, 2005). This paper is a contribution to a body of positive results, identifying strategy-proof mechanisms with desirable distributional properties in economic domains. This is in contrast to abstract social choice where strategy-proofness implies that non-trivial mechanisms are dictatorial (Gibbard, 1973; Satterthwaite, 1975).

Strategy-proofness: For each $(R, m) \in \mathcal{E}^N$, each $i \in N$, and each $R'_i \in \mathcal{R}_i$, $\varphi_i(R, m)\ R_i\ \varphi_i(R'_i, R_{-i}, m)$.

Beyond its implementation appeal, *strategy-proofness* has been advocated on fairness grounds. If a mechanism is not *strategy-proof*, strategic agents can manipulate the at the expense of non-strategic agents (Pathak and Sönmez, 2008).

We move on to consistency, a concept introduced in Nash-bargaining by Harsanyi (1959).[7] Harsanyi argued that if an allocation is viewed as a desirable compromise among a group of agents, then it should not be the case that upon receiving their assignments, two agents pooling their resources will arrive at a different compromise. This idea has been key in the analysis of a wide range of allocation problems.[8]

Consistency: For each $\{N, N'\} \subseteq \mathcal{N}$ such that $N' \subseteq N$, each $(R, m) \in \mathcal{E}^N$, and each $i \in N'$, $\varphi_i(R_{N'}, \sum_{j \in N'} \varphi_j(R, m)) = \varphi_i(R, m)$.

Consistency implies "non-bossiness" (Satterthwaite and Sonnenschein, 1981), a property that has played a role in the study of *strategy-proofness* in economic environments. Non-bossiness requires that an agent is only able to alter another agent's assignment by altering her own. That is, a mechanism φ is **non-bossy** if, for each $(R, m) \in \mathcal{E}^N$, each $i \in N$, and each $R'_i \in \mathcal{R}_i$, $\varphi_i(R, m) = \varphi_i(R'_i, R_{-i}, m)$ implies, for each $j \in N$, $\varphi_j(R, m) = \varphi_j(R'_i, R_{-i}, m)$. *Non-bossiness* has proved useful in the study of classical exchange economies (Barberà and Jackson, 1995; Goswami et al., 2013) and in various one-sided matching problems (Svensson, 1999; Pápai, 2000, 2001).

The last property we consider concerns the changes in a mechanism's recommendations in response to changes in the availability of resources. The property is as follows: if there is more to divide then nobody should get less; equivalently, if there is less to divide then nobody should get more. In the special case of our model when a single resource kind is to be allocated, when K is a singleton, this is exactly the resource-monotonicity property studied by (Moulin, 1999). Moreover, in this context and under efficiency, Moulin's notion coincides with those in Thomson (1994b) and Ehlers (2002).

[7] Harsanyi (1977, Page 196) calls the property "multilateral equilibrium."

[8] As recommended before, see Thomson (2011) for an overview of the extensive literature on consistency and see Balinski (2005) and Thomson (2012) for a discussion of the normative content of consistency. Balinski calls the property "coherence" and argues that it is an important part of what is perceived as just.

Resource-monotonicity: For each $(R, m) \in \mathcal{E}^N$ and each $m' \in M(N)$, $m' \geq m$ implies $\varphi(R, m') \geq \varphi(R, m)$.

5 Separably concave mechanisms

We offer a full description of the class of *strategy-proof*, *unanimous*, *consistent* and *resource-monotonic* mechanisms for the package assignment problem (Theorems 2 and 3 below). As we will show, all these mechanisms maximize a separably concave function over a polyhedral extension of the set of Pareto-efficient allocations.

To illustrate the separably concave mechanisms in the simplest setting, consider the problem of allocating a divisible amount of administrative work m^ℓ to the medical staff in a hospital. Doctors $1, \ldots, n$ would rather do as little of the work as possible and each can do at most $\bar{X}_1^\ell, \ldots, \bar{X}_n^\ell$, respectively. The question of how to allocate m^ℓ among $1, \ldots, n$ has been the subject of a whole strand of research since it was formulated in the context of the adjudication of conflicting claims (O'Neill, 1982).[9] Specific examples and proposed awards can be found in the Babylonian Talmud. However, a systematic procedure or mechanism yielding the awards in these scriptures remained elusive until Aumann and Maschler (1985) succeeded in providing one. Young (1987) then observed that the recommendations made by this mechanism can be computed as solutions to the following optimization problem:

$$\max \textstyle\sum_{i=1}^n u_i(z_i) \quad \text{subject to } \textstyle\sum_{i=1}^n z_i = m^\ell \text{ and } 0 \leq z_i \leq \bar{X}_i^\ell,$$

where

$$u_i(z_i) \equiv \begin{cases} \ln z_i & \text{if} \quad 0 \leq z_i \leq \frac{\bar{X}_i^\ell}{2}, \\ \ln(\bar{X}_i^\ell - z_i) & \text{if} \quad \frac{\bar{X}_i^\ell}{2} \leq z_i \leq \bar{X}_i^\ell. \end{cases}$$

Note that u_i is concave. In fact, the central solutions to this problem can be described as solutions to optimization problems analogous to the one above: each "parametric" mechanism (Young, 1987) can be defined by appropriately choosing the u_i functions.[10] Another central allocation mechanism in the parametric class, "constrained equal awards," is obtained by setting $u_i(z_i) = -z_i^2$.

We have assumed that each doctor prefers as small a share of the administrative work as possible. Note that these preferences are single-peaked and that any division

[9] Claims problems have several interpretations (taxation, bankruptcy, rationing, etc.) and are the most thoroughly studied problems in fair allocation. See Thomson (2003) for a survey.

[10] Young (1987) considers a less general class of concave functions where the function u_i is specific to agent i only in that $\bar{X}_i^\ell$ may be specific to i.

of the administrative workload among the doctors is efficient with respect to these preferences. Thus, we could also define the mechanism rationalizing the awards in the Talmud as the solution to a maximization problem over the set of efficient allocations:

$$\max \textstyle\sum_{i=1}^{n} u_i(z_i) \quad \text{subject to } (z_1, \ldots, z_n) \in P(R_1, \ldots, R_n, m^\ell),$$

where K is assumed to be the singleton $\{\ell\}$ and preferences $R_1, \ldots, R_n$ are assumed to be monotone.

A somewhat surprising observation is that if we were to drop the assumption that $R_1, \ldots, R_n$ are monotone preference relations—and just assume single-peakedness—we could compute the recommendations made by the uniform rule (Sprumont, 1991) by solving[11]

$$\max \textstyle\sum_{i=1}^{n} -z_i^2 \quad \text{subject to } (z_1, \ldots, z_n) \in P(R_1, \ldots, R_n, m^\ell). \tag{1}$$

The uniform rule is *strategy-proof* (Bénassy, 1982; Sprumont, 1991). An insight of this paper is that replacing any $-z_i^2$ by any strictly concave u_i in the above program defines a *strategy-proof* mechanism. This observation extends to our general package assignment model.

All of our mechanisms are obtained as solutions to optimization problems similar to the ones above. The main difference, is that when more than one resource is to be allocated (when $|K| \geq 2$), optimization is no longer defined over the efficient set but over a set containing it. In the single resource case, the two sets coincide (see Remark 1 in the Appendix).

Informally, the mechanisms introduced here are specified as follows: for each resource kind ℓ, each agent i is equipped with a pair $(u_i^{xd,\ell}, u_i^{xs,\ell})$ of concave functions over her possible assignments of resource ℓ. The allocation is computed as follows: in situations of **excess demand (*xd*) for resource $\boldsymbol{\ell}$**,[12] the allocation of resource ℓ is chosen so as to maximize $\sum_i u_i^{xd,\ell}$ while insuring no agent receives more than her preferred consumption of ℓ. In situations of **excess supply (*xs*) for resource $\boldsymbol{\ell}$**,[13] the allocation of resource ℓ is chosen so as to maximize $\sum_i u_i^{xs,\ell}$ while insuring no agent receives less than her preferred consumption of ℓ.

[11] This can be derived from either of the following facts: the allocation recommended by the uniform rule can be obtained by choosing the unique allocation that minimizes the variance among all efficient allocations (Schummer and Thomson, 1997). The allocation recommended by the uniform rule is the Lorenz dominant element among all efficient allocations (De Frutos and Massó, 1995).

[12] The sum of the preferred consumptions of resource ℓ exceeds the available amount.

[13] The sum of the preferred consumptions of resource ℓ is less than the available amount.

Formally, let $\boldsymbol{\mathcal{U}}$ denote the profiles of functions $u \equiv \{(u_i^{xd,\ell}, u_i^{xs,\ell}) : i \in A, \ell \in K\}$ where $u_i^{xd,\ell}, u_i^{xs,\ell} : [0, \bar{X}_i^\ell] \to \mathbb{R}$ are strictly concave and continuous. A mechanism φ is **separably concave** if there is $u \in \mathcal{U}$ such that, for each $N \in \mathcal{N}$, each $(R, m) \in \mathcal{E}^N$, and each $\ell \in K$,[14]

$$\varphi(R,m)|^\ell = \arg\max\{\textstyle\sum u_i^{xd,\ell}(z_i) : z \leq p(R)|^\ell, z \in Z(N,m)|^\ell\} \text{ if } \textstyle\sum p(R_i)|^\ell \geq m^\ell,$$
$$\varphi(R,m)|^\ell = \arg\max\{\textstyle\sum u_i^{xs,\ell}(z_i) : z \geq p(R)|^\ell, z \in Z(N,m)|^\ell\} \text{ if } \textstyle\sum p(R_i)|^\ell \leq m^\ell,$$

where summations are taken over $i \in N$. Let $\boldsymbol{\varphi^u}$ denote the separably concave mechanism specified by $u \in \mathcal{U}$.

Theorem 1. *Every strategy-proof, unanimous, consistent, and resource-monotonic mechanism is separably concave.*

Thus, if the mechanism φ is *strategy-proof*, *unanimous*, *consistent*, and *resource-monotonic*, there is a profile of concave functions u in $\mathcal{U}$ such that $\varphi = \varphi^u$. Conversely, when all resouces are divsible, every profile u also induces a *strategy-proof*, *unanimous*, *consistent*, and *resource-monotonic* mechanism (Theorem 2, below). However, when there are indivisibilities, not every profile u will induce a well defined mechanism. Additional necessary and sufficient structure on the profiles in $\mathcal{U}$ ensuring this is established shortly (Theorem 3, below).

Divisible resources

If all resources are divisible, the set of feasible allocations is compact and convex. Moreover, for each N in $\mathcal{N}$ and each m in $M(N)$, the set of feasible allocations $Z(N, m)$ coincides with the product of the $Z(N, m)|^\ell$ sets taken over ℓ in K. Thus, if all resources are divisible, the separably concave mechanisms are well defined in that they select a single feasible allocation.

Theorem 2. *If all resources are divisible, the separably concave mechanisms are the only strategy-proof, unanimous, consistent, and resource-monotonic mechanisms.*

Theorem 2 can be restated as follows: if all resources are divisible, so K coincides with D, a mechanism φ is *strategy-proof*, *unanimous*, *consistent*, and *resource-monotonic* if and only if there is $u \in \mathcal{U}$ such that $\varphi = \varphi^u$.

[14]Here, $\varphi(R,m)|^\ell, Z(N,m)|^\ell$, and $p(R)|^\ell$ denote the projections of $\varphi(R,m), Z(N,m)$, and $p(R)$, all in $\times_{i\in N} X_i$, onto $\times_{i\in N} X_i^\ell$. Similarly, $p(R_i)|^\ell$ denotes the projection of $p(R_i)$ in X_i onto X_i^ℓ.

Indivisible and divisible resources

We now introduce additional necessary and sufficient structure on $\mathcal{U}$ ensuring that each of its profiles of concave functions induces a well defined mechanism.

For each indivisible resource kind, $\ell \in I$, let $\boldsymbol{\mathcal{I}^\ell}$ denote the profiles of functions $(u_i)_{i\in A}$ such that each $u_i : [0, \bar{X}_i^\ell] \to \mathbb{R}$ is strictly concave and, for each $m^\ell \in M^\ell(A)$,[15]

$$\arg\max\left\{\textstyle\sum_A u_i(z_i) : \sum_A z_i = m^\ell, 0 \leq z_i \leq \bar{X}_i^\ell\right\} \text{ is a profile of integers.} \qquad (2)$$

Let $\boldsymbol{\mathcal{U}^*}$ denote the profiles of functions $u \equiv \{(u_i^{xd,\ell}, u_i^{xs,\ell}) : i \in A, \ell \in K\}$ in $\mathcal{U}$ such that, for each $\ell \in I$, $(u_i^{xd,\ell})_{i\in A}$ and $(u_i^{xs,\ell})_{i\in A}$ are in $\mathcal{I}^\ell$.

A mechanism φ is **separably concave** if there is $u \in \mathcal{U}^*$ such that $\varphi = \varphi^u$. This definition of the separably concave mechanisms *refines* the previous one: when there are indivisibilities, Theorem 2 below establishes that $\mathcal{U}^*$ contains all of the profiles of concave functions that induce well defined mechanisms. Moreover, when all resources are divisible, $\mathcal{U} = \mathcal{U}^*$ and the two definitions are identical.

It is not immediately obvious that the separably concave mechanisms are well defined when there are indivisibilities. Though the optima of the maximization problem defining a separably concave mechanism's recommendation are guaranteed to be feasible allocations, the optima may contain more than one allocation. This possibility is ruled out by the following lemma. It establishes that an allocation chosen by the maximization problem defining a separably concave mechanism is, in fact, the unique solution to a maximization problem over the convex hull of the feasible set. Since this convex hull contains the feasible set, there is in fact a single solution to the optimization problem over the feasible set.

Lemma 1. *For each $u \in \mathcal{U}^*$, each $N \in \mathcal{N}$, each $(R, m) \in \mathcal{E}^N$, and each $\ell \in K$,*[16]

$$\varphi(R,m)|^\ell = \arg\max\{\textstyle\sum u_i^{xd,\ell}(z_i) : z \leq p(R)|^\ell, z \in \text{co}Z(N,m)|^\ell\} \textit{ if } \sum p(R_i)|^\ell \geq m^\ell,$$
$$\varphi(R,m)|^\ell = \arg\max\{\textstyle\sum u_i^{xs,\ell}(z_i) : z \geq p(R)|^\ell, z \in \text{co}Z(N,m)|^\ell\} \textit{ if } \sum p(R_i)|^\ell \leq m^\ell,$$

where summations are taken over $i \in N$ and $\text{co}Z(N,m)$ is the convex hull of $Z(N,m)$.

We can now state the main result:

Theorem 3. *The separably concave mechanisms are the only strategy-proof, unanimous, consistent, and resource-monotonic mechanisms.*

[15] The optimization problem has a unique solution because we are maximizing a strictly concave function and the constraints define a convex and compact set. Note also that there is no preference data in (2).

[16] Here, $\varphi(R,m)|^\ell, Z(N,m)|^\ell$, and $p(R)|^\ell$ denote the projections of $\varphi(R,m), Z(N,m)$, and $p(R)$, all in $\times_{i\in N} X_i$, onto $\times_{i\in N} X_i^\ell$. Similarly, $p(R_i)|^\ell$ denotes the projection of $p(R_i)$ in X_i onto X_i^ℓ.

Theorem 3 can be restated as follows: a mechanism φ is *strategy-proof*, *unanimous*, *consistent*, and *resource-monotonic* if and only if there is $u \in \mathcal{U}^*$ such that $\varphi = \varphi^u$. Theorem 2 is a corollary of Theorem 3.

To understand the full breadth of Theorem 3, we characterize the structure of the profiles of concave functions in $\mathcal{I}^\ell$. Lemma 2, below, shows that we can restate the optimization problem in (2) as an integer linear program in a higher dimensional space. Lemma 2 is also used to prove Lemma 1 (see the Appendix). In effect, Lemma 2 shows that a profile in $\mathcal{I}^\ell$ can be approximated by a profile of piece-wise linear functions with decreasing slopes.

To state Lemma 2, define, for each $\ell \in I$, $\boldsymbol{h(\ell)} \equiv \max_{i \in A} \bar{X}_i^\ell$, and let $\boldsymbol{\mathcal{C}^\ell}$ denote the class of matrices $\boldsymbol{c} \equiv \{c_{ik} \in \mathbb{R}_+ : i \in A; k = 1, 2, \ldots, h(\ell)\}$ such that: (i) for each $i \in A$, $c_{i1} > c_{i2} > \cdots > c_{i\bar{X}_i^\ell} > 0$ and, if $k > \bar{X}_i^\ell$, then $c_{ik} = 0$; and (ii) all non-zero entries in matrix c are distinct.

Lemma 2 (**Linear approximation**). *If $(u_i)_{i \in A} \in \mathcal{I}^\ell$ $(c \in \mathcal{C}^\ell)$, then there is $c \in \mathcal{C}^\ell$ $((u_i)_{\in A} \in \mathcal{I}^\ell)$ such that, for each $m^\ell \in M^\ell(A)$, if*

$$x = \arg\max \left\{ \textstyle\sum_{i \in A} u_i(z_i) : \sum_{i \in A} z_i = m^\ell, 0 \le z \le \bar{X}^\ell \right\} \quad \textit{and}$$
$$y = \arg\max \left\{ \textstyle\sum_{i \in A} \sum_{k=1}^{h(\ell)} c_{ik} z_{ik} : \sum_{i \in A} \sum_{k=1}^{h(\ell)} z_{ik} = m^\ell, 0 \le z_{ik} \le 1 \right\}$$

then, for each $i \in A$, $x_i = \sum_{k=1}^{h(\ell)} y_{ik}$ and $y \in \{0,1\}^{|A| \times h(\ell)}$.

Further results for the single resource case

The case where a single resource is to be allocated has received considerable attention, starting with Sprumont (1991). Examples of allocation problems that fit this description include the division of a partnership's output according to the partners' time investments in the partnership, rationing a commodity under disequilibrium prices (Sprumont, 1991), the assignment of workloads to fixed wage employees (Thomson, 1994b), and distributing a scarce product among retailers in a supply chain (Cachon and Lariviere, 1999).

Some properties of the separably concave mechanisms that hold in this case do not hold in general. For example, all separably concave mechanisms are *efficient* and are immune to coalitional manipulation (Section 6).

By Theorem 3, the separably concave mechanisms are the only *strategy-proof*, *unanimous*, *consistent*, and *resource-monotonic* mechanisms. Moreover, these properties imply efficiency when K is a singleton.

Lemma 3. *If K is a singleton, a strategy-proof, unanimous, and consistent mechanism is efficient.*

Therefore, by Theorem 3, we obtain the following:

Corollary 1. *If K is a singleton, the separably concave mechanisms are the only strategy-proof, efficient, consistent, and resource-monotonic mechanisms.*

For the single resource case, Corollary 1 establishes the coincidence of the separably concave mechanisms with the "fixed-path" mechanisms of Moulin (1999). Apart from representing the class of *strategy-proof*, *efficient*, *consistent*, and *resource-monotonic* mechanisms as solutions to optimization problems, our characterization is tighter: Theorem 3 weakens *efficiency* to *unanimity*.

An additional insight from our results is that they bridge a gap with the literature on claims problems discussed in the beginning of this Section. Claims problems can be formally embedded as special cases of our model, where K is a singleton, preferences are monotone, and the upper capacity constraints are interpreted as claims. As we saw, the separably concave mechanisms subsume the parametric mechanisms of Young (1987), some of which date back to the Babylonian Talmud (Aumann and Maschler, 1985; Young, 1987). Thus, the class of separably concave mechanisms can be viewed as extending and generalizing the parametric mechanisms to the package assignment problems studied here.

6 Applications

We now illustrate the breadth and flexibility of the separably concave mechanisms to accommodate individual rationality and various distributional objectives. For simplicity, we discuss applications in the case of divisible resources.

Equity

A central equity notion in fair allocation is "no-envy" (Foley, 1967). A mechanism φ satisfies no-envy if, for each $N \in \mathcal{N}$, each $(R, m) \in \mathcal{E}^N$, and each pair of agents $i, j \in N$, $\varphi_i(R, m)\ R_i\ \varphi_j(R, m)$. That is, the recommended allocations are such that each agent finds her assignment to be at least as desirable as that of any other agent. Equal treatment of equals, a much weaker property, requires that identical agents receive identical assignments. That is, for each $N \in \mathcal{N}$, each $(R, m) \in \mathcal{E}^N$, and each pair of agents $i, j \in N$ such that $R_i = R_j$, $\varphi_i(R, m) = \varphi_j(R, m)$. Note that these properties require that agents have the same assignment spaces.

There is a unique separably concave mechanism satisfying either of these properties. The usual definition of this mechanism (Amóros, 2002; Adachi, 2010; Morimoto et al., 2013), the **commodity-wise uniform rule**, $\boldsymbol{U}$, is as follows: for each $N \in \mathcal{N}$, each $(R, m) \in \mathcal{E}^N$, and each $\ell \in K$,

$$U(R,m)|^\ell = \begin{cases} \min\{p(R_i)|^\ell, \lambda_\ell\} & \text{if } \sum_{i \in N} p(R_i)|^\ell \geq m^\ell, \\ \max\{p(R_i)|^\ell, \lambda_\ell\} & \text{if } \sum_{i \in N} p(R_i)|^\ell \leq m^\ell, \end{cases}$$

where, for each $\ell \in K$, λ_ℓ is the solution to $\sum_{i \in N} \min\{p(R_i)|^\ell, \lambda_\ell\} = m^\ell$ if the first case above holds and otherwise is the solution to $\sum_{i \in N} \max\{p(R_i)|^\ell, \lambda_\ell\} = m^\ell$.

An alternative definition of the commodity-wise uniform rule, emphasizing its membership in the separably concave class, is as follows: $U = \varphi^u$ where $u \in \mathcal{U}^*$ is such that, for each $i \in A$, each $\ell \in K$, and each $z \in X_i^\ell$, $u_i^{xd,\ell}(z) = u_i^{xs,\ell}(z) = -z^2$. The arguments establishing this coincidence are the same, repeated resource by resource, as those used to establish the coincidence of the usual definition of the uniform rule (Sprumont, 1991) and that in the optimization problem in (1).

When all resources are divisible and all agents share the same assignment spaces, the commodity-wise uniform rule is the only *strategy-proof*, *unanimous*, and *non-bossy* mechanism recommending allocations satisfying equal treatment of equals (Morimoto et al., 2013). It is straightforward to verify that *consistency* implies *non-bossiness* (see Lemma 6 in the Appendix). Thus, the commodity-wise uniform rule is singled out, within the separably concave mechanisms, by equal treatment of equals. Since the commodity-wise uniform rule satisfies no-envy (Adachi, 2010), which implies equal treatment of equals, it is also the only separably concave mechanism satisfying no-envy.

Priorities

Suppose that we need to prioritize the agents in A, which we label $\{1, 2, \ldots, n\}$, so that agent 1 has the highest priority, agent 2 has the second highest priority, and so forth. This means that, if agent 1 is not being assigned her ideal assignment or peak, then there should be no other allocation improving upon her current assignment. Conditional on this being achieved, if agent 2 is not being assigned her peak, then there should be no alternative allocation improving upon her current assignment, and so forth. There is a $u \in \mathcal{U}^*$ such that φ^u implements this priority scheme.[17]

[17] For each agent, $i \in \{1, 2, \ldots, n-1\}$, each $\ell \in K$, and each $(x_i, x_{i+1}), (y_i, y_{i+1}) \in X_i^\ell \times X_{i+1}^\ell$, let $u = \{(v_k^\ell, w_k^\ell) : k \in A, \ell \in K\} \in \mathcal{U}$ be such that, for each i and each ℓ, v_i^ℓ and w_i^ℓ are differentiable, and, dropping the ℓ superscript to avoid clutter, $[v_i'(x_i) > v_{i+1}'(x_{i+1})$ and $v_i'(y_i) > v_{i+1}'(y_{i+1})]$, and $[w_i'(x_i) < w_{i+1}'(x_{i+1})$ and $w_i'(y_i) < w_{i+1}'(y_{i+1})]$. Then, for each $(R, m) \in \mathcal{E}^A$, if $x = \varphi^u(R, m)$, it is straightforward to verify that, for each $z \in Z(N, m)$, $x_1 \; R_1 \; z_1$, that for each $z \in Z(N, m)$ such that $z_1 = x_1$, $x_2 \; R_2 \; z_2$, and so forth.

Note that this mechanism is *efficient*. In fact, this is the sequential dictatorship mechanism discussed in the Introduction.

Individual endowments

To discuss individual rationality and other properties specific to situations with individual endowments, we now account for this data. We specify that each agent $i \in A$ has an endowment of resources $\boldsymbol{\omega_i}$ in her assignment space X_i. Then, for each $N \in \mathcal{N}$, we consider the subclass of economies $(R, m) \in \mathcal{E}^N$ such that, $\sum_N \omega_i = m$. These are the economies where the resources to be allocated among the agents in N are precisely the sum of their endowments. For each $N \in \mathcal{N}$, let $\boldsymbol{\mathcal{E}_\omega^N}$ denote this subclass of economies.

A mechanism φ is individually rational if, for each $N \in \mathcal{N}$, each $(R, m) \in \mathcal{E}_\omega^N$, and each $i \in N$, $\varphi_i(R, m)\ R_i\ \omega_i$. That is, we acknowledge each agent's right to receive an assignment at least as desirable as her endowment. Many mechanisms in our class are individually rational. For example, the **generalized commodity-wise uniform rule** is the separably concave mechanism specified by $u \in \mathcal{U}^*$ such that, for each $i \in A$, each $\ell \in K$, and each $z \in X_i^\ell$, $u_i^{xd,\ell}(z) = u_i^{xs,\ell}(z) = -(z - \omega_i^\ell)^2$.

The notion of fair net trades (Schmeidler and Vind, 1972) extends no-envy to situations with individual endowments. It requires that the way the allocation we recommend *adjusts* over endowments satisfies no-envy. A mechanism φ satisfies fair net trades if, for each $N \in \mathcal{N}$, each $(R, m) \in \mathcal{E}_\omega^N$, and each pair of agents $i, j \in N$, $\varphi_i(R, m)\ R_i\ (\omega_i + \varphi_j(R, m) - \omega_j)$, where $\varphi_j(R, m) - \omega_j$ is j's "adjustment." The requirement is mute when these welfare comparisons are not well defined. The generalized commodity-wise uniform rule satisfies fair net trades.

Group strategy-proofness

A distinguishing feature of the case where a single resource is to be allocated is that the separably concave mechanisms are immune to manipulations by groups of agents. A mechanism φ is **group strategy-proof** if, for each $N \in \mathcal{N}$, each $(R, m) \in \mathcal{E}^N$, and each $N' \subseteq N$, there is no $R'_{N'} \in \mathcal{R}^{N'}$ such that (i) for each $i \in N'$, $\varphi_i(R'_{N'}, R_{N \setminus N'}, m)\ R_i\ \varphi_i(R, m)$ and, (ii) for some $i \in N'$, $\varphi_i(R'_{N'}, R_{N \setminus N'}, m)\ P_i\ \varphi_i(R, m)$.

Proposition 1. *If K is a singleton, the separably concave mechanisms are group strategy-proof.*

If more than one resource is to be allocated (when the cardinality of K is greater than one), the separably concave mechanisms are not necessarily *group strategy-proof*. The commodity-wise uniform rule is not (Morimoto et al., 2013).

Appendix

The following notation will be used in this Appendix: for each $N \in \mathcal{N}$, each $Y \subseteq \times_{i\in N} X_i$, and each $\ell \in K$, $\boldsymbol{Y|^{\ell}}$ denotes the projection of Y onto $\times_{i\in N} X_i^{\ell}$. When Y is the singleton $\{y\}$ we let $\boldsymbol{y^{\ell}} = Y|^{\ell}$. For each $\ell \in K$, let $\boldsymbol{p^{\ell}(R_i)}$ denote the projection of $p(R_i)$ onto X_i^{ℓ}. For each $N \in \mathcal{N}$, each $R \in \mathcal{R}^N$, and each $\ell \in K$, let $\boldsymbol{p^{\ell}(R)} \equiv (p^{\ell}(R_i))_{i\in N}$. For each $N \in \mathcal{N}$, each $\ell \in K$, each $r \in \times_{i\in N} X_i^{\ell}$, and each $\alpha \in M^{\ell}(N)$, let

$$\boldsymbol{S^{\ell}(r,\alpha)} \equiv \ \{z \in \times_{i\in N} X_i^{\ell} : \textstyle\sum_N z_i = \alpha, z \leq r\}.$$

Given a set B, let $\mathbf{co}\boldsymbol{B}$ denote its convex hull. Given a function $f : \mathbb{R} \to \mathbb{R}$, for each $x \in \mathbb{R}$, let $\boldsymbol{\partial_+ f(x)}$ and $\boldsymbol{\partial_- f(x)}$ denote the right hand and left hand derivatives of f at x, respectively.

A.1 Proof of Lemma 2 (linear approximation)

Firstly, we show that, given a $u \in \mathcal{I}^{\ell}$, an appropriate $c \in \mathcal{C}^{\ell}$ can be constructed. Secondly, we show that, given a $c \in \mathcal{C}^{\ell}$, an appropriate $u \in \mathcal{I}^{\ell}$ can be constructed.

Let $\ell \in I$, $h \equiv h(\ell) = \max_{i\in A} \bar{X}_i^{\ell}$, and $\bar{m} \equiv \sum_{i\in A} \bar{X}_i^{\ell}$. For each $u \equiv (u_i)_{i\in A} \in \mathcal{I}^{\ell}$, each $c \in \mathcal{C}^{\ell}$, and each $\nu \in M^{\ell}(A)$ let

$$x(\nu, u) = \arg\max\left\{\textstyle\sum_{i\in A} u_i(z_i) : \sum_{i\in A} z_i = \nu, 0 \leq z \leq \bar{X}^{\ell}\right\} \quad \text{and} \tag{3}$$

$$y(\nu, c) = \arg\max\left\{\sum_{i\in A}\sum_{k=1}^{h} c_{ik} z_{ik} : \sum_{i\in A}\sum_{k=1}^{h} z_{ik} = \nu, 0 \leq z_{ik} \leq 1\right\}. \tag{4}$$

Constructing $c \in \mathcal{C}^{\ell}$ from $u \in \mathcal{I}^{\ell}$

Let $u \in \mathcal{I}^{\ell}$. By the definition of $\mathcal{I}^{\ell}$, $u \in \mathcal{I}^{\ell}$ implies that, for each $\nu \in M^{\ell}(A)$, $x(\nu, u) \in \mathbb{Z}^A$ and $\sum_{i\in A} x_i(\nu, u) = \nu$.

Step 1. *For each $\nu \in M^{\ell}(A)$, $x(\nu, u) \geq x(\nu - 1, u)$.*

Proof. Suppose not, then there is $i \in A$ such that $x_i(\nu, u) < x_i(\nu-1, u)$ and $j \in A\backslash\{i\}$ such that $x_j(\nu, u) > x_j(\nu-1, u)$. A necessary conditions for optimality of $x(\nu-1, u)$ is that, $\partial_+ u_j(x_j(\nu-1, u)) \leq \partial_- u_i(x_i(\nu-1, u))$. By strict concavity we obtain the first and last inequality,

$$\partial_- u_j(x_j(\nu, u)) < \partial_+ u_j(x_j(\nu-1, u)) \leq \partial_- u_i(x_i(\nu-1, u)) < \partial_+ u_i(x_i(\nu, u)).$$

Thus, $\partial_- u_j(x_j(\nu, u)) < \partial_+ u_i(x_i(\nu, u))$, but this contradicts optimality of $x(\nu, u)$ which requires that $\partial_- u_j(x_j(\nu, u)) \geq \partial_+ u_i(x_i(\nu, u))$. Hence, $x(\nu, u) \geq x(\nu-1, u)$. ∎

Summing up we have that

$$x_i(\nu, u) = \begin{cases} x_i(\nu-1, u) + 1 & \text{if, for each } j \in A \setminus i, x_j(\nu, u) = x_j(\nu-1, u) \\ x_i(\nu-1, u) & \text{otherwise.} \end{cases}$$

We can now define the entries in the matrix c: for each $i \in A$ and each $j \in M^\ell(A)$, let

$$c_{ij} \equiv \max_{\nu \in \{0,1,\ldots,\bar{m}\}} \left\{ \bar{m} - \sum_{i \in A} x_i(\nu, u) : x_i(\nu, u) = j, j \leq \bar{X}_i^\ell \right\}, \text{ if } j > \bar{X}_i^\ell \text{ let } c_{ij} \equiv 0.$$

Step 2. $c \in \mathcal{C}^\ell$.

Proof. By definition, if $j > \bar{X}_i^\ell$, then $c_{ij} = 0$. We will now show that $c_{ij} > c_{ij'}$ whenever $j < j'$. For each $\nu \in M^\ell(A)$ let $g(\nu) \equiv \bar{m} - \sum_{i \in A} x_i(\nu, u)$. Let $j < j' \leq \bar{X}_i^\ell$. Note that, for each $\nu \in M^\ell(A)$, $g(\nu) = g(\nu - 1) - 1$. Hence g is a strictly decreasing function. By definition of c, $x_i(\nu, u) = j < j' = x_i(\nu', u)$. Hence $\nu < \nu'$, since $x(\nu, u) \geq x(\nu-1, u)$. Thus, $g(\nu) > g(\nu')$ and we have shown that $c_{ij} > c_{ij'}$.

Next we will show that each positive entry in c is distinct. Let $i, i' \in A$ and $j, j' \in M^\ell(A)$. Suppose that $c_{ij} = c_{i'j'} > 0$. By previous argument $i \neq i'$, since $c_{ij} < c_{i'j'}$ or $c_{ij} > c_{i'j'}$ if $i = i'$. By assumption $c_{ij} = c_{i'j'}$ therefore by definition of c, $g(\nu) = g(\nu')$. Thus, $\sum_{i \in A} x_i(\nu, u) = \sum_{i \in A} x_i(\nu', u)$. But this equality can only hold if $\nu = \nu'$. By optimality of g at ν, and also at ν' since $\nu = \nu'$, and the fact that g is strictly decreasing it follows that $x_i(\nu, u) = x_i(\nu-1, u) + 1$ and $x_{i'}(\nu, u) = x_{i'}(\nu-1, u) + 1$. Summing over all agents and noting that $x(\nu, u) \geq x(\nu-1, u)$ gives us $\sum_{i \in A} x_i(\nu, u) \geq \sum_{i \in A} x_i(\nu-1, u) + 2$. But this contradicts feasibility, since $\sum_{i \in A} x_i(\nu, u) = \sum_{i \in A} x_i(\nu-1, u) + 1$. Thus, each positive entry in c is distinct. We have now established that $c \in \mathcal{C}^\ell$. ∎

Let $m' \in M^\ell(A)$, $y' \equiv y(m', c)$ and $x' \equiv x(m', u)$. The following step concludes the first part of the Lemma.

Step 3. *For each $i \in A$, $x'_i = \sum_{k=1}^h y'_{ik}$.*

Proof. By way of contradiction, suppose that there is $i \in A$ such that $x'_i > \sum_{k=1}^h y'_{ik}$. Then there is $j \in A$ such that $x'_j < \sum_{k=1}^h y'_{jk}$. Let $a \equiv \sum_{k=1}^h y'_{ik}$ and $b \equiv \sum_{k=1}^h y'_{jk}$. By assumption $x'_i > a$ and $x'_j < b$, hence it is possible to give more to agent i and less to agent j. Thus, a necessary condition for optimality at y is that $\partial_+ u_i(a) \leq \partial_- u_j(b)$. By strict concavity, the fact that $a < x'_i$ and $b > x'_j$ it follows that,

$$\partial_- u_i(x'_i) < \partial_+ u_i(a) \leq \partial_- u_j(b) < \partial_+ u_j(x'_j).$$

Thus, $\partial_- u_i(x'_i) < \partial_+ u_j(x'_j)$. But this contradicts optimality of x', since a necessary condition for optimality at x' is that $\partial_+ u_j(x'_j) \leq \partial_- u_i(x'_i)$. Thus, $x'_i = \sum_{k=1}^h y'_{ik}$. ∎

Constructing $\boldsymbol{u} \in \mathcal{I}^\ell$ from $\boldsymbol{c} \in \mathcal{C}^\ell$

Let $c \in \mathcal{C}^\ell$ and

$$\gamma \equiv \frac{\min\{|c_{ik} - c_{jl}| : i, j \in A; k, l = 1, \ldots, h; c_{ik} \neq c_{jl}\}}{2}.$$

Note that $\gamma > 0$. For each $i \in A$, let $f_i : [0, \bar{X}_i^\ell] \to \mathbb{R}$ be such that $f_i(0) = c_{i1}$ and, for each $k = 0, 1, \ldots, \bar{X}_i^\ell - 1$ and each $x_i \in (k, k+1]$, $f_i(x_i) = -\gamma(x_i - k) + c_{i(k+1)}$. For each $i \in A$, each $x_i \in [0, \bar{X}_i^\ell]$, let $u_i(x_i) \equiv \int_0^{x_i} f_i(t)dt$. Since each f_i is strictly decreasing, each u_i is strictly concave. Let $u \equiv (u_i)_{i \in A}$. Let $\mu \in M^\ell(A) = \{0, 1, \ldots, \sum_{i \in A} \bar{X}_i^\ell\}$, $x \equiv x(\mu, u)$, and $y \equiv y(\mu, c)$. Let B and C denote the constraint set in the optimization problems defining x and y in equation (3) and (4), respectively.

Step 4. $y \in \{0,1\}^{|A| \times h}$.

Proof. Because $c \in \mathcal{C}^\ell$, the solution to problem $\max\{\sum_{i \in A} \sum_{k=1}^h c_{ik} z_{ik} : z \in C\}$ is unique and is a vertex of the polyhedron C. Moreover, each vertex of C is in $\{0,1\}^{|A| \times h}$. Thus, $y \in \{0,1\}^{|A| \times h}$. ∎

Step 5. *Let $D \equiv \{ik : (i,k) \in A \times \{1, \ldots, h\}, c_{ik} > 0\}$ be labelled according to*

$d_1 = i'k'$ if, for each $ik \in D \setminus \{i'k'\}$, $c_{i'k'} > c_{ik}$;

$d_2 = i''k''$ if, for each $ik \in D \setminus \{d_1, i''k''\}$, $c_{i''k''} > c_{ik}$, and so forth.

Then, $c_{d_1} > c_{d_2} > \cdots > c_{d_{|D|}}$ and

for each $d \in \{d_1, \ldots, d_\mu\}$, $y_d = 1$ and, for each $d \in \{d_{\mu+1}, \ldots, d_{|D|}\}$, $y_d = 0$.

Proof. The fact that $c_{d_1} > c_{d_2} > \cdots > c_{d_{|D|}}$ follows immediately from the labeling and the fact that all the positive entries in matrix c are distinct. Because y is maximizes $\sum_{i\in A}\sum_{k=1}^{h} c_{ik}z_{ik}$ over C, for each pair $i,j\in A$, and each pair $k,k'\in\{1,\ldots,h\}$,

$$[\text{there is } \alpha>0 \text{ such that } y+\alpha(\mathbf{e}_{jk'}-\mathbf{e}_{ik})\in C] \text{ implies } c_{jk'}\le c_{ik},$$

and, because $c\in\mathcal{C}^\ell$, if (j,k') and (i,k) are distinct,

$$[\text{there is } \alpha>0 \text{ such that } y+\alpha(\mathbf{e}_{jk'}-\mathbf{e}_{ik})\in C] \text{ implies } c_{jk'}< c_{ik}. \tag{5}$$

Then, the desired conclusion follows from Step 4. ∎

Before proceeding, note that, since x maximizes $\sum_{a\in A} u_a$, for each pair $i,j\in A$,

$$[\text{there is } \alpha>0 \text{ such that } x+\alpha(\mathbf{e}_j-\mathbf{e}_i)\in B] \text{ implies } \partial_+u_j(x_j)\le\partial_-u_i(x_i). \tag{6}$$

Also, note the following: if i,j and k,k' denote indexes such that $c_{ik}\neq c_{jk'}$, then, for each $z_i\in(k-1,k]$ and each $z_j\in(k'-1,k']$, the definition of γ and the fact $|(z_j-(k'-1))-(z_i-(k-1))|\le 1$ imply $-\gamma[(z_j-(k'-1))-(z_i-(k-1))]\le\gamma<|c_{jk}-c_{ik'}|$. Thus, if $c_{ik}>c_{jk'}$, rearranging yields

$$-\gamma(z_j-(k'-1))+c_{jk'}<-\gamma(z_i-(k-1))+c_{ik}. \tag{7}$$

Step 6. $x\in\mathbb{Z}^A$.

Proof. If not, there are $i\in A$ and $k\in\{0,1,\ldots,\bar{X}_i^\ell-1\}$ such that $k<x_i<k+1$. Since μ is an integer, this implies there are $j\in A\setminus\{i\}$ and $k'\in\{0,1,\ldots,\bar{X}_i^\ell-1\}$ such that $k'<x_j<k'+1$. Thus,

$$f_i(x_i)=-\gamma(x_i-k)+c_{i(k+1)} \text{ and } f_j(x_j)=-\gamma(x_j-k')+c_{j(k'+1)}.$$

Thus, there is $\alpha>0$ such that $x+\alpha(\mathbf{e}_i-\mathbf{e}_j)\in B$ and $x+\alpha(\mathbf{e}_j-\mathbf{e}_i)\in B$. Thus, by (6), $\partial_+u_i(x_i)\le\partial_-u_j(x_j)$ and $\partial_+u_j(x_j)\le\partial_-u_i(x_i)$. Moreover, because f_i and f_j are smooth at x_i and x_j, respectively, $\partial_-u_i(x_i)=\partial_+u_j(x_j)$ and $\partial_-u_j(x_j)=\partial_+u_j(x_j)$. Thus,

$$-\gamma(x_j-k')+c_{j(k'+1)}=\partial_-u_j(x_j)=\partial_-u_i(x_i)=-\gamma(x_i-k)+c_{i(k+1)},$$

contradicting (7) since, without loss of generality, $c_{i(k+1)}>c_{j(k'+1)}$. Thus, $x\in\mathbb{Z}^A$. ∎

Step 7. *Let $i,j\in A$ be such that $i\neq j$. By Step 6, there are $k\in\{0,1,\ldots,\bar{X}_i^\ell\}$ and $k'\in\{0,1,\ldots,\bar{X}_j^\ell\}$ such that $x_i=k$ and $x_j=k'$. Then,*

$$[\textit{there is } \alpha>0 \textit{ such that } x+\alpha(\mathbf{e}_j-\mathbf{e}_i)\in B] \textit{ implies } 0<c_{j(k'+1)}<c_{ik},$$

Proof. Note that [there is $\alpha > 0$ such that $x + \alpha(\mathbf{e}_j - \mathbf{e}_i) \in B$] implies $k' = x_j < \bar{X}_j^\ell$. Thus, since $c \in \mathcal{C}^\ell$, $0 < c_{j(k'+1)}$. Then, by (6) and the definitions of u_i, u_j,

$$c_{j(k'+1)} = -\gamma(k' - k') + c_{j(k'+1)} = \partial_+ u_j(x_j) \leq \partial_- u_i(x_i) = -\gamma(k - k - 1) + c_{ik} < c_{ik},$$

as desired. ∎

Step 8. *For each $i \in A$, $x_i = \sum_{k=1}^h y_{ik}$.*

Proof. If not, there are $i, j \in A$ such that $x_i < \sum_{l=1}^h y_{il}$ and $x_j > \sum_{l=1}^h y_{jl}$. For each $g \in A$, let k_g denote the largest k such that $y_{gk} = 1$. Since, for each $g \in A$, $c_{g1} > \cdots > c_{gk_g}$, Step 5 implies that, for each $k \in \{1, \ldots, k_g\}$, $y_{gk} = 1$. By Step 6, there are integers k, k' such that $x_i' = k$ and $x_j = k'$. Then, by Step 4,

$$k = x_i < \textstyle\sum_{l=1}^h y_{il} = k_i \text{ and } k' = x_j > \sum_{l=1}^h y_{jl} = k_j.$$

Moreover, $\sum_{k=1}^h y_{jk} \geq 0$ and $\sum_{k=1}^h y_{ik} \leq |\{k : c_{ik} > 0\}| \leq \bar{X}_i^\ell$ imply that $k < \bar{X}_i^\ell$ and $k' > 0$. Thus, there is $\alpha > 0$ such that $x + \alpha(\mathbf{e}_i - \mathbf{e}_j) \in B$. Thus, by Step 5, $0 < c_{i(k+1)} < c_{jk'}$. Since $k + 1 \leq k_i$, $c_{i(k+1)} \geq c_{ik_i}$. Now, recall that $k' > k_j$ implies $y_{jk'} = 0$. Thus, by Step 5, $c_{jk'} < c_{ik_i}$. Combining these inequalities,

$$c_{i(k+1)} < c_{jk'} < c_{ik_i} \leq c_{i(k+1)}.$$

This contradiction establishes the desired conclusion. ∎

A.2 Proof of Lemma 1

Before proving Lemma 1 we need the following result.

Lemma 4. *If $(u_i^\ell)_{i\in A} \in \mathcal{I}^\ell$, then for each $N \in \mathcal{N}$ and each $m^\ell \in M^\ell(N)$,*

$$\arg\max\left\{\textstyle\sum_N u_i(z_i) : \sum_N z_i = m^\ell, 0 \leq z \leq \bar{X}_N^\ell\right\} \in \mathbb{Z}^N.$$

Proof. Let $(u_i^\ell)_{i\in A} \in \mathcal{I}^\ell$, $N \in \mathcal{N}$ and $m^\ell \in M^\ell(N)$. Let $x(m^\ell, u)$ and $y(m^\ell, c)$ be defined as in equation (3) and (4). By Lemma 2 there is $c \in \mathcal{C}^\ell$ such that, for each $\nu \in M^\ell(A)$ and for each $i \in A$, $x_i(\nu, u) = \sum_{k=1}^{h(\ell)} y_{ik}(\nu, c)$

First we prove that there is an $\hat{m} \in M^\ell(A)$ such that $\sum_N x_i(\hat{m}, u) = m^\ell$: Let $m \in M^\ell(A)$. Suppose there is $i \in A$ such that $x_i(m, u) > x_i(m+1, u)$. Then there is $j \in A \setminus \{i\}$ such that $x_j(m+1, u) \geq x_j(m, u) + 1$. By optimality at $x(m, u)$, $c_{i(x_i(m,u))} > c_{j(x_j(m,u)+1))}$. By optimality at $x(m+1, u)$, $c_{i(x_i(m,u))} < c_{j(x_j(m,u)+1)}$. But this is a contradiction. Thus, for each $i \in A$, $x_i(m, u) \leq x_i(m+1, u)$. By feasibility $\sum_A x_j(m+1, u) = \sum_A x_j(m, u) + 1$. Hence there is $i \in A$ such that $x_i(m+1, u) =$

$x_i(m,u)+1$ and, for each $j \in A \setminus \{i\}$, $x_j(m+1,u) = x_j(m,u)$. Restricting attention to N we have that $\sum_N x_i(m+1,u) = \sum_N x_i(m,u)+1$, if there is $j \in N$ such that $x_j(m+1,u) = x_j(m,u)+1$, otherwise $\sum_N x_i(m+1,u) = \sum_N x_i(m,u)$. Consider the sequence $(\sum_N x_i(k,u))_{k \in M^\ell(A)}$. It is a weakly increasing sequence bounded by $\sum_N \bar{X}_i^\ell$ and, for each $k \in M^\ell(A)$, $\sum_N x_i(k+1,u) - \sum_N x_i(k,u) \leq 1$. Thus, there is $\hat{m} \in M^\ell(A)$ such that $\sum_N x_i(\hat{m},u) = m^\ell$.

Let $\tilde{x} \equiv x(m^\ell,u)$ and suppose, by way of contradiction, that $\tilde{x} \notin \mathbb{Z}^N$. From the previous paragraph, there is $\hat{m} \in M^\ell(A)$ such that $\sum_N x_i(\hat{m},u) = \sum_N \tilde{x}_i$. Let $\hat{x} \equiv x(\hat{m},u)$. By Lemma 2, $\hat{x}_N \in \mathbb{Z}^N$. Thus, $\tilde{x} \neq \hat{x}_N$. By the optimality of $\tilde{x}$, $\sum_N u_i(\tilde{x}) > \sum_N u_i(\hat{x}_i)$. Note that $\sum \tilde{x}_i = \sum \hat{x}_i = m^\ell$. Therefore by optimality of $\hat{x}$, $\sum_{A\setminus N} u_i(\hat{x}_i) + \sum_N u_i(\hat{x}_i) > \sum_{A\setminus N} u_i(\hat{x}_i) + \sum_{A\setminus N} u_i(\tilde{x}_i)$. But this is a contradiction to optimality of $\tilde{x}$. Thus, $\tilde{x} \in \mathbb{Z}^N$. ∎

Proof of Lemma 1. The first part of this proof consists of showing that the solutions of the maximization problems in Lemma 1 are feasible, in particular, that when resources are indivisible the corresponding coordinates of the solutions are integral. The second part of the proof shows that these solutions, in fact, coincide with the recommendations made by the separably concave mechanisms.

Let $u \equiv \{(u_i^\ell, v_i^\ell) : i \in A, \ell \in K\} \in \mathcal{U}^*$, $N \in \mathcal{N}$, $(R,m) \in \mathcal{E}^N$, $\ell \in K$, and $p \equiv p^\ell(R)$. Without loss of generality, suppose that $\sum_N p_i \geq m^\ell$ and let

$$x^\ell \equiv \arg\max\{\textstyle\sum_{i\in N} u_i^\ell(z_i) : z \leq p, z \in \mathrm{co}Z(N,m)|^\ell\}$$

and note that $\mathrm{co}S^\ell(p,m^\ell)$ is equal to constraint set defining x^ℓ.

Part 1. Since $Z(N,m)$ has a product structure, it suffices to show that $x^\ell \in Z(N,m)|^\ell$. If ℓ indexes a divisible resource ($\ell \in D$), then, $Z(N,m)|^\ell$ is itself convex, implying $\mathrm{co}Z(N,m)|^\ell = Z(N,m)|^\ell$. Thus, if all resources are divisible, there is nothing to prove. It remains to prove that, if $\ell \in I$, $x^\ell \in Z(N,m)|^\ell$. Let

$$y \equiv \arg\max\{\textstyle\sum_{i\in N} u_i^\ell(z_i) : z \in \mathrm{co}S^\ell((\bar{X}_i^\ell)_{i\in N}, m^\ell)\}.$$

By Lemma 4, $(u_i^\ell)_{i\in A} \in \mathcal{I}^\ell$ implies $y \in \mathbb{Z}_+^N$. Since $y \in \mathrm{co}Z(N,m)|^\ell$, $y \in Z(N,m)|^\ell$. Since $\mathrm{co}S^\ell(p,m^\ell) \subseteq \mathrm{co}S^\ell((\bar{X}_i^\ell)_{i\in N}, m^\ell)$, $\sum_{i\in N} u_i^\ell(y_i) \geq \sum_{i\in N} u_i^\ell(x_i^\ell)$. Thus, if $y \in \mathrm{co}S^\ell(p,m^\ell)$, since the optimum is unique, $x^\ell = y \in Z(N,m)|^\ell$, as desired. It remains to consider the case where $y \notin \mathrm{co}S^\ell(p,m^\ell)$. Then, there is $h \in N$ such that $y_h > p_h$.

Step 1. *For each $i \in N$, $y_i \geq p_i$ implies $x_i^\ell = p_i \in \mathbb{Z}_+$.*

Otherwise, because $x^\ell \in \mathrm{co}S^\ell(p,m^\ell)$, $x_i^\ell < p_i$ and, since $\sum_{i\in N} y_i = m^\ell = \sum_{i\in N} x_i^\ell$ there is $j \in N$ such that $y_j < x_j^\ell \leq p_j$. This would lead to a contradiction: it implies there is a real number $\varepsilon > 0$ such that

$$x^\ell + \varepsilon(\mathbf{e}_i - \mathbf{e}_j) \in \mathrm{co}S^\ell(p, m^\ell) \text{ and } y + \varepsilon(\mathbf{e}_j - \mathbf{e}_i) \in \mathrm{co}S^\ell((\bar{X}_i^\ell)_{i\in N}, m^\ell).$$

Assume that the above is indeed true. Then, a necessary condition for x^ℓ and y to maximize $\sum_{i\in N} u_i^\ell$ over $\mathrm{co}S^\ell(p, m^\ell)$ and $\mathrm{co}S^\ell((\bar{X}_i^\ell)_{i\in N}, m^\ell)$, respectively, is that, $\partial_+ u_i^\ell(x_i^\ell) \le \partial_- u_j^\ell(x_j^\ell)$ and $\partial_+ u_j^\ell(y_j) \le \partial_- u_i^\ell(y_i)$. On the other hand, since u_i^ℓ and u_j^ℓ are strictly concave, we obtain the first and last inequalities in

$$\partial_+ u_j^\ell(y_j) > \partial_- u_j^\ell(x_j^\ell) \ge \partial_+ u_i^\ell(x_i^\ell) > \partial_- u_i^\ell(y_i),$$

which contradicts $\partial_+ u_j^\ell(y_j) \le \partial_- u_i^\ell(y_i)$. This establishes that, in fact, $x_i^\ell = p_i$. Recall that, since preferences are defined over X_i and $\ell \in I$, $p_i \in \mathbb{Z}_+$. This completes Step 1.

Let $N' \equiv \{i \in N : y_i < p_i\}$ and $m' \equiv \sum_{i\in N'} y_i + \sum_{i\in N\setminus N'}(y_i - p_i)$, and

$$y' \equiv \arg\max\{\textstyle\sum_{i\in N'} u_i^\ell(z_i) : z \in \mathrm{co}S^\ell((\bar{X}_i^\ell)_{i\in N'}, m')\}.$$

Note that $\sum_{N'} p_i \ge m'$ and $x_{N'}^\ell = \arg\max\{\sum_{i\in N'} u_i^\ell(z_i) : z \in \mathrm{co}S^\ell(p_{N'}, m')\}$. By Lemma 4, $(u_i^\ell)_{i\in A} \in \mathcal{I}^\ell$ implies $y' \in \mathbb{Z}_+^{N'}$. Clearly, since $\mathrm{co}S^\ell(p_{N'}, m') \subseteq \mathrm{co}S^\ell((\bar{X}_i^\ell)_{i\in N'}, m')$, if $y' \in \mathrm{co}S^\ell(p_{N'}, m')$, then, for each $i \in N'$, $x_i^\ell = y_i' \in \mathbb{Z}_+$. Combining this with Step 1, would yield $x^\ell \in Z(N, m)|^\ell$ as desired. It remains to consider the case where $y' \notin \mathrm{co}S^\ell(p_{N'}, m')$. Then, there is $h \in N'$ such that $y_h' > p_h$.

Step 2. *For each $i \in N'$, $y_i' \ge p_i$ implies $x_i^\ell \in \mathbb{Z}_+$.*

Step 2 is symmetric to Step 1 and is proven symmetrically. We can then move on to Step 3 and so on. At each step, either we establish that $x^\ell \in Z(N, m)|^\ell$ or decrease the number of coordinates of x^ℓ that are possibly non-integer. Since N is finite, the desired conclusion is eventually reached.

Part 2. Let $A \equiv \{z : z \le p^\ell, z \in \mathrm{co}Z(N, m)|^\ell\}$, $B \equiv \{z : z \le p^\ell, z \in Z(N, m)|^\ell\}$, and $w \equiv \varphi^u(R, m)|^\ell$. By Part 1, $x^\ell \in B$. Thus, by the definition of φ^u, $\sum_{i\in N} u_i^\ell(w_i) \ge \sum_{i\in N} u_i^\ell(x_i^\ell)$. By the definition of x^ℓ, since $w \in A$, $\sum_{i\in N} u_i^\ell(w_i) \le \sum_{i\in N} u_i^\ell(x_i^\ell)$. Thus, w maximizes $\sum_{i\in N} u_i^\ell$ over A. Since the maximizer of $\sum_{i\in N} u_i^\ell$ over A is unique, in fact, $w = x^\ell$. ∎

A.3 Proof of Theorems 1,2, and 3

Theorem 3 implies Theorems 1 and 2. Throughout the rest of the Appendix, we will use the definition of the separably concave mechanisms arrived at in Lemma 1, where the recommendation of the mechanism is obtained by maximizing over the convex hull of the feasible set. Next we establish that each separably concave mechanism satisfies the properties in Theorems 2 and 3.

Lemma 5. *The separably concave mechanisms are strategy-proof, unanimous, consistent and resource-monotonic.*

Proof of Lemma 5: resource-monotonicity. Let $u \equiv \{(u_i^\ell, v_i^\ell) : i \in A, \ell \in K\} \in \mathcal{U}^*$ and $N \in \mathcal{N}$. Let $(R, m) \in \mathcal{E}^N$ and $\hat{m} \in M(N)$ be such that $\hat{m} \geq m$. Let $x \equiv \varphi^u(R, m)$ and $\hat{x} \equiv \varphi^u(R, \hat{m})$. We need to show that $\hat{x} \geq x$. Suppose, instead, that there are $i \in N$ and $\ell \in K$ such that $x_i^\ell > \hat{x}_i^\ell$. By the definition of φ^u, if $m^\ell = \hat{m}^\ell$, $x^\ell = \hat{x}^\ell$. Thus, $\sum_{k \in N} \hat{x}_k^\ell = \hat{m}^\ell > m^\ell = \sum_{k \in N} x_k$. Thus, there is $j \in N$ such that $\hat{x}_j^\ell > x_j^\ell$. Suppose that $\sum_{k \in N} p^\ell(R_k) \geq m^\ell$. Thus, by the definition of φ^u, for each $k \in N$, $x_k^\ell \leq p^\ell(R_k)$. If $\hat{m}^\ell \geq \sum_{k \in N} p^\ell(R_k)$, then, by the definition of φ^u, $\hat{x}_i^\ell \geq p^\ell(R_i) \geq x_i^\ell$. Thus, in the case under consideration, $\sum_{k \in N} p^\ell(R_k) > \hat{m}^\ell > m^\ell$. Then, by the definition of φ^u, x^ℓ and $\hat{x}^\ell$ maximize $\sum_{k \in N} u_k^\ell$ over $\text{co}S^\ell(p^\ell(R), m^\ell)$ and $\text{co}S^\ell(p^\ell(R), \hat{m}^\ell)$ respectively. Since $\hat{x}_j^\ell > x_j^\ell \geq 0$ and $\hat{x}_i^\ell < x_i^\ell < p^\ell(R_i)$, there is $\varepsilon > 0$ such that $\hat{x}^\ell + \varepsilon(\mathbf{e}_i - \mathbf{e}_j) \in \text{co}S^\ell(p^\ell(R), \hat{m}^\ell)$. Thus, a necessary condition for $\hat{x}^\ell$ to maximize $\sum_{k \in N} u_k^\ell$ over $\text{co}S^\ell(p^\ell(R), \hat{m}^\ell)$ is that $\partial_+ u_i^\ell(\hat{x}_i^\ell) \leq \partial_- u_j^\ell(\hat{x}_j^\ell)$. Moreover, since u_i^ℓ and u_j^ℓ are strictly concave, we obtain the first and last inequalities in

$$\partial_+ u_j^\ell(x_j^\ell) > \partial_- u_j^\ell(\hat{x}_j^\ell) \geq \partial_+ u_i^\ell(\hat{x}_i^\ell) > \partial_- u_i^\ell(x_i^\ell).$$

Thus, $\partial_+ u_j^\ell(x_j^\ell) > \partial_- u_i^\ell(x_i^\ell)$. Since $x_i^\ell > \hat{x}_i^\ell \geq 0$ and $x_j^\ell < \hat{x}_j^\ell \leq p^\ell(R_j)$, there is $\varepsilon > 0$ such that $x^\ell + \varepsilon(\mathbf{e}_j - \mathbf{e}_i) \in \text{co}S^\ell(p^\ell(R), m^\ell)$. This, in addition to $\hat{x}_j^\ell > x_j^\ell \geq 0$, implies that x^ℓ does not maximize $\sum_{k \in N} u_k^\ell$ over $\text{co}S^\ell(p^\ell(R), m^\ell)$. This contradiction establishes that, in fact, $\hat{x} \geq x$. ∎

Proof of Lemma 5: consistency. Let $u \equiv \{(u_i^\ell, v_i^\ell) : i \in A, \ell \in K\} \in \mathcal{U}^*$ and $\{N, N'\} \subseteq \mathcal{N}$ be such that $N' \subseteq N$. Let $(R, m) \in \mathcal{E}^N$ and $x \equiv \varphi^u(R, m)$. By way of contradiction, suppose that $y \equiv \varphi^u(R_{N'}, \sum_{N'} x_k) \neq x_{N'}$. Thus, there is $\ell \in K$ such that $x_{N'}^\ell \neq y^\ell$. Suppose that $\sum_{i \in N} p^\ell(R_i) \geq m^\ell$. By the definition of φ^u, for each $i \in N$, $x_i^\ell \leq p^\ell(R_i)$ and, for each $i \in N'$, $y_i^\ell \leq p^\ell(R_i)$. Then, $x_{N'}^\ell \neq y^\ell$, because the maximization problem defining y^ℓ has a unique solution, $\sum_{i \in N'} u_i^\ell(x_i^\ell) < \sum_{i \in N'} u_i^\ell(y_i^\ell)$. Recall that $\sum_{i \in N'} x_i^\ell = \sum_{i \in N'} y_i^\ell$. Thus, $z^\ell \equiv (y^\ell, x_{N \setminus N'}^\ell) \leq p^\ell(R)$ and $\sum_{i \in N} z_i^\ell = m^\ell$. Thus, $\sum_{i \in N} u_i^\ell(x_i^\ell) < \sum_{i \in N} u_i^\ell(z_i^\ell)$. Thus, $x^\ell \neq \varphi^u(R, m)|^\ell$, contradicting the definition of φ^u. A symmetric argument applies if $\sum_{i \in N} p^\ell(R_i) \leq m^\ell$. ∎

Proof of Lemma 5: unanimity. Let $N \in \mathcal{N}$, $(R, m) \in \mathcal{E}^N$ and suppose that $p(R)$ is a feasible allocation. Thus, for each $\ell \in K$, $m^\ell = \sum_{i \in N} p^\ell(R_i)$. Moreover, for each $\ell \in K$, each agent $i \in N$ cannot get more than $p^\ell(R_i)$ nor less than $p^\ell(R_i)$. Thus, each agent is assigned her peak $p(R_i)$. ∎

Proof of Lemma 5: strategy-proofness. Let $u \equiv \{(u_i^\ell, v_i^\ell) : i \in A, \ell \in K\} \in \mathcal{U}^*$. Let $N \in \mathcal{N}$ and $(R, m) \in \mathcal{E}^N$ and $x \equiv \varphi^u(R, m)$. Let $i \in N$ and $\tilde{R} \in \mathcal{R}^N$ be such that, for each $j \in N \setminus \{i\}$, $\tilde{R}_j = R_j$. Let $y \equiv \varphi^u(\tilde{R}, m)$. We will prove that, for each $\ell \in K$, either $y_i^\ell \leq x_i^\ell \leq p^\ell(R_i)$ or $y_i^\ell \geq x_i^\ell \geq p^\ell(R_i)$.

Let $\ell \in K$, $p \equiv p^\ell(R)$, $q \equiv p^\ell(\tilde{R})$, and, for each $z \in \mathbb{R}_+^N$, $f(z) \equiv \sum_{i \in N} u_i^\ell(z_i)$.

Case 1: $\sum_N p_j \geq m^\ell$.

Case 1.1: $x_i^\ell = p_i$.

Suppose that $\sum_N q_k \leq m^\ell$. Then, for each $j \in N \setminus \{i\}$, $y_j^\ell \geq q_j = p_j \geq x_j^\ell$. Thus, since $\sum_{j \in N} x_j^\ell = m^\ell = \sum_{j \in N} y_j^\ell$, $y_i^\ell \leq x_i^\ell \leq p_i$.

Suppose that $\sum_N q_k \geq m^\ell$. If $q_i \leq p_i$, then, by the definition of φ^u, $y_i^\ell \leq q_i$. Thus, $y_i^\ell \leq x_i^\ell = p_i$. When $q_i > p_i$ we will prove that if $y^\ell \neq x^\ell$, then $y_i^\ell \geq x_i^\ell$. Suppose $y^\ell \neq x^\ell$. Since $q_i > p_i$, $\text{co}S^\ell(p, m^\ell) \subseteq \text{co}S^\ell(q, m^\ell)$. Thus $f(y^\ell) > f(x^\ell)$. If $y_i^\ell < x_i^\ell$ then $y^\ell \in \text{co}S^\ell(p, m^\ell)$. This is not possible since x^ℓ is the maximizer for f over $\text{co}S^\ell(p, m^\ell)$. Thus, $y_i^\ell \geq x_i^\ell = p_i$ as desired.

Case 1.2: $x_i^\ell \neq p_i$.

Suppose that $\sum_N q_k \leq m^\ell$. Then, for each $j \in N \setminus \{i\}$, $y_j^\ell \geq q_j = p_j \geq x_j^\ell$. Thus, since $\sum_{j \in N} x_j^\ell = m^\ell = \sum_{j \in N} y_j^\ell$, $y_i^\ell \leq x_i^\ell \leq p_i$.

Suppose that $\sum_N q_k \geq m^\ell$.

- If $q_i \leq x_i^\ell$, then, by the definition of φ^u, $y_i^\ell \leq q_i$. Thus, $y_i^\ell \leq x_i^\ell \leq p_i$.
- If $q_i > x_i^\ell$. We will show that $y_i^\ell = x_i^\ell$. Note first that, by the definition of φ^u, $x^\ell \leq p$, and since $x_i^\ell \neq p_i$, $x_i^\ell < p_i$. Thus, $x^\ell \in \text{co}S^\ell(q, m^\ell)$. By definition, $y^\ell \in \text{co}S^\ell(q, m^\ell)$. Suppose first that $y_i^\ell < x_i^\ell$. Since, for each $j \in N \setminus \{i\}$, $q_j = p_j$ then $y^\ell \in \text{co}S^\ell(p, m^\ell)$. But this contradicts x^ℓ being the maximizer of f over $\text{co}S^\ell(p, m^\ell)$. Thus, $y_i^\ell \geq x_i^\ell$. If $y_i^\ell > x_i^\ell$, by feasibility there is $j \in N \setminus \{i\}$ such that $y_j^\ell < x_j^\ell$. Hence, there is $\varepsilon > 0$ such that $y^\ell + \varepsilon(\mathbf{e}_j - \mathbf{e}_i) \in \text{co}S^\ell(q, m^\ell)$. Thus, a necessary condition for y^ℓ to maximize f over $\text{co}S^\ell(q, m^\ell)$ is that $\partial_+ u_j^\ell(y_j^\ell) \leq \partial_- u_i^\ell(y_i^\ell)$. Moreover, since u_i^ℓ and u_j^ℓ are strictly concave, we obtain the first and last inequalities in

$$\partial_- u_j^\ell(x_j^\ell) < \partial_+ u_j^\ell(y_j^\ell) \leq \partial_- u_i^\ell(y_i^\ell) < \partial_+ u_i^\ell(x_i^\ell).$$

Thus, $\partial_- u_j^\ell(x_j^\ell) < \partial_+ u_i^\ell(x_i^\ell)$. Since $x_i^\ell < p_i$ and $x_j^\ell > y_j^\ell \geq 0$, there is $\varepsilon > 0$ such that $x^\ell + \varepsilon(\mathbf{e}_i - \mathbf{e}_j) \in \mathrm{co}S^\ell(p, m^\ell)$. This implies that x^ℓ does not maximize f over $\mathrm{co}S^\ell(p, m^\ell)$. This contradiction establishes that, in fact, $y_i^\ell = x_i^\ell$.

Case 2: $\boldsymbol{\sum_N p_j \leq m^\ell}$. It can be shown, in an analogous manner to Case 1, that either $y_i^\ell \geq x_i^\ell \geq p_i$ or $y_i^\ell \leq x_i^\ell \leq p_i$.

From Cases 1 and 2 it follows that, for each $\ell \in K$, either $y_i^\ell \geq x_i^\ell \geq p_i$ or $y_i^\ell \leq x_i^\ell \leq p_i$. Note that R_i is multidimensional-single-peaked. Thus, $x_i \; R_i \; y_i$. ∎

The proof that the properties in Theorem 2 jointly single out the class of separably concave mechanisms consists of a number of lemmas (Lemmas 6 through 14 below). These Lemmas are then used to establish the Theorem via the so-called "Elevator Lemma" that has been important in the study of the consistency principle.[18]

Lemma 6. *A consistent mechanism is non-bossy.*

Proof of Lemma 6: non-bossiness. Let $N \in \mathcal{N}$, $(R, m) \in \mathcal{E}^N$, and $i \in N$. Let $R' \in \mathcal{R}^N$ be such that, for each $j \in N \setminus \{i\}$, $R'_j = R_j$. Let $x \equiv \varphi(R, m)$ and $x' \equiv \varphi(R', m)$. Suppose, as in the hypothesis of *non-bossiness*, that $x'_i = x_i$. Then, $\sum_{N\setminus\{i\}} x'_j = \sum_{N\setminus\{i\}} x_j$. Thus, $(R'_{N\setminus\{i\}}, \sum_{N\setminus\{i\}} x'_j) = (R_{N\setminus\{i\}}, \sum_{N\setminus\{i\}} x_j)$. Thus, by *consistency*, for each $j \in N \setminus \{i\}$, $x'_j = \varphi_j(R_{N\setminus\{i\}}, \sum_{N\setminus\{i\}} x_j) = x_j$. Thus, $x' = x$. ∎

The following lemma establishes that a mechanism in Theorem 2 satisfies another weak efficiency property, strengthening on *unanimity*: if the amount of one of the resource kinds exactly matches the aggregate demand for this resource, then every agent ought to receive her preferred amount of this resource.

Lemma 7. *Let φ be a mechanism satisfying the properties in Theorem 2. For each $N \in \mathcal{N}$, each $(R, m) \in \mathcal{E}^N$, and each $\ell \in K$, if $\sum_N p^\ell(R_i) = m^\ell$, then $\varphi(R, m)|^\ell = p^\ell(R)$.*

Proof. Let $N \in \mathcal{N}$, $(R, m) \in \mathcal{E}^N$, $x \equiv \varphi(R, m)$ and $p \equiv p^\ell(R)$. Let $\ell \in K$ and suppose that $\sum_N p_i = m^\ell$. By way of contradiction, assume that $x^\ell \neq p$. Let $\underline{m} \in M(N)$ be such that $\underline{m}^\ell = m^\ell$ and, for each $k \in K \setminus \{\ell\}$, $\underline{m}^k = 0$. Let $\underline{x} \equiv \varphi(R, \underline{m})$. By feasibility, for each $k \in K \setminus \{\ell\}$ and each $i \in N$, $\underline{x}_i^k = 0$. By *resource-monotonicity*, $\underline{x}^\ell = x^\ell$. By assumption, $x^\ell \neq p$. Thus, $\underline{x}^\ell \neq p$ and there is $i \in N$ such that $\underline{x}_i^\ell \neq p_i$. Let $R'_i \in \mathcal{R}_i$ be such that $p^\ell(R'_i) = p_i$ and, for each $k \in K \setminus \{\ell\}$, $p^k(R'_i) = 0$. By *strategy-proofness* at $(R_i, R_{N\setminus\{i\}}, \underline{m})$, $\varphi_i(R'_i, R_{N\setminus\{i\}}, \underline{m})|^\ell \neq p_i$. Since

[18] See Thomson (2011) for a survey.

$\sum_N p_k = m^\ell$ there is $j \in N \setminus \{i\}$ such that $\varphi_j(R'_i, R_{N\setminus\{i\}}, \underline{m})|^\ell \neq p_j$. Let $R'_j \in \mathcal{R}_j$ be such that $p^\ell(R'_j) = p_j$ and, for each $k \in K \setminus \{\ell\}$, $p^k(R'_j) = 0$ By *strategy-proofness* at $(R_j, R'_i, R_{N\setminus\{i,j\}}, \underline{m})$, $\varphi_j(R'_j, R'_i, R_{N\setminus\{i,j\}}, \underline{m})|^\ell \neq p_j$. Let

$$y \equiv \varphi(R'_i, R'_j, R_{N\setminus\{i,j\}}, \underline{m}). \text{ By } \textit{consistency},\ y_{\{i,j\}} = \varphi(R'_i, R'_j, y_i + y_j).$$

Suppose that $y^\ell_i + y^\ell_j = p_i + p_j$. By *unanimity*, $y_j = p(R'_j)$ and $y_i = p(R'_i)$. However, $y^\ell_j \neq p_j$, a contradiction to *unanimity*. Thus, for this case, $x^\ell = p$ and we have established Lemma 7.

Suppose instead that $y^\ell_i + y^\ell_j \neq p_i + p_j$. Since $\sum_N p_k = m^\ell$ there is $k \in N \setminus \{i, j\}$ such that $y^\ell_k \neq p_k$. Let $R'_k \in \mathcal{R}_k$ be such that $p^\ell(R'_k) = p_k$ and, for each $k \in K \setminus \{\ell\}$, $p^k(R'_k) = 0$. By *strategy-proofness* at $(R_k, R'_j, R'_i, R_{N\setminus\{i,j,k\}}, \underline{m})$, $\varphi_k(R'_i, R'_j, R'_k, R_{N\setminus\{i,j,k\}}, \underline{m})|^\ell \neq p_k$. Let

$$z \equiv \varphi(R'_i, R'_j, R'_k, R_{N\setminus\{i,j,k\}}, \underline{m}). \text{ By } \textit{consistency}\ \varphi(R'_i, R'_j, R'_k, z_i + z_j + z_k) = z_{\{i,j,k\}}.$$

Suppose that $z^\ell_i + z^\ell_j + z^\ell_k = p_i + p_j + p_k$. By *unanimity*, $z_i = p(R'_i)$, $z_j = p(R'_j)$ and $z_k = p(R'_k)$. However, $z^\ell_k \neq p_k$, a contradiction to *unanimity*. Similarly to before this contradiction implies that $x^\ell = p^\ell(R)$.

Suppose instead that $z^\ell_i + z^\ell_j + z^\ell_k \neq p_i + p_j + p_k$. Since $\sum_N p_i = m^\ell$ there is $g \in N \setminus \{i, j, k\}$ such that $z^\ell_g \neq p_g$. From here on we repeat an analogous argument to the one given above when $y^\ell_i + y^\ell_j \neq p_i + p_j$. We will either derive a contradiction to *unanimity* or conclude that there is another agent in $N \setminus \{i, j, k, g\}$ that is not assigned her peak amount of resource ℓ.

Since the set of agents is finite and $\sum_N p_i = m^\ell$ there is $N' \subseteq N$ such that,

$$\textstyle\sum_{N'} \varphi_i(R'_{N'}, R_{N\setminus N'}, \underline{m})|^\ell = \sum_{N'} p_i.$$

Let $w \equiv \varphi(R'_{N'}, R_{N\setminus N'}, \underline{m})$. By *consistency*, for each $i \in N'$, $\varphi_i(R'_{N'}, R_{N\setminus N'}, \sum_{N'} w_i) = \varphi_i(R'_{N'}, R_{N\setminus N'}, \underline{m})$. Note that, for each $k \in K \setminus \{\ell\}$, $\sum_{N'} p^k(R'_i) = 0$. By *unanimity*, for each $i \in N'$, $w_i = p(R'_i)$. However, consider the last agent, call her $h \in N$, equipped with the preferences $R'_h \in \mathcal{R}_h$ such that $p^\ell(R'_h) = p_h$ and, for each $k \in K \setminus \{\ell\}$, $p^k(R'_h) = 0$. By *strategy-proofness* at $(R_h, R'_{N'\setminus\{h\}}, R_{N\setminus N'}, \underline{m})$, $w^\ell_h \neq p_h$. But this contradicts *unanimity*. Thus, $x^\ell = p^\ell(R)$. ∎

Next we introduce a technical property. A mechanism φ satsifies **same-sidedness** if, for each $N \in \mathcal{N}$ and $(R, m) \in \mathcal{E}^N$, if $\sum_{i\in N} p^\ell(R_i) \leq m^\ell$, then $\varphi(R, m)|^\ell \leq p^\ell(R)$; if $\sum_{i\in N} p^\ell(R_i) \geq m^\ell$, then $\varphi(R, m)|^\ell \geq p^\ell(R)$.

Building on Lemma 7, we now prove that the properties in Theorem 2 imply *same-sidedness*.

Lemma 8. *Let φ be a mechanism satisfying the properties in Theorem 2. Then φ satisfies same-sidedness.*

Proof. Let $N \in \mathcal{N}$, $(R, m) \in \mathcal{E}^N$ and $x \equiv \varphi(R, m)$. By way of contradiction assume that φ does not satisfy *same-sidedness* at x. This means that there are two agents, say agent i and j, and one resource $\ell \in K$ such that $x_i^\ell < p^\ell(R_i)$ and $x_j^\ell > p^\ell(R_j)$. Let $p_i \equiv p^\ell(R_i)$ and $p_j \equiv p^\ell(R_j)$

Let $\underline{m} \in M(N)$ be such that $\underline{m}^\ell = m^\ell$ and, for each $k \in K \setminus \{\ell\}$, $\underline{m}^k = 0$. Let $\underline{x} \equiv \varphi(R, \underline{m})$. By feasibility, for each $k \in K \setminus \{\ell\}$ and each $i \in N$, $\underline{x}_i^k = 0$. By *resource-monotonicity*, $\underline{x}^\ell = x^\ell$. By *consistency*, $\underline{x}_{\{i,j\}} = \varphi(R_i, R_j, \underline{x}_i + \underline{x}_j)$. Since $\underline{x}_i^\ell \neq p_i$ and $\underline{x}_j^\ell \neq p_j$, by Lemma 7, $\underline{x}_i^\ell + \underline{x}_j^\ell \neq p_i + p_j$.

Suppose that $\underline{x}_i^\ell + \underline{x}_j^\ell > p_i + p_j$. Let $R_i' \in \mathcal{R}_i$ be such that $p^\ell(R_i') = \underline{x}_i^\ell + \underline{x}_j^\ell - p_j$ and, for each $k \in K \setminus \{\ell\}$, $p^k(R_i') = p^k(R_i)$. Let $y \equiv \varphi(R_i', R_j, \underline{x}_i + \underline{x}_j)$. By feasibility, for each $k \in K \setminus \{\ell\}$, $y_i^k = 0$. Since $p^\ell(R_i') + p_j = \underline{x}_i^\ell + \underline{x}_j^\ell$, by Lemma 7, $y_i^\ell = p^\ell(R_i')$. By construction $p^\ell(R_i') = \underline{x}_i^\ell + \underline{x}_j^\ell - p_j$. By assumption $\underline{x}_j^\ell > p_j$. Thus, $p_i \geq y_i^\ell > \underline{x}_i^\ell$. Since, for each $k \in K \setminus \{\ell\}$, $\underline{x}_i^k = y_i^k = 0$, $y_i \; P_i \; \underline{x}_i$. However this contradicts *strategy-proofness* since agent i can manipulate at $(R_i, R_j, \underline{x}_i + \underline{x}_j)$. Thus, x_i^ℓ must be on the same side of the peak for each $i \in N$.

If $\underline{x}_i^\ell + \underline{x}_j^\ell < p_i + p_j$, a symmetric argument applies and we reach again a contradiction. ∎

The following is a key step in the proof. We use our previous lemmas to show that, in allocating resource of kind ℓ, a mechanism satisfying the properties in Theorem 2 only uses information pertaining to ℓ. Thus, the availability of other kinds of resources does not affect how resource ℓ is allocated.

Lemma 9. *Let φ be a mechanism satisfying the properties in Theorem 2. Then, for each $N \in \mathcal{N}$, each pair $(R, m), (\tilde{R}, \tilde{m}) \in \mathcal{E}^N$, and each $\ell \in K$,*

$$\text{if } p^\ell(R) = p^\ell(\tilde{R}) \text{ and } m^\ell = \tilde{m}^\ell, \text{ then } \varphi(R, m)|^\ell = \varphi(\tilde{R}, \tilde{m})|^\ell.$$

Proof. Let $N \equiv \{1, 2, \ldots, n\} \in \mathcal{N}$. Let $(R, m), (\tilde{R}, \tilde{m}) \in \mathcal{E}^N$, and $\ell \in K$ be such that $p^\ell(R) = p^\ell(\tilde{R})$, $m^\ell = \tilde{m}^\ell$. Let $q \equiv p^\ell(R)$, $x \equiv \varphi(R, m)$, and $\tilde{x} \equiv \varphi(\tilde{R}, \tilde{m})$. We will prove that $x^\ell = \tilde{x}^\ell$.

First suppose that $\sum_N q_i > m^\ell$. Assume, by way of contradiction, that $x^\ell \neq \tilde{x}^\ell$. Then, there is an agent, say 1, such that $x_1^\ell > \tilde{x}_1^\ell$. By feasibility, since $m^\ell = \tilde{m}^\ell$, there is another agent, say 2, such that $x_2^\ell < \tilde{x}_2^\ell$. By *same-sidedness* (Lemma 8), $x^\ell \leq q$ and $\tilde{x}^\ell \leq q$. Thus, $\tilde{x}_1^\ell < x_1^\ell \leq q_1$ and $x_2^\ell < \tilde{x}_2^\ell \leq q_2$. Let $\underline{m} \in M(N)$ be such that $\underline{m}^\ell = m^\ell$ and, for each $k \in K \setminus \{\ell\}$, $\underline{m}^k = 0$. Let $\underline{x} \equiv \varphi(R, \underline{m})$ and $\underline{\tilde{x}} \equiv \varphi(\tilde{R}, \underline{m})$. For each $i \in N \setminus \{1\}$, let $R_i' \in \mathcal{R}_i$ be such that $p^\ell(R_i') = q_i$ and, for each $k \in K \setminus \{\ell\}$, $p^k(R'_i) = 0$. Let $z \equiv \varphi(R'_2, R_{N\setminus\{2\}}, \underline{m})$. By *resource-monotonicity*, $\underline{x}^\ell = x^\ell$ and $\underline{\tilde{x}}^\ell = \tilde{x}^\ell$. By feasibility,

$$\text{for each } k \in K \setminus \{\ell\} \text{ and each } i \in N, \quad \underline{x}_i^k = \underline{\tilde{x}}_i^k = z_i^k = 0. \tag{8}$$

By *same-sidedness* (Lemma 8), $z_2^\ell \leq q_2$. Thus, by *strategy-proofness* at $(R'_2, R_{N\setminus\{2\}}, \underline{m})$, $z_2 \; R'_2 \; \underline{x}_2$, implying $\underline{x}_2^\ell \leq z_2^\ell$. Similarly, by *strategy-proofness* at $(R_2, R_{N\setminus\{2\}}, \underline{m})$, $\underline{x}_2^\ell \geq z_2^\ell$. Thus, $\underline{x}_2^\ell = z_2^\ell$. By *non-bossiness* (Lemma 6), $z = \underline{x}$. Repeating this argument, for agents 3 through n, yields $\underline{x} = \varphi(R_1, R'_{N\setminus\{1\}}, \underline{m})$. Similarly, we can show that $\underline{\tilde{x}} = \varphi(\tilde{R}_1, R'_{N\setminus\{1\}}, \underline{m})$. Recall that $\underline{\tilde{x}}_1^\ell < \underline{x}_1^\ell \leq q_1$ and, for each $k \in K \setminus \{\ell\}$, $\underline{x}_1^k = \underline{\tilde{x}}_1^k = 0$. Thus, $\underline{x}_1 \; \tilde{P}_1 \; \underline{\tilde{x}}_1$. This contradicts *strategy-proofness* at $(\tilde{R}_1, R'_{N\setminus\{1\}}, \underline{m})$. Thus, in fact, $x^\ell = \tilde{x}^\ell$.

If $\sum_N q_i < m^\ell$, then, an analogous argument again reaches a contradiction. Thus again, $x^\ell = \tilde{x}^\ell$.

If $\sum_N q_i = m^\ell$, then, by *same-sidedness* (Lemma 8), $x^\ell = \tilde{x}^\ell$. ∎

Next, we use Lemma 9 to "decompose" our multidimensional allocation problem into $|K|$ uni-dimensional allocation problems. The result is reminiscent of the decomposition of strategy-proof social choice functions into "marginal" strategy-proof social choice functions in choice problems where preferences have some degree of separability over a set of alternatives with a product structure (Barberà, Gul, and Stacchetti, 1993; Le Breton and Sen, 1999).

Formally, for each $N \in \mathcal{N}$ and each $\ell \in K$, define the mapping $\boldsymbol{\psi^\ell}$, specifying, for each

$$(p^\ell, m^\ell) \in [\times_{i \in N} X_i^\ell] \times M^\ell(N),$$

a feasible division of m^ℓ among the agents in N,

$$\psi^\ell(p^\ell, m^\ell) \in \{x \in \times_{i\in N} X_i^\ell : \textstyle\sum_N x_i = m^\ell\}.$$

Let $\boldsymbol{\Psi}$ denote the class of profiles of such mappings, one of each $\ell \in K$, $\{\psi^\ell : \ell \in K\}$.

Lemma 10. *Let φ be a mechanism satisfying the properties in Theorem 2. Then, there is $\{\psi^\ell : \ell \in K\} \in \Psi$ such that, for each $N \in \mathcal{N}$, and each $(R, m) \in \mathcal{E}^N$,*

$$\varphi(R, m) = \{\psi^\ell(p^\ell(R), m^\ell) : \ell \in K\}.$$

Proof. This follows immediately from Lemma 9. ∎

In the following lemma, we will use the properties in Theorem 2 and Lemma 10 to construct a profile of concave functions, two concave functions for each resource-agent pair.

Lemma 11. *Let φ be a mechanism satisfying the properties in Theorem 3. By Lemma 10, there is $\{\psi^\ell : \ell \in K\} \in \Psi$ such that, for each $N \in \mathcal{N}$ and each $(R, m) \in \mathcal{E}^N$, $\varphi(R, m) = \{\psi^\ell(p^\ell(R), m^\ell) : \ell \in K\}$. Moreover, for each $\ell \in K$, ψ^ℓ satisfies the following properties:*

(i) For each $i \in A$, there is a strictly concave function $u_i^{xd,\ell}$ with domain X_i^ℓ such that, for each $m^\ell \in M^\ell(A)$,

$$\psi^\ell((\bar{X}_i^\ell)_{i\in A}, m^\ell) = \arg\max \left\{ \sum_{k\in A} u_k^{xd,\ell}(z_k) : z_i \in X_i^\ell, \sum_{k\in A} z_k = m^\ell \right\}.$$

Moreover, if $\ell \in I$, $(u_i^{xd,\ell})_{i\in A} \in \mathcal{I}^\ell$.

(ii) For each $i \in A$, there is a strictly concave function $u_i^{xs,\ell}$ with domain X_i^ℓ such that, for each $m^\ell \in M^\ell(A)$,

$$\psi^\ell((0)_{i\in A}, m^\ell) = \arg\max \left\{ \sum_{k\in A} u_k^{xs,\ell}(z_k) : z_i \in X_i^\ell, \sum_{k\in A} z_k = m^\ell \right\}.$$

Moreover, if $\ell \in I$, $(u_i^{xs,\ell})_{i\in A} \in \mathcal{I}^\ell$.

Proof. We prove (i) in the Lemma. (The proof of (ii) is symmetric.) Let $\ell \in K$, $\bar{X}^\ell \equiv (\bar{X}_i^\ell)_{i\in A}$, $X^\ell \equiv \times_{i\in A} X_i^\ell$, $M \equiv M^\ell(A)$, and $\sigma \equiv \sum_{k\in A} \bar{X}_k^\ell$.

Without loss of generality, when ℓ is divisible, we can assume that, for each $i \in A$, the interior of X_i^ℓ relative to $\mathbb{R}$ is non-empty.[19] Similarly, when ℓ comes in indivisible units we assume that the convex hull of X_i^ℓ has non-empty interior relative to $\mathbb{R}$. In the arguments that follow, when ℓ comes in indivisible units, we proceed as if it were divisible identifying X_i^ℓ with its convex hull.

Step 1. *Constructing a monotone path g.*

For each $m \in M$, let $g(m) \equiv \psi^\ell(\bar{X}, m)$. By the *resource-monotonicity* of φ, for each pair $m, m' \in M^\ell$, $m' \geq m$ implies,

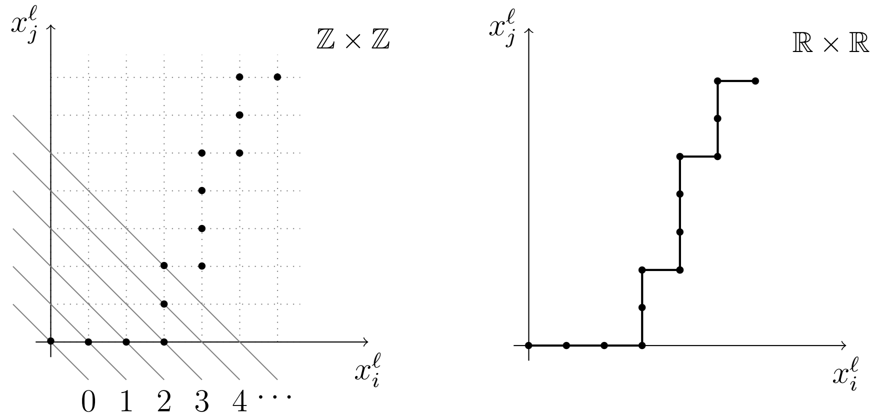
$$g(m') = \psi^\ell(\bar{X}^\ell, m') \geq \psi^\ell(\bar{X}^\ell, m) = g(m).$$

Thus, for each $k \in A$, each g_k is increasing in m. By feasibility, $g(0) = (0)_{k\in A}$ and $g(\sigma) = \bar{X}^\ell$.

When ℓ is divisible, it is clear that g is continuous on the interval M.[20]

[19] Otherwise we could replace A by $A^\ell \equiv \{i \in A : \bar{X}_i^\ell \neq 0\}$ in all the arguments in the proof and attribute, to each $i \in A \setminus A^\ell$ any finite function $u_i^{xd,\ell}$ with domain $X_i^\ell = \{0\}$.

[20] See the first step in the proof of Theorem 2 of Thomson (1994b).



(a) (b)

Figure 2: An illustration of the construction of a monotone path in the the integral case: Suppose that $A = \{i, j\}$ and that agent's have increasing preferences over their consumption of resource a resource kind ℓ, available in indivisible units. (a) As the amount of the resource increases from 0 to 1 (1 to 2, 2 to 3, ...), by *resource-monotonicity*, the allocation recommended by the mechanism is above or to the right of (0,0) ((1,0), (2,0), ...). (b) We obtain a continuous monotone path by connecting the allocations of resource ℓ obtained in part (a). The arguments in Lemma 11, for the divisible resource case, can then be applied to construct a profile of concave functions.

When ℓ comes in indivisible units, the allocations of resource ℓ can be "connected" uniquely by a continuous monotone path as is illustrated in going from Figure (2a) to Figure (2b). Abusing notation, we will denote this continuous path by g as well.

Step 2. *Constructing* $(u_i^{xd,\ell})_{i \in A}$ *from* g.

For each $m \in M$ and each $i \in A$, let $h_i : X_i^\ell \to \mathbb{R}$ denote a strictly increasing such that $h_i(0) = 0$, $h_i(X_i^\ell) = \sigma$,

$$\begin{aligned} \text{for each } x_i \in \text{int}X_i^\ell, \quad x_i = g_i(m) \quad &\text{if and only if} \quad \lim_{z \uparrow x_i} h_i(z) \leq m \leq \lim_{z \downarrow x_i} h_i(z), \\ 0 = g_i(m) \quad &\text{if and only if} \quad 0 \leq m \leq \lim_{z \downarrow 0} h_i(z), \text{ and} \\ \bar{X}_i^\ell = g_i(m) \quad &\text{if and only if} \quad \lim_{z \uparrow \bar{X}_i^\ell} h_i(z) \leq m \leq \sigma. \end{aligned} \tag{9}$$

For each $x_i \in X_i^\ell$, let $f_i(x_i) \equiv \int_0^{x_i} h_i(t)dt$. Then, $f_i : X_i^\ell \to \mathbb{R}$ is a well-defined, closed, and proper convex function.[21] Additionally, because h_i is strictly increasing,

[21]See Theorem 24.2 in (Rockafellar, 1970, page 230). In this context, the closedness of f_i is equivalent to its lower semi-continuity.

$f_i : X_i^\ell \to \mathbb{R}$ is strictly convex. For each $i \in A$, let $\boldsymbol{u_i^{xd,\ell}} \equiv -f_i$. Hence, each $u_i^{xd,\ell} : X_i^\ell \to \mathbb{R}$ is strictly concave.

Step 3. *Verifying that $(u_i^{xd,\ell})_{i\in A}$ is as claimed in the Lemma.*

For each $i \in A$, let $f_i \equiv -u_i^{xd,\ell}$. It suffices to establish that, for each $m \in M$,

$$g(m) = \arg\min \left\{ \sum_A f_k(z_k) : \sum_A z_k = m, z \in X^\ell \right\}. \tag{10}$$

Case 1: $\boldsymbol{m = 0}$ or $\boldsymbol{m = \sigma}$. If $m = 0$, $\{z \in X^\ell : \sum_A z_k = 0\} = (0)_{k\in A}$. Thus, $\arg\min \left\{\sum_A f_k(z_k) : z \in X^\ell, \sum_A z_k = 0\right\} = (0)_{k\in A} = g(0)$, as desired. A symmetric argument establishes (10) when $m = \sigma$.

Case 2: $\boldsymbol{\sigma > m > 0}$. Let $a \equiv \arg\min \left\{\sum_A f_k(z_k) : \sum_A z_k = m, z \in X^\ell\right\}$. For each $k \in A$, let $F_k : \mathbb{R} \to \mathbb{R}$ be the function such that, for each $x_k \in X_k^\ell$, $F_k(x_k) = f_k(x_k)$ and, for each $x_k \notin X_k^\ell$, $F_k(x_k) = \infty$. Note that, under the standard convention that the convex combination of a finite number and ∞ is itself ∞, $F_k : \mathbb{R} \to \mathbb{R}$ is convex, closed, and proper. Moreover,

$$a = \arg\min \{\textstyle\sum_A F_k(z_k) : \sum_A z_k = m\}.$$

Clearly, there is x in the relative interior of X^ℓ such that $\sum_A x_i = m$ and $\sum_A F_k(x_k) = \sum_A f_k(x_k) \neq -\infty$. Thus, by Corollary 28.2.2 in Rockafellar (1970), there is a Kuhn-Tucker coefficient $\lambda^* \in \mathbb{R}$ for the optimization problem $\min\{\sum_A F_k(z_k) : \sum_A z_k = m\}$. For each $(x, \lambda) \in \mathbb{R}^A \times \mathbb{R}$, let $L(x, \lambda) \equiv \sum_A F_k(x_k) + \lambda[m - \sum_A x_k]$. By Theorem 28.3 in Rockafellar (1970),

$$\begin{aligned} \min_{x\in\mathbb{R}^A} L(x, \lambda^*) &= \lambda^* m + \sum_{k\in A} \min\{F_k(x_k) - \lambda^* x_k : x_k \in \mathbb{R}\} \\ &= \lambda^* m + \sum_{k\in A} \{F_k(a_k) - \lambda^* a_k\}. \end{aligned} \tag{11}$$

Thus, for each $k \in A$ and each $x_k \in \mathbb{R}$,

$$F_k(x_k) \geq F_k(a_k) + \lambda^*(x_k - a_k).$$

Thus, λ^* is in the sub-differential of F_k at a_k. That is, for each $k \in A$, $\lambda^* \in \partial F_k(a_k)$. By the definitions of f_k and F_k in Step 2 and Theorem 24.2 in Rockafellar (1970), (i) if a_k is in the interior of X_k^ℓ, $\partial F_k(a_k) = [\lim_{z\uparrow a_k} h_k(z), \lim_{z\downarrow a_k} h_k(z)]$, (ii) if $a_k = 0$, $\partial F_k(a_k) = (-\infty, \lim_{z\downarrow a_k} h_k(z)]$, and (iii) if $a_k = \bar{X}_k^\ell$, $\partial F_k(a_k) = [\lim_{z\uparrow a_k} h_k(z), \infty)$.

Moreover, since $\sigma > m > 0$, there are $i, j \in A$ such that $a_i < \bar{X}_i^\ell$ and $0 < a_j$. Thus, since h_i and h_j are strictly increasing,

$$h_i(a_i) \leq \lim_{z \downarrow a_i} h_i(z) < h_i(\bar{X}_i^\ell) = \sigma \text{ and } 0 = h_j(0) < \lim_{z \uparrow a_j} h_j(z) \leq h_j(a_j).$$

Thus, $0 < \lambda^* < \sigma$. Thus, by (9), for each $k \in A$, $g_k(\lambda^*) = a_k$. Thus, $m = \sum_A a_k = \sum_A g_k(\lambda^*) = \lambda^*$. Thus, $m = \lambda^*$ and $g(m) = a$, confirming (10).

Finally, by construction, if $\ell \in I$, $(u_i^{xd,\ell})_{i \in A} \in \mathcal{I}^\ell$. ∎

Next, using the profile of concave function constructed in Lemma 11, we prove Theorems 2 and 3 for economies involving two agents.

Lemma 12. *Let φ denote a mechanism satisfying the properties in Theorem 3. Then, there is $u \in \mathcal{U}^*$ such that, for each $N \in \mathcal{N}$* ***consisting of two agents*** *and each $(R, m) \in \mathcal{E}^N$, $\varphi(R, m) = \varphi^u(R, m)$.*

Proof. Let $\ell \in K$. Let $\{i, j\} \in \mathcal{N}$ and $(R, m) \in \mathcal{E}^{\{i,j\}}$ be such that $p^\ell(R_i) + p^\ell(R_j) \geq m^\ell$. We will prove that there is $u \in \mathcal{U}^*$ such that $\varphi(R, m)|^\ell = \varphi^u(R, m)|^\ell$. (A symmetric argument establishes the same conclusion if instead $p^\ell(R_i) + p^\ell(R_j) \leq m^\ell$.)

Let $\bar{X}^\ell \equiv (\bar{X}_i^\ell)_{i \in A}$, $X^\ell \equiv \times_{i \in A} X_i^\ell$, and $\bar{R} \in \mathcal{R}^A$ be such that $p^\ell(\bar{R}) = (\bar{X}_i^\ell)_{i \in A}$ and, for each $k \in K \setminus \{\ell\}$, $p^k(\bar{R}) = p^k(R)$. Let $x \equiv \varphi(\bar{R}_i, \bar{R}_j, m)$. By Lemma 11, there are

$$\{\psi^k : k \in K\} \in \Psi \text{ and } u = \{(u_i^{xd,k}, u_i^{xd,k}) : i \in A, k \in K\} \in \mathcal{U}$$

such that, for each $\tilde{m} \in M(A)$,

$$\begin{aligned}
\varphi(\bar{R}, \tilde{m})|^\ell &= \psi^\ell(\bar{X}^\ell, \tilde{m}^\ell) \\
&= \arg\max \left\{ \sum_{i \in A} u_i^{xd,\ell}(z_i) : z \in X^\ell, \sum_{i \in A} z_i = \tilde{m}^\ell \right\} \\
&\equiv \varphi^u(\bar{R}, \tilde{m})|^\ell.
\end{aligned}$$

Thus, by the *consistency* of φ and φ^u (Lemma 5), for each $\{i, j\} \in \mathcal{N}$ and each $m \in M(\{i, j\})$,

$$\begin{aligned}
x^\ell &= \psi^\ell(\bar{X}_i^\ell, \bar{X}_j^\ell, m^\ell) \\
&= \arg\max \left\{ u_i^{xd,\ell}(z_i) + u_j^{xd,\ell}(z_j) : z_i \in X_i^\ell, z_j \in X_j^\ell, z_i + z_j = m^\ell \right\} \\
&\equiv \varphi^u(\bar{R}_i, \bar{R}_j, m)|^\ell.
\end{aligned}$$

For each $R' \in \mathcal{R}^{\{i,j\}}$, let $S(R') \equiv \{z \in \mathbb{R}_+^{\{i,j\}} : z \leq p^\ell(R'), z_i + z_j = m^\ell\}$. We establish the first equality in

$$\varphi(R,m)|^\ell = \arg\max \left\{ u_i^{xd,\ell}(z_i) + u_j^{xd,\ell}(z_j) : (z_i, z_j) \in S(R) \right\} = \varphi^u(R,m)|^\ell, \quad (12)$$

where the second inequality follows from the definition of φ^u.

Case 1: $\boldsymbol{x^\ell \in S(R)}$. Since $p^\ell(R) \leq p^\ell(\bar{R}_{\{i,j\}})$, $S(R) \subseteq S(R_i, \bar{R}_j) \subseteq S(\bar{R}_i, \bar{R}_j)$. Thus, establishing the first equality in (12) amounts to proving that $\varphi(R,m)|^\ell = x^\ell$. Let $w \equiv \varphi(R_i, \bar{R}_j, m)$ and $y \equiv \varphi(R,m)$. By Lemma 10, for each $k \in K \setminus \{\ell\}$, $w^k = x^k$ and $y^k = x^k$. Thus, suppose that $w^\ell \neq x^\ell$. If $x_i^\ell < w_i^\ell$, $w^\ell \in S(R_i, \bar{R}_j)$ implies $x_i^\ell < w_i^\ell \leq p^\ell(R_i) \leq p^\ell(\bar{R}_i)$. Then, $w_i \; \bar{P}_i \; x_i$, contradicting *strategy-proofness* at $(\bar{R}_i, \bar{R}_j, m)$. If $x_i^\ell > w_i^\ell$, $x^\ell \in S(R)$ implies $w_i^\ell < x_i^\ell \leq p^\ell(R_i)$. Then, $x_i \; P_i \; w_i$, contradicting *strategy-proofness* at $(R_i, \bar{R}_j, m)$. Thus, $w = x$. Using a similar argument we go from $(R_i, \bar{R}_j, m)$ to (R,m), arriving at $y = w = x$. Thus, inf fact, $\varphi(R,m)|^\ell = x^\ell$, as desired.

Case 2: $\boldsymbol{x^\ell \notin S(R)}$. Then, without loss of generality, i is such that $x_i^\ell > p^\ell(R_i)$. Let $y = \varphi(R,m)$. Suppose that $y_i^\ell \neq p^\ell(R_i)$. Thus, by Lemma 8 and since $p^\ell(R_i) + p^\ell(R_j) \geq m^\ell$, $y_i^\ell < p^\ell(R_i)$. Let $R_i' \in \mathcal{R}_i$ be such that $p(R_i') = p(R_i)$ and $x_i \; P_i' \; y_i$. Let $w \equiv \varphi(R_i', R_j, m)$. By Lemma 9, $w = y$. Note that $p^\ell(R_i) + p^\ell(R_j) \geq m^\ell = x_i^\ell + x_j^\ell$ and $p^\ell(\bar{R}_i) \geq x_i^\ell > p^\ell(R_i)$ implies $p^\ell(R_j) > x_j^\ell$. Thus, $x^\ell \in S(\bar{R}_i, R_j)$. Thus, by Case 1, $x^\ell = \varphi(\bar{R}_i, R_j, m)|^\ell$. Moreover, by Lemma 9, since, for each $k \in K \setminus \{\ell\}$, $p^k(\bar{R}_i, R_j) = p^k(\bar{R}_i, \bar{R}_j)$, $x^k = \varphi(\bar{R}_i, R_j, m)|^k$. Thus,

$$\varphi_i(\bar{R}_i, R_j, m) = x_i = y_i \; P_i' \; w_i = \varphi_i(R_i', R_j, m),$$

which contradicts *strategy-proofness* at (R_i', R_j, m). Thus, $y_i = p^\ell(R_i)$. Thus, y is the allocation in $S(R)$ that is closest to x. This establishes (12). ∎

The rest of the proof of Theorem 2 relies on the fact that the separably concave mechanisms are "conversely consistent." This property requires it to be possible to deduce if an allocation is desirable for an economy if the restriction of this allocation is itself considered desirable for each two agent sub-economy.[22] A mechanism φ is **conversely consistent** if, for each $N \in \mathcal{N}$ and each $(R,m) \in \mathcal{E}^N$,

$$[x \in Z(N,m) \text{ and, for each } \{i,j\} \subseteq N, \; x_{\{i,j\}} = \varphi(R_{\{i,j\}}, x_i + x_j)] \Rightarrow x = \varphi(R,m).$$

[22]See Thomson (2011) for a detailed discussion of the property.

Lemma 13. *The separably concave mechanisms are conversely consistent.*

Proof. Let $u \equiv \{(u_i^{xd,\ell}, u_i^{xs,\ell}) : \ell \in K, i \in A\} \in \mathcal{U}$. Since the separably concave mechanisms are *strategy-proof*, *unanimous*, *resource-monotonic*, and *consistent*, by Lemma 10, there is $\{\psi^\ell : \ell \in K\} \in \Psi$ such that, for each $N \in \mathcal{N}$, and each $(R, m) \in \mathcal{E}^N$,

$$\varphi^u(R, m) = \{\psi^\ell(p^\ell(R), m^\ell) : \ell \in K\}.$$

By Lemma 5, φ^u is *resource-monotonic*. Using Lemma 9, this implies that, for each $N \in \mathcal{N}$, each $R \in \mathcal{R}^N$, and for each pair $\hat{m}^\ell, \tilde{m}^\ell \in M^\ell(N)$,

$$\hat{m}^\ell > \tilde{m}^\ell \quad \Rightarrow \quad \psi^\ell(p^\ell(R), \hat{m}^\ell) \geq \psi^\ell(p^\ell(R), \tilde{m}^\ell). \tag{13}$$

Let $N \in \mathcal{N}$, and $(R, m) \in \mathcal{E}^N$. Let $x \in Z(N, m)$ be such that, for each $\{i, j\} \subseteq N$, $x_{\{i,j\}} = \varphi^u(R_i, R_j, x_i + x_j)$. Let $y \equiv \varphi^u(R, m)$. Since φ^u is *consistent*, $y_{\{i,j\}} = \varphi^u(R_i, R_j, y_i + y_j)$. Thus, by *consistency*, if there is $\{i, j\} \subseteq N$ such that $y_{\{i,j\}} \neq x_{\{i,j\}}$, $y_i + y_j \neq x_i + x_j$. If so, without loss of generality, there is $\ell \in K$ such that $y_i^\ell + y_j^\ell > x_i^\ell + x_j^\ell$. Thus, by (13),

$$y^\ell_{\{i,j\}} = \psi^\ell(p^\ell(R_i, R_j), y_i^\ell + y_j^\ell) \geq \psi^\ell(p^\ell(R_i, R_j), x_i^\ell + x_j^\ell) = x^\ell_{\{i,j\}}$$

and, without loss of generality, i is such that $y_i^\ell > x_i^\ell$. Thus, since $\sum_N x_h^\ell = m^\ell = \sum_N y_h^\ell$, there is $k \in N \setminus \{i, j\}$ such that $y_k^\ell < x_k^\ell$. By *consistency*, $y_{\{i,k\}} = \varphi^u(R_i, R_k, y_i + y_k)$ and, by assumption, $x_{\{i,k\}} = \varphi^u(R_i, R_k, x_i + x_k)$. Thus, if $y_i^\ell + y_k^\ell \geq x_i^\ell + x_k^\ell$, by (13), $y_k^\ell \geq x_k^\ell$, which is not the case. Thus, $y_i^\ell + y_k^\ell < x_i^\ell + x_k^\ell$. Thus, by (13), $y^\ell_{\{i,k\}} \leq x^\ell_{\{i,k\}}$, contradicting $y_i^\ell > x_i^\ell$. Thus, φ^u is *conversely consistent*. ∎

As we mentioned in the beginning of this Section, the last step in the proof of Theorem 3 and consists of the following "Elevator Lemma." This step establishes that, if a *consistent* mechanism coincides with a *conversely consistent* mechanism for two agent economies then, in fact, they coincide in general. Recall that, in Lemma 12, we proved that a mechanism satisfying the properties in Theorems 2 coincides with a separably concave mechanism. The following results "elevates" this coincidence from two agent economies to those with any finite number of agents.

Lemma 14. *Let φ denote a mechanism satisfying the properties in Theorem 3. Then, there is $u \equiv \{(u_i^{xd,\ell}, u_i^{xs,\ell}) : \ell \in K, i \in A\} \in \mathcal{U}^*$ such that $\varphi = \varphi^u$.*

Proof. Let $N \in \mathcal{N}$, $(R, m) \in \mathcal{E}^N$, and $x \equiv \varphi(R, m)$. By *consistency*, for each $\{i, j\} \subseteq N$, $\varphi(R_i, R_j, x_i + x_j) = x_{\{i,j\}}$. By Lemma 12, there is $u \equiv \{(u_i^{xd,\ell}, u_i^{xs,\ell}) : \ell \in K, i \in A\} \in \mathcal{U}^*$ such that for each $\{i, j\} \subseteq N$, $\varphi(R_i, R_j, x_i + x_j) = \varphi^u(R_i, R_j, x_i + x_j)$. By Lemma 13, φ^u is *conversely consistent*. Thus, $\varphi^u(R, m) = x = \varphi(R, m)$, as desired. ∎

A.4 Results for the single resource case

We first recall useful facts (Sprumont, 1991) that will be helpful in the proof of Lemma 3.

Remark 1. *Suppose that K is a singleton. Let $N \in \mathcal{N}$ and $(R,m) \in \mathcal{E}^N$.*
(i) $P(R,m)$ is a compact and convex set.
(ii) Allocation $x \in Z(N,m)$ is efficient at $(R,m) \in \mathcal{E}^N$ if and only if $\sum_N p(R_i) \geq m$ implies $x \leq p(R)$, and $\sum_N p(R_i) \leq m$ implies $x \geq p(R)$.

Proof of Lemma 3. Suppose that K is a singleton. Let φ denote a *unanimous*, *consistent*, and *strategy-proof* mechanism. We will prove that φ is *efficient*. If φ is not *efficient*, there are $N \in \mathcal{N}$ and $(R,m) \in \mathcal{E}^N$ such that $x \equiv \varphi(R,m)$ is not *efficient* at (R,m), $x \notin P(R,m)$. Thus, by (ii) in Remark 1, there is a pair $i,j \in N$ such that $x_i < p(R_i)$ and $x_j > p(R_j)$. By *consistency*, $x_{\{i,j\}} = \varphi(R_i, R_j, x_i+x_j)$. By *unanimity*, either $x_i + x_j < p(R_i) + p(R_j)$ or $x_i + x_j > p(R_i) + p(R_j)$. Without loss of generality, assume the former. Let $R'_i \in \mathcal{R}_i$ be such that $p(R'_i) = x_i + x_j - p(R_j) > 0$ and $y \equiv \varphi(R'_i, R_j)$. Since, $p(R'_i) + p(R_j) = x_i + x_j$, $p(R'_i, R_j) \in Z(\{i,j\}, x_i + x_j)$. Thus, by *unanimity*, $y_i = p(R'_i)$ and $y_j = p(R_j)$. However, by feasibility, $x_i < y_i < p(R_i)$. Thus, $y_i \; P_i \; x_i$, contradicting *strategy-proofness* at (R_i, R_j). This contradiction establishes that φ is *efficient*. ∎

Finally we establish that the separably concave mechanisms are *group strategy-proof* when K is a singleton.

Proof of Proposition 1. Let $N \in \mathcal{N}$, $u \equiv \{(u_i^{xd}, u_i^{xs}) : i \in A\} \in \mathcal{U}^*$, $(R,m) \in \mathcal{E}^N$, and $x \equiv \varphi^{(l,r)}(R,m)$. If $\sum_N p(R_i) = m$, then, for each $i \in N$, $x_i = p(R_i)$ and no agent has an incentive to misreport her preferences. Suppose that $\sum_N p(R_i) > m$.

Let $M \subseteq N$ and $(R', m) \in \mathcal{E}^N$ be such that, for each $j \in N \setminus M$, $R'_j = R_j$. Let $x' \equiv \varphi^u(R', m)$ and assume that

$$\text{for each } i \in N,\; x'_i \; R_i \; x_i. \tag{14}$$

We will prove that (14) implies $x = x'$. (This implies that the group of agents M cannot manipulate at (R,m).) Because preferences are single-peaked and $x \leq p(R)$ (by Remark 1), a necessary condition for (14) is that

$$\text{for each } i \in M,\; x_i \leq x'_i. \tag{15}$$

Case 1: $\boldsymbol{\sum_N p(R'_j) \leq m}$. Since φ^u is *efficient*, for each $j \in N \setminus M$, $x'_j \geq p(R'_j) = p(R_j) \geq x_j$. Thus, $\sum_M x_k \geq \sum_M x'_k$. Thus, by (15) and because preferences are single-peaked, if there is $i \in M$ such that $x'_i \; P_i \; x_i$ there is also $j \in M$ such that $x_j \; P_i \; x'_j$. Thus, (15) requires that $\sum_N p(R'_j) > m$.

Case 2: $\sum_N p(R'_j) > m$ and $x \in P(R', m)$. Thus, by the definition of φ^u, $x' \neq x$ requires $x' \in P(R', m) \setminus P(R, m)$. Then, by Remark 1, there is $i \in N$ such such that $p(R'_i) \geq x'_i > p(R_i) \geq x_i$. Thus, $P(R'_i, R_{N\setminus\{i\}}, m) \supseteq P(R, m)$ and, since $R_{N\setminus M} = R'_{N\setminus M}$, $i \in M$. Additionally, because preferences are single-peaked, (14) and $x'_i > p(R_i) \geq x_i$ requires $p(R_i) > x_i$. Thus, from the definition of φ^u and by Remark 1, $x = \varphi^u(R'_i, R_{N\setminus\{i\}}, m)$. Thus, $x' \in P(R', m) \setminus P(R'_i, R_{N\setminus\{i\}}, m)$. Thus, there is $j \in N\setminus\{i\}$ such that $p(R'_j) \geq x'_j > p(R_j) \geq x_j$. Thus, $P(R'_i, R'_j, R_{N\setminus\{i\}}, m) \supseteq P(R, m)$ and, since $R_{N\setminus M} = R'_{N\setminus M}$, $j \in M$. Additionally, because preferences are single-peaked, (14) and $x'_j > p(R_j) \geq x_j$ requires $p(R_j) > x_j$. Thus, from the definition of φ^u and by Remark 1, $x = \varphi^u(R'_i, R'_j, R_{N\setminus\{i,j\}}, m)$. Clearly, we can continue in this way until we exhaust M. Thus, (15) implies $x = x'$.

Case 3: $\sum_N p(R'_j) > m$ and $x \notin P(R', m)$. Then, by Remark 1, there is $i \in N$ such that $p(R_i) \geq x_i > p(R'_i) \geq x'_i$. Since $R_{N\setminus M} = R'_{N\setminus M}$, $i \in M$. This contradicts (15). Thus, (15) implies $x = x'$.

Cases 1 through 3 establish that, when $\sum_N p(R_i) \geq m$, (14) implies $x = x'$. A symmetric argument yields the same conclusion when $\sum_N p(R_i) \leq m$. Clearly, this conclusions hold for each $(R, m) \in \mathcal{E}^N$ and each group of agents $M \subseteq N$. Thus, φ^u is *group strategy-proof*. ∎

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