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STRATEGYPROOF ASSIGNMENT BY HIERARCHICAL EXCHANGE

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We give a characterization of the set of group-strategyproof, Pareto-optimal, and reallocation-proof allocation rules for the assignment problem, where individuals are assigned at most one indivisible object, without any medium of exchange. Although there are no property rights in the model, the rules satisfying the above criteria imitate a trading procedure with individual endowments, in which individuals exchange objects from their hierarchically determined endowment sets in an iterative manner. In particular, these assignment rules generalize Gale's top trading cycle procedure, the classical rule for the model in which each individual owns an indivisible good.

KEYWORDS: Strategyproof, housing market, indivisible goods.

1. INTRODUCTION

CONSIDER THE PROBLEM of allocating heterogeneous indivisible goods to individuals so that each individual receives at most one good without any monetary compensation. For example, a manager may want to allocate offices to employees, or tasks to workers, or a professor may intend to assign projects to students. This study investigates whether there are some "nice" solutions to this pure distribution assignment problem[2] that take into account the individuals' strategic behavior. In particular, we are interested in strategyproof assignment rules, rules that don't permit successful manipulation of the outcome via misrepresentation of privately known preferences.

An early paper by Hylland and Zeckhauser (1979) proposed a lottery mechanism for assigning positions to individuals, in which the individuals "purchase" probability shares for obtaining positions. Although they were concerned with eliciting honest preferences, their rule is not entirely immune to cheating. More recently, Zhou (1990) proved Gale's conjecture that, allowing for lotteries over pure assignments, there do not exist allocation rules that satisfy symmetry, Pareto-optimality, and strategyproofness. For the deterministic setting Svensson (1994) proposed a strategyproof and Pareto-optimal queue allocation procedure that accommodates indifferences in the valuations of the indivisible objects. This procedure is essentially what is known as a serial dictatorship,[3] one in which the individuals choose their favorite object among the "remaining" objects, accord-

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[2]An introduction to the results and literature on assignment problems with money can be found in Roth and Sotomayor (1990).

[3]See, for example, Satterthwaite and Sonnenschein (1981).

ing to a hierarchy of the individuals. A recent paper by Svensson (1999) showed that the set of strategyproof, nonbossy, and neutral allocation rules is the set of serial dictatorships, if every individual is assumed to be assigned an object.

Although there are no strategyproof and Pareto-optimal assignment rules that treat individuals equally, the question arises whether there can be found more flexible and less discriminating non-probabilistic rules than serial dictatorships. The current paper answers this question by identifying a larger class of solutions, to be called *hierarchical exchange rules*, which are strategyproof and Pareto-optimal for the pure distribution assignment problem, when the number of objects and the number of individuals are not necessarily the same. The main result we present is that, even though there are no property rights in our model, the assignment rules satisfying group-strategyproofness, Pareto-optimality, and reallocation-proofness (a criterion that rules out an obvious case of manipulation via misrepresenting preferences and swapping objects ex post) can be described by an iterative procedure in which, at each stage, the individuals exchange objects according to Gale's top trading cycle algorithm, as if the objects were owned by them. Thus, a striking feature of our result is that, for this pure distribution problem with collective endowments, the assignment rules satisfying our criteria imitate a market procedure with individual property rights. To put it more sharply, one can draw the conclusion that if we want to efficiently distribute heterogeneous indivisible goods, and nobody is to receive more than one good, all we can (and need to) do is to assign individual property rights and let people trade. Of course, since the objects need not be given as endowments on a one-to-one basis, and the individuals receive at most one object, "property rights" will have to be assigned in a hierarchical manner.

As indicated above, the hierarchical exchange rules can be regarded as generalizations of Gale's well-known top trading cycle procedure. This procedure was first described by Shapley and Scarf (1974), who introduced the housing market, a model in which each individual owns an indivisible object (a house) initially. Gale's procedure gives a constructive way of finding a core allocation of a housing market, which, subsequently, was shown to be unique by Roth and Postlewaite (1977), when indifferences among objects are not allowed. Roth (1982) proved that the core solution, and thus the top trading cycle procedure, is strategyproof. Furthermore, Ma (1994) and Svensson (1999) showed that it is the only Pareto-optimal, individually rational, and strategyproof rule for the housing market. Group-strategyproofness of the core solution was proved by Bird (1984) and Moulin (1995). Recently, Abdulkadiroğlu and Sönmez (1998) proposed the 'core from random endowments' rule for the house allocation problem (the pure distribution assignment problem with an equal number of objects and individuals), a rule that selects the core of a random housing market, and showed that it is equivalent to the random serial dictatorship rule. Another related result is presented in Abdulkadiroğlu and Sönmez (1999) who investigated a mixed model of a house allocation and a housing market. They propose a Pareto-optimal, individually rational, and strategyproof procedure for their model, which accommodates an exogenous hierarchy, subject to existing property rights. The proposed class of procedures corresponds to a special class of hierarchical exchange rules. The results of the present study and those of Abdulkadiroğlu and Sönmez (1999) were obtained independently.

2. HIERARCHICAL EXCHANGE RULES—AN INFORMAL DISCUSSION

First, we describe Gale's top trading cycle procedure. Given a housing market, let everyone point to the person who owns their favorite house. A top trading cycle consists of individuals such that each individual in the cycle points to the next individual. A single person may also constitute a cycle, by pointing to herself, if her top choice is the house she owns. Note that there is at least one top trading cycle, since there are a finite number of individuals. Give every individual in each top trading cycle their top choice, and remove them from the market with their assigned houses. Repeat the process until each individual receives her assignment. The resulting allocation is unique if preferences are strict, and it is the core allocation.

The allocation obtained by a hierarchical exchange rule can be described by the following iterative procedure. Individuals have an initial individual "endowment" of objects such that each object is exactly one individual's endowment. Otherwise, the distribution of initial endowments is arbitrary. It is important to note that some individuals may not be endowed with any objects. Now apply the top trading cycle procedure to this market with individual endowments. Notice that individuals who don't have endowments cannot be part of a top trading cycle, since nobody points to them, and therefore they need not point. Given that multiple endowments are allowed, after the individuals in top trading cycles leave the market with their top-ranked object, unassigned objects in the initial endowment sets of individuals who received their assignment may be left behind. These objects are reassigned as endowments to individuals who are still in the market, that is, they are "inherited" by individuals who have not yet received their assignments. Furthermore, the objects in the initial endowment sets of individuals who are still in the market remain the individual endowments of these individuals. Thus, notice that each unassigned object is the endowment of exactly one individual who is still in the market. Now apply the top trading cycle procedure to this reduced market with the new endowments. Repeat this procedure until every individual has her assignment or all the objects are assigned. Since there exists at least one top trading cycle at every stage, this procedure, similarly to Gale's original procedure, leads to an allocation of the objects in a finite number of steps. In particular, there are at most as many stages as there are individuals or objects, whichever number is smaller, since at each stage at least one person receives her assignment. Furthermore, for any strict preferences of the individuals, the resulting allocation is unique.

A hierarchical exchange rule is determined by the initial endowments and the hierarchical endowment inheritance at later stages. While the initial endowment sets are given a priori, the hierarchical endowment inheritance may be endogenous. In particular, the inheritance of endowments may depend on the assignments made at earlier stages. The exact "rules" of endowment inheritance will be explained in Section 4, where we provide a formal definition of hierarchical exchange rules, and in Section 5 we will illustrate the definition by examples. For now, we would like to point out an intuitive and simple rule of endowment inheritance, the Assurance Rule, which is always satisfied by hierarchical exchange rules. The Assurance Rule requires that individuals keep their endowed objects until they leave the market, and thus endowments can only be inherited from people who are exiting the market. We give an example below to illustrate how a hierarchical exchange rule works, without specifying how the endowments are determined at the noninitial stages of the iterative procedure.

EXAMPLE 1: *A hierarchical exchange rule* (*without specifying endowment inheritance*). We have six individuals, numbered from 1 to 6, and seven objects, denoted $a, \dots, g$. Let the initial endowments be the following:

1	2	3	4	5	6
a, b, c	d, e	—	—	f	g

Consider the preference profile below, which shows the individuals' rankings of objects (from top to bottom):

R_1	R_2	R_3	R_4	R_5	R_6
d	c	c	e	e	d
$\vdots$	$\vdots$	f	g	$\vdots$	c
		$\vdots$	$\vdots$		b
					$\vdots$

The following figures illustrate the iterative procedure for this preference profile. The arrows in the figures indicate who points to whom, based on the endowments and the top choices of the individuals among the objects that are still in the market at the corresponding stage.

At stage 1 (Figure 1) there is one top trading cycle, which consists of individuals 1 and 2. Thus, 1 is assigned object d, and 2 is assigned object c. Since individuals 1 and 2 have received their assignments, objects a and b from 1 and object e from 2 come up for inheritance. Without specifying the endowment inheritance, let's assume that 3 inherits b, 4 inherits a, and 5 inherits e.

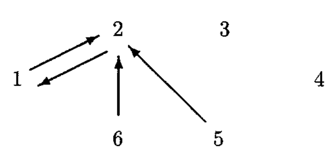

FIGURE 1.—Stage 1.

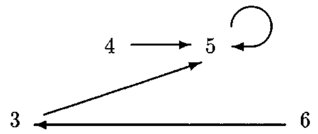

FIGURE 2.—Stage 2.

Then we have the following endowments at the beginning of stage 2:

3	4	5	6
b	a	e, f	g

Note that f and g remain 5's and 6's endowments, respectively, given the Assurance Rule.

The top choices at stage 2 (Figure 2) are f for 3, e for 4 and 5, and b for 6. The only top trading cycle at this stage consists of 5, who is assigned e. Assuming that 4 inherits f from 5, the Assurance Rule implies that we have the following endowments at the beginning of stage 3:

3	4	6
b	a, f	g

At stage 3 (Figure 3) there is a single top trading cycle that includes all three remaining individuals. Thus, the assignments of 3, 4, and 6 are f, g, and b, respectively, the respective top choices at this stage. The procedure stops here since all the individuals have received their assignments. In sum, the assignments are: $1-d$, $2-c$, $3-f$, $4-g$, $5-e$, $6-b$. Object a remains unassigned.

To complete our informal discussion of hierarchical exchange rules, let us point out two special types of hierarchical exchange rules that are well studied in the literature, namely, Gale's top trading cycle rules and serial dictatorships. For problems with no more objects than individuals, a hierarchical exchange rule in which each individual has at most one object as initial endowment is just Gale's top trading cycle procedure applied to the corresponding housing market.[4] On the other hand, serial dictatorships are characterized by a single individual being endowed with all the unassigned objects at each round. For these

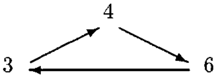

FIGURE 3.—Stage 3.

[4] If there are more objects than individuals, then there is no hierarchical exchange rule that imitates a housing market, since in this case at least one individual is endowed with more than one object.

assignment rules there is one top trading cycle at each stage, which consists of a single person, the "dictator" for that stage, where the sequence of dictators is determined for all preference profiles by an exogenously given hierarchy of the individuals. These two types of rules, the ones that imitate housing markets and the serial dictatorships, can be regarded as two opposite extremes within the class of hierarchical exchange rules: one distributes priorities for objects among the individuals entirely, so that there is no inheritance, while the other one gives the priorities for all objects, at every stage, to one individual, so that all the objects remaining in the market are inherited at each stage.

3. NOTATION AND DEFINITIONS

There are $n \geq 3$ individuals and $k \geq 3$ objects.[5] We denote the set of individuals by N and the set of objects by K. Individuals are numbered from 1 to n, and are denoted, in general, by i, j, h, l, while objects are denoted by a, b, c, d, etc.

An *allocation* $x = (x_1, \ldots, x_n)$ is a list of the assignments for the n individuals, where $x_i \in K$ or $x_i = 0$ for each i. If $x_i = a$ where $a \in K$, then individual i is assigned object a in allocation x. If individual i is not assigned any object in x, we write $x_i = 0$, where 0 is an artificial null object which can be "assigned" to any number of individuals. An allocation x is *feasible* if none of the objects is assigned to more than one individual.

An individual's preferences over allocations are selfish and the preferences over assignments are strict. Thus, individual i's *preferences*, denoted by R_i, are given by a reflexive (for all a, aR_ia), transitive (for all a, b, c, aR_ib and bR_ic imply aR_ic), complete (for all a, b, aR_ib or bR_ia), and antisymmetric (for all a, b, aR_ib and bR_ia imply $a = b$) binary relation over K. The associated strict relation is denoted by P_i. We write aP_ib if aR_ib and $a \neq b$. A *preference profile* (or simply a *profile*) is an n-tuple of preferences, denoted by R. As usual, R_{-i} is a profile of all the individuals except for i. Also, for $M \subset N$, let R_M denote a profile of all individuals in M, and let R_{-M} denote a profile of all the individuals in $N - M$. Let $\text{top}(R_i)$ denote the top-ranked object according to R_i. Furthermore, for all $K' \subseteq K$, let $\text{top}(R_i, K') = a$ if $a \in K'$, and for all $b \in K'$, aR_ib. Besides R_i, i's preferences will also be denoted by $\tilde{R}_i, \overline{R}_i$, and $\hat{R}_i$, with corresponding strict relations $\tilde{P}_i, \overline{P}_i$, and $\hat{P}_i$, respectively.

An *assignment rule* f associates an allocation with each profile. Denoting the set of profiles by $\mathscr{R}$ and the set of feasible allocations by X, an assignment rule is a function $f: \mathscr{R} \mapsto X$. Let $f_i(R)$ denote the assignment prescribed to individual i by f at R, and let $f_M(R)$ denote the assignments prescribed to individuals in $M \subseteq N$.

[5] Our characterization result also holds for the cases where $n = 2$ or $k = 2$. We rule them out in order to ease the exposition, since these cases entail special relationships that don't hold in general.

4. HIERARCHICAL EXCHANGE RULES—A FORMAL DEFINITION

4.1. *Inheritance Trees*

First we define an inheritance tree Γ_a for each object $a \in K$, where Γ_a shows how a is inherited. A *list of inheritance trees* $\Gamma = (\Gamma_a)_{a \in K}$ determines a hierarchical exchange rule f^{Γ}, which we will define afterwards.

An *inheritance tree* $\Gamma_a = (V, Q)$ *for object a* is a *rooted tree*, where V is the set of vertices, and $Q \subset V \times V$ is the set of arcs; if $(v_i, v_j) \in Q$ for $v_i, v_j \in V$, then there is an arc from v_i to v_j. A *Q-path* from v_1 to v_r is a sequence $\{v_s\}_{s=1}^r$, where $r \geq 2$, such that for all $s = 1, \ldots, r-1$, $(v_s, v_{s+1}) \in Q$. The *length* of a Q-path is the number of the connecting arcs.

Since Γ_a is a rooted tree, Q is *acyclic*: if there is a Q-path from v_i to v_j, then $(v_j, v_i) \notin Q$, i.e., there are no cycles. Furthermore, for all $v_1, v_r \in V$ there is at most one Q-path from v_1 to v_r. Thus, if $\{v_s\}_{s=1}^r$ is a Q-path, the *distance* of v_r from v_1 is unambiguously defined by the length of the Q-path: $d(v_1, v_r) = r - 1$. Finally, Γ_a has a unique *root* $v_0 \in V$, that is, v_0 is the only vertex such that there is no $v \in V$ with $(v, v_0) \in Q$.

The inheritance tree Γ_a shows how a is inherited as a function of the assignments of the individuals who leave the market before a is inherited. Thus, the vertices of Γ_a correspond to the individuals, and the arcs from a vertex are labeled by the possible assignments of the individual who is represented by the vertex. More precisely, Γ_a has the following properties.

Properties (A.1) and (A.2) concern the labeling of vertices:

(A.1) *All vertices are labeled by individuals*: *for all* $v \in V, \mathscr{L}(v) \in N$.

(A.2) *Every vertex of a Q-path represents a different individual*: *for all* $v_i, v_j \in V$ *such that there is a Q-path from* v_i *to* v_j, *we have* $\mathscr{L}(v_i) \neq \mathscr{L}(v_j)$.

Properties (B.1), (B.2), and (B.3) concern the labeling of arcs:

(B.1) *All arcs are labeled by objects other than a*: *for all* $(v_i, v_j) \in Q, \mathscr{H}(v_i, v_j) \in K - \{a\}$.

(B.2) *Every arc of a Q-path represents a different object*: *for all* $v_1, v_r \in V$ *such that there is a Q-path* $\{v_s\}_{s=1}^r$ *from* v_1 *to* v_r, *where* $r \geq 3$, *we have* $\mathscr{H}(v_1, v_2) \neq \mathscr{H}(v_{r-1}, v_r)$.

(B.3) *Arcs from the same vertex represent different objects*: *for all* $v_i, v_j, v_l \in V$ *such that* $(v_i, v_j) \in Q$, $(v_i, v_l) \in Q$ *and* $j \neq l$, *we have* $\mathscr{H}(v_i, v_j) \neq \mathscr{H}(v_i, v_l)$.

Properties (C.1), (C.2), and (C.3) define the structure and dimensions of an inheritance tree as a function of the number of individuals (n) and objects (k):

(C.1) $\max_{v \in V} d(v_0, v) = m - 1$, *where* $m = \min\{n, k\}$.

(C.2) *The number of arcs starting from* v_0 *is* $k - 1$.

(C.3) *For all* $v \in V$ *such that there is a Q-path from* v_0 *to* v, *with* $d(v_0, v) = r < m - 1$, *the number of arcs starting from* v *is* $k - r - 1$.

The inheritance tree Γ_a and the properties listed above can be interpreted as follows. Object a is individual $\mathscr{L}(v_0)$'s initial endowment; hence, this initial endowment is independent of any assignments made. Depending on the assignment of individual $\mathscr{L}(v_0)$, the next individual to inherit a, or at least influence its inheritance, is determined by following the arc labeled by the object that $\mathscr{L}(v_0)$ is assigned. Continuing in this way we can determine all the potential inheritances by following the appropriate arcs. Thus, we can interpret each Q-path from the root as an inheritance path. Therefore, property (A.2) insures that there is no repetition in the hierarchy of inheritors, following any Q-path from v_0. Moreover, property (A.2) combined with (B.2) and (B.3) insures that there is no incompatibility among the required assignments for a particular inheritance. Properties (B.2), (B.3), (C.2), and (C.3) together imply that for every possible assignment of earlier potential owners there is a designated inheritor. Note also that object a may not be inherited if it is assigned to somebody, and thus none of the arcs are labeled a, as required by (B.1). Finally, it is clear that there is no need to specify more than m individuals in any of the hierarchies when there are more individuals than objects (where m is the minimum of the number of the individuals and the number of the objects), which is why (C.1) says that each (inheritance) path starting from the root should have a length of $m-1$.

Let G_a denote the set of inheritance trees for a that satisfy all of the properties listed above, and let $G = \times_{a \in K} G_a$. Then for each list of inheritance trees $\Gamma \in G$, a *hierarchical exchange rule* f^{Γ} *associated with* Γ is defined using the endowments that we specify next. In general, we say that an assignment rule f is a *hierarchical exchange rule* if there exists $\Gamma \in G$ such that f is the hierarchical exchange rule associated with Γ. In this case we will also say that Γ is an *underlying inheritance tree list for* f.

In what follows we describe how a hierarchical exchange rule f^{Γ} associated with Γ chooses the allocation at each preference profile.

4.2. *Endowments*

For all $M \subset N$ and $L \subset K$ such that $|M| = |L| \neq 0$, let

$$\Phi_{M,L} = \{\varphi_{M,L} : M \mapsto L : \varphi_{M,L} \text{ is a bijection}\}$$

be the set of bijections from M to L. Here M refers to the set of individuals who have already received their assignments, and L is the set of objects that have already been assigned; furthermore, $\varphi_{M,L}$ shows the assignments.

Now we can define the endowments $\mathscr{E}_i^{\Gamma}$ for each individual $i \in N$, which is based on the list of inheritance trees Γ. First, the *initial endowments* $\mathscr{E}_i^{\Gamma}(\varnothing)$, when $M = L = \varnothing$, are given by

$$(1) \qquad \mathscr{E}_i^{\Gamma}(\varnothing) = \{a \in K : \mathscr{L}(v_0) = i \text{ in } \Gamma_a\}.$$

Thus, each individual i's initial endowment is just the set of objects for which the root v_0 in each of the corresponding inheritance trees designates i as the first owner. Note that the initial endowments are determined by Γ a priori and they do not depend on the preference profile.

The *noninitial endowments* $\mathscr{E}_i^{\Gamma}(M, L, \varphi_{M,L})$,[6] for $M \subset N$, $L \subset K$ such that $|M| = |L| \neq 0$, $\varphi_{M,L} \in \Phi_{M,L}$, and for each individual $i \in N - M$ are given by

$$(2) \qquad \mathscr{E}_i^{\Gamma}(M, L, \varphi_{M,L}) = \Big\{a \in K - L \colon \mathscr{L}(v_0) = i \text{ in } \Gamma_a \text{ or there exists a } Q\text{-path } \{v_s\}_{s=0}^{r} \text{ in } \Gamma_a \text{ from } v_0 \text{ to } v_r \text{ such that } \mathscr{L}(v_r) = i, \text{ and for all } s = 0, \dots, r-1, \text{ we have } \mathscr{L}(v_s) \in M, \mathscr{H}(v_s, v_{s+1}) \in L, \text{ and } \varphi_{M,L}(\mathscr{L}(v_s)) = \mathscr{H}(v_s, v_{s+1})\Big\}.$$

Note that the definition of $\mathscr{E}_i^{\Gamma}$ implies that the hierarchical endowment inheritance of each object a is determined in accordance with the interpretation of Γ_a that we gave before. The definition also makes explicit that the noninitial endowments depend only indirectly on the preference profile: namely the endowments at any noninitial stage of the iterative procedure are a function of only the set of individuals who already left the market (M), the set of objects they are assigned (L), and the assignments made $(\varphi_{M,L})$.

Finally, note that the endowments $\{\mathscr{E}_i^{\Gamma}(M, L, \varphi_{M,L})\}_{i \in N-M}$ partition the set of unassigned objects $K - L$: we have $\bigcup_{i \in N-M} \varepsilon_i^{\Gamma}(M, L, \varphi_{M,L}) = K - L$, and for all $i, j \in N - M$ such that $i \neq j$, $\mathscr{E}_i^{\Gamma}(M, L, \varphi_{M,L}) \cap \mathscr{E}_j^{\Gamma}(M, L, \varphi_{M,L}) = \varnothing$. Thus, at any stage of the iterative procedure all the unassigned objects are distributed as endowments to individuals who are still in the market.

4.3. *The Iterative Procedure*

Once the inheritance of the objects is defined, that is, once we have the definition of the endowments $\mathscr{E}_i^{\Gamma}$ based on $\Gamma \in G$, we apply an iterative procedure, essentially the top trading cycle procedure, to these endowments in order to find the assignments prescribed by the hierarchical exchange rule f^{Γ} associated with Γ.

Fix a preference profile $R \in \mathscr{R}$. Then $f^{\Gamma}(R)$ can be defined by an iterative procedure with a finite number of stages, which is at most $m = \min\{n, k\}$. For each (remaining) individual i we give recursive definitions of the *hierarchical endowments* $E_t(i, R)$, the *top choices* $T_t(i, R)$, and the *trading cycles* $S_t(i, R)$; then we define recursively the set of *assigned individuals* $W_t(R)$, the *assignments* of these individuals $f_i^{\Gamma}(R)$ for all $i \in W_t(R)$, and the set of *assigned objects* $F_t(R)$. All of these are indexed by the corresponding stage $t \in \{1, \dots, m\}$.[7]

[6] With a slight abuse of notation, we denote both the initial and noninitial endowments of i by $\mathscr{E}_i^{\Gamma}$.

[7] We suppress the reference to Γ since it will be obvious to which hierarchical exchange rule we are referring.

Stage 1

Hierarchical Endowments (*Initial Endowments*): For all $i \in N$, individual i's hierarchical endowment at stage 1 is just her initial endowment:

$$E_1(i, R) = \mathscr{E}_i^{\Gamma}(\varnothing). \tag{3}$$

Note that $E_1(i, R)$ is independent of the profile R, and we use this notation for convenience.

Top Choices: Each individual $i \in N$ "points" to the person who is endowed with her favorite object, namely, her top choice at stage 1:

$$T_1(i, R) = \operatorname{top}(R_i).$$

Trading Cycles: Since the number of individuals and the number of objects are finite, there is always at least one trading cycle, that is, a set of individuals who form a cycle by "pointing" to the person endowed with their top choice. A trading cycle, thus, consists of individuals who would like to exchange objects from their initial endowments according to this cycle, so that each would receive her top choice. Formally, i's trading cycle at stage 1, $S_1(i, R)$, is comprised of the set of individuals in her trading cycle at stage 1, if she is part of one, and $S_1(i, R)$ is defined to be the empty set otherwise. Thus, for all $i \in N$,

$$S_1(i, R) = \begin{cases} \{j_1, \dots, j_g\} & \text{if there exist } j_1, \dots, j_g \in N \\ & \text{such that for all } s = 1, \dots, g, \\ & T_1(j_s, R) \in E_1(j_{s+1}, R), \\ & \text{where we let } j_{g+1} = j_1, \text{ and } i = j_s \\ & \text{for some } s = 1, \dots, g; \\ \varnothing & \text{otherwise.} \end{cases}$$

Note that $S_1(i, R) = \{i\}$ if $T_1(i, R) \in E_1(i, R)$. That is, if i's favorite object is in her endowment, then i's "trading cycle" consists of herself.

Individuals in a trading cycle are assigned their top choices and are removed from the market with their assigned objects.

Assigned Individuals: $W_1(R) = \{i : S_1(i, R) \neq \varnothing\}$.

Assignments: For each $i \in W_1(R)$, $f_i^{\Gamma}(R) = T_1(i, R)$.

Assigned Objects: $F_1(R) = \{T_1(i, R) : i \in W_1(R)\}$.

This procedure is repeated iteratively in the remaining reduced market. Having determined iteratively, for stages 1 to t ($t \in \{1, \dots, m-1\}$), the sets of assigned individuals $W_1(R), \dots, W_t(R)$, their assignments $T_1(i, R), \dots, T_t(i, R)$ for all i in $W_1(R), \dots, W_t(R)$, respectively, and the set of assigned objects $F_1(R), \dots, F_t(R)$, we will show next how the procedure works at stage $t+1$, assuming that $N - \bigcup_{z=1}^{t} W_z(R) \neq \varnothing$ and $K - \bigcup_{z=1}^{t} F_z(R) \neq \varnothing$. For each stage $t' \in \{1, \dots, m\}$, we use the notation $W^{t'}(R) = \bigcup_{z=1}^{t'} W_z(R)$ and $F^{t'}(R) = \bigcup_{z=1}^{t'} F_z(R)$.

Stage $t+1$

Hierarchical Endowments (*Noninitial Endowments*): First, let $\varphi_{W^t(R), F^t(R)} \in \Phi_{W^t(R), F^t(R)}$ such that for all $i \in W^t(R)$,

$$\varphi_{W^t(R), F^t(R)}(i) = f_i^{\Gamma}(R).$$

For all $i \in N - W^t(R)$, the hierarchical endowments at stage $t+1$ are based on the set of assigned individuals in the first t stages, namely, $M = W^t(R)$, the set of objects they are assigned, namely, $L = F^t(R)$, and the assignments made at the first t stages, namely, $\varphi_{M,L} = \varphi_{W^t(R), F^t(R)}$. That is, for all $i \in N - W^t(R)$,

$$E_{t+1}(i, R) = \mathscr{E}_i^{\Gamma}\left(W^t(R), F^t(R), \varphi_{W^t(R), F^t(R)}\right). \tag{4}$$

Top Choices: Each remaining individual $i \in N - W^t(R)$ "points" to the person who is endowed with her favorite object among the remaining objects $K - F^t(R)$, namely, her top choice at stage $t+1$:

$$T_{t+1}(i, R) = \operatorname{top}(R_i, K - F^t(R)).$$

Trading Cycles: Again, there is at least one trading cycle, that is, a set of individuals who form a cycle by "pointing" to the person endowed with their top choice at stage $t+1$. A trading cycle, thus, consists of individuals in the reduced market at stage $t+1$ that would like to exchange objects from their hierarchical endowments at this stage according to this cycle. As a result of such a trade each individual in the cycle would receive her top choice among the remaining objects. Thus, for all $i \in N - W^t(R)$,

$$S_{t+1}(i, R) = \begin{cases} \{j_1, \dots, j_g\} & \text{if there exist } j_1, \dots, j_g \in N - W^t(R) \\ & \text{such that for all } s = 1, \dots, g, \\ & T_{t+1}(j_s, R) \in E_{t+1}(j_{s+1}, R), \\ & \text{where we let } j_{g+1} = j_1, \text{ and } i = j_s \\ & \text{for some } s = 1, \dots, g; \\ \varnothing & \text{otherwise}. \end{cases}$$

Note that $S_{t+1}(i, R) = \{i\}$ if $T_{t+1}(i, R) \in E_{t+1}(i, R)$.

Individuals in a trading cycle are assigned their top choices among the remaining objects and are removed from the market with their assigned objects.

Assigned Individuals: $W_{t+1}(R) = \{i\colon S_{t+1}(i, R) \neq \varnothing\}$.

Assignments: For each $i \in W_{t+1}(R), f_i^{\Gamma}(R) = T_{t+1}(i, R)$.

Assigned Objects: $F_{t+1}(R) = \{T_{t+1}(i, R)\colon i \in W_{t+1}(R)\}$.

This procedure is repeated iteratively until there are no individuals or objects remaining in the market. Note that the procedure terminates in a finite number of stages that is no more than m, since at every stage at least one individual is assigned an object. More precisely, for every profile R there exists a *last stage* $t^* \in \{1, \dots, m\}$ such that either $W^{t^*}(R) = N$ or $F^{t^*}(R) = K$, and for all $t < t^*$, $N - W^t(R) \neq \varnothing$ and $K - F^t(R) \neq \varnothing$.

If there are more individuals than objects, then at the last stage t^* we have $F^{t^*}(R) = K$, i.e., all the objects are assigned, and the remaining individuals do not receive any object: for all $i \in N - W^{t^*}(R)$, we have $f_i^{\Gamma}(R) = 0$.

In sum, for all $\Gamma \in G$, the *hierarchical exchange rule* f^{Γ} *associated with* Γ is defined as follows. For all $R \in \mathscr{R}$ and for all $i \in N$,

$$f_i^{\Gamma}(R) = \begin{cases} T_t(i, R) & \text{if } j \in W_t(R) \text{ for some } t \in \{1, \dots, m\}; \\ 0 & \text{otherwise}. \end{cases}$$

Note that f^{Γ} is unambiguously defined, since for every profile R and individual i there exists at most one stage t such that $i \in W_t(R)$.

5. ENDOWMENT INHERITANCE

The inheritance of endowments is the key feature of hierarchical exchange rules that goes beyond Gale's top trading cycle procedure, and thus makes this class of rules a generalization of the top trading cycle algorithm, one which even includes the seemingly unrelated serial dictatorships. This new feature, endowment inheritance, is also the most complex feature of hierarchical exchange rules, and thus it deserves further illustrations and remarks. We collect these in this section. First, in Subsection 5.1, we focus on rules of inheritance that are always satisfied by hierarchical exchange rules, such as the Assurance Rule mentioned in Section 2. Then, in Subsection 5.2 we provide examples of hierarchical exchange rules, complete with inheritance trees, and discuss some special cases. In particular, we introduce the subclass of fixed endowment hierarchical exchange rules for which the endowment inheritance is independent of the assignments made at earlier stages. Finally, in Subsection 5.3 we will make some remarks about the interpretation and uniqueness of the inheritance tree lists underlying hierarchical exchange rules.

5.1. *Inheritance Rules*

Two important rules of inheritance that are always satisfied by hierarchical exchange rules are the Assurance Rule, which pertains to inheritance across stages, and the Twin Inheritance Rule, which pertains to inheritance across preference profiles. These inheritance rules provide some intuition about endowment inheritance, and they are extensively used in the proof of our characterization theorem. We first give their formal definitions.

ASSURANCE RULE: Let $i \in N$ and $R \in \mathscr{R}$ such that $a \in E_t(i, R)$ for some $t < m$ and $a \in K$, and let $i \in N - W^t(R)$. Then $a \in E_{t+1}(i, R)$.

TWIN INHERITANCE RULE: Let $R, \bar{R} \in \mathscr{R}$ such that, for some $t < m$,

(i) for all $z \leq t, W^z(R) = W^z(\bar{R})$,

(ii) for all $j \in W^t(R), f_j(R) = f_j(\bar{R})$,

(iii) $R_{W^t(R)} = \bar{R}_{W^t(R)}$.

Then for all $i \in N - W^t(R), E_{t+1}(i, R) = E_{t+1}(i, \bar{R})$.

As already discussed in Section 2, the Assurance Rule ensures that once an individual is endowed with an object, then this object remains the individual's endowment as long as she is in the market. We can see that the Assurance Rule holds, given any $\Gamma \in G$, as follows. Firstly, (2) implies that for all $i \in N - M'$, $\mathscr{E}_i^{\Gamma}(M, L, \varphi_{M,L}) \subseteq \mathscr{E}_i^{\Gamma}(M', L', \varphi_{M',L'})$, where $M \subset M'$, $L \subset L'$, and for all $i \in M$, $\varphi_{M',L'}(i) = \varphi_{M,L}(i)$. Then, since for any two stages t and t' such that $t < t'$, $W^t(R) \subset W^{t'}(R)$, the Assurance Rule follows from (4).

The Twin Inheritance Rule says that if we have two profiles at which the iterative procedures associated with the hierarchical exchange rule f^{Γ} are the same up to stage t (i.e., we have the same sets of assigned individuals (i), the same assignments (ii), and the same preferences for the assigned individuals (iii) at stages 1 to t) then the endowments are the same at stage $t+1$ for these two profiles. It follows immediately from (4) that the Twin Inheritance Rule is satisfied by any hierarchical exchange rule f^{Γ} associated with $\Gamma \in G$. In fact, (4) reveals, in accordance with our interpretation of $\mathscr{E}_i^{\Gamma}$, that a stronger version of the Twin Inheritance Rule also holds: for all $R, \bar{R}$ and $t, \bar{t}$, if $W^t(R) = W^{\bar{t}}(\bar{R})$, and for all $j \in W^t(R)$, $f_j(R) = f_j(\bar{R})$, then for all $i \in N - W^t(R)$, $E_{t+1}(i, R) = E_{t+1}(i, \bar{R})$. We focus on the weaker version stated above, since it is sufficient for use in the proofs, and it covers the most relevant cases when comparing endowment inheritance between preference profiles.

5.2. *Examples and Special Cases*

First we give an example of a hierarchical exchange rule.

EXAMPLE 2: *A hierarchical exchange rule* (*with a list of inheritance trees*). We have five individuals and three objects, denoted a, b, c. The inheritance trees are those shown in Figure 4.

The list of inheritance trees $(\Gamma_a, \Gamma_b, \Gamma_c)$ determines the hierarchical exchange rule f^{Γ}. We will illustrate the use of inheritance trees by showing how to find the assignments for the preference profile below (the prescribed assignments are

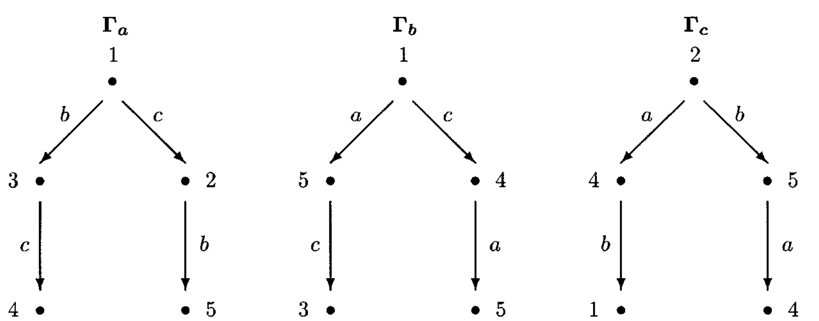

FIGURE 4.

indicated by squares):

R_1	R_2	R_3	R_4	R_5
$\boxed{c}$	$\boxed{b}$	c	a	b
b	a	b	b	$\boxed{a}$
a	c	a	c	c

- *Initial endowments*: $E_1(1, R) = \{a, b\}$, $E_1(2, R) = \{c\}$, $E_1(3, R) = E_1(4, R) = E_1(5, R) = \varnothing$. The initial endowments are found from the label of the root of each tree.
- *Stage* 1: The only trading cycle consists of 1 and 2, and the assignments are $1 - c$ and $2 - b$.
- *Endowments at Stage* 2: $E_2(3, R) = E_1(4, R) = \varnothing$, $E_1(5, R) = \{a\}$. The only remaining object is a, which comes up for inheritance since individual 1 has left the market. Object a is inherited by individual 5, as can be seen by following the right-hand-side branch of Γ_a, labeled c $(1 - c)$, and then, given that 2 has also left, following the second arc $(2 - b)$.
- *Stage* 2: Individual 5 receives a.

Note that at the displayed preference profile both 1 and 2 left the market after stage 1 and thus individual 2 did not inherit object a; instead, a was given to 5, as an endowment, immediately. On the other hand, it is clear that not all the assignments made at an earlier stage will necessarily affect the inheritance of a particular object. For instance, if we change individual 2's preferences so that her top-ranked object becomes a, then 1 and 2 would trade objects a and c, and object b would be inherited. Then, following the right-hand-side branch in Γ_b $(1 - c)$ we get to 4, who is still in the market. Thus, 4 inherits b in this case, regardless of what 2's assignment is.

A special class of hierarchical exchange rules, known as *sequential dictatorships* (see, for example, Ehlers and Klaus (1999) and Pápai (2000)) are similar to serial dictatorships, but allow the choice of the dictator at any noninitial stage to depend on the assignments chosen by the dictators at earlier stages. In order to further illustrate the use of inheritance trees, we provide an example of a sequential dictatorship below.

EXAMPLE 3: *A sequential dictatorship*. There are three individuals and four objects. The inheritance trees in Figure 5 define a sequential dictatorship where the first dictator is individual 1; if 1 chooses object a, then the second dictator is individual 2, and otherwise the second dictator is individual 3. In other words, if 1 receives a, the hierarchy is (1, 2, 3), and if 1 receives an object other than a, then the hierarchy is (1, 3, 2).

As illustrated by Example 2 and Example 3, the inheritance trees allow for the inheritance to depend on the assignments of the individuals who own (or are at

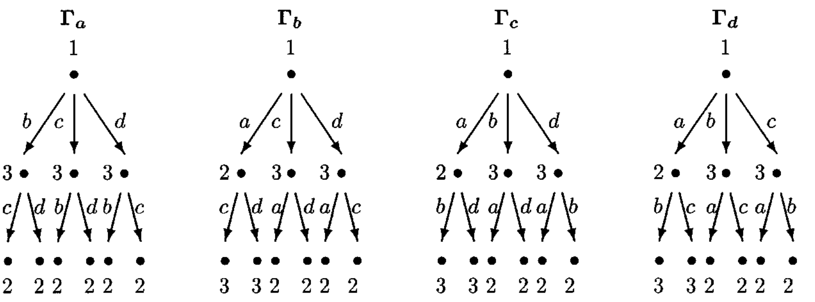

FIGURE 5.

least "qualified" to own) the object at prior stages. This flexibility may be useful in some situations. For example, one may want to use the sequential dictatorship in Example 3 in a situation where 1 is senior to 3 and 3 is senior to 2, but 2 is connected to object a so that if 1 likes a then she promotes 2 to a more senior position than 3 has. As another example, one may consider a situation in which property rights are secondary to seniority, and it is desirable to give priority over (some) remaining objects, as a compensation, to less senior people, whose objects were chosen by their more senior peers.

In situations where it is not desirable or necessary to base the inheritance on the choices of the more favored or more senior individuals, it is natural to use the simpler "exogenous" inheritance rules that only depend on who has left the market and which objects have already been assigned, but not on the assignments themselves. We refer to this subclass of hierarchical exchange rules as *fixed endowment hierarchical exchange rules*.

A fixed endowment hierarchical exchange rule can formally be defined by the following restriction on the inheritance trees. We call $\Gamma \in G$ a *fixed endowment inheritance tree list* if for all $a \in K$, for all $v, v' \in V$ such that $d(v_0, v) = d(v_0, v')$, we have $\mathscr{L}(v) = \mathscr{L}(v')$ in Γ_a. An assignment rule f is a *fixed endowment hierarchical exchange rule* if there exists a fixed endowment inheritance tree list $\Gamma \in G$ such that f is the hierarchical exchange rule associated with Γ.

In a fixed endowment inheritance tree list, for each inheritance tree, vertices that have the same distance from the root must be labeled by the same individual. This means that the inheritance is independent of the assignments made at earlier stages, and that the endowments $\mathscr{E}_i^{\Gamma}$ are only a function of the set of assigned individuals M and the set of assigned objects L. Accordingly, each inheritance tree reduces to a permutation of m individuals, i.e., to a hierarchy $(\mathscr{L}(v_0), \mathscr{L}(v_1), \ldots, \mathscr{L}(v_{m-1}))$, where $d(v_0, v_s) = s$ for all $s = 1, \ldots, m - 1$. Therefore, a fixed endowment inheritance tree list can be replaced by a single *endowment inheritance table* consisting of a hierarchy of endowment inheritance for each object, as illustrated by the next example.

EXAMPLE 4: *A fixed endowment hierarchical exchange rule* (*with an endowment inheritance table*). Consider the following endowment inheritance table:

a	b	c	d	e	f	g
1	1	1	2	2	5	6
2	3	2	1	5	4	2
4	2	4	3	3	6	3
3	4	5	6	4	1	4
5	5	6	5	1	2	5
6	6	3	4	6	3	1

We have six individuals and seven objects, denoted $a, \ldots, g$. Each object corresponds to a column. Moreover, since $m = 6$, each column is a permutation of the set of individuals, indicating the hierarchy of inheritance for the corresponding object.

This endowment inheritance table generates the endowments given in Example 1, and thus we will demonstrate the use of the table for the preference profile specified in Example 1.

- *Initial endowments*: $E_1(1, R) = \{a, b, c\}$, $E_1(2, R) = \{d, e\}$, $E_1(3, R) = E_1(4, R) = \varnothing$, $E_1(5, R) = \{f\}$, $E_1(6, R) = \{g\}$. The initial endowments are given by the first row of the table.
- *Endowments at Stage* 2: $E_2(3, R) = \{b\}$, $E_2(4, R) = \{a\}$, $E_2(5, R) = \{e, f\}$, $E_2(6, R) = \{g\}$. Object a becomes 4's endowment, since individual 4 is the next highest ranked person after 1 in the first column of the table (corresponding to object a) who is still in the market. Object b becomes 3's endowment, since 3 is listed below 1 in the second column (corresponding to object b). Similarly, e is inherited from 2 by 5.
- *Endowments at Stage* 3: $E_3(3, R) = \{b\}$, $E_3(4, R) = \{a, f\}$, $E_3(6, R) = \{g\}$. Individual 4 inherits f from 5, as seen from the penultimate column of the table.

Note that an arbitrary list of permutations of m individuals, that is, any endowment inheritance table, defines a fixed endowment hierarchical exchange rule, which follows from the definition of hierarchical exchange rules.

The two special classes of hierarchical exchange rules mentioned earlier, serial dictatorships and Gale's top trading cycle procedure applied to an arbitrary housing market (when $k \leq n$), are both subclasses of the set of fixed endowment inheritance rules. Thus, for example, the endowment inheritance table associated with the hierarchical exchange rule that corresponds to the serial dictatorship with the exogenously given hierarchy $(1, \ldots, m)$ is the following:

a	b	c	$\ldots$
1	1	1	$\ldots$
2	2	2	$\ldots$
$\vdots$		$\vdots$	
m	m	m	$\ldots$

Now consider the inheritance table:

a	b	c	d	$\ldots$
1	2	3	4	$\ldots$

This table completely determines a top trading cycle rule, given that if each individual is endowed with no more than one object, i.e., if we have a housing market, then there is no endowment inheritance.

It is interesting to note that the solution proposed by Abdulkadiroğlu and Sönmez (1999) to a problem that allows for initial property rights of houses can be interpreted as a combination of these two extreme procedures, serial dictatorships and housing markets, and hence they are also a special subset of the set of fixed endowment hierarchical exchange rules. In particular, existing tenants are endowed with their own house, and the rest of the houses are the endowment of a single person at every stage, according to an a priori given hierarchy of seniority. Thus, the inheritance tables corresponding to the proposed assignment rules are restricted to have the same hierarchy of endowment inheritance for each object (as in a serial dictatorship), except for the objects that are initially owned, for which this hierarchy is modified by lifting the owner to the top of the hierarchy (as in a housing market).

5.3. *On the Interpretation and Uniqueness of Inheritance Trees*

Inheritance tables have a somewhat different interpretation from that of inheritance trees, which follows logically from the nature of the collapse of an inheritance tree to a single inheritance hierarchy. Individuals associated with vertices in a path of an inheritance tree need not inherit the object in question at any preference profile; it may very well be the case that an individual necessarily leaves the market by the time it would be her turn to inherit the object. Nonetheless, the assignment of such an individual may affect the choice of the person inheriting the object. Hence, the individuals in any particular path of an inheritance tree are a mixture of potential inheritors of the object (potential in the sense that actual inheritance depends on the preference profile) and of individuals who don't inherit themselves at any preference profile but whose assignments may influence the identity of subsequent individuals in that path, just as earlier inheritors' assignments may have an effect.

By contrast, individuals listed in an endowment inheritance table as part of the hierarchy for a particular object may only be interpreted as potential inheritors, since the assignments themselves cannot influence the hierarchy of inheritance for fixed endowment hierarchical exchange rules (which is why a single inheritance table suffices to identify such an assignment rule). This difference in interpretation sheds some light on the complexity of non-fixed-endowment hierarchical exchange rules; indeed, the inheritance trees are deceptively simple. Since it is tempting to interpret the inheritance trees simply as a collection of hierarchies of potential inheritors, which allows the hierarchy to vary with the assignments of earlier inheritors, one may easily lose sight of the fact that the identity of an inheritor of an object may be influenced by not only the assignments of earlier owners of the object but also by the assignments of all the individuals who trade with these earlier owners.

The interpretation of inheritance trees and tables leads naturally to the issue of uniqueness. Can we specify an inheritance tree list that includes some individual whose position in all the trees implies that she never inherits a particular object for which she is listed, and whose assignment never affects the subsequent hierarchy for this object? If so, deleting the vertex associated with such an individual (and the arcs starting from it) from the inheritance tree of the object in question does not yield a different hierarchical exchange rule, or leave the assignment rule unspecified. The answer is yes, which can easily be seen from a housing-market-type inheritance table, for which specifying more than the first row is redundant. Indeed, the underlying list of inheritance trees $\Gamma \in G$ is typically not unique for a hierarchical exchange rule. It may contain some redundant information, so that another list $\Gamma' \in G$ may exist that leads to the same allocation at every preference profile as Γ does.[8] That is, we may have $f^{\Gamma}(R) = f^{\Gamma'}(R)$ for all $R \in \mathscr{R}$. It is important to note, however, that the iterative procedures for f^{Γ} and $f^{\Gamma'}$ are identical at every profile. Thus, for a given hierarchical exchange rule f, it is only the underlying inheritance tree list describing endowment inheritance that may not be unique, as opposed to the iterative procedure, which is necessarily unique, since the realized endowment inheritance is the same for all underlying inheritance tree lists for f.

Determining which parts of a particular inheritance tree list, if any, are redundant may not be a trivial task. The only exception to this, in general, is the label of the root of each tree, i.e., the initial endowments, which are never redundant. The simple inheritance tree list in Example 2 can be used to demonstrate the difficulties. Since object c is 2's initial endowment, individual 1 can only receive c if 2 leaves the market at the same stage as 1, given that, by the Assurance Rule, 2 keeps c as long as she is in the market. Thus, the specification of Γ_a implies that 2 cannot inherit a from 1 at any profile. Moreover, 2 does not appear in Γ_b, and thus 2's only endowment is object c at all the profiles. Therefore, 2 exits the market with object c (by taking it or exchanging it), which reveals that specifying more than 2's initial ownership in Γ_c is redundant.

The detection of equivalences is not intractable, as it turns out. We can find the equivalence classes by using the "*canonical form*" of hierarchical exchange rules, or more precisely, their canonical inheritance tree lists. To determine if two inheritance tree lists are equivalent in the sense that they both define the

[8]This type of redundancy is not to be confused with the trivial redundancies that follow from indicating the last object or individual. Namely, when $n > k$ (as in Example 2), the labeling of the arcs leading to the terminal vertices can be omitted. Similarly, when $n \leq k$ (as in Example 3), labeling the terminal vertices by the last individual, and thus indicating the terminal arcs, is not necessary. We defined the dimensions and labeling of inheritance trees uniformly for the sake of simplicity.

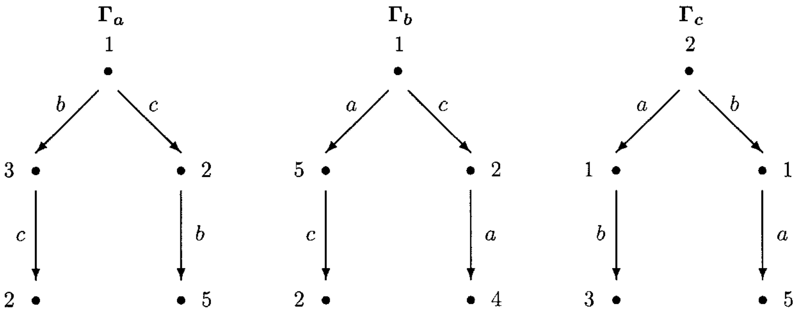

FIGURE 6.

same hierarchical exchange rule, we need only construct the canonical form for each and check if these are the same.

Here is how one can construct the canonical form of a hierarchical exchange rule.[9] Fix a vertex in Γ_a, and assume that the labels of all the vertices preceding this vertex are already determined, given an appropriate labeling of the arcs. Consider a preference profile with the following specifications. For each individual in the path preceding this vertex (the "preceding individuals," which may be the empty set), let the top choice of the individual be the object that labels the corresponding arc in Γ_a. For all the other individuals, let the top choice be object a. It can be checked that any hierarchical exchange rule assigns the top-ranked object to each "preceding individual" at this profile, and thus some individual among the remaining ones receives a. Label the vertex by this individual. We can construct the entire tree in this manner, as well as the other trees. Without going into details of the construction, we provide the canonical inheritance tree list for Example 2.

EXAMPLE 5: *The canonical form of the hierarchical exchange rule specified in Example* 2 (Figure 6). Note that Γ_c in the canonical form is completely different from the original Γ_c, with the exception of the label of the root, indicating that the rest of the tree is redundant, in accordance with our discussion above.

It is natural to ask why we focus on the canonical form in each equivalence class of inheritance tree lists. Without providing a formal definition, we may remark that the canonical form of a hierarchical exchange rule is the only underlying inheritance tree list that is internally consistent, in the sense that the redundant parts reflect inheritances that would yield, if ever realized, assignments resulting from other nonredundant parts of the inheritance trees via trading. Interestingly, this also implies that the canonical form of an inheritance table is typically not a fixed endowment inheritance tree list, and therefore a single specification of a hierarchical exchange rule may not reveal immediately

[9]A formal description of the construction of the canonical form is given in Step 1 of the necessity proof in the Appendix.

whether it is a fixed endowment hierarchical exchange rule. One may gain a better understanding of the internal consistency property of the canonical form, besides inspecting Example 5, by constructing the canonical form of a housing-market-type hierarchical exchange rule.

Finally, it is useful to remark that specifying every detail of all the inheritance trees (with the exception of the trivial redundancies mentioned in footnote 8) is necessary in some cases: for example, every sequential and serial dictatorship is uniquely determined by its canonical form.

6. THE CRITERIA FOR ASSIGNMENT RULES

The main property we require of assignment rules is *group-strategyproofness*. It ensures that no subset of the individuals can gain by reporting false preferences. More precisely, by colluding and jointly misrepresenting preferences, no individual among the deviators can be made better off without hurting at least one other deviator.

DEFINITION 1: An assignment rule f is *group-strategyproof* if for all R, there do not exist $M \subseteq N$ and $\tilde{R}_M$ such that for all $i \in M$, $f_i(\tilde{R}_M, R_{-M}) R_i f_i(R)$ and for some $j \in M$, $f_j(\tilde{R}_M, R_{-M}) P_j f_j(R)$.

Group-strategyproofness is a stricter requirement than *strategyproofness*: the latter rules out individual manipulations only. In the current context, group-strategyproofness is equivalent to strategyproofness and *nonbossiness* (Satterthwaite and Sonnenschein (1981)). Nonbossiness, a criterion frequently used in the context of strategyproof allocation, ensures that individuals cannot be bossy, that is, change the assignment for others, by reporting different preferences, without changing their own. Since it will be convenient in the following to split up group-strategyproofness into these two properties, we prove this equivalence below.

DEFINITION 2: An assignment rule f is *strategyproof* if for all R, $i \in N$, and $\tilde{R}_i$, $f_i(R) R_i f_i(\tilde{R}_i, R_{-i})$.

If f is not strategyproof then it is *manipulable*. Then there exist R, $i \in N$, and $\tilde{R}_i$ such that $f_i(\tilde{R}_i, R_{-i}) P_i f_i(R)$. We then say that individual i can *manipulate at R via $\tilde{R}_i$*.

DEFINITION 3: An assignment rule f is *nonbossy* if for all R, $i \in N$, and $\tilde{R}_i$, $f_i(R) = f_i(\tilde{R}_i, R_{-i})$ implies $f(R) = f(\tilde{R}_i, R_{-i})$.

We also need the following definitions. For an assignment rule f and for all $i \in N$ and R_{-i}, let $o_i(R_{-i}) = \{a$: there exists R_i such that $f_i(R) = a\}$ be individual i's *option set at R_{-i}*. That is, $o_i(R_{-i})$ is the set of objects that individual i can get as an assignment by changing her messages when the other individuals report R_{-i}. A push-up of a preference ordering for an object a is another preference ordering under which each object that is ranked above a is also ranked above a in the original preference ordering. That is, $\tilde{R}_i$ is a *push-up of* R_i *for* a if for all $b \in K, b\tilde{P}_i a$ implies $bP_i a$. Note that if f is strategyproof, then for all $i \in N$ and R, $f_i(R) = \text{top}(R_i, o_i(R_{-i}))$; otherwise i can manipulate at R.[10] Thus, if $\tilde{R}_i$ is a push-up of R_i for $f_i(R)$ then strategyproofness implies that $f_i(R) = f_i(\tilde{R}_i, R_{-i})$. Furthermore, if f is nonbossy as well, then we also have $f(\tilde{R}_i, R_{-i}) = f(R)$.

LEMMA 1: *An assignment rule is group-strategyproof if and only if it is strategyproof and nonbossy.*[11]

PROOF: It is obvious that group-strategyproofness implies strategyproofness and nonbossiness. We will prove the converse. Let f be strategyproof and nonbossy. Let $M \subseteq N$, R, and $\tilde{R}_M$ be such that for all $i \in M$, $f_i(\tilde{R}_M, R_{-M}) R_i f_i(R)$. Let $M = \{1, \ldots, g\}$. For all $i \in M$, let $\hat{R}_i$ preserve the ordering of R_i, except, let $\text{top}(\hat{R}_i) = f_i(\tilde{R}_M, R_{-M})$. Strategyproofness implies that $f_1(\hat{R}_1, R_{-1}) = f_1(R)$, since $f_1(\tilde{R}_M, R_{-M}) \notin o_1(R_{-1})$ if $f_1(\tilde{R}_M, R_{-M}) P_1 f_1(R)$, and otherwise $f_1(\tilde{R}_M, R_{-M}) = f_1(R)$. Then $f(\hat{R}_1, R_{-1}) = f(R)$, by nonbossiness. Repeating the same argument for individuals $2, \ldots, g$, we get $f(\hat{R}_M, R_{-M}) = f(R)$. Now note that for all $i \in M$, $\hat{R}_i$ is a push-up of $\tilde{R}_i$, for $f_i(\tilde{R}_M, R_{-M})$, so that $f(\hat{R}_M, R_{-M}) = f(\tilde{R}_M, R_{-M})$, by strategyproofness and nonbossiness. Thus, $f(\tilde{R}_M, R_{-M}) = f(R)$, which implies that f is group-strategyproof. *Q.E.D.*

Pareto-optimality, another criterion we impose on assignment rules, ensures the efficiency of the allocation at every profile.

DEFINITION 4: An assignment rule f is *Pareto-optimal* if for all R, there does not exist a feasible allocation x such that for all $i \in N$, $x_i R_i f_i(R)$ and for some $j \in N, x_j P_j f_j(R)$.

Group-strategyproofness and Pareto-optimality still allow for an obvious form of manipulation for assignment problems, namely, where individuals report preferences dishonestly, and exchange their assigned objects.[12] Our last criterion, *reallocation-proofness*, rules out the possibility that two individuals can gain by jointly manipulating the outcome and swapping objects ex post, when the collusion is self-enforcing in the sense that neither party can lose by reporting false preferences in case the other party does not adhere to the agreement and reports honestly. Consequently, reallocation-proofness rules out the most likely

[10] The definition of an option set and the related observation about strategyproofness follow Barberà (1983).

[11] See Barberà and Jackson (1995) for a related result.

[12] See Moulin (1995) for a discussion of manipulation via swapping objects ex ante and ex post in the context of the housing market model.

form of cheating via reallocating, since two individuals would find it the easiest to collude, especially if they stood no risk by doing so.[13]

DEFINITION 5: An assignment rule f is *manipulable through reallocation* if there exist R, $i, j \in N$ and $\tilde{R}_i$, $\tilde{R}_j$ such that $f_j(\tilde{R}_i, \tilde{R}_j, R_{-i,j}) R_i f_i(R)$, $f_i(\tilde{R}_i, \tilde{R}_j, R_{-i,j}) P_j f_j(R)$, and $f_h(R) = f_h(\tilde{R}_h, R_{-h}) \neq f_h(\tilde{R}_i, \tilde{R}_j, R_{-i,j})$ for $h = i, j$. An assignment rule is *reallocation-proof* if it is not manipulable through reallocation.[14]

We give an example of an assignment rule which is group-strategyproof, Pareto-optimal, and *not* reallocation-proof, in order to demonstrate that reallocation-proofness is independent of the other properties.[15] For simplicity, the example is given for three individuals and three objects, but similar examples can be constructed for arbitrary numbers of individuals and objects.

EXAMPLE 6: *A group-strategyproof, Pareto-optimal, and not reallocation-proof assignment rule*. When individual 3 prefers object b to a, let the allocation be determined by the following endowment inheritance table, as if the assignment rule were a fixed endowment hierarchical exchange rule:

a	b	c
1	2	2
3	3	1
2	1	3

Similarly, when individual 3 prefers object a to b, let the allocation be determined by the inheritance table:

a	b	c
1	2	1
3	3	2
2	1	3

Note that this assignment rule is not a hierarchical exchange rule since the initial endowments are not given a priori: the initial endowment of object c depends on individual 3's preferences, who is the last one to inherit c at all the profiles.

[13] Requiring reallocation-proofness in the stronger sense that two individuals cannot gain, *in any case*, by swapping objects ex post, after reporting dishonestly, leads to an impossibility result, if we require group-strategyproofness and Pareto-optimality as well.

[14] For our characterization result we can weaken this criterion by replacing $f_j(\tilde{R}_i, \tilde{R}_j, R_{-i,j}) R_i f_i(R)$ by $f_j(\tilde{R}_i, \tilde{R}_j, R_{-i,j}) = f_i(R)$, as group-strategyproofness and Pareto-optimality imply that it is not possible that both parties strictly gain from exchanging their objects after manipulating the outcome.

[15] This independence does not hold in the special cases: if $k = 2$, group-strategyproofness implies reallocation-proofness, and if $n = 2$, Pareto-optimality implies reallocation-proofness.

It can be verified that this assignment rule is group-strategyproof and Pareto-optimal. In particular, individual 3 is not bossy, since by switching from one inheritance table to the other the allocation remains unchanged, or 3's assignment changes from b to a or vice versa. Moreover, the profiles displayed below demonstrate that it does not satisfy reallocation-proofness (the allocations prescribed by the assignment rule are indicated by squares).

R_1	R_2	R_3
$\boxed{a}$	$\boxed{c}$	c
c	a	$\boxed{b}$
b	b	a

$\tilde{R}_1$	R_2	R_3
c	$\boxed{c}$	c
$\boxed{a}$	a	$\boxed{b}$
b	b	a

R_1	R_2	$\tilde{R}_3$
$\boxed{a}$	$\boxed{c}$	c
c	a	a
b	b	$\boxed{b}$

$\tilde{R}_1$	R_2	$\tilde{R}_3$
$\boxed{c}$	c	c
a	a	$\boxed{a}$
b	$\boxed{b}$	b

Given R as the true preference profile, neither 1 nor 3 can change the allocation alone by dishonestly reporting $\tilde{R}_1$ and $\tilde{R}_3$, respectively. Thus, the allocation at profile $(\tilde{R}_1, R_2, \tilde{R}_3)$ reveals that individuals 1 and 3 can manipulate through reallocation.

It is easy to see that the other criteria for assignment rules are independent. An imposed assignment rule, one that prescribes the same allocation to every profile, is group-strategyproof and reallocation-proof, but not Pareto-optimal. Furthermore, consider an assignment rule that acts as a serial dictatorship with hierarchy $(1, 2, 3, 4, \dots, n)$ at all the profiles where individual 1 prefers object a to b, and otherwise acts as a serial dictatorship with hierarchy $(1, 3, 2, 4, \dots, n)$. This assignment rule is clearly Pareto-optimal. It is also reallocation-proof, since no two individuals can change each other's assignments at any profile. Group-strategyproofness is violated by this rule, however, since individual 1 is bossy.

7. THE CHARACTERIZATION RESULT

THEOREM: *An assignment rule is group-strategyproof, Pareto-optimal, and reallocation-proof if and only if it is a hierarchical exchange rule.*

The proof of the theorem is presented in the Appendix.

It is perhaps not obvious to the reader why the hierarchical exchange rules satisfy our criteria, so let us provide a brief explanation. The intuition behind the *strategyproofness* of hierarchical exchange rules is that an individual, by deviating alone, cannot obtain objects that leave the market before she does. However, objects that are preferred by an individual to her assignment at a given profile are assigned at a stage prior to this individual's departure. This feature is also the key to understanding the *Pareto-optimality* of these rules. Clearly, individual i does not envy individual j who leaves at the same stage or later than i does, where we say that a person envies someone if the envied individual is assigned an object she prefers to her assignment. Thus, for any given profile, we can arrange the individuals in a hierarchy in which nobody envies another person who comes lower in the hierarchy, which means that the rule is Pareto-optimal.

We remark here that strategyproofness is ensured by any endowment inheritance that satisfies the Assurance Rule and the Twin Inheritance Rule. By contrast, *nonbossiness* does not follow from these two inheritance rules alone; it is only implied by further attributes of the endowment inheritance. It can easily be seen that individual i cannot be bossy with individual j if j leaves the market before i or simultaneously with i. It is less transparent that i is not bossy with j when j leaves after i. In order to understand nonbossiness in this case, one needs to take a closer look at the endowment inheritance based on inheritance trees. In particular, in the proof of nonbossiness we appeal to the following attribute of endowment inheritance: individual j, who is endowed with an object at some stage at a particular profile, will eventually inherit this object, if she cares about it, at any other profile where all individuals who leave the market before j at the original profile get the same assignments. This, although not in a straightforward manner, implies that no individual who leaves the market before j can be bossy with j.

Finally, *reallocation-proofness* can be seen as follows. If two individuals can manipulate through reallocation, then they can mutually affect each other's assignments at the manipulated profile. Given that a hierarchical exchange rule is group-strategyproof, this can be shown to imply that the two individuals are in the same trading cycle at that profile. In this case, however, they should be able to trade, which is a contradiction.

It is somewhat more difficult to explain why these are the only rules that satisfy our criteria. The proof is structured as follows. As a preliminary result for this *necessity proof*, we first show that the required properties of assignment rules imply that if individual i envies j, then she cannot affect j's assignment. This result is the key to establishing that the objects are endowments, or in other words, that every object can be commanded at all the preference profiles by some individual. We start the necessity proof by constructing the canonical inheritance trees for a particular assignment rule that satisfies our criteria (Step 1). Then we establish that the assignments of individuals who receive their assignments at the first stage of the hierarchical exchange rule defined by the constructed canonical form in Step 1 are as prescribed by this hierarchical exchange rule (Step 2). This is the basis step for the induction argument that we use. The inductive step (Step 3) completes the proof with a similar argument to that of Step 2, by demonstrating that the assignments at any later stage are also made in accordance with this hierarchical exchange rule.

8. CONCLUSION

In this paper we identified a class of rules for the pure distribution assignment problem that are group-strategyproof, Pareto-optimal, and reallocation-proof. These assignment rules can be interpreted as trading rules with individual property rights over the indivisible goods, and they can be conveniently decentralized in practical use. In a decentralized administration, in which at each stage the remaining individuals are asked to identify their favorite among the remaining objects, the incentive properties are retained, and typically very little information about the preferences is used (see Example 1). Furthermore, the class of hierarchical exchange rules is "rich" in the sense that it allows for substantially more flexibility than serial dictatorships. In particular, these assignment rules can accommodate any existing property rights or naturally arising hierarchies of endowment inheritance. Moreover, these are the only rules that satisfy group-strategyproofness, Pareto-optimality, and reallocation-proofness.

An interesting question for further research is whether similar results can be obtained if there is no restriction on the number of objects received by an individual. Under appropriate assumptions on preferences, there may exist simple trading rules with desirable properties for this related problem as well.

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APPENDIX: PROOF OF THE THEOREM

PRELIMINARIES

An individual affects another individual at a given profile if she can change the other individual's assignment by changing her preferences. Individual *i affects j at R* (with respect to f), if there exists $\tilde{R}_i$ such that $f_j(R) \neq f_j(\tilde{R}_i, R_{-i})$, where $i \neq j$. We write then $iA(R)j$.

Individual *j envies i at R* (with respect to f), if $f_i(R) P_j f_j(R)$. We denote this relationship by $\mathscr{V}(R)$. That is, if j envies i at R, we write $j\mathscr{V}(R)i$.

The following terminology and notation are given with respect to a particular hierarchical exchange rule f. Profiles *R and $\bar{R}$ are equivalent up to (stage) t* if, (i) for all $z \leq t$, $W^z(R) = W^z(\bar{R})$, (ii) for all $j \in W^t(R)$, $f_j(R) = f_j(\bar{R})$, and (iii) $R_{W^t(R)} = \bar{R}_{W^t(R)}$. For convenience, we say that all profiles are equivalent up to zero. We also let $W^0(R) = F^0(R) = \varnothing$, for all R. Profiles *R and $\bar{R}$ have identical endowments at t* if $W^{t-1}(R) = W^{t-1}(\bar{R})$ and for all $i \notin W^{t-1}(R)$, $E_t(i, R) = E_t(i, \bar{R})$. Note that all profiles have identical endowments at the first stage. Furthermore, the Twin Inheritance Rule says, in this terminology, that if two profiles are equivalent up to t, then they have identical endowments at stage $t+1$.

For all i, j, R, and t, let

$$\bar{S}_t(j,i,R) = \begin{cases} \{l_1, \dots, l_g\} & \text{if there exist } l_1, \dots, l_g \in N - W^{t-1}(R) \text{ such that} \\ & E_t(l_1, R) \neq \varnothing, \text{and for all } s = 1, \dots, g-1, \\ & T_t(l_s, R) \in E_t(l_{s+1}, R), \text{where } j = l_1 \text{ and } i = l_g; \\ \varnothing & \text{otherwise.} \end{cases}$$

Note that $\bar{S}_t(i,j,R)=\varnothing$ unless $T_t(j,R)\in E_t(j,R)$.

Note also that for all j, R, and t,

$$S_t(j,R)=\begin{cases}\bar{S}_t(j,i,R)\cup\bar{S}_t(i,j,R) & \text{if there exists } i \text{ such that} \\ & \bar{S}_t(j,i,R)\neq\varnothing \text{ and } \bar{S}_t(i,j,R)\neq\varnothing; \\ \varnothing & \text{otherwise.}\end{cases}$$

LEMMA 2: *Let f be a hierarchical exchange rule. Then, if $i\notin W^t(R)$ and $a\in F^t(R)$, we have $a\notin o_i(R_{-i})$.*

PROOF: Fix R, i, and $t\in\{1,\dots,m\}$ such that $i\notin W^t(R)$. Let $a\in F^t(R)$. Note that $a\neq 0$. Suppose there exists $\tilde{R}_i$ such that $f_i(\tilde{R}_i,R_{-1})=a$. Let $t'\in\{0,\dots,m-1\}$ such that $i\in W_{t'+1}(\tilde{R}_i,R_{-i})$. By the Twin Inheritance Rule, R and $(\tilde{R}_i,R_{-i})$ are equivalent up to $\min\{t,t'\}$. Then $t'<t$, since $a\in F^t(R)$ and $f_i(R)\neq a$. Thus, the Twin Inheritance Rule implies that R and $(\tilde{R}_i,R_{-i})$ have identical endowments at $t'+1$. Then, since $a\notin F^{t'}(\tilde{R}_i,R_{-i})$, we have $a\notin F^{t'}(R)$ and there exists j such that $a\in E_{t'+1}(j,R)$. If $j=i$ then $a\in E_{t+1}(i,R)$, by the Assurance Rule, given that $i\notin W^t(R)$. If $j\neq i$, $\bar{S}_{t'+1}(j,i,(\tilde{R}_i,R_{-i}))\neq\varnothing$. Moreover, since $F^{t'}(\tilde{R}_i,R_{-i})=F^{t'}(R)$, it follows that for all $l\notin W^{t'}(R)$ such that $l\neq i, T_{t'+1}(l,(\tilde{R}_i,R_{-i}))=T_{t'+1}(l,R)$, and thus we have $\bar{S}_{t'+1}(j,i,R)\neq\varnothing$. Then, given that $i\notin W^t(R)$, the Assurance Rule implies that $j\notin W^t(R)$, $\bar{S}_{t+1}(j,i,R)\neq\varnothing$, and $a\in E_{t+1}(j,R)$. Therefore, $a\in E_{t+1}(j,R)$ in either case, which contradicts the fact that $a\in F^t(R)$. *Q.E.D.*

SUFFICIENCY PROOF

Let f be a hierarchical exchange rule. Fix an inheritance tree list $\Gamma\in G$ underlying f. We will show that f is Pareto-optimal, strategyproof, nonbossy, and reallocation-proof.

PARETO-OPTIMALITY: Fix R. Take a hierarchy of the individuals $\sigma\colon N\mapsto\{1,\dots,n\}$, where σ is a bijection, such that for all i,j, and t, $\sigma(i)<\sigma(j)$ if $i\in W_t(R)$ and $j\notin W^t(R)$. Then for all i,j such that $\sigma(i)<\sigma(j)$, $\neg(i\mathscr{V}(R)j)$, which implies that f is Pareto-optimal.

STRATEGYPROOFNESS: Suppose f is manipulable. Then there exist i, R, and $\tilde{R}_i$ such that $f_i(\tilde{R}_i,R_{-i})P_if_i(R)$. Thus, there exists t such that $i\notin W^t(R)$ and $f_i(\tilde{R}_i,R_{-i})\in F^t(R)$. Therefore, $f_i(\tilde{R}_i,R_{-i})\notin o_i(R_{-i})$, by Lemma 2, which is a contradiction.

NONBOSSINESS: Fix R,i, and $\tilde{R}_i$ such that $f_i(R)=f_i(\tilde{R}_i,R_{-i})$. Let t^* be the last stage at R. If $f_i(R)=0$, then R and $(\tilde{R}_i,R_{-i})$ are equivalent up to t^*, by the Twin Inheritance Rule. In this case $F^{t^*}(R)=K$, which implies that $f(R)=f(\tilde{R}_i,R_{-i})$.

If $f_i(R)\neq 0$, let $i\in W_t(R)$ and $i\in W_{t'}(\tilde{R}_i,R_{-i})$. Assume, without loss of generality, that $t\leq t'$. Then the Twin Inheritance Rule implies that R and $(\tilde{R}_i,R_{-i})$ are equivalent up to $t-1$ and that R and $(\tilde{R}_i,R_{-i})$ have identical endowments at stage t. Note that since $F^{t-1}(R)=F^{t-1}(\tilde{R}_i,R_{-i})$, for all $j\notin W^{t-1}(R)$ such that $j\neq i, T_t(j,R)=T_t(j,(\tilde{R}_i,R_{-i}))$. Hence, $(W_t(R)-S_t(i,R))\subseteq W_t(\tilde{R}_i,R_{-i})$, and for all $j\in W_t(R)-S_t(i,R), f_j(R)=f_j(\tilde{R}_i,R_{-i})$.

If $t=t'$ then, since $f_i(R)=f_i(\tilde{R}_i,R_{-i})$, we also have $T_t(i,R)=T_t(i,(\tilde{R}_i,R_{-i}))$. Thus, $W_t(R)=W_t(\tilde{R}_i,R_{-i})$, and for all $j\in W_t(R)$, we have $f_j(R)=f_j(\tilde{R}_i,R_{-i})$. If $t<t'$, then $W_t(\tilde{R}_i,R_{-i})=W_t(R)-S_t(i,R)$. Since $i\in W_{t'}(\tilde{R}_i,R_{-i})$ and $E_t(i,R)=E_t(i,(\tilde{R}_i,R_{-i}))$, the Assurance Rule implies that $E_t(i,R)\subseteq E_{t'}(i,(\tilde{R}_i,R_{-i}))$. Then, given that $f_i(R)=f_i(\tilde{R}_i,R_{-i})$, $S_t(i,R)=S_{t'}(i,(\tilde{R}_i,R_{-i}))$, by the Assurance Rule. Hence, for all $j\in W_t(R), f_j(R)=f_j(\tilde{R}_i,R_{-i})$ in this case as well. If t is the last stage ($t=t^*$), the proof is completed. Thus, assume that $t<t^*$.

Fix $z>t$ such that $W_z(R)=\varnothing$. Assume that for all $j\in W^{z-1}(R)$, $f_j(R)=f_j(\tilde{R}_i,R_{-i})$. Fix $S_z(h_1,R)\neq\varnothing$. Let $S_z(h_1,R)=\{h_1,\dots,h_g\}, g\geq 1$, such that for all $s=1,\dots,g, f_{h_s}(R)\in E_z(h_{s+1},R)$, where we let $h_{g+1}=h_1$. Fix $\bar{s}\leq g$. Let $a=f_{h_{\bar{s}-1}}(R)$. Thus, $a\in E_z(h_{\bar{s}},R)$. Then, there exists a vertex

v_r in Γ_a such that $\mathscr{L}(v_r)=h_{\bar{s}}$, and for all $s=0,\ldots,r-1$ in the Q-path $\{v_s\}_{s=0}^r$ from v_0 to v_r, we have $\mathscr{L}(v_s)\in W^{z-1}(R)$ and $f_{\mathscr{L}(v_s)}(R)=\mathscr{H}(v_s,v_{s+1})$. Then, given that for all $s=0,\ldots,r-1$, $f_{\mathscr{L}(v_s)}(R)=f_{\mathscr{L}(v_s)}(\tilde{R}_i,R_{-i})$, for all $z'\le m$ such that $h_{\bar{s}}\notin W^{z'-1}(\tilde{R}_i,R_{-i})$, either $a\in E_{z'}(h_{\bar{s}},(\tilde{R}_i,R_{-i}))$, or $a\in E_{z'}(\mathscr{L}(v_s),(\tilde{R}_i,R_{-i}))$ for some $s=0,\ldots,r-1$. Since this holds for all h_s with $s=1,\ldots,g$, and for all $z'\le m$ such that $h_s\notin W^{z'-1}(\tilde{R}_i,R_{-i})$, we have for all s and all such z', $f_{h_{s-1}}(R)\notin F^{z'-1}(\tilde{R}_i,R_{-i})$.

Therefore, an immediate implication is that for all $s=1,\ldots,g$, $f_{h_s}(\tilde{R}_i,R_{-i})\neq 0$. Then, for all $s=1,\ldots,g$, we can let $h_s\in W_{z_s}(\tilde{R}_i,R_{-i})$, without loss of generality. Now note that since for all $j\in W^{z-1}(R)$, $f_j(R)=f_j(\tilde{R}_i,R_{-i})$, we have, for all s, $f_{h_s}(R)R_{h_s}f_{h_s}(\tilde{R}_i,R_{-i})$. Hence, for all $s=1,\ldots,g$, individual h_s cannot leave at an earlier stage than h_{s+1} does, given that $f_{h_s}(R)\notin F^{z_{s+1}-1}(\tilde{R}_i,R_{-i})$. This means that for all $s, z_s\ge z_{s+1}$, which in turn implies that there exists $z'\le m$ such that for all $s, h_s\in W_{z'}(\tilde{R}_i,R_{-i})$. Then, since for all s, $f_{h_s}(R)\notin W^{z'-1}(\tilde{R}_i,R_{-i})$ and since $f_{h_s}(R)R_{h_s}f_{h_s}(\tilde{R}_i,R_{-i})$, it must be the case that for all $s, T_{z'}(h_s,(\tilde{R}_i,R_{-i}))=f_{h_s}(R)$. Consequently, for all s, $f_{h_s}(\tilde{R}_i,R_{-i})=f_{h_s}(R)$. Since this holds for every $S_z(h_1,R)\neq\varnothing$, it follows that for all $j\in W^z(R)$, $f_j(R)=f_j(\tilde{R}_i,R_{-i})$. Therefore, $f(R)=f(\tilde{R}_i,R_{-i})$, by induction.

LEMMA 3: *Let f be a hierarchical exchange rule. Then, if i, $j\in W_t(R)$ such that $i\neq j$, we have $f_j(R)\in o_i(R_{-i})$ if and only if $j\in S_t(i,R)$.*

PROOF: Fix R,i,j, and t such that $i,j\in W_t(R)$ and $i\neq j$. Let $\tilde{R}_i$ be defined as follows. For all $a\neq f_j(R)$, let $a\tilde{P}_if_j(R)$ if and only if $a\in F^{t-1}(R)$.[16] Note that for all a such that $a\tilde{P}_if_j(R)$, we have $a\notin o_i(R_{-i})$, by Lemma 2. Then, by the Twin Inheritance Rule, R and $(\tilde{R}_i,R_{-i})$ are equivalent up to $t-1$, and they have identical endowments at t. Then, since $i\in W_t(R)$ and thus $E_t(i,R)\neq\varnothing$, we have $E_t(i,(\tilde{R}_i,R_{-i}))\neq\varnothing$. Thus, $T_t(i,(\tilde{R}_i,R_{-i}))=f_j(R)$ implies that $\bar{S}_t(i,l,(\tilde{R}_i,R_{-i}))\neq\varnothing$, where $f_j(R)\in E_t(l,(\tilde{R}_i,R_{-i}))$. Also, for all $h\notin W^{t-1}(R)$ such that $h\neq i$, we have $T_t(h,(\tilde{R}_i,R_{-i}))=T_t(h,R)$. Then, if $j\in S_t(i,R)$, we have $l\in S_t(i,R)$ and $\bar{S}_t(l,i,(\tilde{R}_i,R_{-i}))\neq\varnothing$. Therefore, $l\in S_t(i,(\tilde{R}_i,R_{-i}))$, and $f_i(\tilde{R}_i,R_{-i})=f_j(R)$. On the other hand, if $j\notin S_t(i,R)$, we have $S_t(j,(\tilde{R}_i,R_{-i}))=S_t(j,R)$, and $f_j(\tilde{R}_i,R_{-i})=f_j(R)$. Then, since strategyproofness implies that $f_i(\tilde{R}_i,R_{-i})=\text{top}(\tilde{R}_i,o_i(R_{-i}))$, we have $f_j(R)\notin o_i(R_{-i})$ if $j\notin S_t(i,R)$. *Q.E.D.*

LEMMA 4: *Let f be a hierarchical exchange rule. Then, if $iA(R)j$ and $i\in W_t(R)$, we have $j\in S_t(i,R)$ or $j\notin W^t(R)$.*

PROOF: Fix R, i, and t such that $i\in W_t(R)$. Fix $b\in o_i(R_{-i})$. Let $\tilde{R}_i$ be defined as follows. For all $a\neq b$, let $a\tilde{P}_ib$ if and only if $a\in F^t(R)$ and $h\notin S_t(i,R)$, where $f_h(R)=a$. Then, for all a such that $a\tilde{P}_ib$, we have $a\notin o_i(R_{-i})$, which is implied by Lemma 2 and Lemma 3. Thus, $f_i(\tilde{R}_i,R_{-i})=b$, by strategyproofness. Furthermore, R and $(\tilde{R}_i,R_{-i})$ are equivalent up to $t-1$, and they have identical endowments at t, by the Twin Inheritance Rule. Therefore, for all $l\in W^{t-1}(R)$, $f_l(\tilde{R}_i,R_{-i})=f_l(R)$. Furthermore, for all $l\in W_t(R)$ such that $l\notin S_t(i,R)$ we have $l\neq i$, so that $T_t(l,(\tilde{R}_i,R_{-i}))=T_t(l,R)$. This implies that for all $l\in W^t(R)$ such that $l\notin S_t(i,R)$, we have $f_l(\tilde{R}_i,R_{-i})=f_l(R)$. Since $b\in o_i(R_{-i})$ was chosen arbitrarily, it follows from nonbossiness that for all $l\in W^t(R)$ such that $l\notin S_t(i,R)$, we have $\neg(iA(R)l)$. *Q.E.D.*

REALLOCATION-PROOFNESS: Suppose there exist R,i,j and $\tilde{R}_i,\tilde{R}_j$, such that $f_j(\tilde{R}_i,\tilde{R}_j,R_{-i,j})R_if_i(R)$, $f_i(\tilde{R}_i,\tilde{R}_j,R_{-i,j})P_jf_j(R)$, and $f_h(R)=f_h(\tilde{R}_h,R_{-h})\neq f_h(\tilde{R}_i,\tilde{R}_j,R_{-i,j})$ for $h=i,j$. Then $iA(\tilde{R}_i,\tilde{R}_j,R_{-i,j})j$ and $jA(\tilde{R}_i,\tilde{R}_j,R_{-i,j})i$. Thus, Lemma 4 implies that there exists t such that $i\in S_t(j,(\tilde{R}_i,\tilde{R}_j,R_{-i,j}))$. This implies, in turn, that $f_i(\tilde{R}_i,\tilde{R}_j,R_{-i,j})\in o_j(\tilde{R}_i,R_{-i,j})$, given Lemma 3. Note that since $f_i(R)=f_i(\tilde{R}_i,R_{-i})$, we have $f_j(R)=f_j(\tilde{R}_i,R_{-i})$, by nonbossiness. Thus, $f_i(\tilde{R}_i,\tilde{R}_j,R_{-i,j})P_jf_j(\tilde{R}_i,R_{-i})$, and j can manipulate at $(\tilde{R}_i,R_{-i})$, which contradicts strategyproofness.

[16] In the following we will say that $\tilde{R}_i$ ranks the objects in $F^{t-1}(R)$ first and then $f_j(R)$.

LEMMAS FOR THE NECESSITY PROOF

LEMMA 5: *Let f be group-strategyproof and Pareto-optimal. Then $j\mathscr{V}(R)i$ and $jA(R)i$ imply that for all $\tilde{R}_j$ such that $f_i(\tilde{R}_j, R_{-j}) \neq f_i(R)$, we have $f_i(R) P_i f_i(\tilde{R}_j, R_{-j})$.*

PROOF: Fix R and i, j such that $j\mathscr{V}(R)i$ and $jA(R)i$. Fix $\tilde{R}_j$ such that $f_i(\tilde{R}_j, R_{-j}) \neq f_i(R)$. Suppose $f_i(\tilde{R}_j, R_{-j}) P_i f_i(R)$. Let $f_j(R) = c, f_i(R) = b, f_j(\tilde{R}_j, R_{-j}) = d$, and $f_i(\tilde{R}_j, R_{-j}) = a$. First note that if $c = 0$, then $d = 0$, by strategyproofness, and then $\neg(jA(R)i)$, by nonbossiness. Thus, $c \in K$, and similarly $d \in K$. Furthermore, $b \in K$, since $bP_j c$, and $a \in K$, since $aP_i b$. Note also that $d \neq b$, otherwise j could manipulate at R via $\tilde{R}_j, d \neq c$, otherwise j would be bossy, and $a \neq c$, otherwise Pareto-optimality would be violated at R. Thus, feasibility implies that a, b, c, and d are distinct.

Let $\overline{R}_j$ rank b first, c second, and d third. Let $\overline{R}_i$ rank a first and b second. Since $\overline{R}_j$ is a push-up of R_j for c, and $\overline{R}_i$ is a push-up of R_i for b, $f(\overline{R}_i, \overline{R}_j, R_{-i,j}) = f(R)$, by group-strategyproofness. Let $\hat{R}_j$ rank b first, d second, and c third. Since $b \notin o_j(R_{-j})$, by strategyproofness, and $d \in o_j(R_{-j})$, we have $f_j(\hat{R}_j, R_{-j}) = d$, by strategyproofness. Then $f_i(\hat{R}_j, R_{-j}) = a$, by nonbossiness. Thus, $f_i(\overline{R}_i, \hat{R}_j, R_{-i,j,}) = a$, by strategyproofness, since $\overline{R}_i$ is a push-up of R_i for a. Therefore, $f_j(\overline{R}_i, \hat{R}_j, R_{-i,j}) = d$, by nonbossiness.

Let $\hat{R}_i$ rank a first, c second, and b third. Since $\hat{R}_i$ is a push-up of $\overline{R}_i$ for a, $f(\overline{R}_i, \hat{R}_j, R_{-i,j}) = f(\hat{R}_i, \hat{R}_j, \overline{R}_{-i,j})$, by group-strategyproofness. Note that $f_j(\hat{R}_i, \overline{R}_j, R_{-i,j}) \in \{c, d\}$, by strategyproofness. Note, furthermore, that given $f_i(\overline{R}_i, \overline{R}_j, R_{-i,j}) = b, f_i(\hat{R}_i, \overline{R}_j, R_{-i,j}) \hat{R}_i b$, by strategyproofness. If $f_j(\hat{R}_i, \overline{R}_j, R_{-i,j}) = c, f_i(\hat{R}_i, \overline{R}_j, R_{-i,j}) = a$, since $f_i(\hat{R}_i, \overline{R}_j, R_{-i,j}) = b$ would violate Pareto-optimality. If $f_j(\hat{R}_i, \overline{R}_j, R_{-i,j}) = d$, then nonbossiness implies that $f_i(\hat{R}_i, \overline{R}_j, R_{-i,j}) = a$. This is a contradiction, however, since it implies that i can manipulate at $(\overline{R}_i, \overline{R}_j, R_{-i,j})$ via $\hat{R}_i$. *Q.E.D.*

LEMMA 6: *Let f be group-strategyproof and Pareto-optimal. Then $j\mathscr{V}(R)i$ and $jA(R)i$ imply that for all $\tilde{R}_j$ such that $f_i(\tilde{R}_j, R_{-j}) \neq f_i(R)$, we have $f_j(R) \in o_i(\overline{R}_j, R_{-i,j})$.*

PROOF: Fix R and i, j such that $j\mathscr{V}(R)i$ and $jA(R)i$. Fix $\tilde{R}_j$ such that $f_i(\tilde{R}_j, R_{-j}) \neq f_i(R)$. Suppose $f_j(R) \notin o_i(\tilde{R}_j, R_{-i,j})$. Let $f_j(R) = c$, $f_i(R) = b$, $f_j(\tilde{R}_j, R_{-j}) = d$, and $f_i(\tilde{R}_j, R_{-j}) = a$. First note that $c, d \in K$, by group-strategyproofness, as in Lemma 5, and $b \in K$, since $bP_j c$. Note also that $d \neq b$, by strategyproofness, and $d \neq c$, by nonbossiness, as in Lemma 5. Furthermore, $a \neq c$, since $c \notin o_i(\tilde{R}_j, R_{-i,j})$. Thus, feasibility implies that a, b, c, and d are distinct.

Let $\overline{R}_j$ rank b first, c second, and d third. Let $\overline{R}_i$ rank b first, c second, and, if $a \neq 0$, a third. Since $\overline{R}_j$ is a push-up of R_j for c, and $\overline{R}_i$ is a push-up of R_i for b, $f(\overline{R}_i, \overline{R}_j, R_{-i,j}) = f(R)$, by group-strategyproofness. Note that Lemma 5 implies that $bP_i a$, and thus $b \notin o_i(\tilde{R}_j, R_{-i,j})$, by strategyproofness. Then, since $c \notin o_i(\tilde{R}_j, R_{-i,j})$ and $a \in o_i(\tilde{R}_j, R_{-i,j})$, if $a \neq 0$, we have $f_i(\overline{R}_i, \tilde{R}_j, R_{-i,j}) = a$, by strategyproofness. If $a = 0$, $f_i(\overline{R}_i, \tilde{R}_j, R_{-i,j}) = 0$, by strategyproofness. Thus, $f_j(\overline{R}_i, \tilde{R}_j, R_{-i,j}) = d$, by nonbossiness.

Let $\hat{R}_j$ rank b first, d second, and c third. Since $b \notin o_j(\overline{R}_i, R_{-i,j})$, by strategyproofness, and $d \in o_j(\overline{R}_i, R_{-i,j})$, we have $f_j(\overline{R}_i, \hat{R}_j, R_{-i,j}) = d$, by strategyproofness. This implies that $f_i(\overline{R}_i, \hat{R}_j, R_{-i,j}) = a$, by nonbossiness.

Let $\hat{R}_i$ rank c first, b second, and, if $a \neq 0$, a third. Since $\hat{R}_i$ is a push-up of $\overline{R}_i$ for a, $f(\overline{R}_i, \hat{R}_j, R_{-i,j}) = f(\hat{R}_i, \hat{R}_j, R_{-i,j})$, by group-strategyproofness. Note that $f_j(\hat{R}_i, \overline{R}_j, R_{-i,j}) \in \{c, d\}$, by strategyproofness. Note, furthermore, that $c \in o_i(\overline{R}_j, R_{-i,j})$, given that $j\mathscr{V}(\hat{R}_i, \overline{R}_j, R_{-i,j})i$ and f is group-strategyproof, otherwise Pareto-optimality would be violated. Thus, $f_i(\hat{R}_i, \overline{R}_j, R_{-i,j}) = c$, by strategyproofness, and therefore $f_j(\hat{R}_i, \overline{R}_j, R_{-i,j}) = d$. However, given that $f_j(\hat{R}_i, \hat{R}_j, R_{-i,j}) = d$ and $f_i(\hat{R}_i, \hat{R}_j, R_{-i,j}) = a$, nonbossiness is violated. *Q.E.D.*

LEMMA 7: *Let f be group-strategyproof, Pareto-optimal, and reallocation-proof. Then $j\mathscr{V}(R)i$ implies $\neg(jA(R)i)$.*

PROOF: Suppose there exist i, j, and R such that $j\mathscr{V}(R)i$ and $jA(R)i$. Let $f_i(R) = a$ and $f_j(R) = b$. Fix $\tilde{R}_j$ such that $f_i(\tilde{R}_j, R_{-j}) \neq a$. Note that $aP_i f_i(\tilde{R}_j, R_{-j})$, by Lemma 5. Let $f_j(\tilde{R}_j, R_{-j})$

$= c$. Then $c \neq b$, by nonbossiness, and $c \neq a$, by strategyproofness. Moreover, $b, c \in K$, by group-strategyproofness, as in Lemma 5, and $a \in K$, since $aP_j b$.

Let $\hat{R}_j$ rank a first, b second, and c third. Let $\hat{R}_i$ rank a first and b second. Then $\hat{R}_j$ is a push-up of R_j for b, and $\hat{R}_i$ is a push-up of R_i for a, so $f(\hat{R}_i, \hat{R}_j, R_{-i,j}) = f(R)$, by group strategyproofness.

Let $\overline{R}_j$ rank a first, c second, and b third. Then $f_i(\hat{R}_i, \overline{R}_j, R_{-i,j}) \in \{a, b\}$, by Lemma 6 and strategyproofness. Suppose $f_i(\hat{R}_i, \overline{R}_j, R_{-i,j}) = a$. Then $f_i(\overline{R}_j, R_{-j}) R_i a$, by strategyproofness. Since $aP_i f_i(\tilde{R}_j, R_{-j})$, by Lemma 5, this means that $f_i(\overline{R}_j, R_{-j}) \neq f_i(\tilde{R}_j, R_{-j})$. Note that $f_j(\overline{R}_j, R_{-j}) = c$, by strategyproofness, since $a \notin o_j(R_{-j})$ and $c \in o_j(R_{-j})$. In this case, however, nonbossiness is violated, given that $f_j(\tilde{R}_j, R_{-j}) = c$. Thus, $f_i(\hat{R}_i, \overline{R}_j, R_{-i,j}) = b$. Then strategyproofness implies that $f_j(\hat{R}_i, \overline{R}_j, R_{-i,j}) = c$.

Let $\overline{R}_i$ rank b first and a second. Note that $\overline{R}_i$ is a push-up of $\hat{R}_i$ for b, so that $f(\overline{R}_i, \overline{R}_j, R_{-i,j}) = f(\hat{R}_i, \overline{R}_j, R_{-i,j})$, by group-strategyproofness. Now consider the profile $(\overline{R}_i, \hat{R}_j, R_{-i,j})$. By strategyproofness, $f_j(\overline{R}_i, \hat{R}_j, R_{-i,j}) \in \{b, c\}$. If $f_j(\overline{R}_i, \hat{R}_j, R_{-i,j}) = b$, then $f_i(\overline{R}_i, \hat{R}_j, R_{-i,j}) = a$, by strategyproofness, and Pareto-optimality is violated. Thus, $f_j(\overline{R}_i, \hat{R}_j, R_{-i,j}) = c$, and $f(\overline{R}_i, \hat{R}_j, R_{-i,j}) = f(\overline{R}_i, \overline{R}_j, R_{-i,j})$, by nonbossiness.

In sum, we have (i) $f_i(\overline{R}_i, \overline{R}_j, R_{-i,j}) = f_j(\hat{R}_i, \hat{R}_j, R_{-i,j}) = b$, (ii) $a\overline{P}_j c$, where $f_i(\hat{R}_i, \hat{R}_j, R_{-i,j}) = a$ and $f_j(\overline{R}_i, \overline{R}_j, R_{-i,j}) = c$, and (iii) $f(\hat{R}_i, \overline{R}_j, R_{-i,j}) = f(\overline{R}_i, \hat{R}_j, R_{-i,j}) = f(\overline{R}_i, \overline{R}_j, R_{-i,j})$. Thus, reallocation-proofness is violated. *Q.E.D.*

Necessity Proof

Let f be group-strategyproof, Pareto-optimal, and reallocation-proof. We will show that f is a hierarchical exchange rule.

Step 1. *Construction of the Canonical Inheritance Trees*.

Fix $a \in K$. We will construct its canonical inheritance tree Γ_a. Let $\Gamma_a = (V, Q)$ be a rooted tree with root v_0 that satisfies properties (C.1), (C.2), and (C.3) listed in Subsection 4.1. Assume that the arcs are labeled as required by (B.1), (B.2), and (B.3). In order to determine the labels of the vertices, we first prove the following claim.

Claim: *Fix $M \subset N$ such that $|M| < k$. For all $i \in M$, let $\tilde{R}_i$ rank object b_i first and object a second, such that for all $i, j \in M$, $b_i \neq b_j$ if $i \neq j$. For all $i \in N - M$, let R_i rank a first. Then there exists $j \in N - M$ such that $f_j(\tilde{R}_M, R_{-M}) = a$.*

Proof: The claim follows from Pareto-optimality. First, since $N - M \neq \varnothing$, Pareto-optimality implies that there exists $j \in N$ such that $f_j(\tilde{R}_M, R_{-M}) = a$. Suppose $j \in M$. Then there exists $i \in N$ such that $f_i(\tilde{R}_M, R_{-M}) = b_j$, by Pareto-optimality. Note, however, that if $i \in M$, then we have $a\tilde{P}_i b_j$, since $b_i \neq b_j$, and if $i \in N - M$, then we have $aP_i b_j$. Given that $j \in M$ and $b_j \tilde{P}_j a$, this violates Pareto-optimality. Thus $j \in N - M$, and the claim is proved.

Now we are ready to determine the labels of the vertices in Γ_a, in accordance with properties (A.1) and (A.2). For all $i \in N$, let R_i rank a first. Then Pareto-optimality implies that there exists $j \in N$ such that $f_j(R) = a$. Let $\mathscr{L}(v_0) = j$. Let $v_r \in V$, with $0 < r < m$, and let $\{v_s\}_{s=0}^r$ be the Q-path from v_0 to v_r. Assume that for all $s = 0, \dots, r-1$, the label of vertex v_s is determined. We will show how to determine the label of v_r.

Let $M = \{i \in N$: there exists $s = 0, \dots, r-1$ such that $\mathscr{L}(v_s) = i\}$. Note that for all $s = 0, \dots, r-1, \mathscr{K}(v_s, v_{s+1}) \neq a$, by (B.1). For all $i \in M$, let $\tilde{R}_i$ rank object $\mathscr{K}(v_s, v_{s+1})$ first and a second, where $\mathscr{L}(v_s) = i$. Note that for all $i, j \in M$, $\mathscr{K}(v_{s_i}, v_{s_i+1}) \neq \mathscr{K}(v_{s_j}, v_{s_j+1})$ if $i \neq j$, by (B.2), where $\mathscr{L}(v_{s_i}) = i$ and $\mathscr{L}(v_{s_j}) = j$. Then the claim above implies that there exists $j \in N - M$ such that $f_j(\tilde{R}_M, R_{-M}) = a$. Label v_r by j, that is, let $\mathscr{L}(v_r) = j$. Note that since $j \in N - M$, this is in accordance with (A.2).

This completes the recursive definition of all the labels in Γ_a. The canonical inheritance tree is constructed similarly for each object $a \in K$. Thus, we have determined $\Gamma \in G$, based on the assignments rule f.

In the rest of this proof, we will show that for all R, $f(R) = f^{\Gamma}(R)$, where f^{Γ} is the hierarchial exchange rule associated with $\Gamma \in G$, as defined in Section 4.

Fix $R \in \mathscr{R}$. Let $t^* \leq m$ be the last stage at profile R. We will show, using induction, that for all $t \leq t^*$ and for all $i \in W_t(R)$, $f_i(R) = f_i^{\Gamma}(R)$.

Step 2. *Basis Step*: *Assignments at Stage* 1.

Step 2.*a*: *For all* $i \in N$ *and for all* $a \in E_1(i, R)$, $a \in o_i(\tilde{R}_{-i})$ *for all* $\tilde{R}$.

Fix $i \in N$ and $a \in E_1(i, R)$. Given (1) and (3), we have $\mathscr{L}(v_0) = i$ in Γ_a. Thus, by the construction of Γ_a in Step 1, $f_i(\overline{R}) = a$, where top $(\overline{R}_j) = a$ for all $j \in N$. Fix $\tilde{R}$ and let $i = n$. Then, since $1\mathscr{V}(\overline{R})n$, we have $f_n(\tilde{R}_1, \overline{R}_{-1}) = a$, by Lemma 7. This implies in turn that, since $2\mathscr{V}(\tilde{R}_1, \overline{R}_{-1})n$, we have $f_n(\tilde{R}_1, \tilde{R}_2, \overline{R}_{-1,2}) = a$, by Lemma 7. Continuing iteratively, we get $f_n(\tilde{R}_{-n}, \overline{R}_n) = a$. Thus, for all $\tilde{R}$, $a \in o_i(\tilde{R}_{-i})$, and Step 2.a is completed.

Note that this step implies that for all $i \in N$, for all $a \in E_1(i, R)$, and for all $\tilde{R}$, $f_i(\tilde{R})\tilde{R}_i a$, given that f is strategyproof.

Step 2.*b*: *For all* $i \in W_1(R)$, $f_i(R) = f_i^{\Gamma}(R)$.

Fix $i \in W_1(R)$. If $|S_1(i, R)| = 1$, then $T_1(i, R) \in E_1(i, R)$, so that $f_i(R) = T_1(i, R)$, by Step 2.a. If $|S_1(i, R)| \geq 2$, let $S_1(i, R) = S = \{j_1, \ldots, j_g\}$, such that for all $s = 1, \ldots, g$, $T_1(j_s, R) \in E_1(j_{s+1}, R)$, where we let $j_{g+1} = j_1$ and $i = j_s$ for some $s = 1, \ldots, g$. For all $s = 1, \ldots, g$, let $\tilde{R}_{j_{s+1}}$ rank top$(R_{j_{s+1}})$ first, and top(R_{j_s}) second. Suppose $f_{j_1}(\tilde{R}_S, R_{-S}) \neq$ top(R_{j_1}). Then, since top$(\tilde{R}_{j_g}) \in E_1(j_1, R)$, we have $f_{j_1}(\tilde{R}_S, R_{-S}) =$ top(R_{j_g}), by Step 2.a. Note that in this case $f_{j_g}(\tilde{R}_S, R_{-S}) \neq$ top(R_{j_g}), and thus $f_{j_g}(\tilde{R}_S, R_{-S}) =$ top$(R_{j_{g-1}})$, by a similar argument. Hence, an iterative repetition of this argument yields that for all $s = 1, \ldots, g$, $f_{j_{s+1}}(\tilde{R}_S, R_{-S}) =$ top(R_{j_s}). This implies, however, that for all $s = 1, \ldots, g$, $j_s\mathscr{V}(\tilde{R}_S, R_{-S})j_{s+1}$, which contradicts Pareto-optimality. Therefore, $f_{j_1}(\tilde{R}_S, R_{-S}) =$ top(R_{j_1}). Since top$(R_{j_1}) \in E_1(j_2, R)$, Step 2.a implies that $f_{j_2}(\tilde{R}_S, R_{-S}) =$ top(R_{j_2}), etc. Thus, for all $s = 1, \ldots, g$, $f_{j_s}(\tilde{R}_S, R_{-S}) =$ top$(R_{j_s}) = T_1(j_s, R)$. Then $f(\tilde{R}_S, R_{-S}) = f(R)$ follows, by group-strategyproofness. Therefore, for all $i \in W_1(R), f_i(R) = T_1(i, R) = f_i^{\Gamma}(R)$. Note, furthermore, that since R was chosen arbitrarily, this means that for all $\hat{R}_{-W_1(R)}$, $f_{W_1(R)}(R_{W_1(R)}, \hat{R}_{-W_1(R)}) = f_{W_1(R)}(R)$.

Step 3. *Inductive Step*: *Assignments at Stage* $t + 1$.

Fix $t < t^*$. Assume that for all $i \in W^t(R)$, $f_i(R) = f_i^{\Gamma}(R)$. Let $M = W^t(R)$ and let $L = F^t(R)$. Assume, furthermore, that for all $\tilde{R}_{-M}$, $f_M(R_M, \tilde{R}_{-M}) = f_M(R)$.

Step 3.*a*: *For all* $i \in N - M$ *and for all* $a \in E_{t+1}(i, R)$, $a \in o_i(R_M, \tilde{R}_{-M,i})$ *for all* $\tilde{R}_{-M}$.

Fix $i \in N - M$ and $a \in E_{t+1}(i, R)$. Given (2) and (4), either $\mathscr{L}(v_0) = i$ in Γ_a or there is a Q-path $\{v_s\}_{s=0}^r$ in Γ_a from v_0 to v_r such that $\mathscr{L}(v_r) = i$, and for all $s = 0, \ldots, r-1$ we have $\mathscr{L}(v_s) \in M$, $\mathscr{H}(v_s, v_{s+1}) \in L$, and $f^{\Gamma}_{\mathscr{L}(v_s)}(R) = \mathscr{H}(v_s, v_{s+1})$. For all $j \in N$, let $\overline{R}_j$ rank a first. We will show first that $f_i(R_M, \overline{R}_{-M}) = a$.

Note that if $\mathscr{L}(v_0) = i$ in Γ_a, then $f_i(R_M, \overline{R}_{-M}) = a$, by Step 2.a. Thus, assume that $\mathscr{L}(v_0) \neq i$ in Γ_a. Let $\overline{V} = \{j \in N :$ there exists $s = 0, \ldots, r-1$ such that $\mathscr{L}(v_s) = j\}$, where $\{v_s\}_{s=0}^r$ is the Q-path in Γ_a from v_0 to v_r. Note that $\overline{V} \subseteq M$. Thus, for all $j \in \overline{V}, f_j(R) = f_j^{\Gamma}(R)$, by assumption, and hence $f_j(R) = \mathscr{H}(v_s, v_{s+1})$, where $j = \mathscr{L}(v_s)$. Therefore, by the construction of Γ_a in Step 1, $f_i(\hat{R}_{\overline{V}}, \overline{R}_{-\overline{V}}) = a$, where $\hat{R}_j$ ranks $f_j(R)$ first and a second, for all $j \in \overline{V}$. Now note that for all $j \in M - \overline{V}$, we have $j\mathscr{V}(\hat{R}_{\overline{V}}, \overline{R}_{-\overline{V}})i$. Thus, Lemma 7 implies that $f_i(\hat{R}_{\overline{V}}, R_{M-\overline{V}}, \overline{R}_{-M}) = a$. Note also that $f_M(R_M, \overline{R}_{-M}) = f_M(R)$, by assumption. Then, since $\overline{V} \subseteq M$, for all $j \in \overline{V}$, $\hat{R}_j$ is a push-up of R_j for $f_j(R_M, \overline{R}_{-M}) = f_j(R)$. This implies that $f(\hat{R}_{\overline{V}}, R_{M-\overline{V}}, \overline{R}_{-M}) = f(R_M, \overline{R}_{-M})$, by group-strategyproofness. In particular, $f_i(R_M, \overline{R}_{-M}) = a$, as desired.

Now we show, similarly to Step 2.a, that for all $\tilde{R}_{-M}$, $a \in o_i(R_M, \tilde{R}_{-M,i})$. Since for all $i, j \in N - M$, $j \neq i$, $j\mathscr{V}(R_M, \overline{R}_{-M})i$, Step 3.a follows from Lemma 7. Note that this step implies that for all $i \in N - M$, for all $a \in E_{t+1}(i, R)$, and for all $\tilde{R}_{-M}$, $f_i(R_M, \tilde{R}_{-M})R_i a$, given that f is strategyproof.

Step 3.*b*: *For all* $i \in W_{t+1}(R)$, $f_i(R) = f_i^{\Gamma}(R)$.

The proof of this step is similar to the proof of Step 2.b, using the assumption that for all $\tilde{R}_{-M}$, $f_M(R_M, \tilde{R}_{-M}) = f_M(R)$. Since R was chosen arbitrarily, this also means that for all $\tilde{R}_{-W^{t+1}(R)}$, $f_{W^{t+1}(R)}(R_{W^{t+1}(R)}, \tilde{R}_{-W^{t+1}(R)}) = f_{W^{t+1}(R)}(R)$, which concludes the inductive step.

Therefore, we have, by induction, that for all $i \in W^{t^*}(R)$, $f_i(R) = f_i^{\Gamma}(R)$. Moreover, if there exists $j \in N - W^{t^*}(R)$, then $n > k$ and $F^{t^*}(R) = K$, which implies that $f_j(R) = 0$. In sum, for all $i \in N$, $f_i(R) = f_i^{\Gamma}(R)$. Given that this holds for all $R \in \mathscr{R}$, it follows that f is a hierarchical exchange rule.

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