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Strategyproof Matching with Minimum Quotas*

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Abstract

We consider the school choice problem allowing schools to have minimum (in addition to the standard maximum) quotas. Standard properties such as strategyproofness, fairness, and nonwastefulness become incompatible with minimum quotas. Taking strategyproofness as necessary, we introduce new definitions of nonwastefulness and fairness that are compatible with minimum quotas. We provide new mechanisms that satisfy all minimum quotas and that are as nonwasteful and as fair as possible. We use computer simulations to show that significantly more students prefer our more flexible mechanisms to the commonly used solution of artificially imposing lower maximum quotas and then using standard mechanisms such as deferred acceptance.

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1 Introduction

The theory of matching has been extensively developed for markets in which the agents (students/schools, hospitals/residents, workers/firms) have *maximum* quotas that cannot be exceeded.[1] However, in many real-world markets, *minimum* quotas may also be relevant, and there is a lack of mechanisms that take minimum quotas into account, and of strategyproof minimum quota mechanisms in particular. This paper fills this gap by providing new strategyproof mechanisms that fill all minimum quotas, while satisfying other important desiderata (fairness and efficiency) as much as possible.

There are many examples of matching problems in which minimum quotas are present. School districts may need at least a certain number of students in each school in order for the school to operate, as in college admissions in Hungary (Biró et al. (2010)). In the market for Japanese medical residents studied by Kamada and Kojima (2011), the Japanese government desires more doctors be assigned to hospitals in rural areas, and imposing minimum quotas on such rural hospitals is one possible approach. In the early 2000s, the United States Military Academy (USMA) solicited cadet preferences over assignments to various branches and imposed minimum and maximum quotas on the number of students who could be assigned to each branch (Sönmez and Switzer (2011); Sönmez (2011)). In the context of schools, minimum quotas may be important not only in assigning students across schools, but also in assigning students to classes *within* schools. For example, computer science students at Kyushu University must all complete a laboratory requirement. Students are able to submit preferences over the labs, but each lab has certain minimum and maximum quotas that must be respected.[2]

The fact that there are few mechanisms that satisfy minimum quotas may lead many markets to artificially lower the maximum quotas so that any standard matching that satisfies such artificial caps will satisfy the true minimum quotas as well. Once this is done, an "off-the-shelf" mechanism that respects only maximum quotas can be used. For example, the Japanese medical resident market imposes explicit regional caps and then uses a DA procedure in the hopes of assigning more doctors to rural hospitals. After reforms in 2007, the Army decided to set the maximum quotas in such a way that there was no longer any flexibility.[3] To the extent that this was done only to ensure the true minimum quotas would be implicitly satisfied, it may lead to efficiency

[1]See Roth and Sotomayor (1990) for a comprehensive survey of many results in this literature.

[2]Diversity constraints at schools can also be considered as a minimum quota problem, where school districts impose a minimum quota for each type of student at each school. While not explored in this paper, this is an extremely important application, and the extension of our results to this class of problems is the subject of ongoing work (Fragiadakis and Troyan (2012)).

[3]See the October 1, 2007 memorandum from the Department of the Army to the USMA superintendent.

losses, as some positions in high demand will end up below their *true* capacity, making it possible to reassign the agents and make everyone better off. In many other markets, we may not see minimum quotas only because designers are not able to handle them, and so choose the simple solution of imposing artificial caps, making it appear as if there were only maximum quotas. Once successful minimum quota mechanisms are available, they may begin to appear in more markets.

The goal of this paper is to recover some of the efficiency losses caused by imposing artificial caps by providing minimum quota mechanisms that allocate the extra seats more flexibly based on realized demand. More specifically, we first show that with minimum quotas, no mechanism can be simultaneously strategyproof, fair,[4] and nonwasteful. Because the matching literature has found strategyproofness to be an extremely important property (see, for example, Pathak and Sönmez (2012)), and there is a lack of strategyproof mechanisms for matching problems with minimum quotas, we take strategyproofness as a necessary criterion.[5] We thus introduce new definitions of fairness and nonwastefulness that take the minimum quota restrictions into account. Since we cannot achieve full fairness, we develop a new fairness criterion by which we can rank mechanisms. We then introduce new mechanisms and show that they are maximally fair in the class of (constrained) nonwasteful and strategyproof mechanisms. We lastly use computer simulations to show that since our mechanisms are able to more flexibly allocate seats based on student demand, they waste far fewer seats than imposing artificial caps and are overwhelmingly preferred by the students, in the sense that the rank distributions for our mechanisms first-order stochastically dominate that of imposing artificial caps.

Without minimum quotas, the standard DA mechanism is widely used because it is strategyproof, fair, and nonwasateful. Thus, adhering closely to this algorithm while still somehow satisfying all minimum quotas seems like a fruitful approach. Our first modification of DA, which we call extended-seat DA (ESDA), does this by dividing the school seats into two classes.[6] At each school, a number of seats equal to the school's minimum quota are assigned according to the school's priority list, while the assignment of seats above the minimum quota must be restricted in such a way that ensures that the minimum quotas at all other schools are also satisfied.

[4]Fairness means that if one student envies the assignment of another, then the second student must have a higher priority at her assigned school than the first. It is also called "no justified envy" or "stability" in the matching literature. See Section 2 for a formal definition.

[5]We do not mean to suggest that strategyproofness should be imposed above all else. While Abdulkadiroğlu et al. (2005), Chen and Sönmez (2006), Pathak and Sönmez (2008), and Ergin and Sönmez (2006) discuss negative consequences of the highly manipulable Boston mechanism and recommend shifting to a strategyproof mechanism in school choice settings, other papers such as Abdulkadiroğlu et al. (2011), Featherstone and Niederle (2011), and Troyan (2012) show that non-strategyproof mechanisms may sometimes outperform strategyproof ones.

[6]We use the language of school choice throughout the paper, but our results can obviously be applied to many other allocation problems.

When the minimum quotas are close to the maximum quotas, ESDA produces an allocation that is close to standard DA. However, when the minimum quotas are low, ESDA tends to ignore the individual school priorities, and the output is similar to a serial dictatorship. For these situations, we introduce a second modification, the multi-stage DA algorithm (MSDA), which first "reserves" a number of students equal to the sum of the minimum quotas and assigns the remaining students according to standard DA. This process is repeated until we reach a final assignment. Opposite to ESDA, when the minimum quotas are low, MSDA will reserve few students, and the outcome will approach standard DA (and will approach a serial dictatorship when the minimum quotas are high). Thus, this provides some guidance to policymakers on mechanism selection depending on the relative sizes of the minimum and maximum quotas.

Because the introduction of minimum quotas leads to the incompatibility of many theoretical properties, we also use computer simulations to study our mechanisms. We show that both ESDA and MSDA waste significantly fewer seats than imposing artificial caps, and are overwhelmingly preferred by the students. In addition, we show that as the minimum quotas approach the maximum quotas, ESDA becomes more and more fair in the traditional sense (i.e., it produces few blocking pairs), while as the minimum quotas approach 0, the same holds for MSDA.[7]

Lastly, we note that when school priorities are given a milder interpretation, markets often use the top-trading cycles (TTC) mechanism of Shapley and Scarf (1974).[8] We show that our extended-seat and multi-stage ideas can be applied to TTC as well, giving the ESTTC and MSTTC mechanisms, which will be similar to the standard TTC mechanism but will satisfy all minimum quotas. ESTTC and MSTTC will be strongly group strategyproof and Pareto efficient. Just as in the standard model, if the designer is more concerned with Pareto efficiency, a TTC-based mechanism should be chosen, while if fairness is a more important criterion, either ESDA or MSDA should be used.

Related Literature

While minimum quotas seem like a natural extension to the standard matching models, there is little work on the topic, most likely because the problem becomes difficult and many of the results from previous literature have been negative. The papers most related to this one are Ehlers

[7]A similar approach is taken by Abdulkadiroğlu et al. (2009), who also use computer simulations and argue that the greater the number of blocking pairs a matching algorithm produces, the less successful it is expected to be empirically. With a similar motivation, Hamada et al. (2011) study the problem of actually minimizing the number of blocking pairs.

[8]See Abdulkadiroğlu and Sonmez (2003) for the first application of TTC to school choice. The New Orleans school district has recently adopted a TTC mechanism to allocate students. Hatfield et al. (2011) study how DA and TTC-based mechanisms affect incentives for school competition.

(2010) and Ehlers et al. (2011). In those papers, the authors study a problem equivalent to ours, allowing for multiple types of students (so that the minimum quotas correspond to affirmative action constraints in school choice). They also note several impossibility results once minimum quotas are introduced, but instead choose to forego strategyproofness and provide mechanisms that will be fair and constrained nonwasteful, though they will also be manipulable. We fill this gap by providing strategyproof mechanisms for problems with minimum quotas. Abdulkadiroğlu (2010) studies another model similar to the one here but chooses to relax feasibility constraints by not requiring that all students be enrolled at a school. Kojima (2012) and Halafir et al. (2011) show that some types of affirmative action quotas may actually hurt the very minorities they are supposed to help.

Biró et al. (2010) study college admissions in Hungary, in which colleges may declare minimum quotas for their programs. They study the difficulty (from a computer science perspective) of finding stable matchings when minimum quotas are introduced, but do not provide explicit mechanisms or consider incentive or efficiency issues, as we do here. Hamada et al. (2011) also study matching with minimum quotas from a computer science perspective, showing that minimizing the number of blocking pairs is an NP-hard problem when minimum quotas are imposed.

While the military cadet-branch matching problem studied in Sönmez and Switzer (2011) and Sönmez (2011) does not have explicit minimum quotas, to the extent that the maximum quotas imposed by the military are artificial caps and not true maximum quotas (see the above discussion and the memorandum cited in footnote 3), our mechanisms may allow for more flexibility and improve outcomes. Additionally, the Army is particularly concerned about diversity constraints, and the extension of our mechanisms to minimum quotas for multiple types of agents is the subject of ongoing work. Kominers and Sönmez (2012) have a related model with slot-specific priorities in which they achieve diversity by giving certain students higher priority for different slots within each branch.

As a final possible application, consider the medical residency market studied first by Roth (1984). In these markets, the shortage of doctors in rural areas is a well-known problem, and the so-called rural hospitals theorem suggests it is difficult to solve (see Roth (1986), Martinez et al. (2000), and Hatfield and Milgrom (2005)). Kamada and Kojima (2011) discuss one possible solution used in Japan: capping the number of residents who can be assigned to a given region. Again, to the extent that these caps are simply an ad-hoc way to ensure the true minimum quotas are satisfied, imposing the minimum quotas directly and using one of our mechanisms is another possible approach. As Kamada and Kojima note in their paper, introducing minimum quotas causes considerable difficulties, as stable matches (according to the classical definition of stability) will generally not exist. However, stable matches also may not exist after regional caps are imposed, which leads Kamada and Kojima to propose weaker definitions of stability (as we must do in this paper as well) and define a new mechanism which takes the regional caps into account. The use of our modified DA-based algorithms will produce outcomes that directly satisfy all of the minimum quotas while still adhering closely to the standard deferred acceptance algorithm. Which mechanisms will work better in practice is an empirical question.

Methodologically, our approach of ranking mechanisms that cannot achieve full fairness on the entire preference domain is similar to approaches taken by Pathak and Sönmez (2012) and Carroll (2011), who introduce methods for ranking nonstrategyproof school choice and voting mechanisms (respectively) by their degree of vulnerability to manipulation.

The remainder of this paper is organized as follows. In Section 2, we present a basic model with minimum and maximum quotas and discuss some common desirable properties such as efficiency, nonwastefulness, and fairness. We show that many of the standard properties are incompatible, which leads us to introduce several new definitions. In Section 3, motivated by our new desiderata, we describe our two modifications of the DA algorithm in detail. In Section 4, we show that our new mechanisms will outperform the ad hoc approach of imposing "artificial caps" to satisfy the minimum quotas. Section 5 applies our minimum quota modifications to the top trading cycles (TTC) algorithm. Section 6 concludes. All proofs are in the appendix, unless otherwise stated.

2 Matching with minimum quotas

2.1 Basic Model

A market consists of $(S, C, p, q, \succ_S, \succ_C)$. $S = \{s_1, s_2, \ldots, s_n\}$ is a set of n students, $C = \{c_1, c_2, \ldots, c_m\}$ a set of m schools ("colleges"), and $p = (p_{c_1}, \ldots, p_{c_m})$ and $q = (q_{c_1}, \ldots, q_{c_m})$ are vectors of minimum and maximum quotas, respectively, for each school. We assume $p_c \geq 0$, $q_c > 0$, and $p_c \leq q_c$ for all $c \in C$ and $\sum_{c\in C} p_c < n < \sum_{c\in C} q_c$ to ensure a feasible matching exists.[9] Define $e = n - \sum_{c\in C} p_c$ to be the number of "excess students" above the sum of the minimum quotas. To make the problem interesting, we additionally assume the following:[10]

(A1) $m > 2$

(A2) $p_c < q_c$ for at least two schools

(A3) $p_c > 0$ for at least two schools

[9]We assume both inequalities strict because if $n = \sum_{c\in C} p_c$ or $\sum_{c\in C} q_c$ there is no flexibility in the seats to be assigned and the standard DA algorithm can be used.

[10]These assumptions are likely to be satisfied in any real-world market of reasonable size. The special cases where these assumptions do not hold are dealt with in Appendix I.

Each student s has a strict preference relation $\succ_s$ over C, while each school c has a strict priority relation $\succ_c$ over S. Vectors of such relations, one for each agent, are denoted $\succ_S = (\succ_s)_{s\in S}$ for the students and $\succ_C = (\succ_c)_{c\in C}$ for the schools. Let $\mathcal{P}$ denote the set of possible preference relations over C, and $\mathcal{P}^{|S|}$ denote the set of all preference vectors for all students. As is standard in the school choice literature, the school priorities are fixed and known to all students (in applications, priorities are often related to such things as the distance a student lives from a school or whether or not a student has a sibling attending the school). We assume that all schools are acceptable to all students and vice versa.[11]

A **matching** is a mapping $\mu : S \cup C \to 2^{S\cup C}$ that satisfies: (i) $\mu(s) \in C$ for all $s \in S$, (ii) $\mu(c) \subseteq S$ for all $c \in C$, and (iii) for any $s \in S$ and $c \in C$, we have $\mu(s) = c$ if and only if $s \in \mu(c)$. A matching is **feasible** if $p_c \leq |\mu(c)| \leq q_c$ for all $c \in C$. Let $\mathcal{M}$ denote the set of feasible matchings.

A **mechanism** $\chi : \mathcal{P}^{|S|} \to \mathcal{M}$ is a function that takes as an input any possible preference profile of the students and gives as an output a feasible matching of students to schools. If the students submit preference profile $\succ_S \in \mathcal{P}^{|S|}$, then $\chi(\succ_S)$ is the assigned matching, and we write $\chi_i(\succ_S)$ for the assignment to agent $i \in S \cup C$. If $i \in S$, then $\chi_i(\succ_S) \in C$ is the school student i is assigned to, while if $i \in C$, then $\chi_i(\succ_S) \subseteq S$ is the set of students assigned to school i.[12]

2.2 Desirable properties of matchings and mechanisms

The matching literature has identified a number of desirable properties of matchings and mechanisms. In general, not all of these properties will be compatible, and this problem is compounded by the introduction of minimum quotas. We introduce these desirable properties here and discuss some impossibility results.

The first natural requirement for any matching is Pareto efficiency.

Definition 1. *A matching μ is **Pareto efficient** if there is no other feasible matching μ' such that $\mu'(s) \succeq_s \mu(s)$ for all $s \in S$ and $\mu'(s) \succ_s \mu(s)$ for some $s \in S$.*

Note that our definition of Pareto efficiency considers only the welfare of the students. This is consistent with the interpretation that the seats are objects to be consumed by the students and that $\succ_C$ represents a priority relation, rather than a preference relation for the schools.

[11] If agents on one side of the market were allowed to report agents on the other side as unacceptable, it would be impossible to guarantee the existence of an individually rational matching that satisfied all minimum quotas. See Ehlers (2010) and Ehlers et al. (2011) who impose the same assumption in a model with minimum quotas.

[12] Since student preferences are the only private information, we only explicitly write this as a function of $\succ_S$; however, this function of course implicitly depends on C, p, q, and $\succ_C$ as well.

Another requirement often studied in the matching literature is fairness or no justified envy (also known as stability in some two-sided matching models). To define fairness, we first introduce the notion of a blocking pair.

Definition 2. *Given a matching μ, student-school pair (s, c) form a* ***blocking pair*** *if $c \succ_s \mu(s)$ and $s \succ_c s'$ for some $s' \in \mu(c)$.*

In words, student s would rather be matched to school c than her current match $\mu(s)$, and she has higher priority at c than some student s' who is currently assigned there; thus, s has a claim on a seat at c over student s'. In some papers, it is said that s has **justified envy** towards s', and we will often use these terms interchangeably. Priorities are often based on criteria such as distance between a student and the school or test scores, and if one student justifiably envies another, she may be able to take legal action against the school district (see Abdulkadiroğlu and Sonmez (2003)). Thus, an important goal of many school districts is for a matching to contain no such blocking pairs. When this is true, we say that the matching is fair.[13]

Definition 3. *A matching μ is* ***fair*** *(or* ***eliminates all justified envy****) if no student-school (s, c) can form a blocking pair.*

Even without minimum quotas, Pareto efficient and fair matchings will not exist in general, as two students may be prevented from engaging in a Pareto improving trade because the new assignment would cause a third student to form a blocking pair with one of the schools that was traded. For this reason, Pareto efficiency is often weakened to nonwastefulness when fairness is an important consideration. The next definition adapts the notion of nonwastefulness to our setting with minimum quotas.

Definition 4. *Fix a matching μ. We say student s* ***claims an empty seat at school*** *c if (i) $c \succ_s \mu(s)$ (ii) $|\mu(c)| < q_c$ and (iii) $|\mu(\mu(s))| > p_{\mu(s)}$. If no student claims an empty seat at any school, then matching μ is* ***nonwasteful****.*

In words, this definition says that if student s prefers school c to her current assignment $\mu(s)$, school c has an empty seat, **and** the number of students assigned to her current school $\mu(s)$ is strictly above its minimum quota, then student s should be moved to that school. Part (iii) differentiates this definition from the standard definition of nonwastefulness. If (iii) is not satisfied, then moving s would violate feasibility.

[13] For example, fairness was an extremely important criterion to administrators of the Boston school district when they were redesigning their school assignment mechanism. See Abdulkadiroğlu et al. (2005).

All of the properties of matchings above have counterparts for mechanisms. We say that mechanism χ is efficient if it produces an efficient matching for every possible preference profile. Similarly, we say χ is fair if for every preference profile it produces a fair matching, and χ is nonwasteful if for every preference profile it produces a nonwasteful matching.

The last property we study concerns the incentives of the students to report their preferences truthfully. Obviously, all else equal, designers would like to use a mechanism that elicits the true preferences of the students. The easiest way to do so is to use a mechanism that is strategyproof.

Definition 5. *A mechanism χ is* ***strategyproof*** *if $\chi_s(\succ_S) \succeq_s \chi_s(\succ'_s, \succ_{S\setminus\{s\}})$ for all $\succ_S \in \mathcal{P}^{|S|}$, $s \in S$, and $\succ'_s \in \mathcal{P}$, .*

In words, a mechanism is strategyproof if no student ever has any incentive to misreport her preferences, no matter what the other students report. As discussed in the introduction, strategyproofness has been found to be a very important property in the success of matching mechanisms, and thus we take it as a necessary criterion in our search for mechanisms. All of the mechanisms we propose will be strategyproof. In addition, our mechanisms will be immune to certain types of group manipulations. We define two forms of group strategyproofness below.

Definition 6. *A mechanism χ is* ***weakly group strategyproof*** *if there does not exist a preference profile $\succ_S \in \mathcal{P}^{|S|}$, a group of students $S' \subseteq S$, and a preference profile $(\succ'_s)_{s\in S'} \equiv \succ'_{S'}$ such that $\chi_s(\succ'_{S'}, \succ_{S\setminus S'}) \succ_s \chi_s(\succ_S)$ for all $s \in S'$.*

Definition 7. *A mechanism χ is* ***strongly group strategyproof*** *if there does not exist a preference profile $\succ_S \in \mathcal{P}^{|S|}$, a group of students $S' \subseteq S$, and a preference profile $(\succ'_s)_{s\in S'} \equiv \succ'_{S'}$ such that $\chi_s(\succ'_{S'}, \succ_{S\setminus S'}) \succeq_s \chi_s(\succ_S)$ for all $s \in S'$ and $\chi_s(\succ'_{S'}, \succ_{S\setminus S'}) \succ_s \chi_s(\succ_S)$ for at least one $s \in S'$.*

Clearly, strong group strategyproofness $\implies$ weak group strategyproofness $\implies$ strategyproofness. It is well known in the literature that without minimum quotas, DA is weakly group strategyproof (Hatfield and Kojima (2009)), while TTC is strongly group strategyproof (Pápai (2000)). Similarly, all of our DA-based mechanisms will be weakly group strategyproof, while all of our TTC-based mechanisms will be strongly group strategyproof.

A necessary condition for strategyproofness which will be useful in proving many of the results below is monotonicity. Informally, this means that if we begin with some matching $\chi(\succ_S)$ and consider another preference profile under which each student's original assignment $\chi_s(\succ_S)$ is promoted in her ranking, then the assignment must remain the same. Formally, say that $\succ'_S$ is a **monotonic transformation of $\succ_S$ with respect to $\chi(\succ_S)$** if, for all c and all s, we have: $\chi_s(\succ_S) \succ_s c \implies \chi_s(\succ_S) \succ'_s c$. The result below is well-known in the literature, but we state it here for completeness (see, for example, Maskin (1999)).

Proposition 1. *Let χ be a strategyproof mechanism. If $\succ'_S$ is a monotonic transformation of $\succ_S$ with respect to $\chi(\succ_S)$, then $\chi(\succ'_S) = \chi(\succ_S)$.*

2.3 Impossibility results

Ideally, we would like a mechanism that is simultaneously strategyproof, fair, and Pareto efficient. However, as mentioned above, even without minimum quotas, Pareto efficiency and fairness alone are in conflict. If we weaken Pareto efficiency to nonwastefulness and consider only maximum quotas, the well-known deferred acceptance algorithm will be strategyproof, fair, and nonwasteful. However, once minimum quotas are introduced, this will no longer be true, as shown in the impossibility result below, which was first proven by Ehlers et al. (2011).

Theorem 1. *With minimum quotas, the set of feasible matchings that are fair and nonwasteful may be empty.*

We repeat the counterexample used to show this result since it is instructive in showing the problems minimum quotas introduce. Consider the following market of two students and three schools:[14]

	$\succ_{c_1}$	$\succ_{c_2}$	$\succ_{c_3}$
	s_2	s_2	s_1
	s_1	s_1	s_2
p_c	1	0	0
q_c	1	1	1

$\succ_{s_1}$	$\succ_{s_2}$
c_2	$\boxed{c_3}$
c_3	c_2
$\boxed{c_1}$	c_1

The minimum quota requirement at c_1 means that one of s_1 or s_2 must be assigned there; nonwastefulness then requires that the other student be assigned his most preferred school. However, the student assigned c_1 will then justifiably envy the other student. For example, in the allocation indicated by the boxes, s_1 prefers c_3 and has higher priority than s_2 there, and thus forms a blocking pair. The case where s_2 is assigned c_1 is similar.

Because of this impossibility, we must weaken either nonwastefulness or fairness. There are many possible ways to do this, and we discuss some alternatives (and why they fail) later. For now, consider a definition of (constrained) nonwastefulness based on the following sequence of events. If a student s_1 claims an empty seat, then, if the school district grants his request, there will be a student s_2 who will request the seat s_1 vacates. If the school district grants s_2's request, there

[14]Though we assume for most of the paper that $p_c > 0$ for at least 2 schools, this is not true in this counterexample for simplicity. The market below can be embedded in a larger market of 3 students where the added student is ranked first on the priority list of every school and school c_2 has $p_{c_2} = 1$ and $q_{c_2} = 2$, and the result still holds.

will be a student s_3 who will request the seat s_2 vacates. Eventually, this chain should end with a student who cannot be moved because the minimum quotas would be violated at his school (school c^K in the definition below). Thus, the school district has some justification for not granting such requests.

Definition 8. *A matching μ is* ***constrained nonwasteful*** *if for every student s^1 who claims an empty seat at some school c^0, there exists a chain of students and schools $(s^1, c^1, s^2, c^2, \ldots, s^K, c^K)$ such that for all $k \in \{1, 2, \ldots, K\}$: (i) $\mu(s^k) = c^k$ (ii) $c^{k-1} \succ_{s^k} c^k$ and (iii) $|\mu(c^K)| = p_{c^K}$.*

We say that mechanism χ is constrained nonwasteful if it produces a constrained nonwasteful matching for every possible preference profile. While constrained nonwastefulness seems like a very weak requirement, it is unfortunately also incompatible with strategyproofness and fairness.

Theorem 2. *With minimum quotas, there is no mechanism that is strategyproof, fair, and constrained nonwasteful.*

The above definition of constrained nonwastefulness may seem weak, but it at least gives school districts some justification for denying student requests for empty seats, as it would leave to an avalanche of requests from other students. Another obvious way to weaken nonwastefulness would be to deny student s's request to move to an empty seat at c if the resulting matching would cause some other student s' to justifiably envy student s's new assignment. Unfortunately, this definition of constrained nonwastefulness is also incompatible with strategyproofness and fairness, and in fact is even incompatible with significant weakenings of fairness, and so we do not pursue such constrained nonwasteful mechanisms further. For further discussion of these alternative definitions, see Appendix J.

Given these impossibilities, we must either weaken nonwastefulness further or, alternatively, weaken fairness. These approaches are next discussed in turn.

2.4 Artificial caps deferred acceptance (ACDA)

If we are willing to give up on nonwastefulness entirely, there is a mechanism that is strategyproof and fair, namely, the artificial caps deferred acceptance (ACDA) described in the introduction.[15] Artificial caps deferred acceptance proceeds by imposing artificial maximum quotas at every school. Then, deferred acceptance is run with only these maximum quotas, ignoring the minimum quotas. By imposing sufficiently stringent artificial caps, we can ensure that no matter how the students

[15] The deferred acceptance algorithm is well-known in the literature, and it is also a special case of the new mechanisms we define in section 3, and so we do not give its definition here. See Gale and Shapley (1962) for the original description, or Abdulkadiroğlu and Sonmez (2003) for a discussion of DA in the context of school choice.

are allocated, the minimum quotas will be satisfied. Because the standard DA algorithm with maximum quotas is strategyproof and fair, ACDA will be strategyproof and fair as well.

As an example, consider a market of $n = 100$ students and $m = 10$ schools, each with $p_c = 5$ and $q_c = 20$. Now, imagine imposing artificial caps of $q_c^* = 10$ at each school and running the standard DA algorithm. By doing so, exactly 10 students will be assigned to each school, thereby satisfying all minimum and maximum quotas. However, this may be very wasteful if, for example, there is high demand for c_1, because many students could be moved there and made better off without violating any quotas. The problem is that the mechanism has no flexibility, and eliminates seats without regard to student preferences. In section 4, we use computer simulations to show that while ACDA is fair, it tends to be extremely wasteful. The new mechanisms that we introduce compromise some fairness, but overall seem to perform much better than ACDA, and will be overwhelmingly preferred by the students.

2.5 σ-fairness

Because the ACDA mechanism may be very wasteful, we search for other mechanisms that will waste less seats. At the same time, we do not want to give up fairness entirely, which requires introducing a new way to think about how to rank mechanisms according to fairness. Theorem 2 shows that any mechanism that is constrained nonwasteful (and strategyproof) tends to produce too many blocking pairs. Therefore, our new notion of fairness, which we will call σ-fairness, must eliminate some potential blocking pairs.[16]

Standard blocking pairs (s, c) are defined with relation to the idiosyncratic school priority lists. However, the idiosyncratic priority lists are insufficient for situations in which the preferred schools of two students are willing to accept them individually, but the mechanism must choose one of these students to fill a minimum quota seat at *another* school. This is the case in the example above: c_2 and c_3 can accommodate s_1 and s_2 individually, but the mechanism must remove one of these students from their most preferred school and assign them to c_1. The within-school priority lists $\succ_c$ cannot be used for this purpose; what we need is an additional ordering that works *across* schools to decide whether s_1 or s_2 will be compelled to attend c_1. The mechanism itself is then in

[16]Another approach is to give up fairness entirely, which will allow us to achieve not only (constrained) nonwastefulness, but in fact full Pareto efficiency by using TTC-based mechanisms. Whether this is acceptable depends on the interpretation of the school priorities. If a high priority at a school is interpreted as an opportunity to get into a school, all else being equal, but not necessarily a right to that school, then using a TTC based mechanism which allows students to trade priorities may be best. Kesten (2010), for example, allows students to trade priorities to achieve efficiency if possible objecting students consent, as this trade does not harm the objecting student, who is assigned the same school whether or not the trade takes place. While our main focus is on situations in which fairness is a relevant concern, we briefly explore such TTC-based mechanisms in Section 5.

some sense giving a lower priority to the student who is assigned to c_1.

For the general case of n students, we introduce a list of all n students which will similarly be used to ensure all minimum quotas are met. We call this ordering the **master list (ML)**, and without loss of generality, we let $s_1 \succ_{ML} s_2 \succ_{ML} \cdots \succ_{ML} s_n$. Intuitively, this ordering can be thought of as a type of tiebreaker when two students cannot both be assigned their preferred schools (even though they have open seats) because of minimum quota restrictions at *other* schools.

Such lists may arise naturally in many matching settings. For example, in the military, all cadets are ordered according to a single order of merit list based on academic performance, physical fitness, and military performance (Sönmez and Switzer (2011); Sönmez (2011)). Irving et al. (2008) note that in the Medical Training Application Service that assigns junior doctors to medical posts in the UK, a master preference list based on academic records and other criteria is used. In many countries, such as China, Australia, Turkey, etc., college admissions are centralized and students all take a common entrance exam, which is a natural candidate for a master list (see Abizada and Chen (2011)). If no objective ranking of the students outside of the individual school priority lists is available, we can form a master list by assigning each student a random number. Such random numbers are often used in school choice problems to eliminate envy when there are ties in priority levels.[17,18]

Given the master list, we can then use it to eliminate potential blocking pairs as follows. Given a matching μ and some integer $\sigma_c \leq q_c$, we say that (s, c) form a σ_c**-blocking pair** if the following three conditions are met for some $s' \in \mu(c)$:

(i) $c \succ_s \mu(s)$

(ii) $s \succ_c s'$

(iii) If $|\mu(c)| > \sigma_c$, then $s \succ_{ML} s'$

Parts (i) and (ii) are the standard definition of a blocking pair. Part (iii) is an additional requirement that restricts when student s's envy towards s' is justified: if more than σ_c students are assigned to school c, then in order to form a blocking pair, s must also be higher than s' on the master list. In effect, the more highly demanded school c is, the more difficult it will be for students to form a blocking pair with c because other students can claim seats at c based on their high priority on the master list.

Note that σ_c-blocking pairs are generalizations of standard blocking pairs. As σ_c is increased, condition (iii) becomes less binding, and it becomes easier for students to form blocking pairs. When $\sigma_c = q_c$, σ_c-blocking pairs reduce to standard blocking pairs.

[17] See also Perach et al. (2007) and Perach and Rothblum (2010) for the use of master lists in problems of allocating university housing.

[18] Alternatively, it may also be possible to define a notion of "ex-ante fairness" similar to Afacan (2012).

Remark 1. *If, for some matching μ, (s, c) is a σ_c-blocking pair for some σ_c, then (s, c) is a σ'_c-blocking pair for all $\sigma'_c \geq \sigma_c$. However, the reverse does not hold: (s, c) may be a σ_c-blocking pair but **not** a σ'_c-blocking pair for $\sigma'_c < \sigma_c$.*

Let Σ denote the set of all vectors of integers $\sigma = (\sigma_{c_1}, \ldots, \sigma_{c_m})$ such that $\sigma_c \leq q_c$ for all $c \in C$.[19] We can now generalize the standard definition of fairness to one that eliminates σ-blocking pairs. In light Remark 1, if a matching eliminates all σ_c-blocking pairs, it also eliminates all σ'_c-blocking pairs for all $\sigma'_c \leq \sigma_c$. Thus, in measuring the fairness of a matching μ (or a mechanism), we only need consider the **largest** σ vector (with respect to the product order) for which there are no σ_c-blocking pairs.

For example, consider Figure 1, which represents a market with 2 schools with minimum quota vector $p = (1, 2)$ and maximum quota vector $q = (3, 5)$. If a matching eliminates all 2-blocking pairs at c_1 and all 4-blocking pairs at c_2, then it will also eliminate all σ-blocking pairs for all σ vectors in the shaded region. If the matching does produce a 3-blocking pair at c_1 and a 5-blocking pair at c_2, then $\sigma' = (2, 4)$ is the largest vector for which all σ-blocking pairs are eliminated, and we will call the matching σ'-fair.

Definition 9. *Fix a matching μ. If $\sigma = \max\{\sigma' \in \Sigma : \mu$ contains no σ'_c-blocking pairs$\}$ exists (where the maximum is taken with respect to the product order), then we say that μ is σ**-fair**.*

Note that such a σ may not exist if the matching contains some σ_c blocking pair for all $\sigma \in \Sigma$. In this case, the matching is not σ-fair for any σ. A similar statement can be applied to mechanisms.

Definition 10. *Consider a mechanism χ. If $\sigma = \max\{\sigma' \in \Sigma : \chi(\succ_S)$ is a $\sigma'-$fair matching for all $\succ_S \in \mathcal{P}^{|S|}\}$ exists, then we say that χ is $\sigma-$**fair**.*

In both definitions above, the maximum is taken with respect to the product order.

Since a σ_c-blocking pair becomes a standard blocking pair when $\sigma_c = q_c$, if $\sigma = (q_{c_1}, \ldots, q_{c_m})$, σ-fairness reduces to the standard notion of fairness from Definition 3. This is the strongest definition of fairness, in that the most student envy is eliminated. As the elements of the vector σ increase, σ-fairness will allow more (potential) student objections, and so if the mechanism itself eliminates all potential blocking pairs, higher σ's mean that the mechanism is more fair. Visually, as a mechanism becomes more fair, the shaded region in Figure 1 grows larger and larger, until $\sigma' = q$.

On the other hand, the weakest definition of fairness is one for which σ is a vector of zeros. Define the notation $\sigma_0 = (0, \ldots, 0)$. Then, σ_0-fairness is the weakest definition fairness, and

[19]For $\sigma_c > q_c$, a σ_c-blocking pair is equivalent to a q_c-blocking pair, and so we restrictσ_c to be weakly less than q_c.

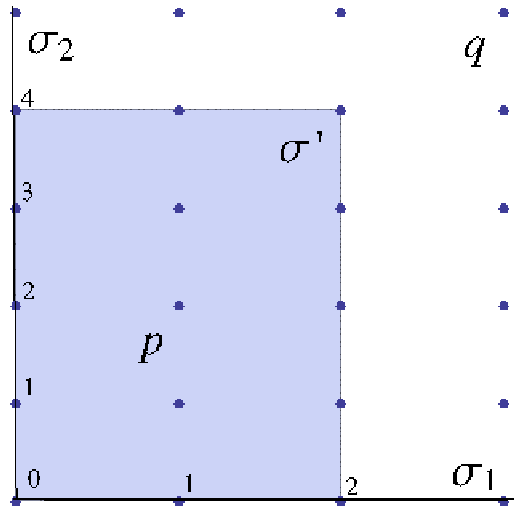


Figure 1: An illustration of σ-fairness.

corresponds to a situation in which, for a student to form a blocking pair with a school c, he must *always* be ranked higher than some s' assigned to c on **both** $\succ_c$ and $\succ_{ML}$, i.e., it is very difficult for students to form blocking pairs.

We should also note that there is some degeneracy in the defintion of σ_0-fairness. Because all mechanisms assign at least p_c students to each school, if a mechanism is σ_0-fair, it is also $\sigma-$fair for $\sigma = (p_{c_1} - 1, \ldots, p_{c_m} - 1)$, and for all σ' in between σ_0 and σ. For ease of notation, we use σ_0 to represent this entire equivalence class, and simply call such mechanisms σ_0-fair.

Another important case is when $\sigma = p$. When $\sigma = p$, this means that to form a blocking pair with c, a student s must be higher than some s' according to $\succ_{ML}$ only if $|\mu(c)| > p_c$, i.e., if any of the flexible seats were given to school c. When c is filled exactly to its minimum quota p_c, only the individual school priority list is relevant for forming blocking pairs. The master list is used only if some of the flexible seats were assigned to school c.

Using these ideas, we can formally rank mechanisms according to fairness. It should be noted that while in general, mechanisms such as DA and our new mechanisms below can be applied to markets of any size, it is only meaningful to compare two mechanisms with respect to fairness for fixed minimum and maximum quota vectors p and q.

Definition 11. *Fix quota vectors p and q, and consider two mechanisms χ and ψ which are $\sigma^\chi-$ and σ^ψ-fair, respectively. If $\sigma^\chi \geq \sigma^\psi$, then we say that χ is **more fair** than ψ.*

Again, the comparison in the above definition is taken with respect to the product order. Because we use the product order, the relation is **not** complete: it may be that given two mechanisms, neither is more fair than the other. While this is allowed a priori, in the next section we are in fact able to find mechanisms that are maximally fair in the classes of strategyproof and nonwasteful/constrained nonwasteful mechanisms.

3 New mechanisms

Recall our original desiderata: strategyproofness, fairness, and nonwastefulness. Without minimum quotas, the standard deferred acceptance (DA) algorithm satisfies all of these properties, and so, even though these properties are incompatible with minimum quotas, developing "DA-like" algorithms that satisfy these properties as much as possible seems to be a sensible approach. Our first mechanism, extended-seat DA, will be strategyproof, constrained nonwasteful, and as fair as possible. Our second mechanism, multistage DA, will perform slightly worse with respect to fairness, but will be fully nonwasteful.

3.1 Extended-seat DA (ESDA)

To define the ESDA algorithm, we take the original market $(S, C, p, q, \succ_S, \succ_C)$ and define a corresponding "extended market": $(S, \tilde{C}, \tilde{q}, \tilde{\succ}_S, \tilde{\succ}_{\tilde{C}}, \succ_{ML})$. When extending the market, the set of students is unchanged. Consider a school c in the original market with a minimum and maximum quota of p_c and q_c, respectively. When extending the market, we divide c into two smaller schools: a "standard school", which, with slight abuse of notation, we label c, and that has a maximum quota of $\tilde{q}_c = p_c$, and an "extended school" c^* which has a maximum quota of $\tilde{q}_{c^*} = q_c - p_c$. Thus, the set of schools is now $\tilde{C} = C \cup C^* = \{c_1, \ldots, c_m, c_1^*, \ldots, c_m^*\}$ and the vector of maximum quotas is $\tilde{q} = \{\tilde{q}_{c'}\}_{c' \in \tilde{C}}$. Note that the extended market has no minimum quotas. By assigning no more than $e = n - \sum_{c \in C} p_c$ students to extended schools, all standard schools will be filled to capacity, thereby satisfying all minimum quotas in the original market.

For the school priorities, if c is a standard school, then $\tilde{\succ}_c = \succ_c$. Because the extended seats are linked in the sense that the total number of extended seats that can be assigned across all schools is e, they must all use the same priority relation, and so if c^* is an extended school, then $\tilde{\succ}_{c^*} = \succ_{ML}$. Thus, the master list can be thought of as a tie-breaker when two students apply to extended seats at different schools.

For the students, preferences over $C \cup C^*$ are created by taking the original preference relation $\succ_s$ and inserting school c_j^* immediately after school c_j. That is,

$$\text{preference relation } \succ_s\text{:} \quad c_j \succ_s c_k \;\cdots \text{ becomes } \tilde{\succ}_s : \quad c_j \,\tilde{\succ}_s\, c_j^* \,\tilde{\succ}_s\, c_k \,\tilde{\succ}_s\, c_k^* \;\cdots$$

The ESDA mechanism then simply applies DA to this fictitious market. The p_c standard seats at each school will be assigned according to $\succ_c$, while all of the extended seats at all schools will be assigned according to $\succ_{ML}$. By limiting the number of extended seats in each round to e, the algorithm will ensure that all of the standard seats will also be assigned and thus all minimum quotas will be satisfied. The master list must be used to decide which students to reject when more than e students apply to all of the extended seats in a given round. Formally:

Extended-seat deferred acceptance

Round 1

1. Each student applies to her most preferred school according to $\tilde{\succ}_s$.

2. Each school $c \in \tilde{C}$ tentatively accepts its most preferred set of students who applied to it up to its quota $\tilde{q}_c$, rejecting the rest.

3. Let $\mu_1(c)$ be the set of students tentatively accepted by school c, and let $\mu_1^* = \cup_{c^* \in C^*} \mu_1(c^*)$ be the set of students tentatively accepted by any extended school $c^* \in C^*$. If $|\mu_1^*| > e$, then the lowest-ranked $|\mu_1^*| - e$ students in μ_1^* (according to $\succ_{ML}$) are rejected from the schools they applied to in round 1.

Round $k > 1$

1. Each student rejected in $k-1$ applies to the most preferred school that has not yet rejected her.

2. Each school $c \in \tilde{C}$ considers all new applicants and those held from round $k-1$, and tentatively accepts its most preferred set of students up to its quota $\tilde{q}_c$, rejecting the rest.

3. Let $\mu_k(c)$ be the set of students tentatively accepted by school c, and let $\mu_k^* = \cup_{c^* \in C^*} \mu_k(c^*)$ be the set of students tentatively accepted by any extended school $c^* \in C^*$. If $|\mu_k^*| > e$, then the lowest-ranked $|\mu_k^*| - e$ students in μ_k^* (according to $\succ_{ML}$) are rejected from the schools they applied to in round k.

Let $\tilde{\mu}$ denote the resulting matching in the extended market. We then define μ, the output of ESDA in the original market, as follows: (i) If $\tilde{\mu}(s) = c_j \in C$, then $\mu(s) = c_j$ and (ii) If $\tilde{\mu}(s) = c_j^* \in C^*$, then $\mu(s) = c_j$. We now show an example of how ESDA runs.

Example 1. There are five students $s_1, \ldots, s_5$ and three schools c_1, c_2, c_3. The preferences and priorities are shown in the table below. Recall that the master list ranks $s_1 \succ_{ML} \cdots \succ_{ML} s_5$.

	$\succ_{c_1}$	$\succ_{c_2}$	$\succ_{c_3}$
	s_5	s_3	s_3
	s_3	s_4	s_4
	s_1	s_1	s_2
	s_2	s_2	s_5
	s_4	s_5	s_1
p	1	1	1
q	2	3	1

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ_{s_4}$	$\succ_{s_5}$
c_2	c_2	c_1	c_2	c_1
c_1	c_1	c_2	c_3	c_2
c_3	c_3	c_3	c_1	c_3

To run ESDA, our extended market uses schools $C \cup C^* = \{c_1, c_2, c_3, c_1^*, c_2^*, c_3^*\}$. The maximum quotas are $\tilde{q}_{c_1} = \tilde{q}_{c_2} = \tilde{q}_{c_3} = 1$, and $\tilde{q}_{c_1^*} = 1, \tilde{q}_{c_2^*} = 2$ and $\tilde{q}_{c_3^*} = 0$. Note that there are no minimum quotas in the extended market. We additionally modify all students' preferences by inserting school c_j^* after school c_j. For example, the modified preferences of student s_1 are as follows:

$$\tilde{\succ}_{s_1}: \ c_2 \ \tilde{\succ}_{s_1} \ c_2^* \ \tilde{\succ}_{s_1} \ c_1 \ \tilde{\succ}_{s_1} \ c_1^* \ \tilde{\succ}_{s_1} \ c_3 \ \tilde{\succ}_{s_1} \ c_3^*.$$

For the school priorities, we set $\tilde{\succ}_c = \succ_c$ for $c \in C$, while for $c^* \in C^*$, we set $\tilde{\succ}_{c^*} = \succ_{ML}$. This leads to the following extended market, where the changes are shown in red:

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ_{s_4}$	$\succ_{s_5}$		$\succ_{c_1}$	$\succ_{c_1^*}$	$\succ_{c_2}$	$\succ_{c_2^*}$	$\succ_{c_3}$	$\succ_{c_3^*}$
c_2	c_2	c_1	c_2	c_1		s_5	s_1	s_3	s_1	s_3	s_1
c_2^*	c_2^*	c_1^*	c_2^*	c_1^*		s_3	s_2	s_4	s_2	s_4	s_2
c_1	c_1	c_2	c_3	c_2		s_1	s_3	s_1	s_3	s_2	s_3
c_1^*	c_1^*	c_2^*	c_3^*	c_2^*		s_2	s_4	s_2	s_4	s_5	s_4
c_3	c_3	c_3	c_1	c_3		s_4	s_5	s_5	s_5	s_1	s_5
c_3^*	c_3^*	c_3^*	c_1^*	c_3^*	q	1	1	1	2	1	0

In round 1 of ESDA, students s_1, s_2 and s_4 apply to school c_2 and students s_3 and s_5 apply to school c_1. Schools c_1 and c_2 tentatively accept s_5 and s_4, respectively. Everyone else is rejected.

In round 2, students s_1 and s_2 apply to c_2^* and s_3 applies to c_1^*. Note that according to the school specific quotas $\tilde{q}_{c_1^*}$ and $\tilde{q}_{c_2^*}$, all three students can be accommodated. However, since only $e = 2$ students can be assigned to extended schools at the final matching, we reject the lowest ranked student who applied to any school in C^* according to $\succ_{ML}$, which is s_3. s_3 then applies to c_2, which rejects s_4, who applies to c_2^*. s_4 is rejected from c_2^*, and finally applies to c_3, is accepted, and the mechanism ends. At the end, the following matching is produced:

$$\begin{pmatrix} c_1 & c_1^* & c_2 & c_2^* & c_3 & c_3^* \\ s_5 & \emptyset & s_3 & \{s_1, s_2\} & s_4 & \emptyset \end{pmatrix}$$

Mapping this back to a matching in the original market:

$$\begin{pmatrix} c_1 & c_2 & c_3 \\ s_5 & \{s_1, s_2, s_3\} & s_4 \end{pmatrix}$$

We now discuss how ESDA performs on each of our metrics. Recall that $p = (p_{c_1}, \ldots, p_{c_m})$. We have the following theorem.

Theorem 3. *The ESDA mechanism is*

(i) (weakly group) strategyproof

(ii) constrained nonwasteful and

(iii) p−fair.

Strategyproofness is readily inherited from the standard DA mechanism. In fact, to prove (group) strategyproofness, we associate the extended market with the matching with contracts model of Hatfield and Milgrom (2005), considering the set of extended schools C^* as one single "umbrella" school which has a capacity of e. A contract then specifies a student and the school to which she is assigned, and the contracts are ranked according to the master list. Hatfield and Milgrom (2005) show that if the choice function of all schools satisfy a substitutes condition, then the DA algorithm in the matching with contracts model is strategyproof, while Hatfield and Kojima (2009) show that it is in addition weakly group strategyproof. The standard schools obviously have substitutable choice functions. In the proof, we show that the "umbrella" extended school also has a substitutable choice function, which implies that ESDA is weakly group strategyproof.

Example 1 shows that ESDA may be wasteful, because we can move student s_3 to an open seat at school c_1 without violating the minimum quotas at the school he is assigned to, c_2. However, if student s_3 is moved, this opens a seat at c_2, which would then be claimed by student s_4. Then, moving s_4 to this open seat would violate the minimum quota at c_3. Thus, ESDA may still be constrained nonwasteful, and this is in fact the case.

What remains is a discussion of fairness. In ESDA, standard schools use the school specific priorities and extended schools use the master list. If s is rejected from a standard school c, every student at that c must have a higher priority at that school than s. If at the completion of the algorithm (and back in the original market) c has exactly p_c students, then no students were assigned to c^*, and hence the master list was never used to allocate students to c. If there are more than p_c students assigned to c, then s has a lower priority on $\succ_{ML}$ than any student who received c^* and a lower priority on $\succ_c$ than any student who received c. Because seats are assigned according to $\succ_{ML}$ at c only if $|\mu(c)| > p_c$, ESDA is p-fair.

Recall from the previous section that as σ is raised, the definition of σ-fairness becomes stronger and moves closer to the full fairness of Definition 3, which is achieved when $\sigma = q$. Since ESDA is only p-fair, the question remains whether we can find another mechanism that is more fair. The following theorem shows that p-fairness is the best we can achieve, if we insist on strategyproofness and even only constrained nonwastefulness. Note that the result is actually quite strong: even though the "more fair" relation is not complete, ESDA does in fact fairness-dominate every other strategyproof and constrained nonwasteful mechanism.

Theorem 4. *Fix quota vectors p and q. If a strategyproof and constrained nonwasteful mechanism is σ-fair, then $\sigma_c \leq p_c$ for all $c \in C$.*

The proof is given in Appendix C. The following corollary is immediate.

Corollary 1. *ESDA is more fair than any other strategyproof and constrained nonwasteful mechanism.*

3.2 Multi-stage DA (MSDA)

Our second modification, the multistage deferred acceptance (MSDA) algorithm, satisfies only a weaker definition of fairness, but will also be fully nonwasteful. As the name suggests, MSDA is run in several stages. At the beginning of any stage we temporarily reserve a group of students from the market and run standard DA on the remaining submarket. The number of students participating in the submarket is never too many to jeopardize the feasibility of the overall match; no matter how they are allocated, the sum of the minimum quotas remaining after the given stage will never exceed the number of students that remain unmatched after that stage. The assignments from the given stage are made final, and we reduce the minimum and maximum quotas accordingly. Now, we are left with a subproblem of unmatched students and updated minimum and maximum quotas. We repeat the process until all students are assigned. The master list is used to determine what students will be reserved.

To formally define MSDA, we return to the original market $(S, C, p, q, \succ_S, \succ_C)$. Recall that without loss of generality, the master list ranks students $s_1 \succ_{ML} s_2 \succ_{ML} \cdots \succ_{ML} s_n$. Start by setting $R^0 = S$, $p_c^1 = p_c$, and $q_c^1 = q_c$ for all $c \in C$. Let $r^1 = \sum_{c \in C} p_c^1$ be the number of students that will be reserved in R^1. Let μ denote the matching that will be produced at the end of MSDA. Initially, set $\mu(i) = \emptyset$ for all $i \in S \cup C$.

Multi-stage deferred acceptance

Stage $k \geq 1$

1. Set $R^k = \{s_{n-r^k+1}, s_{n-r^k+2}, \dots, s_n\}$, i.e., R^k is the is the set of r^k students with the lowest priority according to $\succ_{ML}$.
   (a) If $R^{k-1} \setminus R^k \neq \emptyset$, run the standard DA mechanism on the students in $R^{k-1} \setminus R^k$ with maximum quotas for the schools equal to $(q_c^k)_{c \in C}$.
   (b) If $R^{k-1} \setminus R^k = \emptyset$, run the standard DA mechanism on the students in R^k with maximum quotas for the schools equal to $(p_c^k)_{c \in C}$.
2. Let μ^k be the matching from step 2, and set $\mu = \mu \cup \mu^k$.
3. Define new quotas for each school:
   (a) $q_c^{k+1} = q_c^k - |\mu^k(c)|$,
   (b) $p_c^{k+1} = \max\{0, p_c^k - |\mu^k(c)|\}$.
   (c) $r^{k+1} = \sum_{c \in C} p_c^{k+1}$.

The two substages of step 1 (1(a) and 1(b)) warrant some explanation. Most rounds will use step 1(a), except for possibly the last round of the algorithm. Step 1(b) is inserted to ensure that the algorithm technically finishes. In 1(a), if, once we remove enough students to satisfy the minimum quotas, there are still students left ($R^{k-1} \setminus R^k \neq \emptyset$), we run the standard DA mechanism on those students who were not removed. However, if it happens at any stage that the number of students remaining is *exactly* equal to the minimum quotas, then "removing" these students would leave us with an empty set and the algorithm would run indefinitely. This explains step 1(b). Also note that the first time step 1(b) is executed will be the last round of the algorithm.

In the above description, we determine the number of students who must be removed in each round as the sum of the minimum quotas. We may, however, be able to remove even fewer students and still ensure feasibility. For example, assume there are 15 students and 10 schools, with all minimum quotas equal to 1 and all maximum quotas equal to 2. Then, we actually need only to remove 4 students, not 10, because no matter how the first 11 students are allocated, the minimum quotas of at least 6 schools must be satisfied, and the remaining 4 students can fill the minimum quotas of the other 4 schools. We have thus developed a dynamic programming based method to precisely calculate the smallest possible value for r^k, which will make MSDA close to standard DA as possible. This procedure is described in Appendix H.

We now provide an example of how MSDA works.

Example 2. We use the same instance of Example 1 to illustrate how MSDA works. Since the sum of the minimum quotas is $\sum_{c\in C} p_c = 3$, we temporarily remove students s_3, s_4, and s_5 according to ML. We then run the standard DA mechanism with no minimum quotas on students s_1 and s_2. At the end of this first stage, the assignments are as follows:

$$\begin{pmatrix} c_1 & c_2 & c_3 \\ \emptyset & \{s_1, s_2\} & \emptyset \end{pmatrix}$$

Thus, there are three students remaining, and neither c_1 or c_3 have reached their minimum quotas. We temporarily remove s_4 and s_5, and run DA on s_3 alone. At the end of this stage, the assignments are as follows:

$$\begin{pmatrix} c_1 & c_2 & c_3 \\ s_3 & \{s_1, s_2\} & \emptyset \end{pmatrix}$$

Now, only c_3 has not reached its minimum quota, so we reserve s_5 and run DA on s_4 alone, who chooses to attend c_2. Finally, the only student remaining is s_5, and minimum quota of 1 at c_3 still needs to be filled, so s_5 is assigned c_3. The final outcome is then

$$\begin{pmatrix} c_1 & c_2 & c_3 \\ s_3 & \{s_1, s_2, s_4\} & s_5 \end{pmatrix}$$

Recall the notation that $\sigma_0 = (0, \ldots, 0)$.

Theorem 5. *The MSDA mechanism is:*

(i) (weakly group) strategyproof

(ii) nonwasteful and

(iii) σ_0-fair

Strategyproofness follows because within each stage the standard DA algorithm is fair, and, fixing the preferences of the other students, no student s can affect the stage at which she participates in the algorithm. Nonwastefulness holds because the only time a student would be unable to get into a school with empty seats is the last round of the algorithm, in which case she is forced to go to a school that will be filled exactly to its minimum quota and thus cannot be feasibly moved.

For fairness, consider a student s who prefers c to his assignment. If c was full by the end of a stage before the one in which s participated, then all students at c are higher than s on the master list. Suppose c still has at least one seat available by the stage that s participates, and label this stage k. Since s does not obtain c, c must have rejected s in k. This means that every student matched to c in k must have a higher priority at c than s. Therefore, every student at c must either have a higher priority than s according to either $\succ_c$ or $\succ_{ML}$, and so no student can form a σ_0-blocking pair.

σ_0-fairness is strictly weaker than $p-$fairness, which is satisfied by ESDA. While σ_0-fairness is a weak concept in that it very stringently limits the number of blocking pairs students are allowed to form, it is the price of nonwastefulness. We can once again ask whether we can do better than σ_0-fairnes while still retaining strategyproofness and nonwastefulness. Analogous to the case for ESDA, the result below shows that such a mechanism does not exist. It is interesting to note that simply imposing nonwastefulness alone is enough to limit a mechanism to being at most σ_0-fair; strategyproofness is not used in the result below.

Theorem 6. *Fix quota vectors p and q. If χ is nonwasteful and $\sigma-$fair, then $\sigma = \sigma_0$.*

This again leads to the following corollary:

Corollary 2. *MSDA is more fair than any other nonwasteful mechanism.*

3.3 Relationship to the standard DA and serial dictatorship algorithms

Both the ESDA algorithm and MSDA algorithm are closely related to the standard DA algorithm, and in fact approach the standard DA algorithm as the minimum quotas are varied. For example, for ESDA, as the minimum quotas are increased, more and more seats are assigned according to the idiosyncratic school priority lists and less to the master lists. When $p_c = q_c$ for all $c \in C$, all seats are assigned according to the school priority lists, and ESDA=DA. On the other hand, when the minimum quotas are decreased, more and more seats are assigned according to $\succ_{ML}$. When $p_c = 0$ for all $c \in C$, ESDA assigns all seats according to $\succ_{ML}$, and becomes the simple serial dictatorship (SD) mechanism. When p is close to 0, ESDA is close to the serial dictatorship with minimum quotas (SDMQ), which proceeds just as the standard serial dictatorship, allowing students to choose in some predetermined order ($\succ_{ML}$) their most preferred school with seats remaining, as long as after the student is assigned, there are enough students left to fill all remaining minimum quota seats. If this is not the case, then the student is restricted to choose only from those schools which have not yet reached their minimum quota. (When $p_c = 0$ for all $c \in C$, SD, SDMQ, and ESDA are all equivalent.)

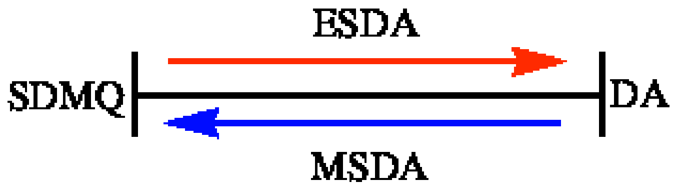


Figure 2: ESDA and MSDA lie on a spectrum from SDMQ to DA. The arrows point in the direction the corresponding mechanism moves as the minimum quotas are increased.

The MSDA algorithm is the counterpart to ESDA: when the minimum quotas are high, many students are reserved in stage 1, and the mechanism approaches SDMQ. On the other hand, when minimum quotas are low, almost all students participate in round 1 of the algorithm, and MSDA approaches standard DA, and the two in fact coincide when $p_c = 0$ for all $c \in C$, since no students are reserved. Thus, ESDA and MSDA try to remain as close to the standard DA algorithm as possible, while making slight adjustments that ensure all minimum quotas are satisfied. It may be helpful to think of the algorithms as part of a spectrum that has SDMQ at one end and standard DA at the other, as in Figure 2. The arrows point in the direction the corresponding mechanism moves as the minimum quotas increase.

4 Comparing with Artificial Caps

The new mechanisms we introduced in the previous section show the boundaries of fairness a mechanism can achieve if we insist on at least some degree of nonwastefulness. As mentioned in section 2.4, if we give up nonwastefulness entirely, we can simply impose artificial maximum quotas (caps) on the schools and run standard DA, which will be strategyproof and fair. This may seem like a natural and easy solution to the problem of minimum quotas, and is in fact used in many matching markets, including the Japanese Medical Resident Program and possibly military cadet matching. It may also be in use in other markets but be unobserved, because it would look like these markets only have maximum quotas. While simple, this ad-hoc way of imposing minimum quotas intuitively seems lacking, as it is very rigid and allows for no flexibility in assigning the "extra" seats depending on student preferences. In this section we show that our more flexible mechanisms which do allow for such flexibility improve upon imposing artificial caps.

The model is essentially the same as before. Once again, p_c denotes the true minimum quota at school c and q_c denotes the true maximum quota at c. However, for each school c, the district may impose an artificial maximum quota q^*_c. By choosing the vector $q^* = (q^*_{c_1}, \ldots, q^*_{c_m})$ wisely, the school district can ensure that no matter how the students are allocated, as long the artificial caps q^* are satisfied, the true quota vectors p and q will be satisfied as well.

Definition 12. *The vector q^** ***ensures a feasible matching*** *if $|\mu(c)| \leq q^*_c \quad \forall c \in C \implies p_c \leq |\mu(c)| \leq q_c \quad \forall c \in C$.*

In words, this definition simply says that q^* ensures a feasible matching if, whenever matching μ satisfies the artificial caps q^*, it also satisfies the true minimum and maximum quotas p and q.

We next identify a necessary and sufficient condition for q^* to ensure the existence of a feasible matching.

Definition 13. *The vector q^* is* ***consistent with quotas*** *p* ***and*** *q if the following hold for all $c \in C$: (i) $p_c \leq q^*_c \leq q_c$ and (ii) $p_c \leq n - \sum_{d \in C \setminus \{c\}} q^*_d$.*

In words, consistency requires that for every school c: (i) the artificial cap is between the true maximum and minimum quotas (an obvious requirement) and (ii) if all other schools d are filled to the artificial cap q^*_d, then there are still enough students remaining to satisfy the minimum quota at c. The lemma below shows that consistency is both a necessary and sufficient condition for q^* to always ensure a feasible matching according to the true quotas p and q.

Lemma 1. *q^* ensures a feasible matching if and only if it is is consistent with quotas p and q.*

Proof. For the "if" direction, assume $p_c \leq q^*_c \leq q_c$ and $p_c \leq n - \sum_{d \in C \setminus \{c\}} q^*_d$ for all $c \in C$, and that $|\mu(c)| \leq q^*_c$ for all c. $|\mu(c)| \leq q_c$ is obvious. Assume that $|\mu(d)| < p_d$ for some d. We know that $\sum_{c \in C} |\mu(c)| = n$, so that $\sum_{c \in C \setminus \{d\}} |\mu(c)| = n - |\mu(d)| > n - p_d$. The first and last expressions imply that $p_d > n - \sum_{c \in C \setminus \{d\}} |\mu(c)| \geq n - \sum_{c \in C \setminus \{d\}} q^*_c$, which is a contradiction.

For the "only if" direction, we first note that $q^*_c \leq q_c$ is immediate from the fact that q^* ensures a feasible matching. So, assume that $|\mu(c)| \leq q^*_c \forall c \in C \implies p_c \leq |\mu(c)| \leq q_c \forall c \in C$. Next, assume that $p_d + \sum_{c \in C \setminus \{d\}} q^*_c > n$ for some $d \in C$ and that $|\mu(c)| = q^*_c \forall c \in C \setminus \{d\}$. This implies $p_d + \sum_{c \in C \setminus \{d\}} |\mu(c)| > n$. Now, this together with $|\mu(d)| \geq p_d$ implies that $\sum_{c \in C} |\mu(c)| > n$, which is a contradiction. Lastly, assume that $p_d > q^*_d$ for some $d \in C$. In this case, it is obvious that $|\mu(c)| \leq q^*_c$ for all $c \in C$ implies $|\mu(d)| \leq q^*_d < p_d$, which is a contradiction. □

We then have the following simple corollary.

Corollary 3. *Let χ^{q^*} be a mechanism such that for all $\succ_S \in \mathcal{P}^{|S|}$, $\chi^{q^*}(\succ_S)$ respects maximum quotas q^* (but doe not necessarily respect any minimum quotas). If q^* is consistent with quotas p and q, then $\chi^{q^*}(\succ_S)$ is a feasible matching for all $\succ_S \in \mathcal{P}^{|S|}$.*

Corollary 3 implies that to ensure a feasible matching, a school district can use, for example, standard DA or TTC with maximum quotas for the schools equal to q^*. Doing so will ensure that all of the true minimum quotas will be satisfied, and so the resulting matching will be feasible. Note that there may be many choices of q^* that are consistent with p and q, and at least one such q^* will always exist.[20]

While imposing artificial caps q^* and using standard DA is feasible and will be fair, it is less flexible than our mechanisms, since it may leave some schools below their true maximum quotas, even though students would prefer to be moved there. This leads us to ask which mechanisms will actually be preferred by the students. The strongest possible result we could hope to achieve is that (either one of) our mechanisms run with the true quotas p and q Pareto dominates standard DA with artificial caps q^*. Unfortunately, this is not the case even if all of the schools use the same priority relation (equal, without loss of generality, to $\succ_{ML}$). When this is true, our mechanisms reduce to the SDMQ mechanism introduced in Section 3.

For any maximum quotas q^*, we can also define the q^*-*serial dictatorship (q^*-SD)*, which just runs the standard serial dictatorship with maximum quotas q^* and no minimum quotas. Note that q^*-SD will only produce a feasible matching if q^* is consistent with p and q. When all schools use the same priority relation, the standard DA algorithm (with no minimum quotas) reduces to the q^*-serial dictatorship.

We now show that with minimum quotas, neither q^*-SD nor SDMQ Pareto dominates the other. Let there be four schools c_1, c_2, c_3 and c_4 with minimum and maximum quota vectors $p = (1, 1, 0, 0)$ and $q = (1, 1, 1, 1)$. Let there be three students s_1, s_2 and s_3. The preferences are as follows:

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$
c_3	c_4	c_2
c_1	c_2	c_1
$\vdots$	$\vdots$	$\vdots$

Imposing caps of $q^* = (1, 1, 0, 1)$ will ensure that, no matter how students are assigned, the true minimum quotas $p = (1, 1, 0, 0)$ will be satisfied. Running the standard serial dictatorship with maximum quotas q^*, we find that the assignment is

$$\begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ s_1 & s_3 & \emptyset & s_2 \end{pmatrix}$$

[20]For example, take any feasible matching μ and set $q^*_c = |\mu(c)|$ for all $c \in C$.

Next, let us run the SDMQ mechanism with the true quota vectors p and q. The outcome in this case is

$$\begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ s_3 & s_2 & s_1 & \emptyset \end{pmatrix}$$

This example shows that neither mechanism Pareto-dominates the other since s_1 strictly prefers the outcome of the SDMQ mechanism, whereas s_2 and s_3 strictly prefer using the serial dictatorship with artificial caps $q^* = (1, 1, 0, 1)$.[21]

Simulations

As the previous section showed, analytical results regarding student welfare are difficult to obtain, and so we turn to computer simulations. To run our computer simulations, we consider a market of $n = 400$ students and $m = 50$ schools. The maximum quotas will be equal to 15 at each school, while the minimum quotas will be the same across schools, and will be varied from 1 to 7.[22]

In order to study how correlation in student preferences affects outcomes, the student preferences are constructed using a linear combination of a common value vector of cardinal utilities and a private vector for each student. More specifically, we draw one common vector of cardinal utilities from the set $[0, 1]^m$ uniformly at random. Label this vector u_c. Then, for each student s, we randomly draw a private vector of cardinal utilities from the same set, again uniformly at random. Label this vector u_s. Then, we construct cardinal utilities for all m schools for student s as $\alpha u_c + (1 - \alpha)u_s$, for some $\alpha \in [0, 1]$. We then convert these cardinal utilities into an ordinal preference relation for each student. The higher the value of α, the more correlated the student preferences. School priorities $\succ_c$ are drawn uniformly at random, and the master list is without loss of generality set to $s_1 \succ_{ML} \cdots \succ_{ML} s_n$.

Our first experiment studies how wasteful the ACDA mechanism is compared to ESDA and MSDA. Figure 3 plots the number of wasted seats as a function of the minimum quotas. For ACDA, we choose artificial caps of $q^* = 8$.[23] The figures show that both ESDA and MSDA are significantly less wasteful than ACDA, for both low and high correlation and for all values of the minimum quota. Of course, MSDA is nonwasteful, and so always produces 0 wasted seats. However, even though ESDA is only constrained nonwasteful, it still performs significantly better

[21] Featherstone (2011) suggests rank efficiency as an alternative welfare criterion, where mechanism χ rank-dominates mechanism ψ if more students get their k^{th} or higher choice under χ than under ψ for all k and all preferences. While in the example above SDMQ rank dominates SD, this will not be true in general. However, our simulation results below do show that on average, our mechanisms do rank-dominate imposing artificial caps.

[22] Minimum quotas of 7 at each school is the largest possible value that still allows for some flexibility. If $p_c = 8$ at all c, then $\sum_{c \in C} p_c = 400 = n$, and standard DA can be used.

[23] This is the largest (symmetric) value for the artificial caps that is consistent with quotas p and q.

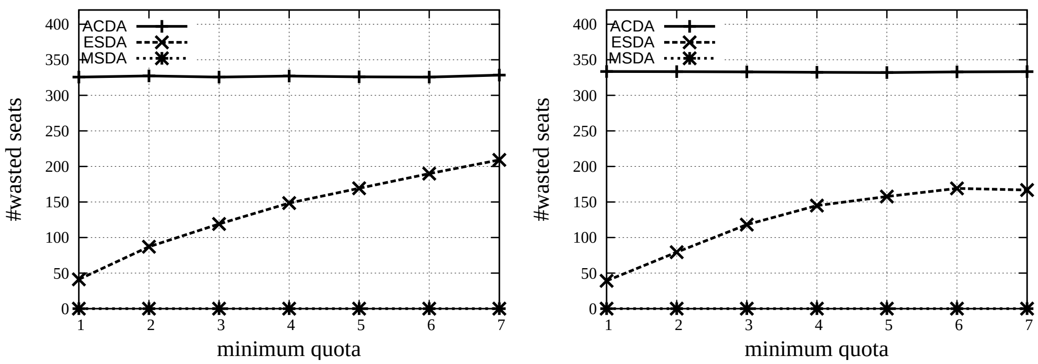


Figure 3: Nonwastefulness for $\alpha = 0.3$ (left) and $\alpha = 0.6$ (right).

than ACDA on this metric.

Perhaps an even better metric by which to compare these mechanisms is to look directly at which mechanism the students prefer, in terms of the ranking of the school they receive. We measure student welfare by plotting cumulative distribution functions of the (average) number of students who received their k^{th} or higher ranked school under each mechanism. For example, in Figure 4, under ACDA with $\alpha = 0.3$, about 10% of students get their first choice, 20% of students get their first or second choice, 30% get their first or second or third choice, and so on. Under ESDA or MSDA, significantly more students get their first choice (about 40% and 50%, respectively, in the left-hand panel), and in fact, the rank distributions for ESDA and MSDA both first-order stochastically dominate that for ACDA.[24] Intuitively, this is happening because ESDA and MSDA are allocating the extra seats more flexibly, taking into account student demand, and so are able to provide students with higher choices, while ACDA simply eliminates these seats ex-ante, forcing these students to be assigned to lower ranked schools. In summary, the students will, on average, unambiguously prefer our more flexible mechanisms to imposing artificial caps.

As a final experiment, we compute the number of standard blocking pairs produced by our mechanisms (in the original sense of Definition 2). Even when it is not possible to eliminate all such blocking pairs (if, for example, we desire even a weak notion of nonwastefulness), the number of blocking pairs is often considered a useful criterion by which to rank mechanisms, because less blocking pairs means there will be less incentive for students to file complaints with the school district and/or circumvent the match. Many papers have, given a set of submitted preferences, simply looked for the allocation that minimizes the number of blocking pairs (e.g., Biró et al. (2010) and Hamada et al. (2011)), while Abdulkadiroğlu et al. (2009) follow a similar approach

[24]For presentation purposes, the figures only show up to $k = 15$, but ESDA and MSDA do in fact first-order stochastically dominate ACDA.

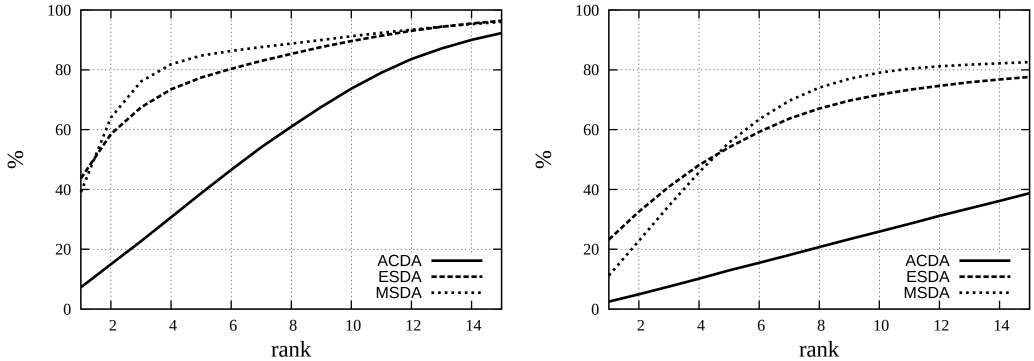


Figure 4: CDFs of student welfare for $\alpha = 0.3$ (left) and $\alpha = 0.6$ (right). The minimum quotas are set to 3 at all schools.

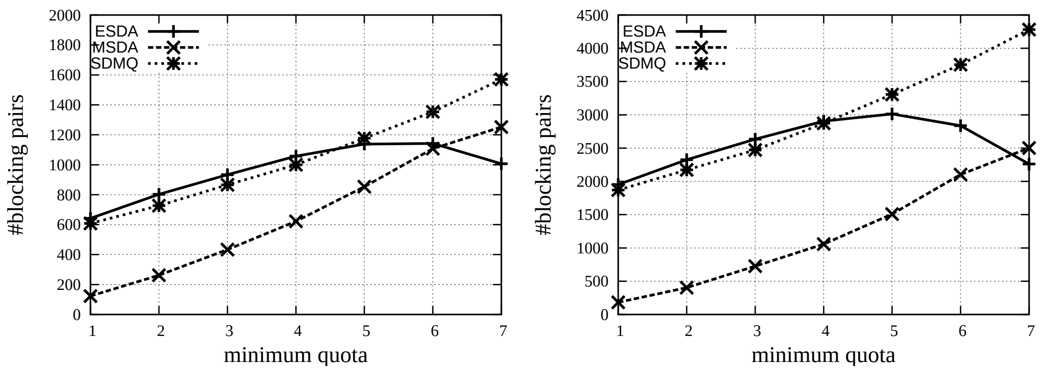


Figure 5: Counting the number of blocking pairs under each mechanism for $\alpha = 0.3$ (left) and $\alpha = 0.6$ (right).

and use the number of blocking pairs produced as a way to rank mechanisms in the context of the New York City High School Match.

The results are shown in Figure 5 for ESDA, MSDA, and, for comparison, SDMQ. (Note that since ACDA is fair, it will always produce 0 blocking pairs, but as discussed above, will be much worse for the students from a welfare perspective.) From the figure, we see that MSDA performs well when the minimum quotas are low, while ESDA performs well when the minimum quotas are high. Both tend to perform much better than SDMQ. The intuition for these results can be summarized by referring back to Figure 2: when the minimum quotas are low, MSDA$\approx$DA, and so tends to produce few blocking pairs, while ESDA$\approx$SDMQ, and so tends to produce many blocking pairs. When the minimum quotas are high, these results are reversed. Thus, one policy recommendation is if blocking pairs are a large concern and the minimum quotas are low in a market, MSDA should be used, while if the minimum quotas are high, ESDA should be chosen.

The results of our simulations can be summarized as follows. While our mechanisms do not eliminate all standard blocking pairs, ESDA produces relatively few blocking pairs when the minimum quotas are high, while MSDA does the same when the minimum quotas are low. As a tradeoff for allowing some standard blocking pairs, we are rewarded with a much smaller number of wasted seats than produced by ACDA. Perhaps most importantly, MSDA and ESDA perform significantly better on welfare grounds, as demonstrated by the CDFs. Thus, by taking into consideration the size of the minimum quotas, we can choose a mechanism that is close to standard DA, wastes few seats, and makes the students (relatively) well off.

5 TTC-based mechanisms

Since Gale's top trading cycle (TTC) algorithm is also a very commonly used mechanism in various allocation problems, and in school choice settings in particular, we now briefly note that our extended-seat and multistage modifications can be easily applied to TTC to ensure that all minimum quotas are satisfied. TTC-based mechanisms are useful when priorities are given a milder interpretation and the designer is more concerned with Pareto efficiency than fairness. In the standard model (without minimum quotas), TTC satisfies strong group strategyproofness and Pareto efficiency. It is interesting to note that, unlike for the DA-based mechanisms, modifying TTC for minimum quotas does not require us to weaken either of these properties: both ESTTC and MSTTC will still be strongly group strategyproof and Pareto efficient.

5.1 Extended-seat TTC (ESTTC)

To define the ESTTC mechanism, we again consider the extended market $(S, \tilde{C}, \tilde{q}, \tilde{\succ}_S, \tilde{\succ}_C)$ from section 3. We endow all of the standard seats at school c to the student with the highest priority according to $\succ_c$, and all of the extended seats **at all schools** to the person with the highest priority according to $\succ_{ML}$. We then run the standard TTC algorithm on this market, with the slight modification that once e extended seats have been assigned, the remaining extended seats are taken off the market, without being assigned to anyone.

Formally, ESTTC is defined as follows.

Round 1

1. Endow all of the seats at each school $c \in C \cup C^*$ to the student with the highest priority according to $\tilde{\succ}_c$. Each student points to the student who is endowed with the seats to her most preferred school according to $\tilde{\succ}_s$.

2. There is at least one cycle. Assign all students involved in each cycle to the their most preferred school and remove them. Reduce the quota of each assigned school by 1.

3. If the sum of the extended school seats assigned is weakly greater than e, then take all remaining extended school seats off the market (without assigning them to anyone).

Round $k > 1$

1. All remaining seats at school $c \in C \cup C^*$ are endowed to the highest priority student according to $\tilde{\succ}_c$. Each remaining student points to the student who is endowed with the seats to her most preferred remaining school according to $\tilde{\succ}_s$.

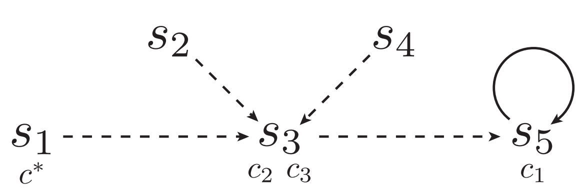
2. There is at least one cycle. Assign all students involved in each cycle to the corresponding schools and remove them. Reduce the quotas of each assigned school by 1.

3. If the sum of extended school seats assigned in rounds 1 through k is weakly greater than e, then take any remaining extended school seats off the market (without assigning them to anyone).

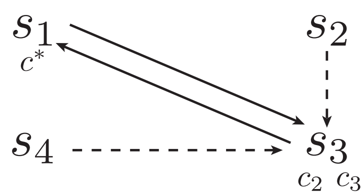
Let $\tilde{\mu}$ denote the resulting matching in the extended market. We then define μ, the output of ESTTC in the original market, as follows: (i) If $\tilde{\mu}(s) = c_j \in C$, then $\mu(s) = c_j$ and (ii) If $\tilde{\mu}(s) = c_j^* \in C^*$, then $\mu(s) = c_j$.

Note that since all $c^* \in C^*$ use the same preference ordering, all seats at all extended schools are assigned to the same student, and so at most one extended seat can be assigned in each round. This is necessary to ensure that ESTTC always produces a feasible matching and assigns exactly e students to extended schools.

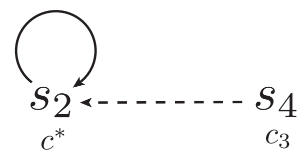
Example 3. We consider the same instance as Example 1 to illustrate the ESTTC mechanism. In round 1 of ESTTC, student s_1 holds all of the extended seats in his endowment, which we denote c^* in the figure below. In round 1, there is only 1 cycle (a self-cycle with s_5), and s_5 is assigned the lone standard seat at c_1:



In round 2 of the ESTTC mechanism, there are no more standard seats at c_1, so s_3 points to c_1^*, which is owned by s_1. Student s_1 points to c_2, which is owned by s_3, and a cycle is formed between s_1 and s_3:



In round 3, since there are no more standard seats at c_2, s_2 points to c_2^* and forms a cycle with herself:



Since two students have been assigned to extended seats, the remaining extended seat at c_3^* is taken off of the market. Thus, in the final round, s_4 is endowed only with a standard seat at c_3. She points to herself and is assigned this seat. This gives the following matching in the extended market

$$\tilde{\mu} = \begin{pmatrix} c_1 & c_1^* & c_2 & c_2^* & c_3 & c_3^* \\ s_5 & s_3 & s_1 & s_2 & s_4 & \emptyset \end{pmatrix}.$$

Mapping this back to the original market, the final output of ESTTC is

$$\mu = \begin{pmatrix} c_1 & c_2 & c_3 \\ \{s_3, s_5\} & \{s_1, s_2\} & s_4 \end{pmatrix}.$$

Just as for standard TTC, ESTTC will be strongly group strategyproof and Pareto efficient. Since Pareto efficiency implies nonwastefulness, ESTTC is also nonwasteful. Of course, ESTTC will not satisfy any of our definitions of fairness.

Theorem 7. *The ESTTC mechanism is:*
(i) strongly group strategyproof and
(ii) Pareto efficient.

The proof of group strategyproofness follows a similar argument to Pápai (2000). We should additionally note that, since TTC-based mechanisms dispense with fairness entirely, the master list need not have any inherent meaning in ranking the students. We simply need such a list to run the mechanism, because we must have an endowment ordering for the extended seats. The master list could then be decided purely randomly to make the mechanism as fair as possible from an ex-ante perspective.

5.2 Multi-stage TTC (MSTTC)

The MSTTC mechanism is defined in exactly the same way as the MSDA mechanism, only replacing "DA" with "TTC" everywhere in the definition of MSDA.[25] Just as for ESTTC, MSTTC will be both strongly group strategyproof and Pareto efficient.

Theorem 8. *The MSTTC mechanism is:*
(i) strongly group strategyproof and
(ii) Pareto efficient.

6 Conclusions

This paper has introduced minimum quotas into a standard school choice model and filled a gap in the literature by providing strategyproof mechanisms that also perform well on other metrics (efficiency, fairness, and nonwastefulness) and are greatly preferred by the students to the commonly used solution of imposing artificial caps.

The matching problem with minimum quotas is complex. No one mechanism will be able to satisfy all desirable properties, and so we conclude by providing some guidance to policymakers on which mechanism to choose, depending on the details of the setting and the goals they have in mind. If fairness is not at all a concern, then either ESTTC or MSTTC are the proper choice, as they will be strongly group strategyproof and Pareto efficient. However, in many settings, fairness is important, and in these situations the choice of mechanism becomes more complicated, as both fairness and nonwastefulness cannot be achieved simultaneously. Our ESDA and MSDA mechanisms were designed to be as fair as possible, while still providing some degree of nonwastefulness. As a simple heuristic, when the minimum quotas tend to be low, the MSDA mechanism should be chosen, as it will be close to the standard DA mechanism; when the minimum quotas tend to be high, on the other hand, the ESDA mechanism approaches the standard DA mechanism, and thus will tend to be more fair.

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[25]In fact, it should be noted that our multi-stage modification can be applied to any mechanism χ which takes only maximum quotas into account in the same manner. Simply replacing "DA" with "χ" everywhere in the definition of MSDA will give the MS-χ mechanism, which will satisfy all minimum quotas.

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A Proof of Theorem 2

The proof is by example. Let there be 3 students s_1, s_2, s_3 and three schools c_1, c_2, c_3. The priorities and quotas at the schools are as follows:

	$\succ_{c_1}$	$\succ_{c_2}$	$\succ_{c_3}$
	s_1	s_1	s_1
	s_3	s_3	s_2
	s_2	s_2	s_3
p_c	1	1	0
q_c	1	2	1

Consider the following preferences:

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$
$\boxed{c_2}$	c_3	c_3
c_1	c_2	$\boxed{c_2}$
c_3	$\boxed{c_1}$	c_1

Since s_1 is ranked highest by each school, fairness requires that he be assigned a seat at his most preferred school c_2. (If he is not, then at least one student is assigned there since $p_{c_2} = 1$, and so (s_1, c_1) would be a blocking pair.) Then, by feasibility, exactly one s_2 or s_3 must be assigned to the remaining minimum quota seat at c_1. First consider the case in which s_2 is assigned to c_1. Then, s_3 cannot be assigned to c_3, because s_2 would then form a blocking pair with c_3 (since she has higher priority than s_3 at c_3). Thus, the assignment must be that shown in the boxes above. Now, consider the following monotonic transformation of these preferences:

$\succ_{s_1}$	$\succ'_{s_2}$	$\succ_{s_3}$
$\boxed{c_2}$	$\boxed{c_1}$	c_3
c_1	c_3	$\boxed{c_2}$
c_3	c_2	c_1

Since this is a monotonic transformation of the original preferences, the assignment cannot change, and must remain that shown by the boxes. However, under these preferences, the boxed assignment is not constrained nonwasteful, which is a contradiction. The case in which s_3 is assigned to c_1 is proved similarly.

B Proof of Theorem 3

Proof of constrained nonwastefulness

To show constrained nonwastefulness, we first generalize Definition 4 to claiming an empty seat in round k of the ESDA algorithm. Recall that μ_k denotes the (tentative) matching at the end of round k, and μ_k^* denotes the set of students assigned to extended seats at the end of round k.

Definition 14. *Say that student s **claims an empty seat at school** c **in round** k if s has been rejected from c by the end of round k and $|\mu_k(c)| < \tilde{q}_c$.*

Additionally, let r be the earliest round for which $|\mu_r^*| = e$, and define $\underline{s}_k = \min_{s \in \mu_k^*} s$ and $\bar{s}_k = \max\{s \in S : s$ claims an empty seat at some school $c$in round $k\}$, where the min and max are both taken with respect to $\succ_{ML}$. That is, $\underline{s}_k$ is the *worst* student (accroding to ML) who is assigned an extended seat in round k, and $\bar{s}_k$ is the *best* student who claims an empty seat at the end of round k.

We note some useful facts which are trivial properties of the ESDA mechanism.

Fact 1. $|\mu_{r'}^*| = e$ for all $r' \geq r$.

Fact 2. If $r'' \geq r' \geq r$, then, $\underline{s}_{r''} \succeq_{ML} \underline{s}_{r'}$.

Fact 3. If $\underline{s}_{r'} \succ_{ML} s$ for some $s \in S$ and some $r' \geq r$, then $\mu_{r''}(s) \in C$ for all $r'' \geq r'$, and, in particular, $\tilde{\mu}(s) \in C$ (i.e., s is never again assigned an extended seat).

To prove part (i), we first show the following lemma.

Lemma 2. *(a) If there exists some student s who claims an empty seat at some school c in round $k \geq r$, then, for all $k' \geq k$, there exists a pair $(s', c') \in S \times \tilde{C}$ such that s' claims an empty seat at school c' in round k'.*

(b) If $k \geq r$, then $\underline{s}_k \succ_{ML} \bar{s}_k$.

(c) $\bar{s}_{k+1} \succeq_{ML} \bar{s}_k$ for all $k \geq r$.

Proof. Part (a): It is clear that no student ever claims an empty seat at any standard school $c \in C$, so we only consider $c^* \in C^*$. The proof is by induction. Consider the first statement. Let s claim an empty seat at c^* in round k, and consider $k' = k + 1$. If $|\mu_{k'}(c^*)| < \tilde{q}_{c^*}$, then the statement is obviously true. Thus, consider the case where $|\mu_{k'}(c^*)| = \tilde{q}_{c^*}$. Some student must have applied (and been accepted) at school c^*, so that $|\mu_{k'}(c^*)| > |\mu_k(c^*)|$. Since $k \geq r$, we have $\sum_{j=1}^{m} |\mu_k(c_j^*)| = e = \sum_{j=1}^{m} |\mu_{k'}(c_j^*)|$. This, together with $|\mu_{k'}(c^*)| > |\mu_k(c^*)|$ implies that there exists some school c'^* such that $\tilde{q}_{c'^*} \geq |\mu_k(c'^*)| > |\mu_{k'}(c'^*)|$, i.e., some student s' was rejected from c'^*, which now has an empty seat. So, s' claims an empty seat at c'^* in round k', which proves the first statement.

Part (b): Let c^* be the school at which $\bar{s}_k$ claims an empty seat in round k. There are two cases. First, assume that $\bar{s}_k$ was rejected from c^* in a round $r' \leq k$ in which c^* was below its capacity $\tilde{q}_{c^*}$. In this case, it must be that $\bar{s}_k$ was rejected because e other students ranked higher on the master list were accepted at other schools in C^*, which means $\underline{s}_{r'} \succ_{ML} \bar{s}_k$. Fact 2 then implies that $\underline{s}_k \succ_{ML} \bar{s}_k$. Next, assume that $\bar{s}_k$ was rejected from c^* in a round $r' \leq k$ in which c^* was exactly at its minimum quota. If $r' \geq r$, then an argument exactly as in the previous case shows the result. Thus, the last case we need to consider is when $r' < r$. When this is true, then $\bar{s}_k$ was rejected from c^* because it was filled with $\tilde{q}_{c^*}$ other students ranked higher than $\bar{s}_k$ on $\succ_{ML}$. There must be some round r'' such that $r' < r \leq r'' \leq k$ at which some student s'' is rejected from c^* and the number of students assigned to c^* drops strictly below $\tilde{q}_{c^*}$. This happens because e other students are assigned to schools in C^*, i.e., $\underline{s}_{r''} \succ_{ML} s''$. However, we also have that $s'' \succ_{ML} \bar{s}_k$, and Fact 2 then gives $\underline{s}_k \succ_{ML} \bar{s}_k$.

Part (c): Again consider some $k \geq r$ and $k' = k+1$, and assume that $\bar{s}_k \succ_{ML} \bar{s}_{k'}$. Let c^* be the school that $\bar{s}_k$ claims an empty seat at in round k. Then, $\bar{s}_k$ must no longer claim an empty seat at this school in round k'. So, $\tilde{q}_{c^*} = |\mu_{k'}(c^*)| > |\mu_k(c^*)|$. By an argument similar to part (a), there is a student s' such that $s' \in \mu_k^*$, but $s' \notin \mu_{k'}^*$ and s' claims an empty seat at some school c'^* in round k'. Part (b) implies that $s' \succ_{ML} \bar{s}_k$. However, the fact that s' claims a seat at school c'^* in round k' imply that $\bar{s}_{k'} \succeq_{ML} s'$. This, together with the assumption that $\bar{s}_k \succ_{ML} \bar{s}_{k'}$, imply that $\bar{s}_k \succ_{ML} s'$, which is a contradiction. $\square$

Consider the student s^1 with the highest priority according to $\succ_{ML}$ who claims an empty seat. (If no such student exists, the matching is clearly nonwasteful.) Let the school he claims a seat at be denoted c^0, and his assignment under the ESDA algorithm be denoted $\mu(s^1) = c^1$. Let r^1 be the round of ESDA at which s^1 is rejected from c^{0*}.

Note that $r^1 \geq r$, and that $\underline{s}_{r^1} \succ_{ML} s^1$. (If neither of these holds, then s^1 was rejected from c^{0*} because it was filled to capacity with $\tilde{q}_{c^{0*}} = q_{c^{0*}} - p_{c^{0*}}$ students ranked higher than s^1 according to $\succ_{ML}$. But, since c^{0*} has an empty seat at the end of the algorithm, one of these students must have been rejected at some round, which contradicts the fact that s^1 is the highest priority student who claims an empty seat.)

If $|\mu(c^1)| = p_{c_1}$, then we are done. If not, then some student is assigned to c^{1*} in the extended market. We first show that this implies that there exists some student s^2 who was rejected from c^{1*}.

Assume not. We know that some student $s^2 \neq s^1$ applies to c^{1*} at some round $r^2 > r^1$.[26] Since we assumed that s^2 was accepted at c^{1*}, we know that some other student s' was rejected at round r^2 from some school c'^* at which he claims an empty seat in round $r^2 + 1$. Note that $s' \succ_{ML} s^1$. Also, by definition, $\bar{s}_{r^2+1} \succeq_{ML} s'$. Let r_{end} be the last round of the ESDA algorithm. By Lemma 2, we have that $\bar{s}_{r_{end}} \succeq_{ML} \bar{s}_{r^2+1} \succeq_{ML} s' \succ_{ML} s^1$. However, this contradicts the fact that s^1 is the highest ranked student on ML that claims an empty seat. So, there exists some student $s^2 \neq s^1$ who was rejected from c^{1*} at some round $r^2 > r^1$.

Let $\mu(s^2) = c^2$. If $|\mu(c^2)| = p_{c_2}$, we are done. If not, then some student is assigned to c^{2*} in the extended market. As previously, we can show that this implies there exists some student $s^3 \neq s^1, s^2$ who was rejected from c^{2*}.

Assume not. Then, we know that some $s^3 \neq s^1, s^2$ applies to c^{2*} in some round $r^3 > r^2$, by a similar argument as in Footnote 26. Since we assumed that s^3 was not rejected from c^{2*}, we know that some other student s' was rejected from some school c'^*, at which he claims an empty seat in round $r^3 + 1$. We again have $s' \succ_{ML} s^1$, and, as above, $\bar{s}_{r_{end}} \succeq_{ML} \bar{s}_{r^3+1} \succeq_{ML} s' \succ_{ML} s^1$, which contradicts the fact that s^1 was the highest ranked student on ML that claims an empty seat. Thus, there exists some $s^3 \neq s^1, s^2$ who is rejected from c^{2*} at some round $r^3 > r^2 > r^1$.

Continuing this line of reasoning, we eventually find a chain of students and schools $(c^0, s^1, c^1, s^2, c^2, \ldots, s^K, c^K)$ such that s^k is rejected from $c^{(k-1)*}$ and $\mu(s^k) = c^k$, where $|\mu(c^K)| = p_{c^K}$.

Proof of strategyproofness

We consider the extended market and allow students to misreport their preferences over both standard and extended seats. If no student has a profitable deviation in the extended market, it immediately follows that no student has a profitable deviation in the original market.

To show strategyproofness in the extended market, we associate it with the many-to-one matching with contracts model of Hatfield and Milgrom (2005). In this model, there are three types of agents: on one side of the market is the students $S = \{s_1, \ldots, s_n\}$. On the other side is a set of schools, which are divided into two types. There are m standard schools in the set $C = \{c_1, \ldots, c_m\}$. There is also one additional school λ, which captures all of the extended seats. Thus, the set of agents in the extended model is $S \cup C \cup \{\lambda\}$. There is in addition a set of contracts,

[26] Let $\hat{r}^1$ be the round at which s^1 applies to his final match c^1 (note that by Fact 3, s^1 must be assigned a standard seat at c^1). If c^1 is filled entering round $\hat{r}^1$, then some s^2 is rejected and applies to c^{1*} in $r^2 = \hat{r}^1 + 1 > r^1$. If c^1 is not full entering $\hat{r}^1$, then no student has yet applied to c^{1*}. However, since we assume that some student is assigned to c^{1*}, we know that some student must apply to c^{1*} at some $r^2 \geq \hat{r}^1 > r^1$.

$X = S \times \{C \cup C^*\}$. Each contract x specifies a student and the school he is assigned. Each contract is *associated* with one student $x_S \in S$ and one school $x_{C\cup\{\lambda\}} \in C \cup \{\lambda\}$.

It is important to note the distinction between *agents* ($S \cup C \cup \{\lambda\}$) and *contracts* ($S \times \{C \cup C^*\}$), even though similar notation is used for both. In particular, while the extended schools C^* may be involved in a contract $x \in X$, they are not agents in this model. For example, consider the contract that assigns student s_1 to a standard seat at school c_1. This contract would be written $x = (s_1, c_1)$, and is associated with student s_1 and school c_1 ($x_S = s_1$ and $x_{C\cup\{\lambda\}} = c_1$). However, now consider the contract $x' = (s_1, c_1^*)$ that assigns student s_1 to an extended-seat at school c_1^*. While student s_1 is assigned to school c_1^* under this contract, contract x' is *associated* with the fictitious school λ, since c_1^* is no longer an agent (that is, $x'_S = s_1$, but $x'_{C\cup\{\lambda\}} = \lambda$, *not* c_1^*).

Given $\tilde{\succ}_s$, we define preferences over contracts in the standard way. For two contracts $x = (s, c)$ and $x' = (s, c')$, then (with slight abuse of notation) we say $x \tilde{\succ}_s x'$ in the model with contracts if and only if $c \tilde{\succ}_s c'$ in the extended market.

For each school $c \in C \cup \{\lambda\}$, we define a choice rule, $\mathrm{Ch}_c(\cdot)$, which specifies the contracts that c would choose when offered a set of contracts. If c is a standard school ($c \in C$), then, for $X' \subseteq X$, $\mathrm{Ch}_c(X')$ is simply the p_c contracts associated with c corresponding to the most preferred students according to $\succ_c$.

If $c = \lambda$, we must be slightly more careful. Given two contracts x and x' associated with λ, define a relation $\tilde{\succ}_\lambda$ lexicographically, first ordering contracts according to the rank of the associated student on $\succ_{ML}$ and breaking ties with any arbitrary ordering of the schools, which we denote $\succ_{\tilde{C}}$.[27] Formally, for any two contracts $x = (s, c)$ and $x' = (s', c')$, define $\tilde{\succ}_\lambda$ as follows:

$$(s, c) \tilde{\succeq}_\lambda (s', c') \iff s \succ_{ML} s' \text{ or } (s = s' \text{ and } c \succeq_{\tilde{C}} c')$$

Then, given any subset of contracts $X' \subseteq X$, $\mathrm{Ch}_\lambda(X')$ is the e highest $\tilde{\succ}_\lambda$−ranked contracts such that:

(i) $x_{C\cup\{\lambda\}} = \lambda$ for all $x \in \mathrm{Ch}_\lambda(X')$

(ii) $|\{(s, c) \in \mathrm{Ch}_\lambda(X') : c = c_j^*\}| \leq \tilde{q}_{c_j^*}$ for all $c_j^* \in C^*$

(iii) $x, x' \in \mathrm{Ch}_\lambda(X')$ and $x \neq x' \implies x_S \neq x'_S$.

In words, (i) says that every contract chosen must be associated with school λ; (ii) says that each extended school can accept no more than its maximum quota of students, $\tilde{q}_{c_j^*}$; and (iii) says that each student can be chosen only once.

The following definitions are taken from Hatfield and Milgrom.

[27] Practically, the ordering $\succ_{\tilde{C}}$ is irrelevant, as each student will only offer one contract at a time, and hence $\succ_{ML}$ is all that will be needed to order the contracts; however, $\succ_{\tilde{C}}$ is needed to formally define $\mathrm{Ch}_\lambda(\cdot)$.

Definition 15. *A choice rule* $Ch_c(\cdot)$ *satisfies the substitutes condition if there do not exist contracts* $x, x' \in X$ *and a set of contracts* $X' \subseteq X$ *such that* $x' \notin Ch_c(X' \cup \{x'\})$ *and* $x' \in Ch_c(X' \cup \{x, x'\})$.

Definition 16. *A choice rule* $Ch_c(\cdot)$ *satisfies the law of aggregate demand if for all* $X' \subseteq X'' \subseteq X$, *we have* $|Ch_c(X')| \leq |Ch_c(X'')|$.

It is clear that if $c \neq \lambda$, $\mathrm{Ch}_c(\cdot)$ satisfies both the substitutes condition and the law of aggregate demand. The following lemma shows the case where $c = \lambda$.

Lemma 3. $Ch_\lambda(\cdot)$ *satisfies both substitutes and the law of aggregate demand.*

Proof. That $\mathrm{Ch}_\lambda(\cdot)$ satisfies the law of aggregate demand is trivial. For substitutes, consider a set X' and a contract x' such that $x' \notin \mathrm{Ch}_\lambda(X' \cup \{x'\})$. Then, consider the set $X' \cup \{x', x\}$. The addition of x means that, when the contracts in $X' \cup \{x', x\}$ are ordered according to $\tilde{\succ}_\lambda$, contract x' is assigned a (weakly) lower ranking than it was when only the contracts in $X' \cup \{x'\}$ were considered. Thus, by the definition of the choice rule, contract $x' \notin \mathrm{Ch}_\lambda(X' \cup \{x', x\})$. □

The lemma above is required to use Theorem 11 of Hatfield and Milgrom (2005). In addition, we must define an appropriate notion of stability and student optimal stable match. Note that the stability defintion below applies only to the model with contracts, and is not related to our fairness criteria. To make this point clear, we use the terminology "Hatfield-Milgrom stable".

Definition 17. *A set of contracts* $X' \subseteq X = S \times \{C \cup C^*\}$ *is Hatfield-Milgrom stable (HM-stable) if (i) it is individually rational, (ii) there exists no standard school* $c \in C$ *and student* $s \in S$ *such that* $(s, c)\tilde{\succ}_s x$ *and* $(s, c) \in Ch_c(X' \cup \{(s, c)\})$, *where* x *is the contract* s *receives in* X' *(if any) and is* $\emptyset$ *otherwise, and (iii) there exists no* $c^* \in C^*$ *and student* $s \in S$ *such that* $(s, c^*)\tilde{\succ}_s x$ *and* $(s, c^*) \in Ch_\lambda(X' \cup \{(s, c^*)\})$, *where* x *is the contract* s *receives in* X' *(if any) and is* $\emptyset$ *otherwise.*

Given a set of individually rational contracts X', define the corresponding matching in the extended model by setting $\tilde{\mu}(s) = c$ if and only if $(s, c) \in X'$. Lastly, we must show that ESDA is equivalent to the student optimal HM-stable mechanism in the associated model with contracts. For a definition of the student optimal HM-stable mechanism, see Hatfield and Milgrom (2005).

Theorem 9. *ESDA produces the same matching as the student optimal HM-stable mechanism in the associated model with contracts.*

Proof. As shown in Hatfield and Milgrom (2005), the student optimal HM-stable match in the matching with contracts model can be found via the cumulative offer process. Then, note that each round of ESDA corresponds to a round of the cumulative offer process. More precisely, if s

applies to school $c \in C \cup C^*$ at some round of ESDA, then at the same round of the cumulative offer process, contract (s, c) is proposed. Lastly, for each standard school c, the set of students held at a round of ESDA corresponds to the set of contracts held by that school at the corresponding round of the cumulative offer process. For school λ, the set of students accepted at the extended schools at a round of ESDA corresponds to the set of contracts held by agent λ at the corresponding round of the cumulative offer process. $\square$

Theorem 10. *The ESDA mechanism is weakly group strategyproof for students.*

Proof. Under substitutes and the law of aggregate demand, any mechanism that selects the student optimal HM-stable match in the associated model with contracts is weakly group strategyproof (Hatfield and Milgrom (2005); Hatfield and Kojima (2009); Hatfield and Kominers (2012)). Since ESDA selects the student optimal HM-stable match, it is weakly group strategyproof for the students. $\square$

Proof of $p-$fairness

We show that no student s can form a $p-$blocking pair with any school c'. Consider any pair of students s and s' such that $c' \succ_s \mu(s)$, where $c' = \mu(s')$.

Case (i): $|\mu(c')| > p_{c'}$. If $\tilde{\mu}(s') \in C$, student s must have applied to school $\tilde{\mu}(s')$ in the extended market and been rejected, which means $s' \succ_{c'} s$. Thus, (s, c') cannot form a p-blocking pair. If student $\tilde{\mu}(s') \in C^*$, then we have $s' \succ_{ML} s$, and, since $|\mu(c')| > p_{c'}$, (s, c') once again do not form a $p-$blocking pair.

Case (ii): $|\mu(c')| = p_c$. Since $|\mu(c')| = p_c$, we know that $\tilde{\mu}(s') \in C$. Thus, student s must have applied to $\tilde{\mu}(s')$ in the extended market and been rejected, which means $s' \succ_{c'} s$ and again (s, c') cannot form a $p-$blocking pair.

C Proof of Theorem 4

The proof proceeds by reducing the market of n students and m schools to a smaller market of two students and three schools. We construct preferences and priorities for students $s_1, \ldots, s_{n-2}$ such that $\sigma-$fairness pins down their assignments. These students can then be removed from the problem, leaving students s_{n-1} and s_n to be assigned. We then examine all possible feasible allocations for these students, and show none of these allocations are simultaneously σ-fair and (constrained) nonwasteful.

Before beginning the main proof, let us show the following lemma, which will allow us to determine in the submarket which schools have minimum quota seats remaining and which schools have flexible seats remaining.

Lemma 4. *Given any school x, there exist distinct schools $y, z \neq x$ such that $p_y > 0$ and $p_z < q_z$.*

Proof. There are three possible cases: (i) $p_x = 0$; (ii) $0 < p_x < q_x$; and (iii) $0 < p_x = q_x$.

First, assume that (i) holds. Since $p_x = 0$, we know that $p_x < q_x$ (or else just delete school x from the problem). By assumptions (A2) and (A3), we know that there exist distinct schools $u, v \neq x$ such that $p_u > 0$ and $p_v > 0$, and that there exists some school $w \neq x$ such that $p_w < q_w$. Define $z := w$. If $w = u$, then let $y := v$. If $w = v$, let $y := u$. If $w \neq u, v$, let $y := w$.

Next, assume (ii) holds. Assumptions (A2) and (A3) imply that there exists a school $u \neq x$ such that $p_u > 0$ and a school $v \neq x$ such that $p_v < q_v$. If the result is false, it must be that $u = v$ and all other schools $w \neq x, u$ are such that both $p_w = 0$ and $p_w = q_w$, i.e., $q_w = 0$, which is a contradiction.

Last, assume $0 < p_x = q_x$. Then, we know that there exist distinct $u, v \neq x$ such that $p_u < q_u$ and $p_v < q_v$, and there exists some school $w \neq x$ such that $p_w > 0$. Define $y := w$. If $w = u$, then define $z := v$; if $w = v$, then let $z := u$.

□

Now we proceed to the main proof.

The proof is by contradiction. Let χ be strategyproof, constrained nonwasteful, and $\sigma-$fair for some σ such that $\sigma_c > p_c$ for at least one $c \in C$, and note that this implies that $p_c < q_c$. Without loss of generality, denote this school as school c_2. From the previous lemma, we know there exist schools two other schools which we call c_1 and c_3 such that $p_{c_1} > 0$ and $p_{c_3} < q_{c_3}$. Consider a market with $n = 1 + \sum_{c \in C} p_c$ students. Recall $s_1 \succ_{ML} \cdots \succ_{ML} s_n$. For the remainder of the proof, consider a priority profile for which $s_1 \succ_{c_i} \cdots \succ_{c_i} s_n$ for $i \neq 2$, and for c_2, we have $s_1 \succ_{c_2} \cdots \succ_{c_2} s_{n-2} \succ_{c_2} s_n \succ_{c_2} s_{n-1}$ (i.e., the positions of s_n and s_{n-1} are reversed).

Let $S' \equiv S \setminus \{s_{n-1}, s_n\}$. We partition the set S' into m sets, $S'_1 \ldots S'_m$, one corresponding to each school. Let S'_1 consist of the $p_{c_1} - 1$ highest ranked students in S' according to $\succ_{c_1}$. Then, let S'_2 consist of the p_{c_2} highest ranked students in $S' \setminus S'_1$ according to $\succ_{c_2}$. Continuing, let S'_i consist of the p_{c_i} highest ranked students in $S'_i = S' \setminus (\cup_{k=1}^{i-1} S'_k)$ for $i = 3, \ldots, m$.

For the remainder of the proof, we consider preference profiles such that for all $i = 1 \ldots m$, if $s \in S'_i$, then s ranks c_i first (the rankings of the remaing schools are irrelevant). Since S'_1 contains the highest ranked students in S according to $\succ_{c_1}$ and all students in S'_1 prefer c_1 the most, $\sigma-$fairness implies that all students in S'_1 must be assigned to c_1. If not, some student $s \in S'_1$ is assigned to a school worse than c_1 for him, and some $s' \notin S'_1$ is assigned c_1 (since we know at least p_{c_1} students are assigned to c_1 in any feasible matching). But then, (s, c_1) form a $\sigma_{c_1}-$blocking pair (for any σ_{c_1}), which contradicts that the mechanism was σ-fair. Then, since S'_2 contains the highest ranked students in $S' \setminus S'_1$ according to $\succ_{c_2}$ and all students in S'_2 rank c_2

first, by σ-fairnesss again, all students in S'_2 must be assigned to c_2. Continuing in this manner, all students in S'_i must be assigned to school c_i for all i, regardless of the preferences of s_{n-1} and s_n.

Last, we consider the assignments of s_{n-1} and s_n under various preferences for these two students. Thus, we effectively have a subproblem with 3 schools with the following quotas: school c_1 has one minimum quota seat that must be filled, while all other schools are filled exactly to their minimum quota. Because we know $p_{c_2} < q_{c_2}$ and $p_{c_3} < q_{c_3}$, schools c_2 and c_3 each have (at least) one seat remaining. Consider the following preferences, and the priorities as described above:[28]

$\succ_{s_{n-1}}$	$\succ_{s_n}$	$\succ_{c_1}$	$\succ_{c_2}$	$\succ_{c_3}$
c_2	c_3	$\vdots$	$\vdots$	$\vdots$
c_3	$\boxed{c_2}$	s_{n-1}	s_n	s_{n-1}
$\boxed{c_1}$	c_1	s_n	s_{n-1}	s_n
$\vdots$	$\vdots$			

Given quotas, it is clear that exactly one student must be assigned to c_1.[29]

Begin with the case that s_{n-1} is assigned to c_1. If s_n were assigned to c_3, then (s_{n-1}, c_3) would form a $\sigma_{c_3}-$blocking pair. Thus, the mechanism must choose the (sub)matching shown in the boxes above. Now, consider the following monotonic transformation of $(\succ_{s_{n-1}}, \succ_{s_n})$:

$\succ'_{s_{n-1}}$	$\succ_{s_n}$
$\boxed{c_1}$	c_3
c_2	$\boxed{c_2}$
c_3	c_1
$\vdots$	$\vdots$

Because the mechanism is strategyproof and this is a monotonic transformation, the assignment must remain that shown in the boxes. However, this is not constrained nonwasteful.

Lastly, consider the case that s_n is assigned to c_1 under the original preferences $(\succ_{s_{n-1}}, \succ_{s_n})$. If s_{n-1} is assigned to c_2, then (s_n, c_2) would form a $\sigma_{c_2}-$blocking pair because $s_n \succ_{c_2} s_{n-1}$ and $\sigma_{c_2} > p_{c_2}$ by assumption, and under the matching indicated, $p_{c_2} + 1$ agents receive c_2 (so part (iii) of the definition of a σ_c-blocking pair holds vacuously). Thus, the choice must be that shown in boxes below:

[28]The ordering of the remaining schools is irrelevant, and so we indicate only the ordering of c_1,c_2 and c_3.

[29]Since c_1 has an empty minimum quota seat, at least one student must be assigned there. However, assigning both students to c_1 would not be constrained nonwasteful. It is also not constrained nonwasteful to assign either of the students to a school other than c_1, c_2, or c_3.

$\succ_{s_{n-1}}$	$\succ_{s_n}$
c_2	c_3
$\boxed{c_3}$	c_2
c_1	$\boxed{c_1}$
$\vdots$	$\vdots$

Once again consider a monotonic transformation of these preferences:

$\succ_{s_{n-1}}$	$\succ'_{s_n}$
c_2	$\boxed{c_1}$
$\boxed{c_3}$	c_3
c_1	c_2
$\vdots$	$\vdots$

By strategyproofness, the assignment must remain that in the boxes. However, this is clearly not constrained nonwasteful.

D Proof of Theorem 5

Proof of weak group strategyproofness

To show weak group strategyproofness, note that in each stage of MSDA, the standard DA algorithm is used, the latter being weakly group strategyproof (Hatfield and Kojima (2009)). Assume the theorem is false, and consider a set of students S' that collectively do strictly better by misreporting $\succ'_{S'}$ than by reporting the truth, $\succ_{S'}$. Let $S'' \subseteq S'$ be the set of students that are matched in the earliest stage under the true preferences $\succ_{S'}$, and denote this stage as stage k. Note that there is no way in which S' can misreport so that a member of S' is matched earlier than stage k. Thus, under any misreport, the students in S'' will be assigned in stage k. However, all of the students in S'' becoming strictly better off by misreporting $\succ'_{S''}$ in stage k is a contradiction to the fact that the standard DA algorithm is weakly group strategyproof. Thus, MSDA is weakly group strategyproof.

Proof of nonwastefulness

Next, we show that MSDA is nonwasteful. Assume not. Then, there exists a student s who justifiably claims an empty seat at some school c. Let k be the stage of the mechanism at which student s participates, and q_c^k be the quotas at school c during that stage. There are two cases.

(i) In stage k, step 1(a) of the algorithm is executed. In this case, the quotas for the standard DA algorithm run in stage k are $q_c^k = q_c^{k-1} - |\mu^{k-1}(c)| = q_c - \sum_{k'<k} |\mu^{k'}(c)|$. Since s is not assigned

a seat at c in k, then, since we are just using DA, it must be that q_c^k students were assigned at stage k to school c. However, adding this to the number of students assigned in stages $k' < k$, we see that $|\mu(c)| = q_c$, and thus there are no empty seats at school c.

(ii) In stage k, step 1(b) of the algorithm is executed. In this case, the quotas are $\max\{0, p_c - \sum_{k'<k} |\mu^{k'}(c)|\}$. In particular, we have $|\mu(\mu(s))| = p_{\mu(s)}$ for all s assigned in stage k. Thus, student s cannot be moved without violating the minimum quotas at $\mu(s)$, and so s cannot justifiably claim an empty seat at c.

Proof of σ_0-fairness

That MSDA is σ_0-fair is shown in the discussion immediately following Theorem 5.

E Proof of Theorem 6

The structure of the proof is similar to Theorem 4: we will reduce the problem to a submarket of 3 students and 3 schools, and show that there is no way to allocate these three students that is nonwasteful and σ-fair for any $\sigma \neq \sigma_0$.

Let χ be nonwasteful and σ-fair, and assume that $\sigma_c > 0$ for at least one $c \in C$. We will show that this leads to a contradiction. Without loss of generality, denote this school c_2, and let $\sigma_{c_2} = \max\{1, p_{c_2}\}$.[30] By Lemma 4 (setting $x = c_2$ in the lemma), we know that there exists two other schools, which we label c_1 and c_3, such that $p_{c_1} > 0$ and $p_{c_3} < q_{c_3}$. Consider a market with $n = 1 + \sum_{c\in C} p_c$ students. Recall that $s_1 \succ_{ML} \cdots \succ_{ML} s_n$. For the remainder of the proof, we consider the following priority profile: $\succ_c = \succ_{ML}$ for all $c \neq c_2$, while $\succ_{c_2}$ is defined as $s_1 \succ_{c_2} s_2 \succ_{c_2} \cdots \succ_{c_2} s_n \succ_{c_2} s_{n-1} \succ_{c_2} s_{n-2}$ (i.e., the priority of the last three students is reversed).

There are two cases, depending on whether p_{c_2} is equal to 0 or not.

Case (i): $p_{c_2} > 0$

Let $S' = S \setminus \{s_{n-2}, s_{n-1}, s_n\}$. We partition S' into m sets, $S'_1, \ldots, S'_m$ as follows: S'_1 consists of the $p_{c_1} - 1$ highest ranked students according to $\succ_{c_1}$ ($=\succ_{ML}$) and S'_2 consists of the $p_{c_2} - 1$ highest ranked students in $S' \setminus S'_1$ according to $\succ_{c_2}$. For the remaining schools, let S'_k consist of the p_{c_k} highest ranked students in the set $S' \setminus \cup_{i=1}^{k-1} S'_k$ according to $\succ_{c_k}$.

For the remainder of the proof, we consider preference profiles such that for all $k = 1, \ldots, m$, if $s \in S'_k$, then s ranks school c_k first. Since S'_1 contains the highest ranked students in S according to $\succ_{c_1}$ and all student in S'_1 rank c_1 first, σ-fairness implies that all students in S'_1 are assigned to c_1. If this were not the case, then some student $s \in S'_1$ is assigned to a school worse than c_1 for him, and some $s' \notin S'_1$ is assigned c_1 (since we know at least p_{c_1} students are assigned to c_1).

[30]The max function is needed here for the case in which $p_{c_2} = 0$. By remark 1, defining σ_{c_2} in this way is sufficient to show the impossibility.

But then, (s, c_1) forms a σ_{c_1}-blocking pair (for any σ_{c_1}), which contradicts that the mechanism was σ-fair. Now, S'_2 contains the highest ranked students in $S' \setminus S'_1$ according to $\succ_{c_2}$, and so, by similar logic, all students in S'_2 must be assigned to c_2. Continuing in this manner, all students in S'_k must be assigned to school c_k for all k, regardless of the preferences of $\{s_{n-2}, s_{n-1}, s_n\}$.

Last, we consider the assignments of $\{s_{n-2}, s_{n-1}, s_n\}$. Now, we know that schools c_1, c_2 and c_3 satisfy the following (where μ' denotes the tenative matching, before $\{s_{n-2}, s_{n-1}, s_n\}$ are assigned, as described in the previous paragraph): $|\mu'(c_1)| = p_{c_1} - 1 < q_{c_1}$, $|\mu'(c_2)| = p_{c_2} - 1 < q_{c_2}$, and $p_{c_3} = |\mu'(c_3)| < q_{c_3}$. So, we have effectively reduced the problem to a subproblem with three students and three schools in which c_1 and c_2 have one minimum quota seat remaining and c_3 has at least one empty seat. Consider the following preferences for $\{s_{n-2}, s_{n-1}, s_n\}$ and the priorities for the schools, as given above:

s_{n-2}	s_{n-1}	s_n	c_1	c_2	c_3
c_2	c_3	c_3	$\vdots$	$\vdots$	$\vdots$
c_3	c_2	c_2	s_{n-2}	s_n	s_{n-2}
c_1	c_1	c_1	s_{n-1}	s_{n-1}	s_{n-1}
$\vdots$	$\vdots$	$\vdots$	s_n	s_n	s_n

Clearly, two of the three students must be assigned to the minimum quota seats, one at c_1 and the other at c_2. There are 3 possible ways to choose these two students:[31]

Subcase (a): s_{n-2} and s_{n-1} are assigned minimum quota seats. Nonwastefulness then requires that s_n be assigned c_3, and so there are two possible allocations here. One is shown by the boxes, the other by the circles.

s_{n-2}	s_{n-1}	s_n
$\boxed{c_2}$	c_3	$\boxed{c_3}$
c_3	c_2	c_2
c_1	$\boxed{c_1}$	c_1
$\vdots$	$\vdots$	$\vdots$

Note, however, that under the circled allocation, both s_{n-2} and s_{n-1} can form a σ_{c_3}-blocking pair with c_3 for any σ_{c_3}, since they are both ranked higher than s_n according to $\succ_{ML}$ and $\succ_{c_3}$. Likewise, under the boxed allocation, s_{n-1} can again form a σ_{c_3}-blocking pair with c_3. Thus, neither matching is σ-fair.

Subcase (b): s_{n-2} and s_n are assigned the minimum quota seats. In this case, s_{n-1} must be assigned to c_3 by nonwastefulness, and there are again two possible allocations:

[31] Note that in what follows, exactly one student must be assigned to c_1, or else the matching would be wasteful.

s_{n-2}	s_{n-1}	s_n
$\boxed{c_2}$	$\boxed{c_3}$	c_3
c_3	c_2	c_2
c_1	c_1	$\boxed{c_1}$
$\vdots$	$\vdots$	$\vdots$

Here, under the circled allocation, s_{n-2} forms a σ_{c_3}-blocking pair with c_3. Under the boxed allocation, s_n forms a σ_{c_2}-blocking pair with c_2, because $s_n \succ_{c_2} s_{n-2}$ and $|\mu(c_2)| = p_{c_2} = \sigma_{c_2}$, and so s_n need not be higher on $\succ_{ML}$ to form a blocking pair. Thus, neither matching is σ-fair.

Subcase (c): s_{n-1} and s_n are assigned the minimum quota seats. Here, nonwastefulness requires s_{n-2} be assigned c_2. Once again, the two possible allocations are boxed/circled.

s_{n-2}	s_{n-1}	s_n
$\boxed{c_2}$	c_3	c_3
c_3	$\boxed{c_2}$	c_2
c_1	c_1	$\boxed{c_1}$
$\vdots$	$\vdots$	$\vdots$

However, here it is simple to see that under either allocation, $|\mu(c_2)| = p_{c_2} + 1$, and thus both matchings are not nonwastaeful, as either s_{n-1} or s_n can be feasibly moved to the more preferred c_3.

Case (ii): $p_{c_2} = 0$

Recall that here, $\sigma_{c_2} = \max\{1, p_{c_2}\} = 1$. Everything is the same as before, except that we let define $S' = S \setminus \{s_{n-1}, s_n\}$ (i.e., s_{n-2} now belongs to S'), and S'_2 is empty. This then leaves us with a smaller submarket of two students, s_{n-1} and s_n, and three schools, c_1, c_2, and c_3, where $|\mu'(c_1)| = p_{c_1} - 1$ (so c_1 has one minimum quota seat left to fill) and $p_{c_2} \leq |\mu'(c_2)| < q_{c_2}$ (so c_2 has no minimum quotas to fill, but has at least one empty seat), and $|\mu'(c_3)| < q_{c_3}$. Again, consider the following preferences for s_{n-1} and s_n, and the priorities from above:

s_{n-1}	s_n	c_1	c_2	c_3
c_2	c_3	$\vdots$	$\vdots$	$\vdots$
c_3	c_2	s_{n-2}	s_n	s_{n-2}
c_1	c_1	s_{n-1}	s_{n-1}	s_{n-1}
$\vdots$	$\vdots$	s_n	s_{n-2}	s_n

There are again two cases, depending who is assigned to the minimum quota seat at c_1. If s_{n-1} is assigned to c_1, then nonwastefulness requires that s_n be assigned to c_3. However, s_{n-1} would then form a σ_{c_3}-blocking pair with c_3 for any σ_{c_3}, because $s_{n-1} \succ_{ML} s_n$ and $s_{n-1} \succ_{c_3} s_n$. If s_n is assigned to c_1, then nonwastefulness requires that s_{n-1} be assigned to c_2. However, s_n then forms a $\sigma_{c_2}-$blocking pair with c_2 because $s_n \succ_{c_2} s_{n-1}$ and $|\mu(c_2)| = 1 = \sigma_{c_2}$, so part (iii) of the definition of a σ_{c_2}-blocking pair holds vacuously (i.e., s_n need not be higher on $\succ_{ML}$).

F Proof of Theorem 7

Part (i)

Before beginning the proof of this theorem, let us introduce some notation similar to Pápai (2000). In the original market, we let $\chi(\succ)$ denote the matching under preferences $\succ$. In the extended market, we represent the corresponding matching as $\psi(\tilde{\succ})$ where $\tilde{\succ}$ are derived from $\succ$. Because $\tilde{\succ}$ and $\succ$ can be unambiguously inferred from each other, we henceforth suppress the notation $\tilde{\succ}$ and exclusively write $\succ$. We keep the two functions ψ and χ to distinguish between assignments in the extended and original markets: $\chi(\succ)$ are the final assignments in the original market and $\psi(\succ)$ are the final assignments in the extended market under $\succ$. Let us define $W_k(\succ)$ as the set of individuals assigned in round k and $W^k(\succ)$ as the set of individuals assigned in all rounds weakly earlier than k. Thus, $W^{k'}(\succ) = \bigcup_{z=1}^{k'} W_z(\succ)$. Let us define $F_k(\succ)$ as the set of seats taken off the market[32] in round k and define the union as $F^{k'}(\succ) = \bigcup_{z=1}^{k'} F_z(\succ)$. Let $T_k(j, \succ)$ be j's most preferred available school in round k under preferences $\succ$. The cycle to which a student s belongs in a round k under preferences $\succ$ is denoted $S_k(s, \succ)$. Suppose there is a cycle involving s and s' under $\succ$ at round k where s points to s'. When we cut the cycle between s and s', we obtain a "chain" starting at s' and ending at s, which we denote as $H_k(s', s, \succ)$. Given profile $\succ$, $o_s(\succ_{S\setminus\{s\}})$ is the set of schools (in the extended market) that s can obtain by reporting any preferences. $E_k(s, \succ)$ denotes the endowment (both extended and standard seats) for individual s at the start of round k given preferences $\succ$. Finally, the function $N(x, y, z)$ is how many seats in school(s) x are off the market by the end of round y under submitted profile z. So, for example, $N(c_a \cup c_a^*, k, \succ) = q_{c_a}$ means that by the end of k, when submitted preferences are $\succ$, there are no more seats available at schools c_a or c_a^*. The following two definitions are from Pápai (2000).

Definition 18. *The Assurance Rule (AR): Suppose under preferences $\succ$, school $c \in \tilde{C}$ is in the endowment of $s \in S$ in round k, i.e. $c \in E_k(s, \succ)$. Then, if s is not assigned in k, $c \in E_{k+1}(s, \succ)$.*

Definition 19. *The Twin Inheritance Rule (TIR): Consider two sets of preferences, $\succ$ and $\succ'$, such that for some k, (i) for all $k' \leq k$, $W^{k'}(\succ) = W^{k'}(\succ')$, (ii) for all $s \in W^k(\succ)$, $\psi_s(\succ$*

[32] A seat is "taken off the market" if it is assigned to an individual or if it is an unassigned extended seat that must be but made unavailable to every unmatched student in order to achieve feasibility.

$) = \psi_s(\succ')$, *and (iii)* $\succ_{W^k(\succ)} = \succ'_{W^k(\succ')}$. *Then, for all* s *unassigned by the end of round* k, $E_{k+1}(s, \succ) = E_{k+1}(s, \succ')$.

AR and TIR are properties of endowment structures. AR states that a student never loses seats from her endowment as long as she is unassigned. She may "inherit" seats as the algorithm proceeds; if so, these seats must not be assigned to any students and come from the endowments of students just assigned in the round prior to the inheritance. TIR states that as long as the matches made under two sets of preferences "look the same" in each round up to the start of a given round k, then the endowments of all unassigned students are the same at the start of round r as well.

Note that TIR implies something in particular for two preferences $\succ \equiv (\succ_s, \succ_{S\setminus\{s\}})$ and $\succ' \equiv (\succ'_s, \succ_{S\setminus\{s\}})$. If $k \equiv \min\{r, r'\} - 1$ where r and r' are the rounds in which s is matched under $\succ$ and $\succ'$ respectively, then since all students other than s have the same preferences under both profiles, (i) for all $k' \leq k$, $W^{k'}(\succ) = W^{k'}(\succ')$, (ii) for all $s \in W^k(\succ)$, $\psi_s(\succ) = \psi_s(\succ')$, and (iii) $\succ_{W^k(\succ)} = \succ'_{W^k(\succ')}$. Hence, by TIR, for all s unassigned by the end of round k, $E_{k+1}(s, \succ) = E_{k+1}(s, \succ')$.

Lemma 5. *The ESTTC mechanism satisfies the assurance rule and the twin inheritance rule.*

Proof. Suppose $c \in E_k(s, \succ)$. By definition, no student unassigned by the start of round k has a higher priority at c than s. As the algorithm proceeds, the subset of unassigned students gets weakly smaller, hence, for all $k' \geq k$, if s is unassigned by the start of k', then s must still have the highest priority at c of all unassigned students at the start of round k', and thus $c \in E_{k'}(s, \succ)$. Suppose $\succ$ and $\succ'$ are two sets of preferences such that for some k, (i) for all $z \leq k$, $W^z(\succ) = W^z(\succ')$, (ii) for all $i \in W^k(\succ)$, $\psi_i(\succ) = \psi_i(\succ')$, and (iii) $\succ_{W^k(\succ)} = \succ'_{W^k(\succ')}$. Furthermore, suppose that $c \in E_{k+1}(s, \succ)$. Then, c still has seats on the market by the start of round $k+1$ under $\succ$. Because the sets of students matched until this point in the algorithm (along with their assignments) are equivalent under both sets of preferences, c still has seats on the market by the start of round $k+1$ under $\succ'$. Furthermore, $c \in E_{k+1}(s, \succ)$ implies no student unassigned by the start of round $k+1$ has a higher priority at c than s under $\succ$. Because the sets of unassigned students by the start of round $k+1$ are equivalent under both sets of preferences, no student unassigned by the start of round $k+1$ has a higher priority at c than s under $\succ'$, thus implying $c \in E_{k+1}(s, \succ')$ by definition of how ESTTC runs. □

Definition 20. *A mechanism* χ *is* ***nonbossy*** *if for any* $\succ$, s, *and* $(\succ'_s, \succ_{S\setminus\{s\}})$ *such that* $\chi_s(\succ) = \chi_s(\succ'_s, \succ_{S\setminus\{s\}})$, $\chi(\succ) = \chi(\succ'_s, \succ_{S\setminus\{s\}})$.

Pápai (2000) proves the following lemma.

Lemma 6. *A mechanism χ is strongly group-strategyproof if and only if it is strategyproof and nonbossy.*

Given the above lemma, we prove that ESTTC is both strategyproof and nonbossy in order to prove strong group strategyproofness. We start by proving that if a student s is unassigned by the end of round k of ESTTC and all of the seats at school c_a are taken off the market by the end of round k given some set of reported preferences by all other students and the preferences reported by s, then c_a is unobtainable for s regardless of any other preferences reported by s provided that the other students do not change their reports. Lemma 7 states this property formally. With this lemma, proving strategyproofness is straightforward. We then proceed to prove nonbossiness.

Lemma 7. *If $s \notin W^k(\succ)$ and $N(c_a \cup c_a^*, k, \succ) = q_{c_a}$, then we have $\{c_a, c_a^*\} \cap o_s(\succ_{S\setminus\{s\}}) = \emptyset$.*

Proof. Fix $\succ$, s, and k as in the statement. Suppose there exists $\succ'_s$ such that $\psi_s(\succ'_s, \succ_{S\setminus\{s\}}) \in \{c_a, c_a^*\}$. Let k' be such that $s \in W_{k'+1}(\succ'_s, \succ_{S\setminus\{s\}})$. Let us show that $k' < k$ always holds. Note that $\succ$ and $(\succ'_s, \succ_{S\setminus\{s\}})$ are equivalent up to the end of round $\min\{k, k'\}$. Assume $k \leq k'$. Then, since s is unmatched by the end of k under both profiles, $N(c_a \cup c_a^*, k, \succ) = q_{c_a}$, implies $N(c_a \cup c_a^*, k, (\succ'_s, \succ_{S\setminus\{s\}})) = q_{c_a}$, which is a contradiction. So $k' < k$.

Second, by TIR, both profiles have identical endowments at the start of $k'+1$. Since $N(c_a \cup c_a^*, k', (\succ'_s, \succ_{S\setminus\{s\}})) < q_{c_a}$, we have $N(c_a \cup c_a^*, k', \succ) < q_{c_a}$ and there must exist s' such that $\{c_a, c_a^*\} \cap E_{k'+1}(s', \succ) \neq \emptyset$. If $s' = s$ so that $\{c_a, c_a^*\} \cap E_{k'+1}(s, \succ) \neq \emptyset$, then by AR, $\{c_a, c_a^*\} \cap E_{k+1}(s, \succ) \neq \emptyset$ since $s \notin W^k(\succ)$. This, however, contradicts the assumption that $N(c_a \cup c_a^*, k, \succ) = q_{c_a}$. If $s' \neq s$, then since s gets c_a or c_a^* in $k'+1$ under $(\succ'_s, \succ_{S\setminus\{s\}})$, there must be a cycle of individuals such that in round $k'+1$, s' and s are removed under $(\succ'_s, \succ_{S\setminus\{s\}})$. The cycle includes s pointing to s'. If we cut the cycle between s and s' so that it becomes a chain with s' at one end and s at the other, the chain is directed from s' all the way to s. Call this chain $H_{k'+1}(s', s, (\succ'_s, \succ_{S\setminus\{s\}}))$.

The same schools have been assigned by the end of round k' under both profiles. As a result, when $k'+1$ starts, everyone other than s has the same top remaining choice under both profiles. Thus, there exists a chain $H_{k'+1}(s', s, \succ)$. AR implies that the chain $H_{k'+1}(s', s, \succ)$ will persist as long as s is unassigned. Given that $s \notin W^k(\succ)$, the chain $H_{k+1}(s', s, \succ)$ will form, meaning $s' \notin W^k(\succ)$. However, $\{c_a, c_a^*\} \cap E_{k+1}(s', \succ) \neq \emptyset$ contradicts $N(c_a \cup c_a^*, k, \succ) = q_{c_a}$. □

Theorem 11. *ESTTC is strategyproof.*

Proof. Suppose that χ is a manipulable ESTTC rule. Then there exist s, $\succ$ and $\succ'_s$ such that $\chi_s(\succ'_s, \succ_{S\setminus\{s\}}) \succ_s \chi_s(\succ)$. Let $\chi_s(\succ'_s, \succ_{S\setminus\{s\}}) = c_a$ and $\chi_s(\succ) = c_b$ for notational simplification. Because $c_a \succ_s c_b$, there exists k such that $s \notin W^k(\succ)$ and $N(c_a \cup c_a^*, k, \succ) = q_{c_a}$. In other words,

s was not able to obtain c_a or c_a^* by the time he was matched. Therefore $\{c_a, c_a^*\} \cap o_s(\succ_{S\setminus\{s\}}) = \emptyset$ by Lemma 7, which is a contradiction. $\square$

Theorem 12. *ESTTC is nonbossy.*

Proof. Fix $\succ$, s, and $\succ'_s$ such that $\chi_s(\succ) = \chi_s(\succ'_s, \succ_{S\setminus\{s\}})$. Let $s \in W_k(\succ)$ and $s \in W_{k'}(\succ'_s, \succ_{S\setminus\{s\}})$. We prove nonbossiness by showing for any $\succ$, s, and $(\succ'_s, \succ_{S\setminus\{s\}})$ such that $\psi_s(\succ) = \psi_s(\succ'_s, \succ_{S\setminus\{s\}})$, we have $\psi(\succ) = \psi(\succ'_s, \succ_{S\setminus\{s\}})$. Without loss of generality, we assume $k \leq k'$. We know that for all $s' \in W^{k-1}(\succ)$, $\psi_{s'}(\succ) = \psi_{s'}(\succ'_s, \succ_{S\setminus\{s\}})$. We will first show that $\psi_{s'}(\succ) = \psi_{s'}(\succ'_s, \succ_{S\setminus\{s\}})$ for all $s' \in W_k(\succ)$. We then proceed by induction to show that for all rounds $z > k$, $\psi_{s'}(\succ) = \psi_{s'}(\succ'_s, \succ_{S\setminus\{s\}})$ for all $s' \in W^z(\succ)$.

Showing $\psi_{s'}(\succ) = \psi_{s'}(\succ'_s, \succ_{S\setminus\{s\}})$ *for all* $s' \in W_k(\succ)$:

We know $F^{k-1}(\succ) = F^{k-1}(\succ'_s, \succ_{S\setminus\{s\}})$ implies that for all $s' \notin W^{k-1}(\succ) \cup \{s\}$ that $T_k(s', \succ) = T_k(s', (\succ'_s, \succ_{S\setminus\{s\}}))$ (i.e. everybody in round k, except student s, points to the same school under $\succ$ and $(\succ'_s, \succ_{S\setminus\{s\}})$ because the same seats and schools are remaining at the start of k under $\succ$ and $(\succ'_s, \succ_{S\setminus\{s\}})$). Hence, $W_k(\succ) \setminus S_k(s, \succ) \subseteq W_k(\succ'_s, \succ_{S\setminus\{s\}})$. (If $k = k'$, then we know that $\chi_s(\succ) = \chi_s(\succ'_s, \succ_{S\setminus\{s\}})$ implies it's a strict subset; if $k < k'$, then the sets are equal.) For all $s' \in W_k(\succ) \setminus S_k(s, \succ)$, $\psi_{s'}(\succ) = \psi_{s'}(\succ'_s, \succ_{S\setminus\{s\}})$.

Case 1: $k = k'$
If $k = k'$, then $\chi_s(\succ) = \chi_s(\succ'_s, \succ_{S\setminus\{s\}})$ implies $\psi_s(\succ) = \psi_s(\succ'_s, \succ_{S\setminus\{s\}})$.[33] Then, we have $W_k(\succ) = W_k(\succ'_s, \succ_{S\setminus\{s\}})$, and for all $s' \in W_k(\succ)$, $\psi_{s'}(\succ) = \psi_{s'}(\succ'_s, \succ_{S\setminus\{s\}})$.

Case 2: $k < k'$
If $k < k'$, then $W_k(\succ'_s, \succ_{S\setminus\{s\}}) = W_k(\succ) \setminus S_k(s, \succ)$. Since $s \in W_{k'}(\succ'_s, \succ_{S\setminus\{s\}})$, $E_k(s, \succ) = E_k(s, (\succ'_s, \succ_{S\setminus\{s\}}))$, and $E_k(s, (\succ'_s, \succ_{S\setminus\{s\}})) \subseteq E_{k'}(s, (\succ'_s, \succ_{S\setminus\{s\}}))$, we have $E_k(s, \succ) \subseteq E_k(s, (\succ'_s, \succ_{S\setminus\{s\}}))$.

Subcase 1: $S_k(s, \succ) = \{s\}$
If $\psi_s(\succ) = \chi_s(\succ)$, then it must be that $\psi_s(\succ'_s, \succ_{S\setminus\{s\}}) = \chi_s(\succ'_s, \succ_{S\setminus\{s\}})$. If $\psi_s(\succ) \neq \chi_s(\succ)$, then it must be that $\psi_s(\succ'_s, \succ_{S\setminus\{s\}}) \neq \chi_s(\succ'_s, \succ_{S\setminus\{s\}})$. Thus, in either case, we have $\psi_s(\succ) = \psi_s(\succ'_s, \succ_{S\setminus\{s\}})$.

[33]Note that this argument rests on $t = t'$. For two profiles $\succ$ and $\succ'$ and student i, it is always the case that $\psi_i(\succ) = \psi_i(\succ')$ implies $\chi_i(\succ) = \chi_i(\succ')$. However, the converse is not always true.

Subcase 2: $S_k(s, \succ) = \{s, \ldots, s'\}$ where $s \neq s'$

Suppose $\psi_s(\succ) \in E_k(s', \succ)$. Then, we know there exists a chain directed from $s' \in S_k(s, \succ)$ all the way to s, and that this chain is the same under both profiles: $H_k(s', s, \succ) = H_k(s', s, (\succ'_s, \succ_{S\setminus\{s\}}))$. By AR, we know that this chain will persist such that $H_k(s', s, (\succ'_s, \succ_{S\setminus\{s\}})) = H_k(s', s, (\succ'_s, \succ_{S\setminus\{s\}}))$. We know that under $\succ$, s pointed to s' in round k and the cycle $S_k(s, \succ)$ was removed. If $\psi_s(\succ) = \chi_s(\succ)$, then we know that $\chi_s(\succ) = \chi_s(\succ'_s, \succ_{S\setminus\{s\}})$ implies that s must point to s' in k' under $(\succ'_s, \succ_{S\setminus\{s\}})$, and thus, $\psi_s(\succ) = \psi_s(\succ'_s, \succ_{S\setminus\{s\}})$. If $\psi_s(\succ) \neq \chi_s(\succ)$, then we know that by the start of round k, there were no more regular seats on the market at school $\chi_s(\succ)$ under both profiles. s points to s' in round k under $\succ$ and the extended seats are in the endowment of s'. Thus, $\chi_s(\succ) = \chi_s(\succ'_s, \succ_{S\setminus\{s\}})$ implies that s must get an extended seat by pointing to s' in k' under $(\succ'_s, \succ_{S\setminus\{s\}})$, and thus, $\psi_s(\succ) = \psi_s(\succ'_s, \succ_{S\setminus\{s\}})$.

Therefore, we have shown that $k < k'$ implies $\psi_s(\succ) = \psi_s(\succ'_s, \succ_{S\setminus\{s\}})$, and thus, that the cycle $S_k(s, \succ)$ carries through and is removed in k'.

Thus, we have shown that $\psi_{s'}(\succ) = \psi_{s'}(\succ'_s, \succ_{S\setminus\{s\}})$ for all $s' \in W_k(\succ)$. Assume k is not the last round under $\succ$, since otherwise the proof would be finished.

Showing that for all rounds $z > k$, $\psi_{s'}(\succ) = \psi_{s'}(\succ'_s, \succ_{S\setminus\{s\}})$ for all $s' \in W^z(\succ)$:

We now show by induction that all assignments made in later rounds are the same under both profiles. Fix $z > k$ such that $W_z(\succ) \neq \emptyset$ (so someone is matched in stage z). Assume that for all $s' \in W^{z-1}(\succ)$, $\psi_{s'}(\succ) = \psi_{s'}(\succ'_s, \succ_{S\setminus\{s\}})$ (inductive step). There exists some cycle $S_z(h_1, \succ) = \{h_1, \ldots, h_g\}$ such that for all $i = 1, \ldots, g$, $\psi_{h_i}(\succ) \in E_z(h_{i+1}, \succ)$, where we let $h_{g+1} = h_1$. Note that $s \notin S_z(h_1, \succ)$ (because s was matched in round k under $\succ$, and $z > k$). Fix $\bar{i} \leq g$ (where $\bar{i} - 1 = g$ if $\bar{i} = 1$). Then $\psi_{h_{\bar{i}-1}}(\succ) \in E_z(h_{\bar{i}}, \succ)$. Then, for any student s' with higher priority at school $\psi_{h_{\bar{i}-1}}(\succ)$ than student $h_{\bar{i}}$, we know that $s' \in W^{z-1}(\succ)$, and hence, $\psi_{s'}(\succ) = \psi_{s'}(\succ'_s, \succ_{S\setminus\{s\}})$.

Then, for all z' such that $h_{\bar{i}} \notin W^{z'-1}(\succ'_s, \succ_{S\setminus\{s\}})$, we either have $\psi_{h_{\bar{i}-1}}(\succ) \in E_{z'}(h_{\bar{i}}, (\succ'_s, \succ_{S\setminus\{s\}}))$ or $\psi_{h_{\bar{i}-1}}(\succ) \in E_{z'}(s', (\succ'_s, \succ_{S\setminus\{s\}}))$ for some $s' \neq h_{\bar{s}}$, where s' is a student with higher priority at $\psi_{h_{\bar{i}-1}}(\succ)$ than $h_{\bar{i}}$. This means that

$$N(\psi_{h_{\bar{i}-1}}(\succ), z' - 1, (\succ'_s, \succ_{S\setminus\{s\}})) < \tilde{q}_{\psi_{h_{\bar{i}-1}}(\succ)}$$

Since this holds for all h_i with $i = 1, \ldots, g$, then for all z' such that $h_i \notin W^{z'-1}(\succ'_s, \succ_{S\setminus\{s\}})$, we have,

$$N(\psi_{h_i}(\succ), z' - 1, (\succ'_s, \succ_{S\setminus\{s\}})) < \tilde{q}_{\psi_{h_i}(\succ)}$$

For all $i = 1, \ldots, g$, we define z_i such that $h_i \in W_{z_i}(\succ'_s, \succ_{S\setminus\{s\}})$ holds. Since for all $s' \in W^{z-1}(\succ)$, $\psi_{s'}(\succ) = \psi_{s'}(\succ'_s, \succ_{S\setminus\{s\}})$, we have for all i that $\psi_{h_i}(\succ) \tilde{\succeq}_{h_i} \psi_{h_i}(\succ'_s, \succ_{S\setminus\{s\}})$. If there was some i such that $\psi_{h_i}(\succ'_s, \succ_{S\setminus\{s\}}) \tilde{\succ}_{h_i} \psi_{h_i}(\succ)$, then it must be that $\psi_{h_i}(\succ'_s, \succ_{S\setminus\{s\}})$ was not available in z under $\succ$. In other words, the assignments of the set of subjects comprising $W^{z-1}(\succ)$ are such that no other student can receive $\psi_{h_i}(\succ'_s, \succ_{S\setminus\{s\}})$ and have the final matching be feasible. Hence, if the students comprising $W^{z-1}(\succ)$ get the same assignments under $(\succ'_s, \succ_{S\setminus\{s\}})$, then there is no way that h_i could obtain $\psi_{h_i}(\succ'_s, \succ_{S\setminus\{s\}})$ under $(\succ'_s, \succ_{S\setminus\{s\}})$, which is clearly a contradiction. Therefore, we must have that $\psi_{h_i}(\succ) \tilde{\succeq}_{h_i} \psi_{h_i}(\succ'_s, \succ_{S\setminus\{s\}})$.

Suppose that $\psi_{h_i}(\succ) \tilde{\succ}_{h_i} \psi_{h_i}(\succ'_s, \succ_{S\setminus\{s\}})$. Then, it must be that $\psi_{h_i}(\succ)$ is no longer available by the time that h_i is matched under $(\succ'_s, \succ_{S\setminus\{s\}})$. This means that $N(\psi_{h_i}(\succ), z_i - 1, (\succ'_s, \succ_{S\setminus\{s\}})) = \tilde{q}_{\psi_{h_i}(\succ)}$. This, however, contradicts our previous result that for all h_i with $i = 1, \ldots, g$ and for all z' such that $h_i \notin W^{z'-1}(\succ'_s, \succ_{S\setminus\{s\}})$, we have

$$N(\psi_{h_i}(\succ), z' - 1, (\succ'_s, \succ_{S\setminus\{s\}})) < \tilde{q}_{\psi_{h_i}(\succ)}$$

Consequently, for all i, $\psi_{h_i}(\succ) = \psi_{h_i}(\succ'_s, \succ_{S\setminus\{s\}})$. Since this holds for every $S_z(h_1, \succ) \neq \emptyset$, it follows that for all $s' \in W^z(\succ)$, $\psi_{s'}(\succ) = \psi_{s'}(\succ'_s, \succ_{S\setminus\{s\}})$. Therefore, $\psi(\succ) = \psi(\succ'_s, \succ_{S\setminus\{s\}})$ by induction, and thus, $\chi(\succ) = \chi(\succ'_s, \succ_{S\setminus\{s\}})$.

□

For the final step, we apply Lemma 6 to conclude that ESTTC is strongly group strategyproof.

Part (ii)

Let μ denote the output of ESTTC, and suppose that it is inefficient. Then, there must be some alternative matching ν such that $\nu(s) \succeq_s \mu(s)$ for all s, strictly for some s. Let $S' = \{s \in S : \nu(s) \succ_s \mu(s)\}$ be the set of students who are strictly better off under ν, and let $S'' \subseteq S'$ be the students such that no student in S' was matched at an earlier round of ESTTC (there may be several such students). Call the round at which the students in S'' were assigned k, choose some $s' \in S''$, and let $\nu(s') = c$. Then, since s' did not receive c under matching μ, it must be that, the following are true (in the extended market):

(i) $|\mu_{k-1}(c)| = \tilde{q}_c$

and either:

(ii) $|\mu_{k-1}(c^*)| = \tilde{q}_{c^*}$ or (iii) $|\mu^*_{k-1}| = e$

Note that both (ii) and (iii) may hold. By (i), there are $\tilde{q}_c$ students assigned to (regular) school c by the beginning of round k. Note that each of these students must also receive c under ν (they must be weakly better off under ν by Pareto dominance, and cannot be strictly so since they are not members of S'', and thus must receive the same school under μ and ν).

Finally, there are two cases, depending on whether (ii) or (iii) holds. First, assume that (ii) holds. By a similar argument, each of the $\tilde{q}_{c^*}$ students assigned to c^* by the beginning of round k must also receive c (in the original market) under ν. Combining this with the previous paragraph, we see that under ν, there are $\tilde{q}_c + \tilde{q}_{c^*} + 1 = q_c + 1$ students assigned to c (where the $+1$ comes from student s'), which is a contradiction to the feasibility of ν.

Last, assume that (iii) holds. Let $C' = \{c' \in C : \exists s \in S \text{ s.t. } \tilde{\mu}(s) = c'^*\}$ be the set of schools such that some student is assigned to them through an extended seat under ESTTC. Since all extended seats are assigned before round k, we know that all students assigned schools in C' must receive the same school under ν. Thus, note that $|\nu(c')| \geq |\mu(c')|$ for all $c' \in C'$ and, at school c, we have $|\nu(c)| \geq |\mu(c)| + 1$. These two inequalities give:

$$|\nu(c)| + \sum_{c' \in C' \setminus \{c\}} |\nu(c')| > |\mu(c)| + \sum_{c' \in C' \setminus \{c\}} |\mu(c')|$$

For all $c' \notin C' \cup \{c\}$, we have $|\mu(c')| = p_{c'}$, and, since ν is feasible, we need $|\nu(c')| \geq |\mu(c')|$ for all such schools. Adding this to the equation above, we get

$$|\nu(c)| + \sum_{c' \in C' \setminus \{c\}} |\nu(c')| + \sum_{c' \notin C' \cup \{c\}} |\nu(c')| > |\mu(c)| + \sum_{c' \in C' \setminus \{c\}} |\mu(c')| + \sum_{c' \notin C \cup \{c\}} |\mu(c')|$$

The RHS is equal to n, and so we have

$$|\nu(c)| + \sum_{c' \in C' \setminus \{c\}} |\nu(c')| + \sum_{c' \notin C' \cup \{c\}} |\nu(c')| > n$$

which is clearly a contradiction.

G Proof of Theorem 8

Part (i)

Lemma 6 shows that strong group strategyproofness is equivalent to strategyproofness and nonbossiness. We first show that MSTTC is strategyproof. Consider a student s and preference profiles $\succ$ and $(\succ'_s, \succ_{S\setminus\{s\}})$. Student s cannot affect the stage in which he is matched,[34] and so is matched in stage k under both profiles. Furthermore, the assignments made in all stages before k are identical under both profiles, and so the submarket at stage k in which s participates is the same under both profiles. It is well-known that TTC is strategyproof on this submarket (see, for example, Abdulkadiroğlu and Sonmez (2003)), and thus MSTTC is strategyproof as well.

For nonbossiness, let χ denote the MSTTC algorithm, and suppose that $\chi_s(\succ'_s, \succ_{S\setminus\{s\}}) = \chi_s(\succ)$. Let k be the stage in which s is assigned when he submits his true preferences $\succ_s$. Note that for all s' assigned in stages $k' < k$, we have $\chi_{s'}(\succ'_s, \succ_{S\setminus\{s\}}) = \chi_{s'}(\succ)$. This implies that the submarkets in which s participates in stage k are equivalent under both profiles. Since the problem that s faces in stage k is equivalent to a standard multiple-copy TTC problem, which is nonbossy, we know that $\chi_s(\succ'_s, \succ_{S\setminus\{s\}}) = \chi_s(\succ)$ implies that, for all s' participating in stage k, $\chi_{s'}(\succ'_s, \succ_{S\setminus\{s\}}) = \chi_{s'}(\succ)$. Finally, this also implies that at the end of stage k, the set of students and schools remaining is identical under the two profiles, and so the algorithm continues in the same manner under $\succ$ or $(\succ'_s, \succ_{S\setminus\{s\}})$, i.e., all students s' assigned in stages $k'' > k$ also have $\chi_{s'}(\succ'_s, \succ_{S\setminus\{s\}}) = \chi_{s'}(\succ)$. So, $\chi(\succ'_s, \succ_{S\setminus\{s\}}) = \chi(\succ)$ (i.e., χ is nonbossy), and, by Lemma 6, χ is strongly group strategyproof.

The proof of part (ii), that MSTTC is efficient, follows standard TTC efficiency arguments, and is thus omitted.

H Procedure for minimizing r^k in the multi-stage mechanisms

We describe a procedure to obtain the smallest r^k for current minimum/maximum quotas $p^k = (p^k_{c_1}, \ldots, p^k_{c_m})$, $q^k = (q^k_{c_1}, \ldots, q^k_{c_m})$, where the number of remaining students is $n^k = n - \sum_c |\mu(c)|$. Our goal is to identify the smallest r^k, such that no matter how $n^k - r^k$ students are allocated, by allocating remaining r^k students appropriately, we can fill all of the remaining minimum quotas. If we set $r^k = \sum_c p^k_c$, it is clear that we can satisfy all of the minimum quotas. Our procedure checks whether we can set r^k smaller than $\sum_c p^k_c$. Let us assume when we decide to allocate n' students, an adversary chooses the worst outcome, i.e., the adversary selects an assignment of n' students such that minimum quotas are least satisfied. For a given n', let us denote $v(n')$ as the total number of students that are effective to reduce minimum quotas in the assignment of the

[34] Note that "stage" refers to the multiple stages in the MSTTC algorithm. Within each stage, there are a series of rounds where we remove cycles, i.e. run standard TTC with multiple copies. Although s may be removed in different *rounds* under the profiles, both of these rounds must be part of stage k.

adversary. If we know $v(n')$ for each n' where $n^k - \sum_c p_c^k \leq n' \leq n^k$, we can select the largest n' such that $\sum_c p_c^k - v(n') \leq n^k - n'$ holds. Then, r^k is chosen as $n^k - n'$. Now, the remaining question is how to obtain $v(\cdot)$. Let us assume the adversary first solves the following optimization problem. For a given p', which is the total number of students that are effective to reduce minimum quotas, find $u(p')$, which is the largest number of students that can be assigned without further reducing minimum quotas. If we know $u(\cdot)$, we can define $v(n')$ as follows: $v(n') = p'$, where p' is the smallest number where $u(p') \geq n'$. We can formalize the optimization problem of finding $u(\cdot)$ as the well-known knapsack problem Kellerer et al. (2004). In this formalization, we assume that p' represents the capacity of a knapsack. Also, we assume each school c is an item, whose capacity is p_c and its value is q_c. Furthermore, since we can partially assign students to a school, we assume there exist enough additional items, where the capacity/value of an additional item is 1. The goal of a knapsack problem is to select items such that the total value is maximized under the capacity constraint. When the capacity of the knapsack is bounded (which is true in our case), this problem can be solved efficiently using a standard dynamic programming procedure. Let us show an example. Assume there are 15 students and 10 schools. For each school c, $p_c^k = 1$ and $q_c^k = 2$. Here, $\sum_c p_c^k = 10$. Then, we obtain $u(0) = 0, u(1) = 2, u(2) = 4, \ldots, u(5) = 10, u(6) = 12, \ldots, u(10) = 20$. Thus, we obtain $v(1) = 1, v(2) = 1, v(3) = 2, v(4) = 2, \ldots, v(11) = 6, v(12) = 6, \ldots$. Thus, if we set $n' = 11$, $\sum_c p_c^k - v(n') = 10 - 6 = 4$. This is equal to $n^k - n' = 15 - 11 = 4$. On the other hand, if we set $n' = 12$, $\sum_c p_c^k - v(n') = 10 - 6 = 4 > 3 = 15 - 12 = n^k - n'$. Thus, the largest n' is 11 and the smallest r^k is 4. In other words, if the adversary allocates 11 students, the adversary ends up filling at least 6 seats that are effective to reduce minimum quotas. Then, the total number of remaining minimum quotas is 4. Thus, we can assign remaining 4 students to fill these remaining minimum quotas.

I Special cases: $m = 2$ or $p_c < q_c$ for at most one school or $p_c > 0$ for at most one school

First, consider the case of $m = 2$. When there are only two schools, we can simply impose artificial caps of $\tilde{q}_{c_1} = \min\{n - p_{c_2}, q_{c_1}\}$ and $\tilde{q}_{c_2} = \min\{n - p_{c_1}, q_{c_2}\}$.[35] Again, fairness and strategyproofness are immediate from the properties of DA. In this special case we also get nonwastefulness, because if a student s is rejected from her first choice c_i, it is because c_i is filled with $\tilde{q}_{c_i}$ students in the

[35]Note that $\tilde{q}$ is consistent with quotas p and q according to Definition 13, and so by Corollary 3, DA with maximum quotas $\tilde{q}$ will always produce a feasible matching. When $m > 2$, the natural extension of these caps, namely $\tilde{q}_{c_j} = \min\{q_{c_k}, n - \sum_{k \neq j} p_{c_k}\}$, will in general not be consistent with p and q.

first round of DA. If $\tilde{q}_{c_i} = q_{c_i}$, there are no empty seats at c_i and so s cannot claim an empty seat; on the other hand, if $\tilde{q}_{c_i} = n - p_2$, then it must be that $|\mu(c_j)| = p_2$, and so s cannot be moved to c_i without violating thte minimum quota at c_j. It is only in the special case of $m = 2$ that we can find artificial caps that will always deliver a nonwasteful assignment: when $m \geq 3$, it is in general not obvious which school should be ex-ante capped to ensure the minimum quotas at other schools are satisfied, and if a popular school is capped, the assignment may be very wasteful.

Next, consider the case $p_{c_i} < q_{c_i}$ for only c_i (and $p_{c_j} = q_{c_j}$ for all $i \neq j$). Here, any feasible matching will be such that $|\mu(c_j)| = q_{c_j}$ for $j \neq i$ and $|\mu(c_i)| = n - \sum_{j \neq i} p_{c_j}$. Since we know exactly how many seats will be assigned at every school in any feasible matching, standard DA with maximum quotas $\tilde{q}_{c_j} = q_{c_j} (= p_{c_j})$ for $j \neq i$ and $\tilde{q}_{c_i} = n - \sum_{j \neq i} p_{c_j}$ will produce a feasible matching for any preference profile. This will clearly be strategyproof and fair; it is nonwasteful as well because the only "empty" seats will be at school c_i. If a student s claims an empty seat, she must be at a school $c_j \neq c_i$ for which $|\mu(c_j)| = \tilde{q}_{c_j} = p_{c_j}$, i.e., s cannot be moved without violating the minimum quota at c_j.

Last, consider the case when $p_c > 0$ for at most one school. When this is the case, consider the following mechanism: run the ESDA mechanism, but, since c is the only school for which $p_c > 0$, let school c consist only of regular seats assigned according to $\succ_c$. Students apply just as for regular ESDA, but now, **at most** $n - \sum_{c \in C} p_c$ seats will be assigned at schools other than c (which all consist of extended seats). In particular, this leaves open the possibility of strictly less than this number of seats being assigned at other schools, if there is high demand for school c. This mechanism will be strategyproof, nonwasteful, and more fair than both ESDA and MSDA when $p_c < q_c$.[36] Of course, such a mechanism is not feasible if there is another school c' such that $p_{c'} > 0$, because when there is high demand for school c, this may not leave enough students to fill the minimum quota slots at c'.

J Alternate Definitions of Constrained Nonwastefulness

As mentioned in section 2.3, there are other natural definitions of constrained nonwastefulness one might consider. In this section, we discuss these definitions, and show that they lead to impossibility results, thus further justifying the definition used in the paper.

The most natural way to weaken nonwastefulness, which is followed by Ehlers et al. (2011), is to deny a student s the opportunity to move to an empty seat if doing so would cause some other student s' to justifiably envy s's new assignment. Call this property **constrained nonwastefulness-**

[36]While this mechanism is more fair than ESDA and MSDA, it is clearly not fully fair, as most of the seats are assigned according to $\succ_{ML}$.

A (cnw-A). Ehlers et al. (2011) shows that this definition is in fact incompatible with strategyproofness and fairness. Here, we strengthen this result by showing that it is in fact incompatible with all but σ_0-fairness. To do so, we add an additional requirement that the mechanism always produce a matching that is Pareto efficient up to fairness: that is, μ is **Pareto efficient up to fairness** if there is no other fair matching ν such that $\nu(s) \succeq_s \mu(s)$ for all $s \in S$, strictly so for some $s \in S$. In words, this says that if two students would like to trade their assignments, and this trade does not violate the priority of any other student (not involved in the trade), then this trade should be allowed to take place. This seems like a reasonable requirement (since the only reason many markets forego Pareto efficiency is to achieve fairness), and indeed the MSDA mechanism will satisfy this property. Then, since MSDA is already a mechanism that is σ_0-fair, Pareto efficient up to fairness, and fully nonwasteful (and thus is also cnw-A), there is no need to search for other cnw-A mechanisms.

Proposition 2. *Let χ be strategyproof, Pareto efficient up to fairness, and cnw-A. If χ is also σ-fair, then $\sigma = \sigma_0$.*

Proof. Assume that χ is σ-fair for some $\sigma \in \Sigma$ such that $\sigma_c > 0$ for some $c \in C$. Label this school c_2, and define $\sigma_{c_2} = \max\{1, p_{c_2}\}$. The problem can be reduced to a submarket the exact same way as in the proof of Theorem 6 (given in Appendix E).

First, consider the case where $p_{c_2} > 0$. The problem can be reduced to a subproblem with three students s_{n-2}, s_{n-1} and s_n and three schools c_1, c_2 and c_3 where c_1 and c_2 each have one minimum quota seat left and c_2 and c_3 each have (at least) one seat remaining overall. Consider the following preferences for $\{s_{n-2}, s_{n-1}, s_n\}$ and the corresponding quotas and priorities for the schools (again, see Appendix E for details):

s_{n-2}	s_{n-1}	s_n	c_1	c_2	c_3
c_2	c_3	c_3	$\vdots$	$\vdots$	$\vdots$
c_3	c_2	c_2	s_{n-2}	s_n	s_{n-2}
c_1	c_1	c_1	s_{n-1}	s_{n-2}	s_{n-1}
$\vdots$	$\vdots$	$\vdots$	s_n	s_{n-1}	s_n

Clearly, two of the three students must be assigned to the minimum quota seats, one at c_1 and the other at c_2. There are 3 possible ways to choose these two students:[37]

Case (i): s_{n-2} and s_{n-1} are assigned minimum quota seats. There are two subcases here. In the first, s_n is assigned to c_3, and there are two possible allocations here. One is shown by the boxes, the other by the circles.

[37]Note that in what follows, we can never assign two agents to c_1 since this would not be cnw-A.

$$
\begin{array}{c|ccc}
 & s_{n-2} & s_{n-1} & s_n \\
\hline
 & \boxed{c_2} & c_3 & \boxed{c_3} \\
 & c_3 & c_2 & c_2 \\
 & c_1 & \boxed{c_1} & c_1 \\
 & \vdots & \vdots & \vdots
\end{array}
$$

Note, however, that under the circled allocation, both s_{n-2} and s_{n-1} can form a σ_{c_3}-blocking pair with c_3 for any σ_{c_3}, since they are both ranked higher than s_n according to $\succ_{ML}$ and $\succ_{c_3}$. Likewise, under the boxed allocation, s_{n-1} can again form a σ_{c_3}-blocking pair with c_3. Thus, neither matching is σ-fair.

In the second subcase, s_n is assigned to c_2. There are again two possible allocations, indicated by boxes and circles:

$$
\begin{array}{c|ccc}
 & s_{n-2} & s_{n-1} & s_n \\
\hline
 & \boxed{c_2} & c_3 & c_3 \\
 & c_3 & c_2 & \boxed{c_2} \\
 & c_1 & \boxed{c_1} & c_1 \\
 & \vdots & \vdots & \vdots
\end{array}
$$

Now, consider the following monotonic transformation of the above:

$$
\begin{array}{c|ccc}
 & s_{n-2} & s_{n-1} & s_n \\
\hline
 & \boxed{c_2} & c_2 & c_3 \\
 & c_1 & \boxed{c_1} & \boxed{c_2} \\
 & c_3 & c_3 & c_1 \\
 & \vdots & \vdots & \vdots
\end{array}
$$

By strategyproofness, the assignments cannot change. But now, neither the boxed nor the circled allocations are cnw-A, since s_n can be feasibly moved to c_3 without causing any justified envy.

Case (ii): s_{n-2} and s_n are assigned the minimum quota seats. Now, there are two possible subcases depending on the assignment for s_{n-1}. First, let s_{n-1} be assigned to c_3. There are two allocations, indicated by boxes and circles:

$$
\begin{array}{c|ccc}
 & s_{n-2} & s_{n-1} & s_n \\
\hline
 & \boxed{c_2} & \boxed{c_3} & c_3 \\
 & c_3 & c_2 & c_2 \\
 & c_1 & c_1 & \boxed{c_1} \\
 & \vdots & \vdots & \vdots
\end{array}
$$

Here, under the circled allocation, s_{n-2} forms a σ_{c_3}-blocking pair with c_3. Under the boxed allocation, s_n forms a σ_{c_2}-blocking pair with c_2, because $s_n \succ_{c_2} s_{n-2}$ and $|\mu(c_2)| = p_{c_2} = \sigma_{c_2}$, and so s_n need not be higher on $\succ_{ML}$ to form a blocking pair. Thus, neither matching is σ-fair.

Now, consider the subcase in which s_{n-1} is assigned to c_2. The two possible allocations are again indicated by circles and boxes:

$$
\begin{array}{c|ccc}
 & s_{n-2} & s_{n-1} & s_n \\
\hline
 & \boxed{c_2} & c_3 & c_3 \\
 & c_3 & \boxed{c_2} & c_2 \\
 & c_1 & c_1 & \boxed{c_1} \\
 & \vdots & \vdots & \vdots
\end{array}
$$

The boxed allocation is not cnw-A because s_{n-1} can be moved to c_3 without causing any justified envy. For the circled allocation, again consider the following monotonic transformation:

$$
\begin{array}{c|ccc}
 & s_{n-2} & s_{n-1} & s_n \\
\hline
 & \boxed{c_2} & \boxed{c_2} & c_3 \\
 & c_1 & c_3 & c_2 \\
 & c_3 & c_1 & \boxed{c_1} \\
 & \vdots & \vdots & \vdots
\end{array}
$$

The circled allocation is now not cnw-A, since s_n can be moved to c_3 without causing any justified envy.

Case (iii): s_{n-1} and s_n are assigned the minimum quota seats. There are again two subcases. First consider that s_{n-2} is assigned to c_2.

$$
\begin{array}{c|ccc}
 & s_{n-2} & s_{n-1} & s_n \\
\hline
 & \boxed{c_2} & c_3 & c_3 \\
 & c_3 & \boxed{c_2} & c_2 \\
 & c_1 & c_1 & \boxed{c_1} \\
 & \vdots & \vdots & \vdots
\end{array}
$$

Under the boxed allocation, s_{n-1} can be moved to c_3 without causing justified envy, and so the matching is not cnw-A. For the circled allocation, consider the following monotonic transformation:

s_{n-2}	s_{n-1}	s_n
$\boxed{c_2}$	c_1	c_3
c_3	$\boxed{c_2}$	c_2
c_1	c_3	$\boxed{c_1}$
$\vdots$	$\vdots$	$\vdots$

By strategyproofness, the allocation must remain the circled one, but now this is not cnw-A, since s_n can be moved to c_3.

Last, consider the subcase where s_{n-2} is assigned to c_3.

s_{n-2}	s_{n-1}	s_n
c_2	c_3	c_3
$\boxed{c_3}$	$\boxed{c_2}$	c_2
c_1	c_1	$\boxed{c_1}$
$\vdots$	$\vdots$	$\vdots$

Under the boxed allocation, s_n forms a σ_{c_2}-blocking pair with c_2, since $|\mu(c_2)| = p_{c_2}$.

As for the circled allocation, consider the following monotonic transformation:

s_{n-2}	s_{n-1}	s_n
c_2	c_1	c_3
$\boxed{c_3}$	$\boxed{c_2}$	c_2
c_1	c_3	$\boxed{c_1}$
$\vdots$	$\vdots$	$\vdots$

Now, the circled allocation is not Pareto efficient up to fairness, because s_{n-2} and s_n would prefer to trade their seats, and this can be done without causing justified envy from s_{n-1}.

Last, we consider the case when $p_{c_2} = 0$, and recall that now, towards a contradiction, we have $\sigma_{c_2} = \max\{1, p_{c_2}\} = 1$. Everything is the same as before, except that we let define $S' = S \setminus \{s_{n-1}, s_n\}$ (i.e., s_{n-2} now belongs to S'), and S'_2 is empty. This then leaves us with a smaller submarket of two students, s_{n-1} and s_n, and three schools, c_1, c_2, and c_3, where $|\mu'(c_1)| = p_{c_1} - 1$ (so c_1 has one minimum quota seat left to fill) and $p_{c_2} \leq |\mu'(c_2)| < q_{c_2}$ (so c_2 has no minimum quotas to fill, but has at least one empty seat), and $|\mu'(c_3)| < q_{c_3}$. Again, consider the following preferences for s_{n-1} and s_n, and the priorities from above:

s_{n-1}	s_n	c_1	c_2	c_3
c_2	c_3	$\vdots$	$\vdots$	$\vdots$
c_3	c_2	s_{n-2}	s_n	s_{n-2}
c_1	c_1	s_{n-1}	s_{n-1}	s_{n-1}
$\vdots$	$\vdots$	s_n	s_{n-2}	s_n

There are again two cases, depending who is assigned to the minimum quota seat at c_1. First, assume that s_{n-1} is assigned to c_1. Then, s_n cannot be assigned c_3, since s_{n-1} would form a $\sigma_{c_3}-$blocking pair. Thus, the allocation must be that shown in the boxes below:

s_{n-1}	s_n
c_2	c_3
c_3	$\boxed{c_2}$
$\boxed{c_1}$	c_1
$\vdots$	$\vdots$

Then, consider the following monotonic transformation:

s_{n-1}	s_n
c_2	c_3
$\boxed{c_1}$	$\boxed{c_2}$
c_3	c_1
$\vdots$	$\vdots$

Now, note that this is not cnw-A because s_n can be moved to c_3 without causing justified envy from s_{n-1}.

Last, assume that we began by assigning the minimum quota seat at c_1 to s_n. Then, s_{n-1} cannot be assigned to c_2, because $|\mu(c_2)| = 1 = \sigma_{c_2}$, and so s_n can form a $\sigma_{c_2}-$blocking pair. Thus, the allocation must be:

s_{n-1}	s_n
c_2	c_3
$\boxed{c_3}$	c_2
c_1	$\boxed{c_1}$
$\vdots$	$\vdots$

However, the following monotonic transformation then is not nonwasteful, as s_{n-1} can now be moved to c_2 without causing justified envy from s_n:

	s_{n-1}	s_n
	c_2	c_3
	$\boxed{c_3}$	$\boxed{c_1}$
	c_1	c_2
	$\vdots$	$\vdots$

□

Nonwastefulness can be weakened even further by denying a student s an opportunity to move to an empty seat if doing so would cause some other student s' to justifiably envy the new assignment of student s according to **either** $\succ_c$ **or** $\succ_{ML}$. Formally, we say that matching μ is **constrained nonwasteful-B (cnw-B)** if, whenever s claims an empty seat at some school c, there exists another student s' such that $c \succ_{s'} \mu(s')$ and (i) $s' \succ_c s$ or (ii) $s' \succ_{ML} s$. Unfortunately, cnw-B is also incompatible with strategyproofness and all but σ_0-fairness:

Theorem 13. *Let χ be strategyproof, Pareto efficient up to fairness, and cnw-B. If χ is also σ-fair, then $\sigma = \sigma_0$.*

Proof. The argument is the same as the proof of Proposition 2. Simply note that every allocation that was not cnw-A in that proof is also not cnw-B.

□

Thus, since MSDA is already a mechanism that is strategyproof, Pareto efficient up to fairness, σ_0-fair, and fully nonwasteful, the above theorems show that there is no need to search for mechanisms that are cnw-A or B. We can weaken nonwastefulness as in the text and find a more fair mechanism than MSDA (namely, ESDA). However, there is no other (obvious) way to weaken nonwastefulness that will allow us to find a more fair mechanism, and so can just use MSDA.