Stratifications of equivariant varieties

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Let G be a compact Lie group and V and W be linear G
spaces. A study is made of the canonical stratification of some
algebraic varieties that arise naturally in the theory of C^∞
equivariant maps from V to W. The main corollary of our
results is the equivalence of Bierstone's concept of "equivariant
general position" with our own of "G transversal". The paper
concludes with a description of Bierstone's higher order
conditions for equivariant maps in the framework of
equisingularity sequences.

Let G be a compact Lie group and V and W be linear G spaces
(that is, G representations). A map $f : V \to W$ is said to be equi-
variant if $f(gv) = gf(v)$ for all $g \in G$ and $v \in V$. In this paper we
study the stratifications of some algebraic varieties that arise naturally
in the theory of C^∞ equivariant maps from V to W. As corollaries of
our results we obtain the equivalence of Bierstone's definition of
"equivariant general position" [1], and our definition of "G transversal"
[3, 4]. We also obtain a reformulation of Bierstone's higher order
conditions described in [2].

1. Stratifications

We start by recalling some facts about stratifications of real
algebraic sets (for more details we refer to Mather's exposition in [6]).

Let X be an algebraic subset of R^m . A stratification of X is a locally finite partition of X into connected submanifolds of R^m , the strata of X , such that the frontier (in R^m) of each stratum is a union of lower dimensional strata. For $0 \le j \le m$, we define X_j to be the union of all strata of dimension j and refer to X_j as the

j-dimensional stratum of X . Whilst X_j is a submanifold of R^m of dimension j (possibly empty), it need not be connected nor need the frontier of X_j be a union of X_k's , $k < j$. In the sequel, we shall always regard the stratification of X as consisting of the set of j-dimensional strata of X , $0 \le j \le m$, and we shall denote the stratification of X by $\{X_j\}$. We say that the stratification is an α-stratification, where α may be C^∞ , analytic, semi-algebraic, and so on, if the strata are all of class α . The stratification is termed Whitney regular if, given any pair (X_j, X_k) , with $j < k$, Whitney's regularity condition (b) holds at every point x of $X_j \cap \overline{X_k}$:

If $\{p_i\}$ and $\{q_i\}$ are sequences of points of X_j and X_k respectively such that both sequences converge to x , the line joining p_i to q_i converges to a line L, and the tangent space $T_{q_i} X_k$ converges in the grassmannian of k-dimensional subspaces of R^m to T , then $T \supset L$.

A minimal α-stratification of X is a Whitney stratification $\{X_j\}$ of X such that if $\{Y_j\}$ is any other Whitney α-stratification of X , then either the two stratifications are the same or there exists j_0 such that $X_j = Y_j$, $j > j_0$, and $X_{j_0} \supset Y_{j_0}$. Clearly if a minimal α-stratification of X exists, it is unique. As described in [6], X has a minimal semi-algebraic stratification - the so called canonical stratification of X . This stratification is also the minimal C^∞ and (strongly) analytic stratification of X .

Suppose that V and W are finite dimensional vector spaces and that

we are given a polynomial map

$$P : V \times R^k \to W .$$

Let $X = P^{-1}(0)$ and, for each $t \in R^k$, set $X_t = \{x \in V : P(x, t) = 0\}$. We are interested in the family of germs of X_t at zero as t varies in R^k (*cf.* Varčenko [9]). Giving X the canonical stratification $\{X_j\}$, we define a Whitney stratification of $V \times R^k$ by taking as top dimensional stratum $(V \times R^k) \backslash X$ and $\{X_j\}$ for the remaining strata. It is convenient to continue to use the notation $\{X_j\}$ for this stratification of $V \times R^k$ and we shall do this in the sequel. Finally, we shall regard R^k as embedded in $V \times R^k$ as $\{0\} \times R^k$. We recall from [4].

DEFINITION. An equisingularity sequence for the family $\left\{ X_t : t \in R^k \right\}$ is a Whitney stratification $\{\Lambda_i\}$ of R^k such that

$$\left\{ X_j \backslash R^k, \, \Lambda_i \, : \, 0 \leq j \leq k+\dim(V), \, 0 \leq i \leq k \right\}$$

is a Whitney stratification of $X \cup R^k$ and $X_j \cap \Lambda_i$ is empty for $j < i$, and an open, possibly empty, subset of Λ_i for $j = i$.

DEFINITION. We say that the equisingularity sequence $\{\Lambda_i\}$ is a fundamental equisingularity sequence for $\left\{ X_t : t \in R^k \right\}$ if it is minimal amongst the set of semi-algebraic equisingularity sequences for $\left\{ X_t : t \in R^k \right\}$.

REMARKS. 1. The methods of [4] show that a fundamental equisingularity sequence for $\left\{ X_t : t \in R^k \right\}$ exists and is in fact minimal amongst C^∞ and strongly analytic equisingularity sequences. Since it is clearly unique we shall in future refer to it as the fundamental equisingularity sequence of $\left\{ X_t : t \in R^k \right\}$. In the next section we shall

give a rather explicit characterization of fundamental equisingularity
sequences for a class of algebraic sets arising in the theory of equi-
variant maps.

2. In [4], an equisingularity sequence was defined to be a sequence
$R^k = A_0 \supseteq A_1 \supseteq \cdots \supseteq A_k$ such that the sets $\Lambda_{k-i} = A_i \backslash A_{i+1}$ satisfied the
first definition above. In other words, it was defined in terms of the
filtration by codimension of the stratification of R^k.

3. The sets Λ_j give a measure of the degree of Whitney equi-
singularity of the family $\left\{ X_t : t \in R^k \right\}$. For example, it follows from
Thom's First Isotopy Lemma that $\left\{ \text{germ}(X_t) \text{ at zero} : t \in R^k \right\}$ is locally
topologically trivial over Λ_k.

We conclude this section with some remarks about the pullback of a
Whitney stratification. Let $P : R^m \times R^n \to R^n$ be a C^∞ map. We say that
P is a projection map if there exists a C^∞ diffeomorphism h of
$R^m \times R^n$ such that the composite Ph is the projection of $R^m \times R^n$ on
R^n. In case P is also a polynomial, we say that P is an algebraic
projection map. We have the following trivial

PROPOSITION. *Let* $P : R^m \times R^n \to R^n$ *be an algebraic projection map
and* X *be an algebraic subset of* R^n *with canonical stratification*
$\{X_j\}$ *. Then* $\left\{ P^{-1} X_j \right\}$ *is the canonical stratification of the algebraic set*
$P^{-1}X$ *.*

REMARK. If we had only required P to be a submersion, it would
follow from [6, Corollary 8.8] that $\left\{ P^{-1} X_j \right\}$ is a Whitney stratification
on $P^{-1}X$. However, it is easy to find examples showing that it need not
be minimal.

2.   Varieties defined by equivariant maps

Let G be a compact Lie group and V and W be linear G-spaces.

We let $P_G(V, W)$ denote the space of equivariant polynomial maps from V to W and $P_G(V)$ denote the space of real valued G-invariant polynomials on V. As is well known, the compactness of G implies that $P_G(V, W)$ is a finitely generated $P_G(V)$-module and $P_G(V)$ is finitely generated as an algebra (see, for example, [7]).

For the remainder of this paper we shall assume that V has no proper subspaces on which G acts trivially (that is, V contains no trivial G representations).

Let $F_1, \ldots, F_s \in P_G(V, W)$ and define $F : V \times R^s \to W$ by

$$F(x, t) = \sum_{j=1}^{s} t_j F_j(x) \ ,$$

where $t = (t_1, \ldots, t_s) \in R^s$ and $x \in V$. Clearly $F \in P_G(V \times R^s, W)$, where we have given $V \times R^s$ the G-action defined by taking the product of the given action on V with the trivial action on R^s.

Set $X = F^{-1}(0)$ and let $\{X_j\}$ denote the canonical stratification of X (the top dimensional stratum is, by the convention of Section 1, $(V \times R^s) \backslash X$). The minimality of the stratification implies that each stratum is a G-invariant submanifold of $V \times R^s$. Let $\Pi : V \times R^s \to R^s$ denote the projection on R^s and define the family $\left\{X_t' : t \in R^s\right\}$ by setting $X_t = \Pi^{-1}(t) \cap X = \{x \in V : F(x, t) = 0\}$.

One further piece of notation before we state our main result. If Y is a connected analytic subset of R^s we define

$$\mathrm{reg}_i Y = \emptyset \ , \quad i \neq \dim(Y)$$
$$= \text{regular part of } Y \ , \quad i = \dim(Y) \ .$$

More generally, if $\cup X_\alpha$ is the decomposition of an analytic subset X of R^s into its connected components, we define

$$\text{Reg}_i X = \bigcup_\alpha \text{reg}_i X_\alpha \ .$$

THEOREM. *A fundamental equisingularity sequence* $\{\Lambda_i\}$ *for*
$\{X_t : t \in R^s\}$ *exists. It is characterised by any of the following equivalent conditions:*

1. $\Lambda_i = \bigcup_j \text{reg}_i \left(X_j \cap R^s \right)$, $\quad 0 \le i \le s$;

2. *if* $t \in X_j \cap R^s$, *then* $t \in \Lambda_i$ *if and only if*
    $\dim \left(T_t X_j \cap R^s \right) = i$;

3. *if* $t \in X_j \cap R^s$, *then* $t \in \Lambda_i$ *if and only if*
    $\dim \left(\Pi \left(T_t X_j \right) \right) = i$.

Finally suppose $t \in \Lambda_i \cap X_j$. *Then the connected component of*
$X_j \cap R^s$ *containing* t *is an open subset of* Λ_i .

REMARK. We must emphasise that this theorem is special to the class of equivariant sets that we are considering. None of the statements of the theorem holds for arbitrary varieties.

Before starting the proof of the theorem we shall prove two lemmas.

LEMMA 1. *Suppose* U· *is a* G*-invariant linear subspace of* $V \times R^s$. *Then*

$$U = (V \cap U) \oplus \left(R^s \cap U \right) \ .$$

In particular,

$$U \cap R^s = \Pi(U) \ .$$

Proof. U has a unique decomposition $A \oplus B$, where A is a sum of (non-trivial) irreducible G-representations and B is a trivial representation. Since V does not contain any trivial representation, $A \subset V$ and $B \subset R^s$.   //

LEMMA 2. *Suppose* $t \in R^s \cap X_j$ *and* $\dim T_t X_j \cap R^s = i$. *Then the connected component of* $X_j \cap R^s$ *containing* t *is a submanifold of* R^s *of dimension* i .

Proof. Let A denote the connected component of X_j containing t and B denote the component of $A \cap R^s$ containing t . By Lemma 1, $T_b A \cap R^s = \Pi(T_b A)$ for $b \in B$ and so a simple semi-continuity argument shows that $\dim\left(T_b A \cap R^s\right) = i$, for all $b \in B$.

A is a G-invariant submanifold of $V \times R^s$ and so, given $b \in B$, there exists an open neighbourhood U of b in $V \times R^s$ and a C^∞ equivariant diffeomorphism $h : U \rightarrow h(U) \subset V \times R^s$ which maps $A \cap U$ onto an open neighbourhood of zero in $T_b A$. Since h is equivariant, $h\left(R^s \cap U\right) \subset R^s$ and so h maps $A \cap U \cap R^s$ onto an open neighbourhood of zero in $T_b A \cap R^s$. Hence B is a submanifold of R^s of dimension i .//

Proof of Theorem. Lemma 1 implies Conditions 2 and 3 are equivalent; Lemma 2 implies Conditions 1 and 2 are equivalent. We claim that $\{\Lambda_i\}$ is an equisingularity sequence. Condition 1 implies that $X_j \cap \Lambda_i$ is empty for $j < i$ and an open subset of Λ_i for $j = i$. It remains to be checked that $\left\{X_j \backslash R^s, \Lambda_i\right\}$ is a Whitney stratification of $X \cup R^s$. Now $\left(X_j \backslash R^s, X_k \backslash R^s\right)$, $j < k$, and $\left(\Lambda_j, R^s \backslash X_i\right)$, $j < i$, satisfy Whitney's regularity condition (b) where applicable since the stratification $\{X_j\}$ is Whitney. Let $t \in \Lambda_j \cap \overline{\Lambda}_k$, $k > j$. We must verify that condition (b) holds at t for the pair (Λ_j, Λ_k) . Let $p_i \rightarrow t$, $q_i \rightarrow t$ be sequences of points of Λ_j and Λ_k respectively such that the line joining p_i to q_i converges to L and $T_{q_i} \Lambda_k$ converges to T , where T is a k-dimensional subspace of R^s . Taking subsequences if necessary, we may

suppose that $p_i \in X_\alpha$, $q_i \in X_\beta$, for some fixed α and β , and that

$T_{q_i} X_\beta$ converges to a β-dimensional subspace $\tilde{T}$ of $V \times R^s$. Now in

general $\tilde{T}$ will not contain T and all that we can say is $\Pi(\tilde{T}) = T$.

But since our stratification is G-invariant and V does not contain any

trivial representation, we may apply Lemma 1 to deduce that $\tilde{T} \cap R^s = T$.

Since $\{X_j\}$ is Whitney regular, $\tilde{T} \supset L$. But $L \subset R^s$ and so $T \supset L$.

Since $\left(X_i \backslash R^s\right) \cap \overline{\Lambda}_j = \emptyset$, for all i and j , condition (b) is vacuously

satisfied for the pair $\left(X_i \backslash R^s, \Lambda_j\right)$. As the frontier conditions clearly

hold, it follows that $\left\{X_j \backslash R^s, \Lambda_i\right\}$ is a Whitney stratification of

$X \cup R^s$. Finally we remark that the equisingularity sequence that we have

constructed is obviously minimal and so the fundamental equisingularity

sequence of $\left\{X_t : t \in R^s\right\}$.   //

REMARK*. If, following [3, 4], we define $A_j = \bigcup_{i \le s-j} \Lambda_i$, $1 \le j \le s$,

it is easy to verify that A_j is a closed semi-algebraic cone, vertex

origin. It is natural to ask whether the sets A_j are projective

algebraic varieties or, what amounts to the same thing, if they are

analytic cones. Indeed, if we had taken the stratification of X defined

by taking the real part of the canonical complex analytic stratification of

the complexification of X the equisingularity sequence that we obtain

does give projective algebraic subsets. However, this stratification need

not coincide with the canonical stratification. For example, the

stratifications differ for the varieties defined by $x\left(x^2+y^2\right)$ or

$y^2 + z^2 x^3 + x^5$. In any case, we ask: Are the varieties A_j projective

algebraic?

Continuing with the notation and assumptions of the theorem, suppose

we are given polynomials $p_{kj} \in P_G(V)$, where $1 \le i \le s$ and, for each

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* I am grateful to T.C. Kuo for pointing out these examples to me.

i , $1 \le j \le m_i$. We shall assume that $p_{ij}(0) = 0$ for all i and j (since V contains no trivial representations, this implies that Dp_{ij} vanishes at zero for all i and j).

Set $m = s + \sum_{i=1}^{s} m_i$ and give R^m the coordinates

$\left(t_1, t_{11}, \ldots, t_{1m_1}; t_2, t_{21}, \ldots, t_{sm_s} \right)$. Let $p : R^m \to R^s$ be the obvious projection map. We also use the notation p for the projection (Id $\times p$) : $V \times R^m \to V \times R^s$. Define $Q : V \times R^m \to W$ by

$$Q(x,\ t) = \sum_{i=1}^{s} \left[t_i + \sum_{j=1}^{m_i} t_{ij} p_{ij}(x) \right] F_i(x) \ .$$

Q is equivariant if we give $V \times R^m$ the G action defined by taking the product of the given action on V with the trivial action on R^m . Denote the fundamental equisingularity sequence associated to the variety defined by Q by $\{\Sigma_i\}$.

PROPOSITION.

$$\Sigma_i = p^{-1} \Lambda_i \ , \quad i = m, \ldots, m-s$$
$$= \emptyset \ , \quad i < m-s \ .$$

Proof. We claim that the map $\tilde{p} : V \times R^m \to V \times R^s$ defined by

$$\tilde{p}\left(x,\ t_i,\ t_{ij} \right) = \left(x,\ t_i + \sum_{j=1}^{m_i} t_{ij} p_{ij}(x) \right)$$

is a projection map in the sense of Section 1. Indeed the map $h : V \times R^m \to V \times R^m$ defined by

$$h\left(x,\ t_i,\ t_{ij} \right) = \left(x,\ t_i - \sum_{j=1}^{m_i} t_{ij} p_{ij}(x),\ t_{ij} \right)$$

is a C^∞ equivariant diffeomorphism and $\tilde{p}h = p$.

Applying the proposition at the end of Section 1, we see that the pull-

back of the canonical stratification of X by $\tilde{p}$ gives the canonical stratification of $Q^{-1}(0)$. The characterisation of a fundamental equisingularity sequence given by Condition 1 of the theorem implies that the fundamental equisingularity sequence for $Q^{-1}(0)$ is given by

$$\left\{\tilde{p}^{-1}(\Lambda_i)\right\} = \left\{p^{-1}(\Lambda_i)\right\} . \qquad //$$

3. Applications to the theory of equivariant maps

Let $C_G^\infty(V, W)$ denote the space of C^∞ equivariant maps from V to W and $C_G^\infty(V)$ denote the space of C^∞ equivariant real valued maps on V . $C_G^\infty(V, W)$ has the structure of a $C_G^\infty(V)$-module. Fix a minimal set of homogeneous polynomials which generate $P_G(V, W)$ as a $P_G(V)$-module, say $\{F_1, \ldots, F_k\}$. Then $\{F_1, \ldots, F_k\}$ generate $C_G^\infty(V, W)$ as a $C_G^\infty(V)$-module (for a proof of this non-trivial fact, see [1, 4, 7]). Moreover, if R^μ is a trivial G-space, $\{F_1, \ldots, F_k\}$ also generate $C_G^\infty(V \times R^\mu, W)$ as a $C_G^\infty(V \times R^\mu)$-module. Let $\{X_j\}$ denote the canonical stratification of the variety $X \subset V \times R^k$ defined by the polynomial $F(x, t) = \sum t_j F_j(x)$ and $\{\Lambda_i\}$ denote the corresponding fundamental equisingularity sequence.

We shall briefly review two definitions of equivariant transversality. Full details are given in [1, 4]. Let $f \in C_G^\infty(V \times R^\mu, W)$. Then there exist $f_j \in C_G^\infty(V \times R^\mu)$ such that

$$f(x, u) = \sum_{j=1}^{k} f_j(x, u)F_j(x) .$$

Whilst the maps f_j are not necessarily uniquely determined by f , the map $\gamma_f : R^\mu \to R^k$ defined by $\gamma_f(u) = (f_1(0, u), \ldots, f_k(0, u))$ depends only on f and the choice of generators $\{F_1, \ldots, F_k\}$. We let $G_f : V \times R^\mu \to V \times R^k$ denote the "graph" map defined by

$G_f(x, u) = \left(x, f_1(x, u), \ldots, f_k(x, u)\right)$. Both γ_f and G_f are C^∞ .

DEFINITION A (adapted from Bierstone [1]). Let $f \in C_G^\infty(V \times R^\mu, W)$.
f is said to be in equivariant general position to $0 \in W$ at
$(0, 0) \in V \times R^k$ if either $f(0, 0) \neq 0$ or $f(0, 0) = 0$ and the map
$G_f : V \times R^\mu \rightarrow V \times R^k$ is transversal to the canonical stratification of X
at zero.

DEFINITION B ([4]). Let $f \in C_G^\infty(V \times R^\mu, W)$. f is said to be
G-transversal to $0 \in W$ at $(0, 0) \in V \times R^k$ if either $f(0, 0) \neq 0$ or
$f(0, 0) = 0$ and $\gamma_f : R^\mu \rightarrow R^k$ is transversal to the fundamental equi-
singularity sequence $\{\Lambda_i\}$ at zero.

We refer to [6] for the definition and basic properties of maps
transversal to a Whitney stratification.

PROPOSITION. *Definitions* A *and* B *are equivalent.*

Proof. Let $f \in C_G^\infty(V \times R^\mu, W)$ and suppose $f(0, 0) = 0$. Let
$t = \gamma_f(0) \in \Lambda_i$ and $G_f(0, 0) \in X_j$. Now if f is in equivariant general
position to $(0, 0)$ at 0 , $T_t X_j$ complements $\left(T_{(0,0)} G_f\right)\left(V \times R^\mu\right)$ in
$V \times R^k$. Since G_f is the identity on $V \times \{0\}$, this is equivalent to
$\left(T_0 \gamma_f\right)\left(R^\mu\right)$ complementing $\Pi\left(T_t X_j\right)$ in R^k . By Condition 3 of our theorem
characterising fundamental equisingularity sequences, this is equivalent to
$\left(T_0 \gamma_f\right)\left(R^\mu\right)$ complementing $T_t \Lambda_i$ in R^k . But this is just the condition
for G-transversality.   //

REMARKS. 1. We have put the definition of equivariant general
position in a form that allows us to prove its equivalence with
G-transversality with the least effort. In [1], the requirement that V
does not contain any trivial representations is dropped. The definition is
then naturally stated as the transversality of the graph map
$x \longmapsto \left(x, f_1(x), \ldots, f_k(x)\right)$ to the canonical stratification of the variety

associated to a minimal set of homogeneous generators of $P_G(V, W)$.

2. Since equivariant general position is an open condition [1], the
proposition implies the openness of G-transversality, answering a question
raised in [4].

For the remainder of this section we intend to reformulate Bierstone's
higher order equivariant general position conditions described in [2] in
the framework of equisingularity sequences.

As Bierstone points out by an example in [1], although Definitions A
and B are quite satisfactory from the point of view of stability of
intersections, the maps that can occur are sometimes undesirably singular.

EXAMPLE. Consider the space of C^∞ S^1 invariant maps from C^2 to
C^2 , where the S^1 action on the domain is multiplication by $(e^{3i\theta}, e^{4i\theta})$
and on the target is multiplication by $(e^{10i\theta}, e^{11i\theta})$. A minimal set of
homogeneous generators for $P_{S^1}(C^2, C^2)$ is $\left\{\left(z_1^2 z_2, 0\right), \left(0, z_1 z_2^2\right)\right\}$ and
the corresponding fundamental equisingularity sequence is

$$\Lambda_4 = \left\{(t_1, t_2) \in C^2 : (t_1, t_2) \neq (0, 0)\right\} ;$$

$$\Lambda_3 = \Lambda_2 = \Lambda_1 = \emptyset ; \quad \Lambda_0 = \{(0, 0)\} .$$

(The subscripts refer to real dimension.) For $(t, s) \in C^2$, let
$f_{t,s}(z_1, z_2) = \left(t z_1^2 z_2, s z_1 z_2^2\right)$. $f_{t,s}$ is C^∞ and S^1 invariant and is
S^1 transversal to zero at zero provided both t and s are not zero.
However, the maps $f_{t,0}$ and $f_{0,s}$ are easily seen to be highly singular
at zero by comparison with $f_{t,s}$, $t \neq 0$, $s \neq 0$. For instance,
$Df_{t,s}(z_1, z_2)$ is a linear isomorphism off the z_1 and z_2 axes if t
and s are non-zero. It is desirable that the transversality definition
exclude the cases when either s or t is zero.

We make no attempt here to give a full description of the higher order
theory developed by Bierstone to overcome this deficiency in the definition
as a full presentation is given in his paper [2]. Instead we confine
ourselfes to outlining the theory to a point where we can state our

formulation of the higher order conditions.

Let $\{p_1, \ldots, p_1\}$ denote a minimal set of homogeneous algebra generators for the R-algebra $P_G(V)$. If we set

$$P = (p_1, \ldots, p_1) : V \to R^1$$

and let $P^* : P(R^1) \to P_G(V)$ denote the map defined by $P^*(q) = qP$, $q \in P(R^1)$, then P^* is surjective (see, for example, [7]).

Schwarz' theorem [8], states that $P^* : C^\infty(R^1) \to C^\infty_G(V)$ is also surjective. That is, given $f \in C^\infty_G(V)$, there exists $g \in C^\infty(R^1)$ such that $f = gP$.

If G acts trivially on R^μ , then $\{p_1, \ldots, p_1, t_1, \ldots, t_u\}$ is a minimal set of homogeneous generators for $P_G(V \times R^\mu)$ and so, given $h \in C^\infty_G(V \times R^\mu)$, there exists $g \in C^\infty(R^1 \times R^\mu)$ such that $h(x, t) = g(P(x), t)$. Consequently, any $f \in C^\infty_G(V \times R^\mu, W)$ may be written, not necessarily uniquely, in the form

$$f(x, t) = \sum_{j=1}^{k} g_j(P(x), t)F_j(x) ,$$

$g_j \in C^\infty_G(R^1 \times R^\mu)$. The higher order conditions described by Bierstone come by differentiating this equation.

We use the notation of [5] to describe jets and jet spaces. Thus, if $f \in C^\infty(V, W)$, we let $j^d f(x) = (f(x), Df(x), \ldots, D^d f(x))$ denote the d-jet of f at x , $d = 0, 1, \ldots$. We denote the corresponding jet space by $J^d(V, W)$ and recall that

$$J^d(V, W) = \bigoplus_{i=0}^{d} L^i_s(V; W) ,$$

where $L_s^i(V; W)$ is the space of i-linear symmetric maps from V to W .

Define $Q : V \times R^\mu \to R^1 \times R^\mu$ by $Q(x, t) = (P(x), t)$.

$DQ : V \times R^\mu \to L(V \times R^\mu, R^1 \times R^\mu)$ and $DQ(x, t) = (DP(x), Id)$. Hence DQ depends only on x, and we shall write $DQ(x)$ for $DQ(x, t)$ in the sequel.

Before we proceed further we need to recall some linear algebra. Suppose E and F are vector spaces and that we are given positive integers $m_1, \ldots, m_s; q_1, \ldots, q_s$. Set $p = m_1 q_1 + \ldots + m_s q_s$ and $q = q_1 + \ldots + q_s$. We define the "symmetrization" map

$$\text{symm}^* : \mathop{\times}_{j=1}^{s} \left(L_s^{m_j}(E; F) \right)^{q_j} \to L_s^p(E; \times^q F)$$

by

$\text{symm}^*\big((A_1), \ldots, (A_s)\big)(e_1, \ldots, e_p) =$
$$1/(q_1! \ldots q_s!) \sum \left[A_1^1\big(e_{\sigma(1)}, \ldots, e_{\sigma(m_1)}\big), \right.$$
$$\left. \ldots, A_s^{m_s}\big(e_{\sigma(p-m_s+1)}, \ldots, e_{\sigma(p)}\big) \right] ,$$

where $(e_1, \ldots, e_p) \in \times^p E$, $(A_j) = \left[A_j^1, \ldots, A_j^{q_j} \right] \in \left(L_s^{m_j}(E; F) \right)^{q_j}$, and the sum is taken over all permutations σ of $\{1, \ldots, p\}$ such that, if we set $M_{ij} = m_1 q_1 + \ldots + m_{j-1} q_{j-1} + i m_j$,

$\sigma(M_{ij}+1) < \sigma(M_{ij}+2) < \ldots < \sigma(M_{ij}+m_j)$,
$$j = 1, \ldots, s-1 , \quad i = 0, \ldots, q_j-1 .$$

For $r = 0, 1, \ldots$ we define

$$T_r^\mu : V \times \mathop{\oplus}_{i=0}^{r} L_s^i(R^1 \times R^\mu; R^k) \to L_s^r(V \times R^\mu; R^k)$$

by

$$T_r^\mu(x,\ a)\ =\ \sum_{s=1}^{r}\ \sum_{m_1 q_1 + \ldots + m_s q_s = r}\ a_q \left[\text{symm}* \left(\left(D^{m_1} Q \right)^{q_1},\ \ldots,\ \left(D^{m_s} Q \right)^{q_s} \right) (x) \right]\ ,$$

where we have set $q = q_1 + \ldots + q_s$ and

$$a\ =\ \left(a_0,\ \ldots,\ a_r \right)\ \in\ \bigoplus_{i=0}^{r}\ L_s^i \left(R^1 \times R^\mu;\ R^k \right)\ .$$

(This rather complicated expression becomes the formula for the rth derivative of the map gQ if we replace a_q everywhere by $D^q g\left(Q(x) \right)$.)

For $d = 0,\ 1,\ \ldots$ we define the equivariant polynomials

$$P_d^\mu\ :\ V \times \bigoplus_{i=0}^{d}\ L_s^i \left(R^1 \times R^\mu;\ R^k \right)\ \to\ L_s^d \left(V \times R^\mu;\ W \right)$$

by

$$P_d^\mu(x,\ a)\ =\ \sum_{r=0}^{d}\ \sum_{i=1}^{k}\ \binom{d}{r} T_r^\mu(x,\ a)_i\ \odot\ D^{d-r} F_i(x)\ ,$$

where "$\odot$" denotes the symmetric tensor product and the subscript i on $T_r^\mu(x,\ a)$ denotes the ith component of $T_r^\mu(x,\ a)$ in R^k .

Setting $U_d^\mu = \left[P_0^\mu,\ \ldots,\ P_d^\mu \right]$, we see that

$$U_d^\mu\ :\ V \times \bigoplus_{i=0}^{d}\ L_s^i \left(R^1 \times R^\mu;\ R^k \right)\ \to\ J^d \left(V \times R^\mu,\ W \right)\ .$$

Let X_d^μ be the zero set of U_d^μ and $\left\{ \Sigma_{d,i}^u \right\}$ denote the corresponding fundamental equisingularity sequence.

Let $\dot{f} \in C_G^\infty \left(V \times R^\mu,\ W \right)$ and write

$$f(x,\ t)\ =\ \sum_{i=1}^{k}\ g_i \left(P(x),\ t \right) F_i(x)\ ,$$

$g_i \in C_G^\infty \left(R^1 \times R^\mu \right)$. Set $g = \left(g_1,\ \ldots,\ g_k \right) \in C^\infty \left(R^1 \times R^\mu,\ R^k \right)$ and define

$$\gamma_d(f) : R^\mu \to \overset{d}{\underset{i=0}{\oplus}} L^i_s(R^1 \times R^\mu; R^k)$$

by

$$\gamma_d(f)(t) = j^d g(0, t) = \bigl(g(0, t), \ldots, D^d g(0, t)\bigr) \ .$$

DEFINITION (Bierstone [2]). Let $f \in C^\infty_G(V \times R^\mu, W)$. f is said to
be in dth approximation to equivariant general position to $0 \in W$ at
$(0, t) \in V \times R^\mu$ if either $f(0, t) \neq 0$ or $f(0, t) = 0$ and $\gamma_r(f)$ is
transversal to the fundamental equisingularity sequence. $\left\{\Sigma^u_{r,i}\right\}$ at t for
$0 \leq r \leq d$.

EXAMPLE. Following the notation of our previous example, the map
$f_{t,s}$ is in dth approximation to equivariant general position to $0 \in C^2$
at zero for all $d \geq 0$ if $t \neq 0$ and $s \neq 0$.

REMARKS. 1. It is easily shown that the above definition is
equivalent to that stated in [2]. Moreover, it is shown in [2] that the
definition is independent of the choice of g implicit in the maps γ_d
and of the choice of generators $\{F_1, \ldots, F_k\}$ and $\{p_1, \ldots, p_1\}$. It is
also shown that there exists d_0 , depending on V, W , and u , such that
d_0th approximation to equivariant general position implies dth
approximation to equivariant general position for $d \geq d_0$. Equivariant
general position is then redefined to be d_0th approximation to equivariant
general position.

2. It should be noted that no coefficients involving powers of the
polynomials $\{p_1, \ldots, p_1\}$ appear in U^μ_d . The inclusion of such higher
order terms has no effect on the fundamental equisingularity sequence — see
the proposition at the end of Section 2.

3. A natural question arising out of our formulation of the higher
order transversality conditions is the extent to which the fundamental
equisingularity sequence $\left\{\Sigma^u_{d,i}\right\}$ depends on u . For example, suppose

$f \in C_G^\infty(V, W)$ is in equivariant general position to $0 \in W$ at $0 \in V$ and define $\tilde{f} : V \times R^\mu \to W$ by $\tilde{f}(x, t) = f(x)$. Is $\tilde{f}$ in equivariant general position to $0 \in W$ on R^μ ?

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