Strengthened Monotonicity of Relative Entropy via Pinched Petz Recovery Map

NEW: 1 July 2015 David Sutter, Marco Tomamichel, Aram W. Harrow arXiv:1507.00303

Result (overview)

- New proof of [Fawzi and Renner, arXiv:1410.0664] based on concavity and monotonicity of the operator logarithm
- Strengthening/generalization of the inequality

Notation

- *Relative entropy*: $D(\rho\|\sigma) := \mathrm{tr}(\rho(\log\rho - \log\sigma))$ if $\mathrm{supp}(\rho) \subseteq \mathrm{supp}(\sigma)$ and $+\infty$ otherwise
- (von Neumann) *entropy*: $H(\rho) := -\mathrm{tr}(\rho\log\rho)$
- *Conditional mutual information* (CMI): $I(A:C|B) := H(AB) + H(BC) - H(ABC) - H(B)$
- *Fidelity*: $F(\rho,\sigma) := \left\|\sqrt{\rho}\sqrt{\sigma}\right\|_1$
- Measured relative entropy $D_{\mathbb{M}}(\rho\|\sigma) := \sup\{D(\mathcal{M}(\rho)\|\mathcal{M}(\sigma)) : \mathcal{M}(\rho) = \sum_x \mathrm{tr}(\rho M_x)|x\rangle\langle x| \text{ with } \sum_x M_x = \mathrm{id}\}$, where $\{|x\rangle\}$ is a finite set of orthonormal vectors

What is known

1973	Strong subadditivity: $I(A:C\|B) \geqslant 0$ [3, 4]
1975	Monotonicity of relative entropy (data processing inequality): $D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \geqslant 0$ [5, 8]
Oct. 2014	For any ρ_{ABC} there exists a recovery map $\mathcal{R}_{B\to BC}$ such that [2] $I(A:C\|B) \geqslant -2\log F\big(\rho_{ABC}, \mathcal{R}_{B\to BC}(\rho_{AB})\big)$ Recovery map has the form of a rotated Petz reocvery map
Nov. 2014	For any ρ_{ABC} there exists a recovery map $\mathcal{R}_{B\to BC}$ such that [1] $I(A:C\|B) \geqslant D_{\mathbb{M}}\big(\rho_{ABC}\|\mathcal{R}_{B\to BC}(\rho_{AB})\big) \geqslant -2\log F\big(\rho_{ABC}, \mathcal{R}_{B\to BC}(\rho_{AB})\big)$
April 2015	For CMI lower bound there exits a *universal* recovery map (that only depends on ρ_{BC}) & unitaries commute [7]
May 2015	For any σ, $\mathcal{N}$ there exists a recovery map $\mathcal{R}_{\sigma,\mathcal{N}}$ (rotated Petz with commuting unitaries) such that $(\mathcal{R}_{\sigma,\mathcal{N}}\circ\mathcal{N})(\sigma) = \sigma$ and [9] $D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \geqslant -2\log F\big(\rho, (\mathcal{R}_{\sigma,\mathcal{N}}\circ\mathcal{N})(\rho)\big)$

Pinched and rotated Petz recovery map

- For $H = \sum_x \lambda_x |x\rangle\langle x|$ let $P_\lambda := \sum_{x:\lambda_x=\lambda} |x\rangle\langle x|$ and define the *pinching map*

$$\mathcal{P}_H \;:\; \mathrm{P}(A) \ni X \mapsto \sum_{\lambda\in\mathrm{spec}(H)} P_\lambda\, X\, P_\lambda \in \mathrm{P}(A)$$

- *Pinching recovery map* for $n \in \mathbb{N}$

$$\mathcal{R}^n_{\sigma,\mathcal{N}} : \; \mathrm{P}(B^{\otimes n}) \to \mathrm{P}(A^{\otimes n})$$

$$X_{B^n} \mapsto (\sigma^{\frac{1}{2}})^{\otimes n}\mathcal{P}_{\sigma^{\otimes n}}\Big((\mathcal{N}^\dagger)^{\otimes n}\Big[(\mathcal{N}(\sigma)^{-\frac{1}{2}})^{\otimes n}\mathcal{P}_{\mathcal{N}(\sigma)^{\otimes n}}(X_{B^n})(\mathcal{N}(\sigma)^{-\frac{1}{2}})^{\otimes n}\Big]\Big)(\sigma^{\frac{1}{2}})^{\otimes n}$$

- For $\sigma = \sum_{k\in[d_1]} \lambda_k P_k$ let $U^\vartheta_\sigma := \sum_{k\in[d_1]} \exp(\mathrm{i}\vartheta_k)P_k$ with $\vartheta \in [0,2\pi]^{\times d_1}$ let us define a *rotated Petz recovery map*

$$\mathcal{T}^{\varphi,\vartheta}_{\sigma,\mathcal{N}} \;:\; \mathrm{P}(B) \to \mathrm{P}(A)$$

$$X_B \mapsto U^\vartheta_\sigma \sigma^{\frac{1}{2}}\mathcal{N}^\dagger\big(\mathcal{N}(\sigma)^{-\frac{1}{2}}U^\varphi_{\mathcal{N}(\sigma)} X_B U^{\varphi\dagger}_{\mathcal{N}(\sigma)}\mathcal{N}(\sigma)^{-\frac{1}{2}}\big)\sigma^{\frac{1}{2}}U^{\vartheta\dagger}_\sigma$$

- For any $n \in \mathbb{N}$

$$\mathcal{R}^n_{\sigma,\mathcal{N}}(\,\cdot\,) = \frac{1}{(2\pi)^{d_1}}\int_{[0,2\pi]^{\times d_1}} \mathrm{d}\vartheta\; \frac{1}{(2\pi)^{d_2}}\int_{[0,2\pi]^{\times d_2}} \mathrm{d}\varphi\; (\mathcal{T}^{\varphi,\vartheta}_{\sigma,\mathcal{N}})^{\otimes n}(\,\cdot\,) \qquad (1)$$

Result (formal)

Let

$$\mathsf{T}_{\sigma,\mathcal{N}} := \mathrm{conv}\Big(\mathcal{T}^{\varphi,\vartheta}_{\sigma,\mathcal{N}} \;:\; \vartheta \in [0,2\pi]^{\times d_1}, \varphi \in [0,2\pi]^{\times d_2}\Big)$$

Main result

For any $\sigma \in \mathrm{P}(A)$, any $\rho \in \mathrm{S}_\sigma(A)$, and any $\mathcal{N} \in \mathrm{TPCP}(A,B)$ there exists a recovery map $\mathcal{R}_{\sigma,\mathcal{N},\rho} \in \mathsf{T}_{\sigma,\mathcal{N}}$, such that

$$D(\rho\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \geqslant D_{\mathbb{M}}\big(\rho\,\|(\mathcal{R}_{\sigma,\mathcal{N},\rho}\circ\mathcal{N})(\rho)\big) \overset{\text{easy}}{\geqslant} -2\log F\big(\rho, (\mathcal{R}_{\sigma,\mathcal{N},\rho}\circ\mathcal{N})(\rho)\big)\,.$$

- The recovery map satisfies $(\mathcal{R}_{\sigma,\mathcal{N},\rho}\circ\mathcal{N})(\sigma) = \sigma$
- For $\rho = \rho_{ABC}$, $\sigma = \rho_{BC}$, and $\mathcal{N}(\cdot) = \mathrm{tr}_C(\cdot)$ we reproduce [2, 1]
- The second inequality was proved in [9] using Hadamard's three line theorem

Proof

Proposition $D(\rho\,\|\sigma) - D(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)) \geqslant \liminf_{n\to\infty} \frac{1}{n} D\big(\rho^{\otimes n}\|(\mathcal{R}^n_{\sigma,\mathcal{N}}\circ\mathcal{N}^{\otimes n})(\rho^{\otimes n})\big)$

Proof sketch for the Proposition.

- $\mathcal{P}_H(X)$ commutes with H
- Pinching inequality: $\mathcal{P}_H(X) \geqslant \frac{1}{|\mathrm{spec}(H)|}X$ for all $X \in \mathrm{P}(A)$
- For any $\rho \in \mathrm{P}(A)$ we have $|\mathrm{spec}(\rho^{\otimes n})| = O(\mathrm{poly}(n))$
- Operator logarithm is concave and monotone

□

With the proposition the main result follows from (1) together with the "Piani"-argument [6], which shows that for any $n \in \mathbb{N}$

$$\frac{1}{n}\min_{\mu\in\mathbb{P}(\mathsf{T}_{\sigma,\mathcal{N}})} D\Big(\rho^{\otimes n}\Big\|\int \mathrm{d}\mu(\varphi,\vartheta)\,(\mathcal{T}^{\varphi,\vartheta}_{\sigma,\mathcal{N}}\circ\mathcal{N})(\rho)^{\otimes n}\Big) \geqslant \min_{\mu\in\mathbb{P}(\mathsf{T}_{\sigma,\mathcal{N}})} D_{\mathbb{M}}\Big(\rho\Big\|\int \mathrm{d}\mu(\varphi,\vartheta)\,(\mathcal{T}^{\varphi,\vartheta}_{\sigma,\mathcal{N}}\circ\mathcal{N})(\rho)\Big)$$

Discussion and open problems

- Universality: a recovery map that does not depend on ρ [to appear] (Here the measured relative entropy bound becomes important)
- Does the Petz recovery map satisfy all these inequalities?
- The pinching recovery map (for $n = 1$) does not satisfy the inequalities

References

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