STRESS ANALYSIS IN VISCO-ELASTIC BODIES*

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Abstract. The analysis of stress and strain in linear visco-elastic bodies is considered when the loading is quasi-static so that inertia forces due to the deformation are negligible. It is shown that removal of the time variable by applying the Laplace transform enables the solution to be obtained in terms of an associated elastic problem. Thus the extensive literature in the theory of elasticity can be utilized in visco-elastic analysis. The operation of the transform on the prescribed boundary tractions and displacements and body forces may completely modify the spatial distribution in the associated problem. For proportional loading, in which the space and time variations of the prescribed quantities separate, the spatial distribution is maintained in the associated problem. A convenient method of treating a common case of non-proportional loading, moving surface tractions, is demonstrated. This work is compared with related approaches to this problem in the literature of visco-elastic stress analysis.

1. Introduction. We are concerned with quasi-static problems in which inertia forces due to the deformation are negligible. Inertia forces due to effectively rigid body motion such as centrifugal forces come within the scope of the analysis. We shall consider materials satisfying the general isotropic linear visco-elastic law:

$$Ps_{ij} = Qe_{ij} , \tag{1a}$$

$$P'\sigma_{ii} = Q'\epsilon_{ii} , \tag{1b}$$

where P, Q, P' and Q' are linear operators of the form $\sum_0^m a_n D^n$, and D is the time derivative $\partial/\partial t$. The coefficients a_n and the numbers m are in general different for each operator, although certain restrictions on the m's are required to determine observed physical characteristics. σ_{ij} and ϵ_{ij} are the stress and strain tensors, the latter considered to be infinitesimal, and s_{ij} and e_{ij} are respectively their deviators defined in the usual way:

$$s_{ij} = \sigma_{ij} - \tfrac{1}{3}\sigma_{kk}\delta_{ij} , \qquad e_{ij} = \epsilon_{ij} - \tfrac{1}{3}\epsilon_{kk}\delta_{ij} , \tag{2}$$

where δ_{ij} is the Kronecker delta $\delta_{ij} = 1, i = j; \delta_{ij} = 0, i \neq j$.

Equations (1) are a generalisation for combined stresses of the well-known visco-elastic law,

$$P\sigma = Q\epsilon, \tag{3}$$

relating stress and strain when only one component of each tensor is needed, as for example in simple tension (see Alfrey [1]†). In developing the most general linear isotropic relation, operator equations of the type (3) can be written between the deviators and the first invariants of σ_{ij} and ϵ_{ij} as detailed in (1). This is analogous to the expression of the general isotropic elastic relation in terms of the two constants, shear modulus and

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†Numbers in square brackets refer to the bibliography at the end of the paper.

bulk modulus. (1) is equivalent to the relations used by Read [2], and is more general than that prescribed by Tsien [3] according to the same premises.

We adopt the form (1), which corresponds to a discrete spectrum of relaxation times, in preference to the hereditary integral method of specifying visco-elastic behaviour, which gives a continuous spectrum, because of simplicity and since we are particularly interested in stress analysis for short loading times. In such a restricted range visco-elastic behaviour can be represented with tolerable accuracy by relations of the form (1) with only a few terms in each operator. This is equivalent to representation by a simple model of dashpots and springs. Methods of determining such simple models to cover a limited frequency range will be described elsewhere. The hereditary integral representation equivalent to (1) has been used by Volterra [4].

2. The equivalent elastic problem. As shown in Fig. 1 we consider bodies subjected

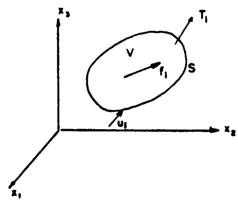

FIG. 1

to prescribed body forces $f_i(x_j, t)$ per unit volume, tractions $T_i(x_j, t)$ on the surface S or surface displacements $u_i(x_j, t)$, including combinations of these in the groupings in which they occur also in elasticity theory. In order to determine a complete solution we must obtain the stresses $\sigma_{ij}(x_k, t)$ and the displacements $u_i(x_j, t)$ throughout the body V to satisfy the stress-strain relations (1), and the equilibrium equations:

$$\sigma_{ij,j} = f_i(x_k, t), \tag{4}$$

where as usual the subscript after the comma indicates differentiation with respect to the corresponding space coordinate. The strain tensor is given in terms of the displacement by

$$\epsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}). \tag{5}$$

The displacement must be compatible with the prescribed surface displacement, and the stresses with the prescribed surface tractions according to:

$$T_i = \sigma_{ij} n_j \text{ on } S, \tag{6}$$

where n_j is the outward normal.

We now remove the time dependence by operating with the Laplace transform on all these equations (see for example Carslaw and Jaeger [5]). The transform is denoted by a star on the corresponding function. We will consider problems in which the bodies are initially undisturbed since this is the common situation, but the Laplace transform technique is also well suited to problems with non-zero initial conditions. With zero initial conditions an operator in D merely becomes the same function of the transform parameter p, so that the required equations become

$$P(p)s^*_{ij} = Q(p)e^*_{ij} , \tag{7a}$$

$$P'(p)\sigma^*_{ii} = Q'(p)\epsilon^*_{ii} , \tag{7b}$$

$$\sigma^*_{ij,j} = f^*_i(x_k , p), \tag{8}$$

$$\epsilon^*_{ij} = \tfrac{1}{2}(u^*_{i,j} + u^*_{j,i}), \tag{9}$$

$$T^*_i(x_j , p) = \sigma^*_{ij}n_j , \tag{10}$$

(7)-(10) represents a stress analysis problem for an elastic body of the same shape as the visco-elastic body with elastic constants a function of the parameter p. The prescribed surface tractions $T^*_i(x_j , p)$, body force $f^*_i(x_j , p)$ and boundary displacement $u^*_i(x_j , p)$ are also functions of p. When the stresses $\sigma^*_{ij}(x_k , p)$ have been determined, inversion of the Laplace transform gives the desired stresses $\sigma_{ij}(x_k , t)$ for the visco-elastic problem, and similarly for other variables. Thus we can utilize the extensive literature of the theory of elasticity to evaluate visco-elastic stress analysis problems.

Although the body shape for the associated elastic problem is the same, the prescribed distribution of surface tractions, displacements, and body forces may be quite different since the Laplace transform $\varphi^*(x_i , p)$ will in general have an entirely different space distribution than $\varphi(x_i , t)$. Thus the associated elastic problem is not in general simply related to the visco-elastic problem. There is however a common type of problem, which we shall call proportional loading, in which the space and time dependence of prescribed quantities separates out with a common time dependence. Thus

$$\begin{aligned} T_i(x_j , t) &= T'_i(x_j)f(t), \\ f_i(x_j , t) &= f'_i(x_j)f(t), \\ u_i(x_j , t) &= u'_i(x_j)f(t). \end{aligned} \tag{11}$$

In this case the Laplace transform simply changes $f(t)$ to $f^*(p)$ leaving the space dependence unchanged. Since $f^*(p)$ then merely appears as a multiplying factor, the associated elastic problem contains the same spatial distribution of prescribed quantities as the visco-elastic problem. This represents a considerable simplification since the associated elastic problem can be solved for T'_i , f'_i and u'_i prescribed, and these are independent of p. The analysis being linear, if the loading falls into groups of proportional loading and prescribed displacements each with different time dependence, they can be considered separately and superposed. An important class of problems which does not fall in this category is associated with moving loads. A particular method of treating these is detailed below.

3. A simple example. Let us consider the problem shown in Fig. 2 of a vertical point force $P(t)$ acting normally at a fixed point on the surface of a semi-infinite visco-

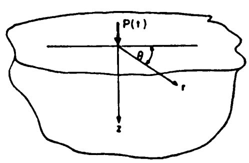

FIG. 2

elastic body. The point of action is the origin of cylindrical coordinates (r, θ, z), with the body occupying the space $z \geq 0$. Since only one point force is specified this is a case of proportional loading, and the associated elastic problem is that of a semi-infinite elastic body with a normal point load of magnitude $P^*(p)$. The solution to this elastic problem is given by Timoshenko ([6], p. 331), and to demonstrate the method we will consider the stress component σ_r^* which is given by

$$\sigma_r^* = \frac{P^*}{2\pi}\left\{(1 - 2\nu)\left[\frac{1}{r^2} - \frac{z}{r^2}(r^2 + z^2)^{-1/2}\right] - 3r^2z(r^2 + z^2)^{-5/2}\right\} \tag{12}$$

where ν is the Poisson's ratio of the elastic body. Transforming the notation of (7) to more familiar elastic constants

$$Q(p)/P(p) = 2G, \tag{13}$$

where G is the shear modulus, and

$$Q'(p)/P'(p) = 3K, \tag{14}$$

where K is the bulk modulus, since σ_{ii}^* is three times the average hydrostatic tension. We can express Poisson's ratio in terms of K and G (see [6] p. 10) according to the relation

$$1 - 2\nu = \frac{3G}{3K + G}. \tag{15}$$

Thus, in the notation of (7), (12) becomes

$$\sigma_r^* = \frac{P^*(p)}{2\pi}\left\{\frac{3Q(p)/P(p)}{[2Q'(p)/P'(p)] + [Q(p)/P(p)]}\left[\frac{1}{r^2} - \frac{z}{r^2}(r^2 + z^2)^{-1/2}\right] - 3r^2z(r^2 + z^2)^{-5/2}\right\}. \tag{16}$$

For a prescribed material, with known operators P, Q, P' and Q', the inverse of (16) gives the radial stress in the visco-elastic problem. Suppose, for example, that in shear the material behaves as a delayed elastic Voigt material, but is perfectly elastic under hydrostatic pressure. (1) would then read

$$s_{ij} = (AD + B)e_{ij}\,, \tag{17a}$$

$$\sigma_{ii} = C\epsilon_{ii}\,, \tag{17b}$$

where A, B and C are constants of the material. Suppose that a constant load P_0 is suddenly applied at $t = 0$, and maintained. Then $P(t) = P_0$, $t > 0$, and $P^*(p) = P_0/p$, the transform of the Heaviside step function (see [5] p. 4). Thus (16) becomes:

$$\sigma_r^* = \frac{P_0}{2\pi p}\left\{\frac{3(Ap + B)}{2C + Ap + B}\left[\frac{1}{r^2} - \frac{z}{r^2}(r^2 + z^2)^{-1/2}\right] - 3r^2z(r^2 + z^2)^{-5/2}\right\}. \tag{18}$$

This can be inverted directly by partial fractions to give:

$$\sigma_r = \frac{P_0}{2\pi}\left\{\frac{3}{2C + B}\,[B + 2C\exp[-(2C + B)t/A]]\left[\frac{1}{r^2} - \frac{z}{r^2}(r^2 + z^2)^{-1/2}\right] - 3r^2z(r^2 + z^2)^{-5/2}\right\}, \qquad t > 0. \tag{19}$$

Thus for the visco-elastic problem with a constant force applied, the stress consists of two terms, one remaining constant, and the other dying out exponentially with time. There is thus a relaxation influence between the shear and the hydrostatic response of the material.

For more complicated materials, Q/P and Q'/P' will still be rational functions of (p), so that this method of inversion can be used, although the manipulation will be more cumbersome.

For more complicated $P(t)$, it might be worthwhile to separate the inversion of $P^*(p)$ and of the function of p arising from the material properties, since these appear as a product. The inversion can then be expressed as a convolution integral (see [5] p. 7) for general $P(t)$. Thus for the material considered in the examples, (18) can be inverted in this way, with the special value P_0/p replaced by $P(p)$, to give:

$$\sigma_r = \frac{1}{2\pi}\left\{\int_0^t 3\left[\delta(0,\tau) - \frac{2C}{A}\exp\left[-(2C+B)\tau/A\right]H(\tau)\right]P(t-\tau)\,d\tau \right. \\ \left. \times\left[\frac{1}{r^2} - \frac{z}{r^2}(r^2+z^2)^{-1/2}\right] - P(t)3r^2z(r^2+z^2)^{-5/2}\right\}, \tag{20}$$

where $\delta(0, \tau)$ is the Dirac delta function, and $H(\tau)$ the Heaviside step function. For the special case $P(t) = P_0$, the integral is simply evaluated to give (19).

4. Moving loads. A common type of loading which falls outside the category of proportional loading is that due to moving surface tractions. Such a situation is often produced mechanically or by the pressure system in a moving fluid. It can be handled directly as described in the previous section by taking the Laplace transform of the moving surface tractions, but an alternative method is described which may be more convenient.

Instead of taking the transform of the prescribed boundary conditions and body forces to determine the associated elastic problem, and then solving this, the prescribed visco-elastic problem could first be solved assuming it to be elastic, and the transform of the resulting stresses would be the desired solution of the associated elastic problem. However the surface tractions vary in time, at each instant the solution of a quasi-static elastic problem is simply the solution of the problem with the current tractions held constant. This is of course not true for a visco-elastic problem in which the history of loading has an influence, as was demonstrated by the example of the previous section. Thus, the solution of the elastic problem for the series of instantaneous distributions of surface tractions gives the sequence of stress values at each point, and the transform of these values gives the required solution of the associated elastic problem. The elastic constants are the functions of the transform parameter p associated with the transformed stress-strain relation (7), but since they are not time dependent this adds no complication to the determination of the transform of the stresses. On inverting to obtain the stress for the visco-elastic problem, the fact that these elastic constants are functions of p modifies the reverse process.

To illustrate the method, let us consider the problem, shown in Fig. 3, of a point load $P(t)$ moving along the x axis on the surface of a semi-infinite visco-elastic body. The motion of the force is prescribed by the displacement from the origin $\xi(t)$, and the force is first applied at $\xi = 0$, $t = 0$ with the body initially undisturbed. At each instant the solution of the elastic problem with the same loading is that used in the previous section, it merely being necessary to change the origin of coordinates. To obtain a simple example for illustrative purposes, let us consider the stress component σ_x at the point $(x, 0, z)$ directly under the path of the load. At any instant the stress σ_x^e in the elastic problem with this point force loading is given by (12), with r replaced by $[x - \xi(t)]$,

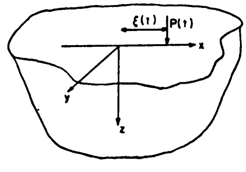

FIG. 3

since the directions of x and r are the same at the point in question, and the radius of the point $(x, 0, z)$ relative to cylindrical coordinates based on the load axis is $[x - \xi(t)]$. Thus:

$$\sigma_x^e = \frac{P(t)}{2\pi}\left\{(1-2\nu)\left[\frac{1}{(x-\xi)^2} - \frac{z}{(x-\xi)^2}\{(x-\xi)^2 + z^2\}^{-1/2}\right]\right. \\ \left. - 3(x-\xi)^2 z\{(x-\xi)^2 + z^2\}^{-5/3}\right\}, \tag{21}$$

where ν is the Poisson's ratio of the elastic material. $\sigma_x^*(x, z, p)$ of the associated elastic solution is the Laplace transform of this, with $1 - 2\nu$ replaced by the transform of the equivalent visco-elastic operator $[3Q(p)/P(p)]/[2Q'(p)/P'(p) + Q(p)/P(p)]$, as in Eq. (16). In carrying out the transform the t appearing in the prescribed $\xi(t)$ must be included. The stress $\sigma_x(x, z, t)$ in the original visco-elastic problem is now obtained by taking the inverse of σ_x^* which contains the transform parameter p both from the transform of σ_x^e (21) and from the visco-elastic operators. The transform of (21) may be quite complicated depending on the particular forms of $P(t)$ and $\xi(t)$. However since the elastic constant ν appears only as a multiplying factor there is no need to carry out the transform of σ_x^e , since the inverse transform can be effected using the product rule in terms of a convolution integral (see [5], p. 7). This gives for the stress component σ_x at the point $(x, 0, z)$ for the particular visco-elastic material considered in the previous section the value:

$$\sigma_x(x, z, t) = \frac{1}{2\pi}\left\{\int_0^t 3\left[\delta(0, \tau) - \frac{2C}{A}\exp\left[-(2C+B)\tau/A\right]H(\tau)\right]P(t-\tau)\right. \\ \cdot\left[\frac{1}{[x-\xi(t-\tau)]^2} - \frac{z}{[x-\xi(t-\tau)]^2}\{[x-\xi(t-\tau)]^2 + z^2\}^{-1/2}\right]d\tau \\ \left. - P(t)3[x-\xi(t)]^2 z\{[x-\xi(t)]^2 + z^2\}^{-5/2}\right\}. \tag{22}$$

In many cases it may be much easier to complete the solution in this form, in place of carrying out the Laplace transform of $T_i(x_j, t)$ to obtain $\sigma_{ij}^*(x_k p)$. In using the method of the present section, the need for determining this quantity is avoided.

Although this alternative method is particularly well suited to a problem with moving tractions which apart from this motion are otherwise unchanged in form, it can also be used for the general case $T_i(x_j, t)$. In this case a series of elastic solutions would be needed, with the space distribution of traction that obtaining in the visco-elastic problem at each instant.

5. Discussion. The analysis given in this paper was developed as an extension of the work of Alfrey [7] and Tsien [3]. Alfrey showed that for an incompressible visco-elastic medium with prescribed surface tractions the stress is the same as that for an elastic body. This result is obtainable from the present analysis by taking P'/Q' equal to zero. Tsien showed the same to be true for a compressible visco-elastic body under proportional loading, but he had assumed a restrictive relation between the operators P, Q, P' and Q' such that the equivalent Poisson's ratio reduced to a constant rather than a rate dependent operator. The example of Sec. 3 shows that in the general case the stress field changes under constant tractions, so that it cannot be equal to the solution of an invariant elastic problem.

In the work of Alfrey and Tsien only a single pair of operators P, Q appeared, and it was possible simply to separate the time and space operators in the differential equations, and to obtain the elastic solution from the space operators. For the general isotropic medium four rate operators P, Q, P', Q' arise, and the Laplace transform assists in separating out the time dependence. The first application of a method of this type which I have seen was the use of Heaviside's operational calculus for the stress analysis of certain visco-elastic bodies by Jeffreys [8]. More recently Read [2] has used the Fourier transform to discuss the general dynamic problem. He obtains an associated elastic problem similar to that developed in Sec. 2 of the present report, but from the standpoint of application this result is extremely restricted in the dynamic case, since the inertia forces lead to body forces in the associated elastic problem which are proportional to displacements. Solutions of this type of problem are not common in the elasticity literature. For the quasi-static case the principle of the present paper is closely related to Read's, but his illustration of the use of the operator method is restricted to the case of proportional loading. For the usual type of problem in which loads are applied with or without an initial disturbance in the medium, it is felt that the Laplace transform is more convenient to use than the Fourier transform. The examples given in this report indicate the convenient form in which solutions are derived.

Papers by Mindlin [9] and Graffi [10, 11] are concerned with the special situation when the visco-elastic stress field is identical with an elastic field. The former studies this question from the standpoint of photo-elastic testing. Special solutions in which the space and time variables separate have been considered by Volterra [4]. The example of Sec. 3 shows that this condition does not obtain for a constant fixed force, and so is quite limited for applications. Another particular group of problems which has received considerable attention is when the time dependence is entirely represented by a factor $e^{i\omega t}$. When this is factored out, an equation in space coordinates only is obtained. The relation of this type of problem to both elastic and compressible-viscous fluid theory is stated by Oestreicher [12], and an example is given.

This assessment of the literature suggests that the generality of the analysis presented in this paper is needed for many problems of visco-elastic stress analysis, and that the use of the Laplace transform is a convenient means of treating such problems. The ease with which solutions can be evaluated depends on the way in which the elastic constants appear in the solution of the associated elastic problem. Much is known about this, for example, as stated by Read [2], that they do not appear in the expressions for stress in the plane problem with prescribed surface tractions, if the tractions on internal boundaries are in equilibrium. In most elementary solutions they appear in a simple form so that the use of these solutions in visco-elastic analysis leads to comparatively simple evaluations.

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Note added in proof: Reference should also be made to a recent article by M. A. Brull, *A structural theory incorporating the effect of time-dependent elasticity*, Proc. 1st Midwest. Conf. Solid Mech., Urbana, Ill., 1953, pp. 141–147.