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Stress fields in composite cross-ply laminates

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Abstract

A formulation based on integral equations is proposed in order to analyse the edge-stress problem in composite laminates subjected to uniaxial extension. The integral equations are obtained by employing a direct approach founded on the reciprocity theorem. Once the fundamental solutions are known, the proposed formulation allows a straightforward solution of the problem by the boundary element method. The enforcement of displacement and traction continuity along the interfaces leads to a set of linear algebraic equations providing the problem solution. Some numerical applications are presented and good agreement between the results obtained with the proposed method and the existing ones is shown. The stress singularity arising at the laminate free edges is investigated determining its power and strength.

Introduction

A great effort has been made in the research field in order to determine the behaviour of interlaminar stresses in composite laminates which are widely employed in the most advanced structural engineering technologies. One of the events most frequently addressed is the interlaminar stress distribution under uniaxial loading and particular care has been devoted to the study of the singular behaviour of the stress field near free edges. Actually, due to the mismatch in the elastic characteristics of the layers, a three-dimensional stress field, showing high stresses at the interfacial zones, arises. Many authors[1-15] have dealt with this matter and many techniques have been used to lead to the solution of the problem. The methods employed to solve the laminate edge-stress problem vary from finite difference[1] or finite elements[2-8] to the Galerkin method[9] or the use of approximate closed-form analyses.[10-15] In this paper, the elastic response of

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a multi-layered beam subjected to uniaxial extension is analysed according to the boundary integral equation theory.[16-17] The beam is supposed to be formed by prismatic plies having different elasticity and general lay-up. Each ply is considered homogeneous and in term of constitutive equation it is described by a generalised orthotropic law having one principal axis parallel to the generators of the beam lateral surface (cross-ply laminate). The boundary integral equations governing the problem are directly obtained by applying the reciprocity theorem[18] using the fundamental solutions of orthotropic elasticity. This approach gives a convenient basis for the application of the boundary element method in order to achieve the solution. The proposed formulation makes it possible to analyse the beam whatever the shape and the composition of the section. Some applications are presented with the aim of testing the method proposed and investigating, in terms of singularity power and strength, the singular behaviour of the stress field at the free edges.[4,5,7,12] The results obtained show the robustness and the efficacy of the present method, in which some computational advantages with respect to other techniques are involved.

Definitions

Let the beam be considered as composed by prismatic layers having different elastic properties and let it be referred to a co-ordinate system x_i, i=1,2,3, with the x_3 axis parallel to the lateral surface generators of the beam having section A and length l. The generic layer having section A_e and length l is considered as homogeneous and orthotropic with material symmetry axes parallel to x_i (cross-ply laminate).

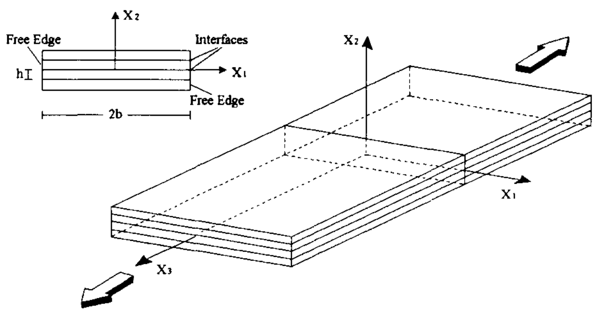

Figure 1: *Laminate configuration.*

When a tensional load is applied at the beam's ends the displacements s_1, s_2 and s_3 at the generic point P of the beam may be assumed as

$$\begin{aligned} s_1 &= u_1(x_1, x_2) \\ s_2 &= u_2(x_1, x_2) \\ s_3 &= \varepsilon_0\, x_3 \end{aligned} \tag{1}$$

where ε_0 is constant all over the beam section. The strain field associated with the displacement system (1) is given by

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$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \end{bmatrix} = \begin{bmatrix} \partial/\partial x_1 & 0 \\ 0 & \partial/\partial x_2 \\ \partial/\partial x_2 & \partial/\partial x_1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \mathbf{D}\mathbf{u} \tag{2}$$

$$\varepsilon_{33} = \varepsilon_0$$

while the stresses are

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & 0 \\ E_{12} & E_{22} & 0 \\ 0 & 0 & G_{12} \end{bmatrix} \boldsymbol{\varepsilon} + \begin{bmatrix} E_{13} \\ E_{23} \\ 0 \end{bmatrix} \varepsilon_0 = \mathbf{E}\boldsymbol{\varepsilon} + \mathbf{Q}\varepsilon_0 \tag{3}$$

$$\sigma_{33} = \begin{bmatrix} E_{13} & E_{23} & 0 \end{bmatrix} \boldsymbol{\varepsilon} + E_{33}\varepsilon_0 = \mathbf{Q}^T \boldsymbol{\varepsilon} + E_{33}\varepsilon_0$$

Finally the governing field equations, *i.e.* the equilibrium equations, are expressed as

$$\mathbf{D}^T \boldsymbol{\sigma} = \mathbf{0} \quad \text{in } A_e \quad ; \quad \mathbf{D}_n \boldsymbol{\sigma} = t \quad \text{in } \Gamma_e \tag{4}$$

where $\mathbf{t}$ are the surface forces applied on the ply boundary Γ_e and

$$\mathbf{D}_n = \begin{bmatrix} \alpha_1 & 0 & \alpha_2 \\ 0 & \alpha_2 & \alpha_1 \end{bmatrix} \tag{5}$$

In the previous relation α_1 and α_2 are the direction cosines of the outwardly directed normal to the boundary.

Integral equations

The generic ply having section A_e is loaded on the surface by the force system $\mathbf{t}$ which is constant along the x_3 axis. Let the elementary solid be subjected to a fictitious system of body forces $\mathbf{f}_j$ constant along x_3, $\mathbf{f}_j = \mathbf{f}_j(x_1, x_2)$, and let $\mathbf{u}_j$ be a particular system of displacements satisfying the equilibrium equations

$$\mathbf{D}^T \mathbf{E} \mathbf{D} \mathbf{u}_j + \mathbf{f}_j = \mathbf{0} \qquad \text{in } A_e \tag{6}$$

Moreover, let ε_j, σ_j and $\mathbf{t}_j$ be the strain, stress and traction fields related to $\mathbf{u}_j$, respectively. By applying the reciprocity theorem one has

$$\int_{\Gamma_e} \left(\mathbf{t}_j^T \mathbf{u} - \mathbf{u}_j^T \mathbf{t} \right) d\Gamma_e + \int_{A_e} \mathbf{f}_j^T \mathbf{u} \, dA_e = \int_{A_e} \left(\sigma_j^T \varepsilon - \varepsilon_j^T \sigma \right) dA_e \tag{7}$$

and, taking into account that $\varepsilon_{33j} = 0$, eqn (7), via eqn (3), becomes

$$\int_{\Gamma_e} \left(\mathbf{t}_j^T \mathbf{u} - \mathbf{u}_j^T \mathbf{t} \right) d\Gamma_e + \int_{A_e} \mathbf{f}_j^T \mathbf{u} \, dA_e = -\varepsilon_0 \int_{A_e} \varepsilon_j^T \mathbf{Q} \, dA_e \tag{8}$$

In the hypothesis that $\mathbf{f}_j$ is a concentrated load applied at the point P_0 and directed along the j direction, from eqn (8) one obtains

$$\mathbf{c}_j^T \mathbf{u}(P_0) + \int_{\Gamma_e} \left(\mathbf{t}_j^T \mathbf{u} - \mathbf{u}_j^T \mathbf{t}\right) d\Gamma_e = -\varepsilon_0 \int_{A_e} \varepsilon_j^T \mathbf{Q}\, dA_e \tag{9}$$

where

$$\mathbf{c}_j = \int_{A_e} \mathbf{f}_j\, dA_e = -\int_{\Gamma_e} \mathbf{t}_j\, d\Gamma_e \tag{10}$$

If the domain integral on the right-hand side of the eqn (9) is transformed into a boundary integral, one obtains the following integral equation

$$\mathbf{c}_j^T \mathbf{u}(P_0) + \int_{\Gamma_e} \left(\mathbf{t}_j^T \mathbf{u} - \mathbf{u}_j^T \mathbf{t}\right) d\Gamma_e = -\varepsilon_0 \int_{\Gamma_e} \mathbf{u}_j^T \mathbf{q}\, d\Gamma_e \tag{11}$$

where

$$\mathbf{q}^T = \left[E_{13}\alpha_1 \quad E_{23}\alpha_2\right] \tag{12}$$

Eqn (11) represents the general relation which gives for j=1,2 the integral equations coupling the tractions and the displacements on the boundary of the generic laminate ply. Once the generic ply has been discretized by means of boundary elements, the displacements and the tractions **u** and **t** on its boundary may be expressed in terms of their nodal values δ and **p** by properly selected shape functions defined along the element

$$\mathbf{u} = \mathbf{N}\delta \quad ; \quad \mathbf{t} = \mathbf{N}\mathbf{p} \tag{13}$$

Therefore eqn (11) becomes

$$\mathbf{c}^*\mathbf{u}(P_0) + \int_{\Gamma_e} \left(\mathbf{t}^*\mathbf{N}\delta - \mathbf{u}^*\mathbf{N}\mathbf{p}\right) d\Gamma_e = -\varepsilon_0 \int_{\Gamma_e} \mathbf{u}^*\mathbf{q}\, d\Gamma_e \tag{14}$$

in which

$$\mathbf{c}^* = \left[c_{ij}\right]^T \; ; \; \mathbf{u}^* = \left[u_{ij}\right]^T \; ; \; \mathbf{t}^* = \left[t_{ij}\right]^T \tag{15}$$

Generally eqn (14) presents four unknowns for each nodal point but nonetheless one can write four integral equations at each nodal point because the latter is common to the boundaries of the contiguous layers. Taking the interfacial continuity conditions into account, one obtains a linear algebraic system the solution of which provides the interlaminar displacements and tractions and hence the stress and the displacements at each point of the beam.

Fundamental solutions

The fundamental solutions of the problem are related to the compliances of the layer considered. Indeed the fundamental solutions depend on the roots of the characteristic equation

$$C_{11}\lambda^2 - 2\left(C_{12} + C_{33}/2\right)\lambda + C_{22} = 0 \tag{16}$$

where the coefficients C_{rs} are the compliances of the ply considered:

$$[C_{rs}] = \mathbf{E}^{-1} \tag{17}$$

Assuming that the roots λ_i of the eqn (16) are distinct and positive in sign, as generally happens, the fundamental solutions $\mathbf{u}_j$ for j=1,2 are

$$\begin{bmatrix} u_{1j} \\ u_{2j} \end{bmatrix} = \begin{bmatrix} \varphi_{1j}B_{11} & \varphi_{2j}B_{12} \\ \psi_{1j}B_{21} & \psi_{2j}B_{22} \end{bmatrix} \begin{bmatrix} A_{1j} \\ A_{2j} \end{bmatrix} \tag{18}$$

in which

$$\begin{aligned} B_{1i} &= C_{11} - C_{12}/\lambda_i \\ B_{2i} &= C_{12} - C_{22}/\lambda_i \end{aligned} \tag{19}$$

In eqn (18) φ_{ij} and ψ_{ij} are defined as

$$\begin{aligned} \varphi_{i1}(P,P_0) &= -(ln\, r_i)/2\pi \\ \psi_{i1}(P,P_0) &= -\left(atn\left(\sqrt{\lambda_i}\, y/x\right)\right)/2\pi\sqrt{\lambda_i} \\ \varphi_{i2}(P,P_0) &= -\left(atn\left(\sqrt{\lambda_i}\, y/x\right)\right)\sqrt{\lambda_i}/2\pi \\ \psi_{i2}(P,P_0) &= (ln\, r_i)/2\pi \end{aligned} \tag{20}$$

where

$$\begin{gathered} x = x_1(P) - x_1(P_0) \quad ; \quad y = x_2(P) - x_2(P_0) \\ r_i = \sqrt{x^2 + \lambda_i y^2} \end{gathered} \tag{21}$$

The fundamental solution stresses and the A_{ij} coefficients are provided by

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \end{bmatrix}_j = \begin{bmatrix} \partial\varphi_{1j}/\partial x_1 & \partial\varphi_{2j}/\partial x_1 \\ -\partial\psi_{1j}/\lambda_1\partial x_2 & -\partial\psi_{2j}/\lambda_2\partial x_2 \\ \partial\varphi_{1j}/\lambda_1\partial x_2 & \partial\varphi_{2j}/\lambda_2\partial x_2 \end{bmatrix} \begin{bmatrix} A_{1j} \\ A_{2j} \end{bmatrix} \tag{22}$$

$$\begin{aligned} \mathbf{A}_1 &= \frac{\lambda_1\lambda_2}{C_{22}(\lambda_2 - \lambda_1)} \begin{bmatrix} B_{22}\sqrt{\lambda_1} \\ B_{21}\sqrt{\lambda_2} \end{bmatrix} \\ \mathbf{A}_2 &= \frac{\lambda_1\lambda_2}{C_{11}(\lambda_1 - \lambda_2)} \begin{bmatrix} -B_{12}/\sqrt{\lambda_2} \\ B_{11}/\sqrt{\lambda_1} \end{bmatrix} \end{aligned} \tag{23}$$

Applications

To check the efficiency of the present method, numerical results are compared to data available in the literature. In all the applications the elastic constants, with respect to the principal material axes of each ply, for the high-modulus

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graphite/epoxy are used

$$E_{LL} = 137.9 \text{ GPa} \quad ; \quad E_{TT} = E_{ZZ} = 14.5 \text{ GPa}$$
$$G_{LT} = G_{LZ} = G_{TZ} = 5.9 \text{ GPa} \qquad (24)$$
$$\nu_{LT} = \nu_{LZ} = \nu_{TZ} = 0.21$$

where L, T and Z refer to along fiber, transverse and thickness directions, respectively. In the discretization linear elements and linear shape functions are employed. The c_{ij} coefficients at the ply corner points are evaluated, according to eqn (10), by using Gauss quadrature formulas in the same way as all the other influence coefficients. Notice that the enforcement of the free edge boundary conditions requires taking into account a linear combination of the two equations obtained for a load applied at the free edge interlaminar point and directed along the x_1 axis.

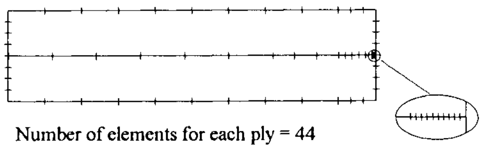

Figure 2: *BEM discretization for a quarter of the beam section.*

Two symmetrical four-ply laminates having width equal to 16·h, where h is the ply thickness, and $[0/90]_S$ and $[90/0]_S$ lay-ups have been analysed. The results, obtained by discretizing a quarter of the laminate section as shown in figure 2, are relative to unit axial extension.

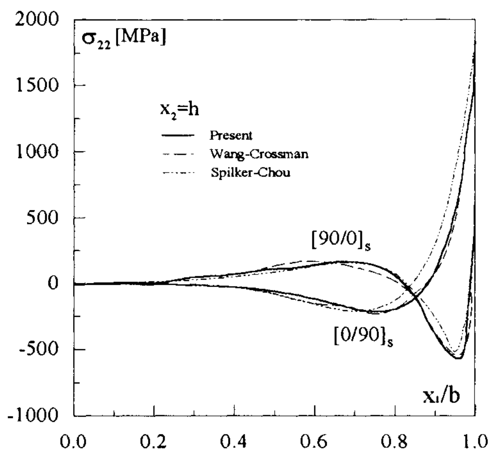

Figure 3: *The σ_{22} distribution for $[0/90]_S$ and $[90/0]_S$ laminates.*

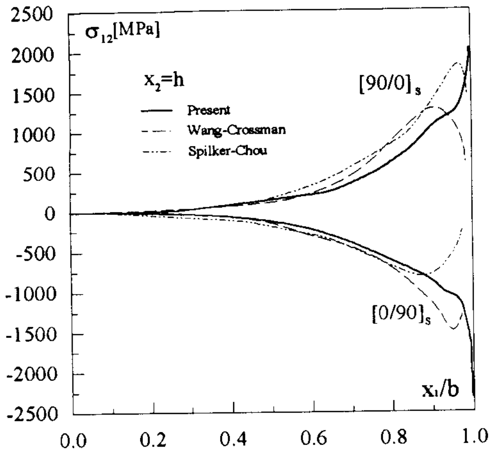

Figure 4: *The* σ_{12} *distribution for* $[0/90]_S$ *and* $[90/0]_S$ *laminates.*

In figures 3 and 4 the σ_{22} and σ_{12} distributions at the 0/90 interface of these cross-ply laminates are plotted, respectively. This kind of laminates exhibits high stress gradients near the free edge indicating that the stresses have the tendency to be singular at the free edge location. It has been pointed out that the singularity behaves like r^{α} and hence near the free edge, where the singularity is dominant, one can write

$$\log\sigma = \log A - \alpha \log r \tag{25}$$

where α and A are two constants indicating the power and the strength of the singularity respectively, whereas r is the distance from the singularity point. In figure 5 the results obtained applying the log-linear procedure to the $[0/90]_S$ laminate are shown and they are compared in terms of singularity power α with those of Raju *et al.*[4] Good agreement between the present results and the existing ones proves the efficacy of the proposed method.

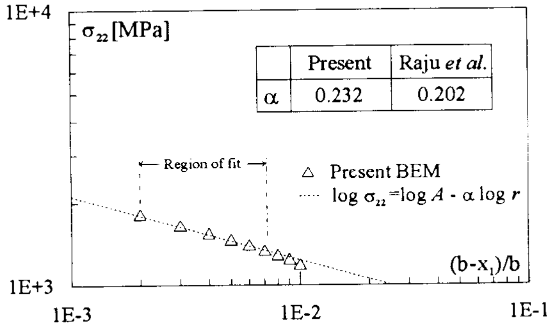

Figure 5: *Log-log plot for* σ_{22} *along* $x_2 = h$ *for a* $[0/90]_S$ *laminate.*

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Conclusions

The present formulation allows one to analyse the multi-layered cross-ply composite laminate subjected to uniaxial extension. The approach is based on the theory of integral equations and it employs the boundary element method to obtain a numerical solution of the problem. The satisfaction of displacement and traction continuity along the interfaces leads to a linear system of algebraic equations. The interlaminar stress field, obtained without any *a priori* assumption, shows high gradients at the laminate free edges. This confirms, in agreement with the literature, the presence of stress singularities at the ply's corners. The determination of singularity power and strength is carried out and good results are obtained despite the relatively small number of elements employed. The future developments of the proposed formulation will focus mainly on its application to angle-ply laminates and to absolutely general lay-ups.

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