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Stress-Tensor Commutators and Schwinger Terms*

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We investigate, in local field theory, general properties of commutators involving Poincaré generators or stress-tensor components, particularly those of local commutators among the latter. The spectral representation of the vacuum stress commutator is given, and shown to require the existence of singular "Schwinger terms" at equal times, similar to those present in current commutators. These terms are analyzed and related to the metric dependence of the stress tensor in the presence of a prescribed gravitational field and some general results concerning this dependence presented. The resolution of the Schwinger paradox for the $T^{\mu\nu}$ commutators is discussed together with some of its implications, such as "nonclassical" metric dependence of $T^{\mu\nu}$. A further paradox concerning the vacuum self-stress—whether the stress tensor or its vacuum-subtracted value should enter in the commutators—is related to the covariance of the theory, and partially resolved within this framework.

I. INTRODUCTION

THE commutation relations among the generators $(P^\mu, J^{\lambda\sigma})$ of the Poincaré group, together with the existence of a unique normalizable vacuum state, require their vacuum expectation values to vanish.[1] Lorentz invariance also dictates the effect of these generators on any tensor, in particular on the symmetric stress tensor $T^{\mu\nu}$ itself, thereby placing requirements on the vacuum expectation value of the latter. While the stress tensor does not in general vanish in the vacuum, one may of course define subtracted stresses, $\bar{T}^{\mu\nu} = T^{\mu\nu} - \langle T^{\mu\nu}\rangle$. However, the commutator of any operator with $\bar{T}^{\mu\nu}$ is equal to that with $T^{\mu\nu}$. In particular, commutators such as $i[T^{00}(\mathbf{r}), T^{00}(\mathbf{r}')]$ are independent of whether T^{00} or $\bar{T}^{00}$ is used. This commutator, one of several which determine the Lorentz covariance of a theory,[2,3] has the particularly simple form $i[T^{00}(\mathbf{r}), T^{00}(\mathbf{r}')] = [T^{0k}(\mathbf{r}) + T^{0k}(\mathbf{r}')]\partial_k\delta(\mathbf{r}-\mathbf{r}')$ for fields of spin ≤ 1. The right-hand sides of such relations, on the other hand, are clearly dependent on whether $T^{\mu\nu}$ or $\bar{T}^{\mu\nu}$ is used. We may see, in going symmetrically from the Poincaré algebra, through relations of the type $[J^{\mu\nu}, T^{\lambda\sigma}]$ to $[T^{\mu\nu}, T^{\lambda\sigma}]$, that the right sides are in fact independent of whether $T^{\mu\nu}$ or $\bar{T}^{\mu\nu}$ is used, provided, as is required by Lorentz invariance, that $\langle T^{\mu\nu}\rangle = -\lambda\eta^{\mu\nu}$ (λ is constant, $\eta^{\mu\nu}$ is the Lorentz metric). Conversely, the connection of the stress tensor commutators to the Poincaré algebra will then also be verifiable in terms of either the original or the subtracted stresses. While these results are satisfactory, they are somewhat formal, for the usual evaluation of $\langle T^{\mu\nu}\rangle$ (even for free fields) yields a divergent, *noncovariant* result. (For example, the Maxwell field has $T^\mu_\mu = 0$, $\langle T^{00}\rangle > 0$.) Taking this noncovariance literally implies that a Wick ordering must be performed not only on $T^{\mu\nu}$ itself, but on all commutation relations involving $T^{\mu\nu}$ or the generators as well. This can be avoided by using extremely *ad hoc* prescriptions, which make $\langle T^{\mu\nu}\rangle$ covariant. These prescriptions are closely related to the necessity (for reasons given below) of redefining $T^{\mu\nu}$ as the limit of a spatially nonlocal operator.

Independently of the operator commutators mentioned above, the vacuum expectation values of local stress-tensor commutators $\langle[T^{\mu\nu}, T^{\lambda\sigma}]\rangle$ may be expressed in Lehmann–Källén (spectral) form solely on covariance grounds. Comparison with the operator expressions then implies, in addition to the above conditions on $\langle T^{\mu\nu}\rangle$, the necessary presence of Schwinger terms[4] [singular terms involving higher derivatives of $\delta(\mathbf{r})$] in the equal time $T^{\mu\nu}$ commutators, in close analogy to the corresponding results for current commutators. The metric dependence (of a fully quantum nature) of $T^{\mu\nu}$ in an external gravitational field implied by these terms is discussed. This dependence is in addition to the "classical" one dictated by general covariance which is also treated here. We give both general results on metric dependence of

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[1] This also follows from the absence of constant vectors or antisymmetric tensors to represent the constants $\langle P^\mu\rangle$ and $\langle J^{\mu\nu}\rangle$. We use the notation $\langle A\rangle$ to denote the vacuum expectation value, where the vacuum is assumed to be unique, normalizable, and invariant under the inhomogeneous Lorentz group.
[2] J. Schwinger, Phys. Rev. 127, 324 (1962); 130, 406, 800 (1963).
[3] P. A. M. Dirac, Rev. Mod. Phys. 34, 1 (1962).
[4] J. Schwinger, Phys. Rev. Letters 3, 259 (1959).

$T^{\mu\nu}$ and also, in the canonical formulation of specific local fields, the explicit (classical) dependence on the components $g_{0\mu}$ needed to evaluate the commutator expressions.

II. COMMUTATORS INVOLVING GENERATORS

Lorentz invariance is established in a field theory when the existence of Poincaré generators can be demonstrated. What is often actually exhibited, in a manifestly covariant theory, is not the Poincaré algebra of the $(P^\mu, J^{\lambda\sigma})$ but rather their effects as generators of field transformations:

$$i[\psi(x), P^\mu] = \partial^\mu \psi(x), \tag{1a}$$

$$i[\psi(x), J^{\mu\nu}] = (x^\mu\partial^\nu - x^\nu\partial^\mu)\psi(x) + iS^{\mu\nu}\psi(x), \tag{1b}$$

the matrices $S^{\mu\nu}$ realizing a finite dimensional representation of the Lorentz group. If Eqs. (1) hold for a complete set of fields ψ, they define the generators uniquely to within an additive c number. We now invoke the group structure implicit in Eqs. (1) and observe that, by the Jacobi identity, the operators $(\bar{P}^\mu, \bar{J}^{\mu\nu})$ defined by the right-hand sides of

$$i[P^\mu, P^\nu] = 0, \tag{2a}$$

$$i[P^\mu, J^{\lambda\sigma}] = \eta^{\mu\lambda}\bar{P}^\sigma - \eta^{\mu\sigma}\bar{P}^\lambda, \tag{2b}$$

$$i[J^{\mu\nu}, J^{\lambda\sigma}] = \eta^{\mu\lambda}\bar{J}^{\sigma\nu} - \eta^{\mu\sigma}\bar{J}^{\lambda\nu} + \eta^{\nu\lambda}\bar{J}^{\mu\sigma} - \eta^{\nu\sigma}\bar{J}^{\mu\lambda}, \tag{2c}$$

generate the same Lorentz transformations [Eqs. (1)] as do $(P^\mu, J^{\lambda\sigma})$. The $(\bar{P}^\mu, \bar{J}^{\lambda\sigma})$ then differ at most by a c number from $(P^\mu, J^{\lambda\sigma})$; further, Eqs. (2), together with the existence of a unique (invariant) normalizable vacuum, require that $\langle\bar{P}^\mu\rangle = 0 = \langle\bar{J}^{\lambda\sigma}\rangle$, but *not*, of course, that $\langle P^\mu\rangle = 0 = \langle J^{\lambda\sigma}\rangle$. The $(\bar{P}^\mu, \bar{J}^{\lambda\sigma})$ are completely fixed by this requirement, for any other set would differ by a c number and hence not vanish in the vacuum. We may then, if we like, rewrite Eqs. (2) as the usual algebra of generators with vanishing vacuum values simply by putting bars over the $(P^\mu, J^{\lambda\sigma})$ on the left sides:

$$i[\bar{P}^\mu, \bar{P}^\lambda] = 0, \tag{3a}$$

$$i[\bar{P}^\mu, \bar{J}^{\lambda\sigma}] = \eta^{\mu\lambda}\bar{P}^\sigma - \eta^{\mu\sigma}\bar{P}^\lambda, \tag{3b}$$

$$i[\bar{J}^{\mu\nu}, \bar{J}^{\lambda\sigma}] = \eta^{\mu\lambda}\bar{J}^{\sigma\nu} - \eta^{\mu\sigma}\bar{J}^{\lambda\nu} + \eta^{\nu\lambda}\bar{J}^{\nu\sigma} - \eta^{\nu\sigma}\bar{J}^{\mu\lambda}. \tag{3c}$$

We emphasize that Lorentz invariance requires not only the vanishing in the vacuum of the right sides of Eqs. (2) [or the members of the algebra of Eqs. (3)], but that the commutators on the left must automatically produce the correct $(\bar{P}^\mu, \bar{J}^{\lambda\sigma})$ generators.

Consider now the effect of the generators on an arbitrary symmetric second-rank tensor $T^{\mu\nu}(x)$; the commutators must take the form

$$i[T^{\mu\nu}(x), P^\lambda] = \partial^\lambda T^{\mu\nu}(x), \tag{4a}$$

$$\begin{aligned} i[T^{\mu\nu}(x), J^{\lambda\sigma}] = (x^\lambda\partial^\sigma - x^\sigma\partial^\lambda)T^{\mu\nu}(x) \\ + \eta^{\mu\lambda}T^{\sigma\nu}(x) - \eta^{\mu\sigma}T^{\lambda\nu}(x) \\ + \eta^{\nu\lambda}T^{\mu\sigma}(x) - \eta^{\nu\sigma}T^{\mu\lambda}(x). \end{aligned} \tag{4b}$$

As before, the left sides of Eqs. (4) must vanish in the vacuum. For consistency then, Eq. (4a) requires that $\langle T^{\mu\nu}\rangle$ be constant,

$$\partial^\lambda\langle T^{\mu\nu}(x)\rangle = 0 \tag{5a}$$

while Eq. (4b) requires in addition that the constant be invariant, namely that

$$\langle T^{\mu\nu}(x)\rangle = -\lambda\eta^{\mu\nu}. \tag{5b}$$

Equations (5) just express the well-known translation and rotation invariance requirements on the vacuum expectation of any local symmetric second-rank tensor. If, in particular, $T^{\mu\nu}(x)$ is chosen to be the stress tensor of a local field theory,[5] we see that Lorentz invariance [as expressed by Eqs. (4)] does not require that $\langle T^{\mu\nu}(x)\rangle$ vanish, but only that it satisfy Eqs. (5). However, precisely the conditions expressed by Eqs. (5) are sufficient for the right sides of Eqs. (4) to have the same form in terms of $\bar{T}^{\mu\nu}(x) = T^{\mu\nu}(x) - \langle T^{\mu\nu}\rangle$, as is easily verified. We may then write

$$i[T^{\mu\nu}(x), P^\lambda] = \partial^\lambda \bar{T}^{\mu\nu}(x), \tag{6a}$$

$$\begin{aligned} i[T^{\mu\nu}(x), J^{\lambda\sigma}] = (x^\lambda\partial^\sigma - x^\sigma\partial^\lambda)\bar{T}^{\mu\nu}(x) + \eta^{\mu\lambda}\bar{T}^{\sigma\nu}(x) \\ - \eta^{\mu\sigma}\bar{T}^{\lambda\nu}(x) + \eta^{\nu\lambda}\bar{T}^{\mu\sigma}(x) - \eta^{\nu\sigma}\bar{T}^{\mu\lambda}(x). \end{aligned} \tag{6b}$$

From Eqs. (6), one may now conclude that if the $T^{\mu\nu}$ on the left are integrated[6] to yield P^λ or $J^{\lambda\sigma}$ [or if one puts $\bar{T}^{\mu\nu}$ on the left and integrates them to $(\bar{P}^\mu, \bar{J}^{\lambda\sigma})$] the corresponding integrals on the right are represented by the correct $(\bar{P}^\mu, \bar{J}^{\lambda\sigma})$ as required by Eqs. (2) or (3). [Some care must be taken in establishing this; if one starts from Eqs. (4) in terms of the original $T^{\mu\nu}$ on the right, the required integrations by parts yield nonvanishing surface terms here since $T^{\mu\nu}$, unlike $\bar{T}^{\mu\nu}$, does not vanish at infinity.[7]]

[5] $T^{\mu\nu}$ is not necessarily a local function of the canonical variables even if the Lagrangian is local (e.g., the Maxwell field with sources or the gravitational field). Even where $T^{\mu\nu}$ is a local function, as for the free spin-two massless field, the commutator $[T^{00}(\mathbf{r}), T^{00}(\mathbf{r}')]$, for example, is not necessarily local. See S. Deser, J. Trubatch, and S. Trubatch, Nuovo Cimento **39**, 1159 (1965).

[6] The generators P^μ and $J^{\lambda\sigma}$ can, of course, be written in terms of the stress tensor $T^{\mu\nu}$ through the relations $P^\mu = \int d^3r T^{0\mu}(x)$, $J^{\lambda\sigma} = \int d^3r[x^\lambda T^{0\sigma}(x) - x^\sigma T^{0\lambda}(x)]$. The same relations then obviously hold.

[7] We here assume that any physical system is sufficiently well localized that $(x^i)^4\bar{T}^{0\nu}(x) \to 0$ as $\mathbf{r} \to \infty$. This will insure that the generators $(\bar{P}^\mu, \bar{J}^{\lambda\sigma})$ have finite matrix elements between physical states.

III. STRESS-TENSOR COMMUTATORS

We now consider a general set of local equal-time commutation relations[8] among the $T^{\mu\nu}(x)$ which, upon integration,[6] yield the Poincaré algebra, Eqs. (2) and (3), as well as Eqs. (4):

$$i[T^{00}(\mathbf{r}), T^{00}(\mathbf{r}')] = (T^{0k}(\mathbf{r}) + T^{0k}(\mathbf{r}'))\partial_k\delta(\mathbf{r}-\mathbf{r}') - \tau^{00,00}(\mathbf{r},\mathbf{r}'), \quad (7a)$$

$$i[T^{00}(\mathbf{r}), T^{0m}(\mathbf{r}')] = (T^{mn}(\mathbf{r}) + T^{00}(\mathbf{r}')\delta^{mn})\partial_n\delta(\mathbf{r}-\mathbf{r}') - \tau^{00,0m}(\mathbf{r},\mathbf{r}'), \quad (7b)$$

$$i[T^{00}(\mathbf{r}), T^{mn}(\mathbf{r}')] = (-\partial^0 T^{mn}(\mathbf{r}) + T^{0m}(\mathbf{r}')\partial^n + T^{0n}(\mathbf{r}')\partial^m)\delta(\mathbf{r}-\mathbf{r}') - \tau^{00,mn}(\mathbf{r},\mathbf{r}'), \quad (7c)$$

$$i[T^{0k}(\mathbf{r}), T^{0m}(\mathbf{r}')] = (T^{0m}(\mathbf{r})\partial^k + T^{0k}(\mathbf{r}')\partial^m)\delta(\mathbf{r}-\mathbf{r}') - \tau^{0k,0m}(\mathbf{r},\mathbf{r}'), \quad (7d)$$

$$i[T^{0k}(\mathbf{r}), T^{mn}(\mathbf{r}')] = (T^{mn}(\mathbf{r})\delta^{kl} - T^{ml}(\mathbf{r}')\delta^{nk} - T^{nl}(\mathbf{r}')\delta^{mk})\partial_l\delta(\mathbf{r}-\mathbf{r}') - \tau^{0k,mn}(\mathbf{r},\mathbf{r}'). \quad (7e)$$

The operators $\tau^{\mu\nu,\lambda\sigma}(\mathbf{r},\mathbf{r}')$ in Eqs. (7) are, in general, model dependent; they are, however, constrained to have certain integrals and moments vanishing. These constraints arise as the explicit $T^{\mu\nu}$ dependence on the right sides of Eqs. (7) is precisely such as to yield Eqs. (4) when integrating (or taking first moments) over $\mathbf{r}$ or $\mathbf{r}'$ [and, of course, yields Eqs. (2, 3) when integrated over both variables]. Thus, for

$$\tau^{00,00}(\mathbf{r},\mathbf{r}') = -\tau^{00,00}(\mathbf{r}',\mathbf{r}),$$

we must have in general that

$$\int d^3r\,\tau^{00,00}(\mathbf{r},\mathbf{r}') = 0 = \int d^3r\,x^k\tau^{00,00}(\mathbf{r},\mathbf{r}').$$

Relations (7) do not form an algebra, partly because of the τ, partly because no condition from the Poincaré relations is available to specify $[T^{kl}, T^{mn}]$ in a model independent way. We are, of course, assured by the earlier discussion that, upon integration of Eqs. (7), the right sides will be expressible in terms of the $\bar{T}^{\mu\nu}$. We may now ask if this is also the case for Eqs. (7) themselves? The condition $\langle T^{\mu\nu}\rangle = -\lambda\eta^{\mu\nu}$ clearly ensures that Eqs. (7a)–(7d) hold also in terms of $\bar{T}^{\mu\nu}$. However, Eq. (7e) changes form, by a term

$$\sim\lambda(\delta^{mn}\delta^{kl} - \delta^{mk}\delta^{nl} - \delta^{ml}\delta^{nk})\partial_l\delta(\mathbf{r}-\mathbf{r}')$$

when $T^{\mu\nu}$ is replaced by $\bar{T}^{\mu\nu} + \lambda\eta^{\mu\nu}$. This difference has a vanishing integral over $\mathbf{r}$ and a vanishing antisymmetric first moment; hence it can be absorbed into the $\tau^{0k,mn}(\mathbf{r},\mathbf{r}')$ term, leaving a formally identical expression for the model-independent stress-tensor parts in terms of the $\bar{T}^{\mu\nu}$ together with an appropriately redefined $\tau^{0k,mn}(\mathbf{r},\mathbf{r}')$. We will see, in fact, in terms of the spectral form for $\langle[T^{\mu\nu}(x), T^{\lambda\sigma}(x')]\rangle$, that a sum rule relates certain integrals of spectral functions for $\langle T^{mn}\rangle$ and $\langle\tau^{0k,mn}\rangle$ or, equivalently, these integrals to $\langle\bar{T}^{mn}\rangle$ and the redefined $\langle\tau^{0k,mn}\rangle$.

[8] Some of these relations are given in Refs. 2 and 3.

IV. SPECTRAL FORM OF VACUUM COMMUTATORS

If the $T^{\mu\nu}(x)$ are local operators and transform as tensors under proper Lorentz transformations, the vacuum expectation of the stress tensor commutators can be given a Lehmann–Källén representation.[9] For an arbitrary conserved symmetric second-rank tensor, there are two independent weight functions specifying the vacuum commutator;

$$\langle 0|\,[T^{\mu\nu}(x), T^{\lambda\sigma}(x')]\,|0\rangle = \int_0^\infty ds\{\rho_2(s)[\theta^{\mu\lambda}\theta^{\nu\sigma} + \theta^{\mu\sigma}\theta^{\nu\lambda} - \tfrac{2}{3}\theta^{\mu\nu}\theta^{\lambda\sigma}] + \rho_0\theta^{\mu\nu}\theta^{\lambda\sigma}\}\Delta(x-x', s), \quad (8)$$

where $\theta^{\mu\nu} \equiv \eta^{\mu\nu} - s^{-1}\partial^\mu\partial^\nu$ is conserved [i.e., $\partial_\nu\theta^{\mu\nu}\times\Delta(x,s) = 0$] and $\Delta(x-x',s)$ is the causal propagator with the property that $\Delta(x-x',s) = 0$ and

$$\partial^0\Delta(x-x',s) = i\delta(\mathbf{r}-\mathbf{r}')$$

for $x^0 = x'^0$. The functions $\rho_2(s)$ and $\rho_0(s)$, representing the contributions of intermediate states of mass $s^{\frac{1}{2}}$ and spin 2 and 0 respectively are nonnegative if the Hilbert space metric is positive definite.[10]

The only nonvanishing equal-time commutators are those with an odd number of temporal indices [since $\Delta(x,s)$ is odd in x^0]:

$$\langle 0|\,[T^{00}(\mathbf{r}), T^{0k}(\mathbf{r}')]\,|0\rangle = -i\int_0^\infty ds\, s^{-2}[\tfrac{4}{3}\rho_2(s) + \rho_0(s)](-\nabla^2)\partial^k\delta(\mathbf{r}-\mathbf{r}') \quad (9a)$$

$$\langle 0|\,[T^{0k}(\mathbf{r}), T^{mn}(\mathbf{r}')]\,|0\rangle = -i\int_0^\infty ds\{s^{-1}\rho_2(s)(\delta^{mk}\delta^{nl} + \delta^{nk}\delta^{ml} - \tfrac{2}{3}\delta^{kl}\delta^{mn}) \times\partial_l\delta(\mathbf{r}-\mathbf{r}') + s^{-1}\rho_0(s)\delta^{kl}\delta^{mn}\partial_l\delta(\mathbf{r}-\mathbf{r}') + s^{-2}[\tfrac{4}{3}\rho_2(s) + \rho_0(s)](-\partial^k\partial^m\partial^n)\delta(\mathbf{r}-\mathbf{r}')\}. \quad (9b)$$

[9] There are systems which violate these assumptions. For zero mass fields with spin $\geq\frac{3}{2}$; the Lorentz transformations induce additional gauge transformations on $\bar{T}^{\mu\nu}$ [see Ref. 5, and C. M. Bender and B. M. McCoy, Phys. Rev. **148**, 1375 (1966)], and so the latter do not transform as Lorentz tensors. There are, however, no restrictions on the singularity of the $\langle T^{\mu\nu}(x)T^{\lambda\sigma}(x')\rangle$ function. If the Wightman function exists, then the spectral form does also; see K. Bardacki and B. Schroer, J. Math. Phys. **7**, 10 (1966).

[10] This condition includes the radiation gauge formulation of electrodynamics which possesses a positive definite metric and a gauge invariant stress tensor.

Comparing the equal-time forms with Eqs. (7) and using $\langle T^{\mu\nu}\rangle = -\lambda\eta^{\mu\nu}$, we find first, from the vanishing components, that

$$\langle\tau^{00,00}\rangle = 0 = \langle\tau^{00,mn}\rangle = 0 = \langle\tau^{0m,0n}\rangle, \quad (10a)$$

while Eq. (9a) yields

$$\langle\tau^{0k,00}(\mathbf{r},\mathbf{r}')\rangle = -\int_0^\infty ds\, s^{-2}[\tfrac{4}{3}\rho_2(s) + \rho_0(s)] \times (-\nabla^2)\partial^k\delta(\mathbf{r}-\mathbf{r}'). \quad (10b)$$

The right side of Eq. (9b) has both a $\partial^k\delta(\mathbf{r}-\mathbf{r}')$ and a $\partial^k\partial^m\partial^n\delta(\mathbf{r}-\mathbf{r}')$ part, and so must $\langle\tau^{0k,mn}(\mathbf{r},\mathbf{r}')\rangle$. Equating first derivatives yields a sum rule between $\int_0^\infty ds\, s^{-1}\rho_2(s)$, $\int_0^\infty ds\, s^{-1}\rho_0(s)$, λ, and the $\partial^k\delta(\mathbf{r}-\mathbf{r}')$ part of $\langle\tau^{0k,mn}\rangle$, or, alternatively, between these integrals and the redefined $\langle\tau^{0k,mn}\rangle$. The part of $\langle\tau^{0k,mn}\rangle$ which is proportional to $\partial^k\partial^m\partial^n\delta(\mathbf{r}-\mathbf{r}')$ satisfies

$$\langle\tau^{0k,mn}(\mathbf{r},\mathbf{r}')\rangle = \int_0^\infty ds\, s^{-2}[\tfrac{4}{3}\rho_2(s) + \rho_0(s)] \times \partial^k\partial^l\partial^m\delta(\mathbf{r}-\mathbf{r}'). \quad (10c)$$

Note that the $(\partial)^3\delta(\mathbf{r}-\mathbf{r}')$ terms in both $\langle\tau^{00,0m}\rangle$ and $\langle\tau^{0k,mn}\rangle$ involve the same nonnegative integral

$$\int_0^\infty ds\, s^{-2}[\tfrac{4}{3}\rho_2(s) + \rho_0(s)].$$

Equations (10) are, for our purposes, the most important consequences of the spectral relations (9). They imply that singular Schwinger terms[4] proportional to $(\partial)^3\delta(\mathbf{r}-\mathbf{r}')$ must be present in the operator relations, Eqs. (7b) and (7e), if the operator $T^{\mu\nu}$ itself is not to vanish. For, since ρ_2 and ρ_0 are separately nonnegative, they would each have to vanish if the $(\partial)^3\delta(\mathbf{r}-\mathbf{r}')$ terms in Eqs. (10) were absent. However, we could then conclude from the Wightman product corresponding to Eq. (8), that $\langle T^{\mu\nu}(x)T^{\lambda\sigma}(x')\rangle = 0$, and hence (since $T^{\mu\nu}$ is Hermitian) that $T^{\lambda\sigma}(x)\,|0\rangle = 0$. This follows from the fact that the Wightman product differs from Eq. (8) only by the replacement of $\Delta(x-x',s)$ by $\Delta^{(+)}(x-x',s)$, but has the same spectral functions. But, by the Federbush-Johnson theorem,[11] $T^{\mu\nu}$ itself must vanish (as an operator) if $T^{\mu\nu}(x)\,|0\rangle$ vanishes.[12] Thus, positive Hilbert space metric, positive energy spectrum, proper Lorentz covariance and locality by themselves require the presence of singular terms in $\tau^{00,0m}$ and $\tau^{0k,mn}$, i.e., in the commutators of Eqs. (7b) and (7e). As in the case of currents, naive application of canonical commutation or anticommutation relations, even for free spin 0, $\frac{1}{2}$, or 1 fields yield, paradoxically, no Schwinger terms. Hence, the singular operator $T^{\mu\nu}$ must be redefined, in analogy with the procedure for currents, as the limit of a nonlocal $T^{\mu\nu}$ in which the constituent field operators are separated by a space like distance and the commutators evaluated before taking the limit. This prescription does yield nonvanishing $(\partial)^3\delta(\mathbf{r}-\mathbf{r}')$ contributions, at least for free systems whose $T^{\mu\nu}$ are bilinear in the fields. For interacting fields, $T^{\mu\nu}$ contains, of course, higher powers of field operators. This case has not been investigated, but it seems likely that the essence of the problem resides in the kinematical free field parts.

[11] P. Federbush and K. Johnson, Phys. Rev. **120**, 1926 (1960). The essential point is that any local operator which annihilates the vacuum must vanish identically. See also R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics and All That* (W. A. Benjamin, Inc., New York, 1964), Chap. 4.

[12] This derivation is quite similar to that used elsewhere for vector currents: D. G. Boulware and S. Deser, Phys. Letters **22**, 99 (1966); Phys. Rev. **151**, 1278 (1966).

V. METRIC DEPENDENCE AND STRESS-TENSOR COMMUTATORS

We discuss here the general dependence of the stress tensor on a weak external metric $g_{\mu\nu}$; our treatment is essentially a generalization of the analysis of the second paper of Ref. 2, which treated the case of a weak external g_{00} (this being sufficient for the $[T^{00}, T^{00}]$ commutator). These considerations bring out some properties of the functions τ of Eqs. (7), constituting, in fact, a derivation of the latter equations. We also remark on a more specific problem: the dependence on an arbitrary metric of the stress tensor for local dynamical fields. The dependence on the four components $g_{0\nu}$, needed to evaluate the right sides of Eqs. (7), is explicitly exhibited for fields of spin ≤ 1, and seen to be in accord with the requirements for a Hamiltonian formulation of the coupled matter and gravitational fields.

We begin with the definition of the stress tensor of a dynamical system as the coefficient of the variation of an external metric in the generally covariant form of its action[13] according to

$$\delta W_M = \int dx\tfrac{1}{2}\delta g_{\mu\nu}(x)\mathcal{T}^{\mu\nu}(x),$$

where $\mathcal{T}^{\mu\nu}(x)$ is the metric dependent symmetric tensor density. Thus a general matrix element in a prescribed classical external $g_{\mu\nu}$ obeys

$$-2i[\delta\langle a\mid b\rangle/\delta g_{\mu\nu}(x)] = \langle a|\,\mathcal{T}^{\mu\nu}(x)\,|b\rangle \quad (11a)$$

and a second variation then yields the stress-tensor correlation function

$$2[\delta\langle a|\,\mathcal{T}^{\mu\nu}(x)\,|b\rangle/\delta g_{\lambda\sigma}(x')] = i\,\langle a|\,[\mathcal{T}^{\mu\nu}(x)\mathcal{T}^{\lambda\sigma}(x')]_+\,|b\rangle + 2\,\langle a|\,\delta\mathcal{T}^{\mu\nu}(x)/\delta g_{\lambda\sigma}(x')\,|b\rangle. \quad (11b)$$

[13] A complementary problem is the use of prescribed external $T^{\mu\nu}$ in probing the properties of a dynamical metric field, which has been discussed by the authors, Nuovo Cimento **30**, 1009 (1963).

The last term takes into account the explicit $g_{\lambda\sigma}$ dependence of $\mathcal{T}^{\mu\nu}$ (in analogy with terms $\delta j^\mu/\delta A_\lambda$ in electrodynamics[2]). Note the reciprocity

$$\delta\mathcal{T}^{\mu\nu}(x)/\delta g_{\lambda\sigma}(x') = \delta\mathcal{T}^{\lambda\sigma}(x')/\delta g_{\mu\nu}(x).$$

The conservation law for $\mathcal{T}^{\mu\nu}$ is now the covariant one,

$$\mathcal{T}^{\mu\nu};\nu \equiv \mathcal{T}^{\mu\nu},\nu + \mathcal{T}^{\alpha\beta}\Gamma^\mu_{\alpha\beta} = 0. \tag{12}$$

If we vary a matrix element of this equation, we obtain from

$$\delta/\delta g_{\lambda\sigma}\langle a|\,\mathcal{T}^{\mu\nu};\nu\,|b\rangle \equiv 0$$

and from Eqs. (11) and (12), the relation

$$\partial_\nu\langle a|\,i[\mathcal{T}^{\mu\nu}(x)\mathcal{T}^{\lambda\sigma}(x')]_+\,|b\rangle + \partial_\nu\langle a|\,2\frac{\delta\mathcal{T}^{\mu\nu}(x)}{\delta g_{\lambda\sigma}(x')}|b\rangle + 2\frac{\delta\Gamma^\mu_{\alpha\beta}(x)}{\delta g_{\lambda\sigma}(x')}\langle a|\,\mathcal{T}^{\alpha\beta}(x)\,|b\rangle + \Gamma^\mu_{\alpha\beta}2\frac{\delta\langle a|\,\mathcal{T}^{\alpha\beta}(x)\,|b\rangle}{\delta g_{\lambda\sigma}(x')} = 0. \tag{13}$$

While relation (13) holds in the presence of an arbitrary metric, we are primarily interested here in the flat space limit $g_{\mu\nu}\to\eta_{\mu\nu}$. Then Γ is zero and the variation of Γ,

$$\delta\Gamma^\mu_{\alpha\beta} = -g^{\mu\lambda}\delta g_{\lambda\sigma}\Gamma^\sigma_{\alpha\beta} + \tfrac{1}{2}g^{\mu\rho}(\partial_\alpha\delta g_{\rho\beta} + \partial_\beta\delta g_{\rho\alpha} - \partial_\rho\delta g_{\alpha\beta}), \tag{14}$$

reduces to the three $\partial\delta g$ terms. We may then conclude from Eqs. (13) and (14) that, in the flat space limit,

$$\partial_\nu\left\{i[T^{\mu\nu}(x)T^{\lambda\sigma}(x')]_+ + 2\frac{\delta\mathcal{T}^{\mu\nu}(x)}{\delta g_{\lambda\sigma}(x')}\right\} + [\eta^{\mu\lambda}T^{\sigma\nu}(x) + \eta^{\mu\sigma}T^{\lambda\nu}(x) - \eta^{\mu\nu}T^{\lambda\sigma}(x)] \times \partial_\nu\delta(x-x') = 0. \tag{15}$$

In Eq. (15) we have returned to the flat space tensor $T^{\mu\nu}$ (which is, of course, identical to the tensor density $\mathcal{T}^{\mu\nu}$ in the limit), except in $\delta\mathcal{T}/\delta g$ where the distinction must be kept. On the other hand, the discontinuity of the time-ordered product at $x^0 = x'^0$ now yields

$$i[T^{0\mu}(x), T^{\lambda\sigma}(x')]\delta(x^0 - x'^0) = [\eta^{\mu\nu}T^{\lambda\sigma}(x) - \eta^{\mu\lambda}T^{\nu\sigma}(x) - \eta^{\mu\sigma}T^{\nu\lambda}(x)] \times \partial_\nu\delta(x-x') - 2\partial_\nu[\delta\mathcal{T}^{\mu\nu}(x)/\delta g_{\lambda\sigma}(x')]. \tag{16}$$

The absence of a $[T^{kl}, T^{mn}]$ relation here reflects the fact that T^{kl} does not obey a (partial) conservation law. The commutator terms arise exclusively from the discontinuities of the time-ordered products, which yield commutators when differentiated with respect to time in the course of applying Eq. (15).

Equation (16) is nearly of the form of Eqs. (7) with $2\partial_\nu(\delta\mathcal{T}^{\mu\nu}/\delta g_{\lambda\sigma})$ playing the role of the (model-dependent) $\tau^{0\mu,\lambda\sigma}$; however, the right side of Eq. (16) contains explicit time derivatives of the delta function which are inconsistent with the equal time nature of the commutator. There must therefore be terms in $\delta\mathcal{T}^{0\mu}/\delta g_{\lambda\sigma}$ which cancel these time derivatives (there are also other, time local, parts of $\delta\mathcal{T}/\delta g$). The analysis of the various terms can most easily be presented by defining functions $t^{\mu\nu,\lambda\sigma}$:

$$t^{0\nu,00}(x,x') = 2[\delta\mathcal{T}^{0\nu}(x)/\delta g_{00}(x')] + T^{00}(x)\eta^{0\nu}\delta(x-x'), \tag{17a}$$

$$t^{0\nu,0m}(x,x') = 2[\delta\mathcal{T}^{0\nu}(x)/\delta g_{0m}(x')] + T^{00}(x)\eta^{\nu m}\delta(x-x'), \tag{17b}$$

$$t^{0\nu,mn}(x,x') = 2[\delta\mathcal{T}^{0\nu}(x)/\delta g_{mn}(x')] + [T^{0m}(x)\eta^{\nu n} + T^{0n}(x)\eta^{\nu m} - T^{0m}(x)\eta^{0\nu}] \times \delta(x-x'), \tag{17c}$$

$$t^{k\nu,0m}(x,x') = 2[\delta\mathcal{T}^{k\nu}(x)/\delta g_{0m}(x')] + [T^{0k}(x)\eta^{\nu m} + T^{0\nu}(x)\eta^{km}] \times \delta(x-x'), \tag{17d}$$

$$t^{k\nu,mn}(x,x') = 2[\delta\mathcal{T}^{k\nu}(x)/\delta g_{mn}(x')]. \tag{17e}$$

The $t^{\mu\nu,\lambda\sigma}(x,x')$ are symmetric,

$$t^{\mu\nu,\lambda\sigma}(x,x') = t^{\lambda\sigma,\mu\nu}(x',x)$$

and, comparing with Eqs. (7), we have the relation

$$\partial_\nu t^{\mu\nu;\lambda\sigma}(x,x') = \tau^{0\mu,\lambda\sigma}(x,x') \tag{18}$$

As an example of how these equations are derived, we consider Eq. (17a). Equation (16) states that

$$i[T^{00}(x), T^{00}(x')]\delta(x^0 - x'^0) = 2T^{0k}(x)\partial_k\delta(x-x') + T^{00}(x)\partial_0\delta(x-x') - 2\partial_0[\delta\mathcal{T}^{00}(x)/\delta g_{00}(x')] - 2\partial_k[\delta\mathcal{T}^{0k}(x)/\delta g_{00}(x')].$$

Then, the definition

$$2[\delta\mathcal{T}^{00}(x)/\delta g_{00}(x')] = T^{00}(x)\delta(x-x') + t^{00,00}(x,x')$$

explicitly cancels the undesirable $T^{00}(x)\partial_0\delta(x-x')$ term. To see whether a similar redefinition is needed for the $\delta\mathcal{T}^{0k}(x)/\delta g_{00}(x')$ term, consider

$$i[T^{0k}(x), T^{00}(x')]\delta(x^0 - x'^0) = T^{00}(x)\partial^k\delta(x-x') - 2\partial_\nu[\delta\mathcal{T}^{k\nu}(x)/\delta g_{00}(x')].$$

Clearly, none is required, since there are no explicit $\partial_0\delta$ terms on the right. Thus, we arrive at Eq. (17a),

$$t^{0\nu,00}(x,x') = 2[\delta\mathcal{T}^{0\nu,00}(x)/\delta g_{00}(x')] + T^{00}(x)\eta^{0\nu}\delta(x-x')$$

and obtain the expression

$$i[T^{00}(x), T^{00}(x')]\delta(x^0 - x'^0) = [T^{0k}(x) + T^{0k}(x')]\partial_k\delta(x-x') - \partial_\nu t^{0\nu,00}(x,x') \tag{19}$$

A similar analysis yields the remainder of Eqs. (17) together with the analogs of Eq. (19).

The $\bar{\tau}$ of Eqs. (7) are antisymmetric, $\bar{\tau}^{\mu\nu,\lambda\sigma}(x,x') = -\bar{\tau}^{\lambda\sigma,\mu\nu}(x',x)$, hence, using Eq. (18), the symmetry of the t, the antisymmetry of the $\bar{\tau}$, and the integral conditions which enforce the vanishing of the moments of $\bar{\tau}$, the following expressions are obtained.

$$t^{00,00}(x,x') = \partial_k\partial_l\partial'_m\partial'_n\sigma^{kl,mn}(x,x'), \tag{20a}$$

$$t^{0k,00}(x,x') = \partial_l\partial'_m\partial'_n[\tau_1^{kl,mn}(x,x') - \partial_0\sigma^{kl,mn}(x,x')], \tag{20b}$$

$$t^{0k,0m}(x,x') = \partial_l\partial'_n[\tau_2^{kl,mn}(x,x') + \tfrac{1}{2}(\partial_0 - \partial'_0) \times \tau_1^{kl,mn}(x,x') + \partial_0\partial'_0\sigma^{kl,mn}(x,x')], \tag{20c}$$

$$t^{00,mn}(x,x') = -\partial_k\partial_l[\tau_2^{kl,mn}(x,x') - \tfrac{1}{2}(\partial_0 + 3\partial'_0) \times \tau_1^{kl,mn}(x,x') - \partial_0'^2\sigma^{kl,mn}(x,x')], \tag{20d}$$

$$t^{0k,mn}(x,x') = -\partial_l[\tau_3^{kl,mn}(x,x') + \partial'_0(\tau_2^{kl,mn}(x,x') + \tfrac{1}{2}(\partial_0 - \partial'_0)\tau_1^{kl,mn}(x,x') + \partial_0\partial'_0\sigma^{kl,mn}(x,x'))], \tag{20e}$$

$$t^{kl,mn}(x,x') = \tau_4^{kl,mn}(x,x') + \tfrac{1}{2}(\partial_0 - \partial'_0)\tau_3^{kl,mn}(x,x') + \partial_0\partial'_0[\tau_2^{kl,mn}(x,x') + \tfrac{1}{2}(\partial_0 - \partial'_0) \times \tau_1^{kl,mn}(x,x')] + \partial_0^2\partial_0'^2\sigma^{kl,mn}(x,x'), \tag{20f}$$

where $\sigma^{kl,mn}(x,x')$, $\tau_2^{kl,mn}(x,x')$, and $\tau_4^{kl,mn}(x,x')$ are symmetric under $\tau^{kl,mn}(x,x') \to \tau^{mn,kl}(x',x)$ and τ_3 and τ_1 are antisymmetric. Furthermore,

$$\tau_i^{kl,mn}(x,x') = \tau_i^{lk,mn}(x,x').$$

We have inferred from the integral statements [$\int d^3r\bar{\tau}^{0k,mn}(x,x')$, for example] that $\bar{\tau}^{0k,mn}(x,x') = \partial_l\bar{\tau}^{lk,mn}(x,x')$. This conclusion holds if $\bar{\tau}(x,x')$ is local, as we assume here. For then, the matrix element,

$$\langle p|\,\bar{\tau}(x,x')\,|0\rangle = \exp(-i)p\tfrac{1}{2}(x+x')\sum_n f^{(n)}(p)\delta^{(n)}(x-x'),$$

where $\langle p|$ is an arbitrary state (by the Federbush–Johnson theorem, we do not need to consider more general matrix elements[11]), is a finite sum of derivatives of $\delta(x-x')$. Then the Fourier transform with respect to x at $x'=0$ is $\sum_n[i(k+\tfrac{1}{2}p)]^n f_{(p)}^{(n)}$, a finite polynomial in k. If $\int d^3r\bar{\tau}(x,x') = 0$, then the leading term must be k, and we can re-express $\bar{\tau}^{0\cdots}$ as $\partial_l\bar{\tau}^{l\cdots}$. For $\bar{\tau}^{00,00}$ we can similarly conclude that $\bar{\tau}^{00,00} = \partial'_m\partial'_n\partial_k\partial_l\bar{\tau}^{kl,mn}$. If the $\bar{\tau}$'s are nonlocal,[9] the argument breaks down and one can no longer assume the derivative form in all cases.

Equations (16)–(18) and Eq. (20) may then be used to determine the equal-time commutators

$$i[T^{00}(x), T^{00}(x')]\delta(x^0 - x'^0) = [T^{0k}(x) + T^{0k}(x')]\partial_k\delta(x-x') - \partial_k\partial_l\partial'_m\partial'_n\tau_1^{kl,mn}(x,x'), \tag{21a}$$

$$i[T^{00}(x), T^{0m}(x')]\delta(x^0 - x'^0) = [T^{ml}(x) + T^{00}(x')\delta^{ml}]\partial_l\delta(x-x') - \partial_k\partial_l\partial'_n[\tau_2^{kl,mn}(x,x') - \tfrac{1}{2}(\partial_0+\partial'_0)\tau_1^{kl,mn}(x,x')], \tag{21b}$$

$$i[T^{00}(x), T^{mn}(x')]\delta(x^0 - x'^0) = [-\partial^0T^{mn}(x) + T^{0m}(x')\partial^n + T^{0n}(x')\partial^m]\delta(x-x') + \partial_k\partial_l[\tau_3^{kl,mn}(x,x') + (\partial_0+\partial'_0)\tau_2^{kl,mn}(x,x') - \tfrac{1}{2}(\partial_0+\partial'_0)^2\tau_1^{kl,mn}(x,x')], \tag{21c}$$

$$i[T^{0k}(x), T^{0m}(x')]\delta(x^0 - x'^0) = [T^{0m}(x)\partial^k + T^{0k}(x')\partial^m]\delta(x-x') - \partial_l\partial'_n\tau_3^{kl,mn}(x,x'), \tag{21d}$$

$$i[T^{0k}(x), T^{mn}(x')]\delta(x^0 - x'^0) = [T^{mn}(x)\delta^{kl} - T^{ln}\delta^{mk} - T^{lm}\delta^{nk}]\partial_l\delta(x-x') - \partial_l[\tau_4^{kl,mn}(x,x') - \tfrac{1}{2}(\partial_0+\partial'_0)\tau_3^{kl,mn}(x,x')]. \tag{21e}$$

This is the most general form of the stress-tensor commutation relations consistent with the Poincaré algebra and locality. The form of the relations has been obtained here from the metric dependence of $\mathcal{C}^{\mu\nu}$, the τ_i functions representing model-dependent parts. The structures are consistent with the time locality of the commutators since time derivatives only occur in the combination $\partial_0 + \partial'_0$ which cannot generate any derivatives of a delta function $\delta(x^0 - x'^0)$, but only the time derivative (or commutator with P^0) of the operator coefficient of the delta function. Hence, we conclude that the functions τ_i must be local in time. There is no such direct requirement on σ since it does not appear in any of the commutators. These statements do not imply that $\mathcal{C}^{\mu\nu}$ is independent of time derivatives of $g_{\lambda\sigma}$, but only restrict the form of the dependence to that implicit in Eqs. (17) and (20). The spectral functions ensure that τ_2 and τ_4 cannot be zero. Their vacuum expectation values must be

$$\partial_k\partial_l\partial'_n\langle\tau_2^{kl,mn}(x,x')\rangle = \int_0^\infty ds s^{-2}[\tfrac{4}{3}\rho_2(s) + \rho_0(s)]\nabla^2\partial^m\delta(x-x'), \tag{22a}$$

$$\partial_l\langle\tau_4^{kl,mn}(x,x')\rangle = -\int_0^\infty ds\{s^{-1}\rho_2(s)[\delta^{mk}\delta^{nl} + \delta^{ml}\delta^{nk} - \tfrac{2}{3}\delta^{mn}\delta^{lk}] + s^{-1}\rho_0(s)\delta^{kl}\delta^{mn}\}\partial_l\delta(x-x') + \int_0^\infty ds s^{-2}[\tfrac{4}{3}\rho_2(s) + \rho_0(s)]\partial^k\partial^m\partial^n\delta(x-x'). \tag{22b}$$

The time derivative terms cannot contribute in the vacuum, since they are commutators with P^0, hence only the above terms survive if we express the relations in terms of $\bar{T}$. The single derivative term is highly model dependent and occurs "classically" in the spin $\frac{1}{2}$ case, for example.

The general results embodied in Eq. (16) and the subsequent form Eq. (21) determine the equal-time commutators once the metric dependence (both classical and quantum) of the stress tensor of a particular system is known. It is interesting that for an important class of systems, namely local dynamical fields of low spin (≤ 1), this dependence (more precisely, its classical part) can be inferred explicitly in a uniform way. One takes the field's flat space action in terms of canonical variables[14] (π_A, ϕ_A) and expresses it in a generally covariant form. It is then possible to redefine the canonical variables in the presence of $g_{\mu\nu}$ such that the flat space canonical form

$$W_M = \int dx[\sum \pi_A \partial_0 \phi_A - \mathcal{H}(\pi_A, \phi_A; \eta)] \quad (23)$$

only changes by $\mathcal{H}(\pi, \phi; \eta) \to \mathcal{H}(\pi, \phi, g)$. This may be accomplished[14] essentially by defining π_A so as to absorb the $(-g_4)^{\frac{1}{2}}$ of the volume element.

The energy density $\mathcal{H}$ now takes the form

$$\mathcal{H}(\pi_A, \phi_A; g) = -N\Theta_0^0(\pi_A, \phi_A, g_{ij}) - N^i\Theta_i^0(\pi_A, \phi_A, g_{ij})$$

in terms of the convenient notation $N_i \equiv g_{0i}$, $N^i \equiv {}^3g^{ij}N_j$, $N \equiv (-g^{00})^{-\frac{1}{2}} = (N_iN^i - g_{00})^{\frac{1}{2}}$, where the contravariant metric ${}^3g^{ij}$ is the inverse of the spatial part of $g_{\mu\nu}$: ${}^3g^{ij}g_{jk} = \delta_k^i$. The fundamental point is that the Θ_μ^0 are functions only of the spatial components and *not* of the $g_{0\mu}$, the full dependence on the latter being through the linear coefficients N, N^i. In the flat space limit, the Θ_μ^0 are just the energy momentum density components. Thus, for the Maxwell field, $-\Theta_0^0 = \frac{1}{2}g^{-\frac{1}{2}}[g_{ij}(\epsilon^i\epsilon^j + \mathcal{B}^i\mathcal{B}^j)]$, $\Theta_i^0 = \epsilon_{ijk}\epsilon^j\mathcal{B}^k$, and $\epsilon^i = (-g_4)^{\frac{1}{2}}F^{0i}$, $\mathcal{B}^i = \epsilon^{ijk}\partial_jA_k$. The correct variables here are the contravariant densities ϵ^i and $\mathcal{B}^i$ while g is the three-dimensional determinant and $-g_4$ represents the four-dimensional one.

Now, if one varies the combined Einstein-matter action,

$$W = W_E + W_M, W_E = K^{-1}\int dx(-g_4)^{\frac{1}{2}}R$$

the quantities Θ_μ^0 are precisely the sources of the G_μ^0 components of the Einstein tensor, referred to a time constant surface. For W_E itself may be written in the form[14]

$$W_E = \int dx[\pi^{ij}\partial_0 g_{ij} - NR_0(\pi^{ij}, g_{ij}) - N^iR_i(\pi^{ij}, g_{ij})],$$

the R_μ being linear combinations of the G_μ^0 and depending only on g_{ij} and its conjugate variable π^{ij} but not on N or N_i. The four equations $R_\mu = -K\Theta_0^\mu$ are in fact the four constraint equations corresponding to $\nabla \cdot E = j^0$ in electrodynamics and Θ_μ^0 are then clearly linear combinations of the correct energy momentum density source of the Einstein field.

The energy momentum density Θ_μ^0 depends only, as it must for a correct formulation of the initial value problem (Cauchy data), on quantities which transform as tensors under coordinate transformations within the t = const surface and are invariant under coordinate transformations off the surface, namely on π_A, ϕ_A, and g_{ij}. The gauge quantities N, N^i (or, equivalently, the $g_{0\mu}$), on the other hand, are altered by coordinate changes off the surface (they correspond to the gauge variable A^0 in electrodynamics) which is why they are not desirable in a correct Θ_μ^0.[14]

The Θ_μ^0 may now be used to evaluate the stress-density $\mathcal{T}^{\mu\nu}$ defined according to $W_M = \frac{1}{2}\int dx\delta g_{\mu\nu}\mathcal{T}^{\mu\nu}$, which enters in the general commutation relations. The Θ_μ^0 and $\mathcal{T}^{0\mu}$ are not identical since they are the coefficients of (N, N^i) and (g_{00}, g_{0i}), respectively, in the action. Thus we find from $-\frac{1}{2}\mathcal{T}^{\mu\nu} = \delta\mathcal{H}/\delta g_{\mu\nu}$ that

$$\mathcal{T}^{00} = -N^{-1}\Theta_0^0, \quad \mathcal{T}^{0i} = {}^3g^{ij}[\Theta_j^0 + N_jN^{-1}\Theta_0^0], \quad (24)$$

which gives the explicit dependence of the $\mathcal{T}^{0\mu}$, for example, on $g_{0\mu}$ and thus also defines the ("classical" part of) $\delta\mathcal{T}^{0\mu}/\delta g_{\lambda\sigma}$. Note that in the limit $g_{\mu\nu} = \eta_{\mu\nu}$ the $\mathcal{T}^{0\nu}$ and $\Theta^{0\mu}$ coincide. However, in computing the $\delta\mathcal{T}/\delta g$ terms, the relations (24) must be used. For fields of spin ≤ 1, including electrodynamics, these results (which hold for arbitrary g) may be used to calculate the $[T^{0\mu}, T^{\lambda\sigma}]$ relations. They agree with direct calculations using canonical commutation

[14] These results arise from the canonical analysis of coupled gravitational and matter fields: R. Arnowitt, S. Deser, and C. W. Misner, J. Math. Phys. **1**, 434 (1960), and Phys. Rev. **120**, 313 (1960) for derivations and explicit examples (including the Maxwell field). For the canonical form of the spinor field (which is somewhat more complicated, involving essentially derivative coupling to the metric) see T. W. B. Kibble, J. Math. Phys. **4**, 1433 (1963). Higher spin cases, where the constraints among matter field components complicate matters are dealt with in their g_{00} dependence, which is relevant to the $[T^{00}, T^{00}]$ relation in Ref. 2. Here, the process of eliminating constraints to reach canonical form in terms of the independent modes may bring in more metric dependence than that given in the text for low spins. In particular, the Θ_μ^0 may acquire N, N^i dependence when expressed in terms of the reduced variables. In view of the subsequent discussion, this may be regarded as a strong argument against the physical significance of elementary higher spin systems; for the latter would then not have the desired property of a system in time development from a given set of Caudy data, the energy momentum density being dependent at any instant on the physically meaningless choice of coordinates N, N^i, as well as on the dynamical variables. This would also raise analogous difficulties in the Einstein constraint equations $R_\mu = -K\Theta_\mu^0$.

relations and keeping $g_{\mu\nu}=\eta_{\mu\nu}$.[15] One may also recover the results of Ref. 2 for a weak external g_{00}. In particular these forms imply that there are no additional $\tau^{00,00}$ terms in the $[T^{00},T^{00}]$ relations for low spins.

We emphasize that the general metric dependence obtained here is the classical one and does not include the purely quantum dependence on the metric which is required to yield the Schwinger terms. Indeed, there is here a curious contrast to the situation for currents. There,[12] "classical" dependence of the current on the corresponding external field (e.g., the Maxwell field) may or may not be present, depending on whether or not the system has spin $\frac{1}{2}$. If there is classical (A^2) dependence, it automatically gives rise to Schwinger type terms. Here, on the other hand, there is always classical metric dependence on $\mathcal{T}^{\mu\nu}$, irrespective of spin, but this dependence turns out never to be sufficient to yield Schwinger terms (at least for spin ≤ 1). Thus, for all fields, one must redefine $\mathcal{T}^{\mu\nu}$ as the limit of a spatially nonlocal operator to obtain the terms.

In our framework, involving an external (or dynamical) metric, one must simultaneously insert an appropriate quantum metric dependence in this redefined $\mathcal{T}^{\mu\nu}$. The necessity for this prescription may also be inferred either from general covariance (for a "split" $\mathcal{T}^{\mu\nu}$ without extra dependence no longer transforms as a coordinate tensor) or, in terms of a dynamical gravitational field, along lines similar to those of Ref. 12 for currents coupled to a Bose field. The Schwinger terms will then correspond to the nonclassical part of $\delta\mathcal{T}/\delta g$. The elaboration of these remarks regarding the nonclassical metric dependence and nonlocal $\mathcal{T}^{\mu\nu}$ constitutes a separate program, which we do not pursue here.

Some general conclusions may be drawn, however, from the $\partial^3\delta$ nature of the Schwinger terms, together with the fact that they must arise from $\partial_j\delta\mathcal{T}^{ij}/\delta g_{00}$ or $\partial_i(\delta\mathcal{T}^{0i}/\delta g_{0j})$ and $\partial_j(\delta\mathcal{T}^{ij}/\delta g_{kl})$ in $[T^{00},T^{0j}]$ and $[T^{0i},T^{kl}]$, respectively. There must be at least the following nonclassical dependence: $\mathcal{T}^{0i}[\partial^2_{kl}g_{0i}]$, $\mathcal{T}^{00}[\partial^2_{kl}g_{mn}]$, and $\mathcal{T}^{ij}[\partial^2_{kl}g_{mn}]$, and $\mathcal{T}^{ij}[\partial^2_{kl}g_{00},\partial^2_{kl}g_{mn}]$.

An alternate argument leading to these dependences in the $\mathcal{T}^{\mu\nu}$ is as follows. In electrodynamics, $[j^0,j^i]\neq 0$ and Gauss's equation $\nabla\cdot E=j^0$ implies that $[\mathbf{E}^L,j^i]\neq 0$, where $\mathbf{E}^L$ is the longitudinal electric filed. Lorentz invariance then requires that the transverse part $\mathbf{E}^T$ also fail to commute, i.e., that $[\mathbf{E}^T,j]\neq 0$, and hence that $\mathbf{j}=\mathbf{j}(A^T)$. Similarly the constraint equations $G^0_\mu=-\kappa T^0_\mu$ require that $[G^0_0,T^{0i}]$, $[G^0_i,T^{00}]$ and $[G^0_i,T^{kl}]$ not vanish. In the linearized approximation, where $G^0_0\approx\nabla^2 g_{ij}$ and $G^0_i\sim\pi^{ij},j$ Lorentz invariance then requires that $\mathcal{T}^{0i}$ depend on the variables π^{ij} conjugate to g_{ij} which means in particular that it involves $\partial^2_{kl}g_{0j}$ (since π^{ij} is by its definition proportional to $g_{0i,j}$). Likewise $\mathcal{T}^{00}$ and $\mathcal{T}^{kl}$ must depend on $\partial^2_{kl}g_{ij}$. It is hoped to return to these questions elsewhere.

[15] Explicit calculations on these questions have been carried out by J. Trubatch (unpublished).

VI. SUMMARY

We have examined a number of consistency conditions on the commutation relation among the Poincaré generators and the stress-tensor components in local field theory. In particular, the apparent difficulty that, while the right sides of such relations should vanish in vacuum, they actually involve the unsubtracted (nonvanishing in vacuum) stresses or their integrals, was resolved by the Lorentz covariance requirement that $\langle T^{\mu\nu}\rangle=-\lambda\eta^{\mu\nu}$. The latter ensured that the right side could simultaneously satisfy both these apparently contradictory conditions.

The general form of the equal-time stress-tensor commutation relations compatible with the Poincaré algebra was exhibited, and compared with the Lehmann–Källén representation[16] for

$$\langle 0|\,[T^{\mu\nu}(x),T^{\lambda\sigma}(x')]\,|0\rangle.$$

The latter depends only on the locality and Lorentz transformation properties of $T^{\mu\nu}$, and involves two nonnegative weight functions for conserved $T^{\mu\nu}$ when the Hilbert space metric is positive. The main result of the spectral representation (and hence a consequence of only locality, proper Lorentz covariance, positive energy spectrum, and positive Hilbert space metric) was the necessary existence of Schwinger terms, of the form $\partial^3\delta(\mathbf{r}-\mathbf{r}')$ in the equal-time commutators $[T^{00}(\mathbf{r}),T^{0m}(\mathbf{r}')]$, and $[T^{0k}(\mathbf{r}),T^{mn}(\mathbf{r}')]$.

Paradoxically, straightforward calculations from canonical commutation relations (even for free fields) yields neither Schwinger terms nor the covariant form $\lambda\eta^{\mu\nu}$ for $\langle T^{\mu\nu}\rangle$. If the stress tensor is defined as the limit of a spatially nonlocal operator, the Schwinger terms required by the spectral forms appear. However, this prescription does not simultaneously reinstate the covariance of $\langle T^{\mu\nu}\rangle$. We have been able to achieve the latter only by extremely artificial means, such as regularization with indefinite weight functions which would probably introduce negative energy states or a limiting process in which the spacelike separation was not along a $t=$ const surface. Thus, while it is likely

[16] The Lehmann–Källén representation is, of course, valid only in the flat space limit $g\rightarrow\eta$. We have used the more general metric as a device for studying the flat space limit, but many of our results will be reflected in the full nonlinear theory.

that the singularity of the strictly local product is responsible both for loss of Lorentz covariance and the Schwinger paradox, a unified prescription for removing both problems has not been found. Incidentally, the above difficulties are most apparent in the vacuum expectation values, since the operator products are most singular when associated with creation and annihilation of excitations at the same point. For a free field, however, it is possible to calculate $\langle 0| T^{\mu\nu}(x)T^{\lambda\sigma}(x')|0\rangle$ for unequal times. This form is manifestly covariant (with the exception of the $\langle 0| T^{\mu\nu}|0\rangle\langle 0| T^{\lambda\sigma}|0\rangle$ terms) and satisfies the Lehmann–Källén representation. If this is used to calculate the commutators $\langle[T^{\mu\nu}, T^{\lambda\sigma}]\rangle$, we find that the right sides have all the requisite properties in the vacuum. It is, of course, impossible to calculate $\langle \bar{T}^{\mu\nu}\rangle = -\lambda\eta^{\mu\nu}$ by this method, but it does establish the form for $\langle \bar{T}^{\mu\nu}\rangle$. It is clear from the discussion in Sec. II that $\bar{T}^{\mu\nu}$ rather than $T^{\mu\nu}$ is the tensor, otherwise J would have to be expressed in terms of T rather than $\bar{T}$. The source of the difficulty can be understood somewhat better by considering the case of a free spin 0 field. The term from which the trouble stems is $\phi^\mu(x)\phi^\nu(x)$, which must be written

$$\phi^\mu(x+\tfrac{1}{2}\xi)\phi^\nu(x-\tfrac{1}{2}\xi) \equiv \mathbf{T}^{\mu\nu}(x,\xi).$$

Then

$$\begin{aligned} i[\mathbf{T}^{\mu\nu}(x,\xi), J^{\lambda\sigma}] &= (x^\lambda\partial_x^\sigma - x^\sigma\partial_x^\lambda)\mathbf{T}^{\mu\nu}(x,\xi) + g^{\mu\lambda}\mathbf{T}^{\sigma\nu}(x,\xi) \\ &\quad - g^{\mu\sigma}\mathbf{T}^{\lambda\nu}(x,\xi) + g^{\nu\lambda}\mathbf{T}^{\mu\sigma}(x,\xi) - g^{\nu\sigma}\mathbf{T}^{\mu\lambda}(x,\xi) \\ &\quad + (\xi^\lambda\partial_\xi^\sigma - \xi^\sigma\partial_\xi^\lambda)\mathbf{T}^{\mu\nu}(x,\xi). \end{aligned}$$

In the limit $\xi \to 0$, the last term never appears; however, in the vacuum expectation value, that term is essential for the proper covariance, *even in the limit* $\xi \to 0$. Thus, the noncovariance of

$$T^{\mu\nu}(x) = \lim_{\xi\to 0} T^{\mu\nu}(x,\xi),$$

is due to extra terms which are not transformed properly as $\xi \to 0$. Once these terms are subtracted, the remainder $\bar{T}^{\mu\nu}$ does transform correctly.

We have further exhibited the dependence of the stress tensor on $g_{\mu\nu}$ which is forced by the structure constants of the Poincaré algebra and compatible with the most general additional "nonalgebra" terms. These considerations are consistent with the (classical) explicit metric dependence of $\mathcal{T}^{\mu\nu}$ which was obtained in the generally covariant canonical formulation of matter fields of spin ≤ 1.

The nonlocal prescription for $\mathcal{T}^{\mu\nu}$ requires, in order to maintain general covariance, that explicit dependence on the metric be inserted into the "spread" $\mathcal{T}^{\mu\nu}$, which would otherwise no longer transform as a tensor under general coordinate transformations. Now, by direct calculation[15] in terms of canonical commutation relations with $g_{\mu\nu} = \eta_{\mu\nu}$, spreading the points in $\mathcal{T}^{\mu\nu}$ is actually sufficient to produce terms proportional to $(\partial)^3\delta(\mathbf{r}-\mathbf{r}')$ in the $[T^{00}, T^{0m}]$ and $[T^{0k}, T^{mn}]$ commutators. In the presence of an external metric (or in terms of the general $\delta\mathcal{T}/\delta g$), the additional metric dependence must of course be used. The specific form of this dependence {which corresponds to the definition

$$j^\mu(x) = e\bar{\psi}(x+\epsilon)\gamma^\mu \exp\left[ie\int_x^{x+\epsilon} dy_\mu A^\mu(y)\psi(x)\right]$$

in electrodynamics} and the (presumed) consistency of the general covariance and Schwinger term requirements are separate questions which we have not studied in detail here. We have only given necessary conditions of the dependence of $\mathcal{T}^{\mu\nu}$ on second derivatives of $g_{\mu\nu}$.

However, from purely geometrical considerations, it may be shown that the necessary nonclassical dependence on the metric appears in restoring the coordinate tensor nature of the "split" $T^{\mu\nu}$, say $\phi_\mu(x+\epsilon)\phi_\nu(x)$, by use of parallel transfer to make it a tensor at one point. An operator ${}^\mu D(x,x')_\nu$ such that ${}^\mu D(x,x')_\nu\phi^\nu(x')$ is a vector at x may be defined and is essentially a path integral over the affinity

$${}^\mu D(x,x')_\nu = {}^\mu\left\{\exp\left[\int_{x'}^{x} dy^\alpha\Gamma_\alpha(y)\right]\right\}_{+\nu}.$$

It is hoped to return to this elsewhere.

Some speculations on the role of these terms when the gravitational field itself is dynamical and quantized may be of interest, however. In electrodynamics, the additional A_μ dependence ensures the preservation of gauge invariance in at least two situations.[17] The first is in the maintenance of zero photon self-mass in the closed loop diagram, the second the elimination of finite, but gauge dependent, terms in the "box" diagram (scattering of light by light). Similarly, it may be that some of the difficulties encountered in renormalizing the interaction of a scalar field with the quantized Einstein field can be avoided if the correct form for $\mathcal{T}^{\mu\nu}[g]$ is employed. In electrodynamics, where the current correlation function $\delta\langle j\rangle/\delta A$ differs from the time-ordered product $i\langle(jj)_+\rangle$ by the explicit dependence $\langle\delta j/\delta A\rangle$,[18] the additional term is needed both for covariance and for charge conservation.

[17] D. G. Boulware, Phys. Rev. **151**, 1024 (1966).

[18] See Ref. 2, K. A. Johnson, Nucl. Phys. **31**, 464 (1962); L. S. Brown, Phys. Rev. **150**, 1338 (1966). The additional dependence discussed here may or may not be reflected in the Feynman rules of the resultant theory; this question can only be decided by a detailed analysis of the role of the extra terms.

The latter property ensures a vanishing photon self mass. It seems likely that there is a closed analogy in our case, where (covariant) conservation requires the explicit $\delta\mathcal{T}/\delta g$ term of Eq. (11b); a nonconserved correlation function would correspond to a graviton mass (in the language of the linearized theory at least). There are probably also terms in the graviton–graviton scattering through virtual matter pairs with difficulties similar to those of the box diagram in quantum electrodynamics. Certainly, unless the metric dependence is inserted, no interaction is possible with the gravitational field at all, just as the $\exp\,(ie \int dy_\mu A^\mu)$ term is essential for a nonvanishing current in electrodynamics. Another interesting problem has to do with the resulting lack of commutation, at equal times, between the matter $\mathcal{T}^{\mu\nu}$ and the gravitational field variables. For, just as in electrodynamics, where [**E**, **j**] fails to vanish as a consequence of the A dependence of j, the corresponding commutators between $\mathcal{T}^{\mu\nu}$ and the canonical Einstein variables will be nonzero. Since the Einstein equations are nonlinear, the computation of this noncommutativity is not so direct as for vector currents coupled to, say, a spin one field[12]; also it is presumably necessary to split the points in the nonlinear terms of the Einstein equations (which correspond to the $\mathcal{T}^{\mu\nu}$ of the gravitational field) in order to avoid similar paradoxes for the Einstein field itself.

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Note on the Kerr Metric and Rotating Masses

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Kerr's metric is often said to describe the geometry exterior to a body whose mass and rotation are measured by Kerr's parameters m and a, respectively, even though no interior solution is known. In this paper we give an interior solution valid in the limit when the rotation parameter a is sufficiently small so that terms of higher power than the first are negligible, but the mass parameter m is allowed to be large. This is accomplished by bringing Kerr's exterior metric into the form of the metric for a slowly rotating mass shell. Also, the connection is found between Kerr's parameters and the physical parameters characterizing the rotating body.

I. INTRODUCTION

IN 1963, Kerr[1] gave the exact stationary but not static exterior solution to Einstein's equations:

$$ds^2 = \Sigma(d\theta^2 + \sin^2\theta\, d\bar{\phi}^2) + 2(dU + a\sin^2\theta\, d\bar{\phi})(dr + a\sin^2\theta\, d\bar{\phi}) - (1 - 2mr\Sigma^{-1})(dU + a\sin^2\theta\, d\bar{\phi})^2, \quad (1)$$

where

$$\Sigma = R^2 + a^2\cos^2\theta, \quad (2)$$

$$U = \bar{t} + R, \quad (3)$$

and m and a are constants. Kerr claims that this metric (1) is the metric exterior to a rotating body. The parameter a is related to the rate of rotation, and m is the mass parameter. Since the appearance of Kerr's paper there has been a search for an interior solution. If any interior solution exists, there must in particular be interior solutions in the case when a is sufficiently small that terms of higher power than the first in a can be neglected. The purpose of this paper is to provide such an interior solution which matches the Kerr solution at a radius r_0 to first order in a, *but for any m* whose gravitational radius does not exceed r_0.

This is accomplished in Sec. III via coordinate transformations which bring the Kerr exterior metric into the form of the metric for a thin slowly rotating mass shell.[2] For completeness, the exterior and interior metrics associated with a thin slowly rotating mass shell are given in Sec. II.

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[1] R. Kerr, Phys. Rev. Letters **11**, 237 (1963).
[2] D R. Brill and J. M. Cohen, Phys. Rev. **143**, 1011 (1966).