

Stretched Coherent States

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Abstract

The fundamental properties of recently introduced stretched coherent states are investigated. It has been shown that stretched coherent states retain the fundamental properties of standard coherent states and generalize the resolution of unity, or completeness condition, and the probability distribution that n photons are in a stretched coherent state.

The stretched displacement and stretched squeezing operators are introduced and the multiplication law for stretched displacement operator is established. The results of the action of the stretched displacement and stretched squeezing operators on the vacuum and the Fock states are presented.

Stretched squeezed stretched coherent states and stretched squeezed stretched displaced number states are introduced and their properties are studied.

The inner product of two quantum mechanical vectors was defined in terms of their stretched coherent state representations, and functional Hilbert space was introduced.

PACS numbers: 05.10.Gg; 05.45.Df; 42.50.-p.

Keywords: Quantum coherent states, Displacement operator, Squeezed coherent states

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1 Introduction

Standard coherent states were discovered by Schrödinger [1] in his search for quantum states in whose representation the diagonal matrix element of the evolution operator of a quantum mechanical oscillator exhibits the same temporal behavior as a classical mechanical oscillator. The term "*coherent states*" was coined by Glauber [2], who introduced these states as superpositions of Fock states of the quantized electromagnetic field that are not modified by the action of photon annihilation operators. In other words, Glauber found that the quantum state of a coherent quantized field has to be an eigenvector of the boson-annihilation operator with complex eigenvalue.

In this note we explore the fundamental properties of recently introduced stretched coherent states which generalize Glauber's coherent states framework. We have applied the concept of stretched coherent states to design stretched squeezed stretched coherent states, and stretched squeezed stretched displaced number states.

The motivation for introducing and developing these states is twofold. First, we expand the foundations of quantum optics: quantum coherent and squeezed coherent states, as well as coherent displaced number states. Second, presented generalizations of the well-known fundamental concepts open up new horizons for researchers working in the field of quantum optics to search for manifestations and applications of the new fundamentals in optical experiments. The first step in this direction has been made in the seminal paper by Longhi [4], where the optical realization of fractional quantum mechanics [5] was proposed based on transverse light dynamics in aspherical optical cavities. As a laser implementation of the fractional quantum harmonic oscillator it has been found that dual Airy beams can be selectively generated under off-axis longitudinal pumping. Another interesting realization of fractional quantum mechanics was implemented in [6], based on similarity between the standard Schrödinger equation and paraxial wave-propagation equation in optics. The authors of the paper [6] proposed a protocol that uses the transverse dynamics of light in aspherical optical cavities designed in the $4f$ configuration. In the paraxial approximation and without taking into account losses on optical elements and diffraction, the output transverse modes correspond to the eigenfunctions of the fractional Schrödinger equation [7].

It was shown in [4] and [6] that the use of fractional models in the field of optics allows to manage diffraction of light and design novel signal-processing schemes and beam solutions.

Therefore, we hope that this note will initiate a search for whether stretched quantum coherent states play the same role in fractional quantum mechanics as standard coherent states in quantum mechanics, and will stimulate new developments and experiments in the field of quantum optics.

We present the proof that stretched coherent states are indeed generalized coherent states and explore their properties. The stretch displacement operator and newly introduced stretched squeezing operator have been studied. It has been shown that stretched coherent states retain the fundamental properties of standard coherent states and generalize the resolution of unity or completeness condition, the probability distribution that n photons are in a stretched coherent state, the multiplication law for stretched displacement operator, and the results of action of the stretched displacement and stretched squeezing operators on the vacuum and the Fock states.

The inner product of two quantum mechanical vectors was defined in terms of their stretched coherent state representations, and functional Hilbert space was introduced.

The presented new concepts of quantum optics include the parameters σ , $0 < \sigma \leq 1$ and v , $0 < v \leq 1$. In the limiting case, when $\sigma = 1$ and $v = 1$, all our new results turn into known equations of the theory of standard coherent states.

2 Stretched coherent states

The *stretched coherent states* $|\zeta\rangle_\sigma$, which are a generalization of standard coherent states, were introduced as follows [8],

$$|\zeta\rangle_\sigma = \exp\left(-\frac{|\zeta|^{2\sigma}}{2}\right) \sum_{n=0}^{\infty} \frac{\zeta^{\sigma n}}{\sqrt{n!}} |n\rangle, \quad 0 < \sigma \leq 1, \quad (1)$$

and the adjoint states ${}_\sigma\langle\zeta|$

$${}_\sigma\langle\zeta| = \exp\left(-\frac{|\zeta|^{2\sigma}}{2}\right) \sum_{n=0}^{\infty} \frac{(\zeta^{\sigma*})^n}{\sqrt{n!}} \langle n|, \quad 0 < \sigma \leq 1, \quad (2)$$

where a complex number ζ stands for labelling the stretched coherent states, vector $|n\rangle$ is an eigenvector of the photon number operator $\hat{n} = a^+a$,

$$\hat{n}|n\rangle = n|n\rangle, \quad (3)$$

$\langle n|m \rangle = \delta_{n,m}$, the operators a^+ and a are photon field creation and annihilation operators that satisfy the Bose-Einstein commutation relation $[a, a^+] = aa^+ - a^+a = 1$, and $[a^+, a^+] = 0$, $[a, a] = 0$. The action of the operators a^+ and a on the number state $|n \rangle$ reads

$$a^+|n \rangle = \sqrt{n+1}|n+1 \rangle \quad \text{and} \quad a|n \rangle = \sqrt{n}|n-1 \rangle. \quad (4)$$

It is easy to see that stretched quantum coherent state $|\zeta \rangle_\sigma$ is eigenstate of photon field annihilation operator a . Indeed, we have

$$\begin{aligned} a|\zeta \rangle_\sigma &= \exp\left(-\frac{|\zeta|^{2\sigma}}{2}\right) \sum_{n=0}^{\infty} \frac{\zeta^{\sigma n}}{\sqrt{n!}} \sqrt{n}|n-1 \rangle = \\ & \exp\left(-\frac{|\zeta|^{2\sigma}}{2}\right) \sum_{n=0}^{\infty} \frac{\zeta^{\sigma(n+1)}}{\sqrt{n!}} |n \rangle = \zeta^\sigma |\zeta \rangle_\sigma, \end{aligned} \quad (5)$$

that is

$$a|\zeta \rangle_\sigma = \zeta^\sigma |\zeta \rangle_\sigma, \quad (6)$$

which shows that the eigenvalue of the photon field annihilation operator a is ζ^σ . Therefore, we have

$$\sigma < \zeta |a|\zeta \rangle_\sigma = \zeta^\sigma, \quad \text{and} \quad \sigma < \zeta |a^+|\zeta \rangle_\sigma = \zeta^{\sigma*}.$$

Using the following representation for $|n \rangle$

$$|n \rangle = \frac{(a^+)^n}{\sqrt{n!}} |0 \rangle, \quad (7)$$

where $|0 \rangle$ is the vacuum state with $\zeta = 0$, we obtain the alternative expression for the stretched quantum coherent states

$$|\zeta \rangle_\sigma = \exp\left(-\frac{|\zeta|^{2\sigma}}{2}\right) \sum_{n=0}^{\infty} \frac{(\zeta^\sigma a^+)^n}{\sqrt{n!}} |0 \rangle, \quad 0 < \sigma \leq 1. \quad (8)$$

3 Fundamental properties of the stretched coherent states

Let us answer the question whether the states $|\varsigma\rangle_\sigma$ are generalized coherent states. Quantum mechanical states are generalized coherent states if they [10]:

- (i) are parameterized continuously and normalized;
- (ii) admit a resolution of unity with a positive weight function;
- (iii) provide temporal stability, that is, a coherent state that evolves over time belongs to the family of coherent states.

To prove (i), we note that the stretched coherent states $|\varsigma\rangle_\sigma$ introduced by Eq.(1) are evidently parametrized continuously by their label ς which is a complex number $\varsigma = \xi + i\eta$, with $\xi = \text{Re}\varsigma$ and $\eta = \text{Im}\varsigma$. The square of the modulus $|\langle n|\varsigma\rangle_\sigma|^2$ of projection of the stretched coherent state $|\varsigma\rangle_\sigma$ onto the number state $|n\rangle$ gives us the probability $P_\sigma(n, \varsigma)$ that n photons will be found in a coherent state $|\varsigma\rangle_\sigma$,

$$P_\sigma(n, \varsigma) = |\langle n|\varsigma\rangle_\sigma|^2 = \frac{|\varsigma|^{2\sigma n}}{n!} \exp(-|\varsigma|^{2\sigma}), \quad 0 < \sigma \leq 1. \quad (9)$$

Therefore, the stretched quantum coherent states $|\varsigma\rangle_\sigma$ are normalized due to the normalization condition for the probability distribution $P_\sigma(n, \varsigma)$,

$$\sum_{n=0}^{\infty} |\langle n|\varsigma\rangle_\sigma|^2 = \sum_{n=0}^{\infty} P_\sigma(n, \varsigma) = 1. \quad (10)$$

To prove (ii), that is, the coherent states $|\varsigma\rangle_\sigma$ admit a resolution of unity with a positive weight function, we introduce a function $W_\sigma(|\varsigma|^2) > 0$ which obeys the equation

$$\int_{\mathbb{C}} d^2\varsigma |\varsigma\rangle_\sigma W_\sigma(|\varsigma|^2) \langle \varsigma| = I, \quad (11)$$

where $d^2\varsigma = d(\text{Re}\varsigma)d(\text{Im}\varsigma)$ and the integration extends over the entire complex plane \mathbb{C} . This equation with yet unknown function $W_\sigma(|\varsigma|^2)$ is the resolution of unity for stretched coherent states $|\varsigma\rangle_\sigma$. Introducing new integration variables r and φ by $\varsigma = re^{i\varphi}$, $d^2\varsigma = r dr d\varphi$ and making use of Eqs.(1) and (2) yield

$$\int_{\mathbb{C}} d^2\varsigma |\varsigma\rangle_{\sigma} W_{\sigma}(|\varsigma|^2) {}_{\sigma}\langle \varsigma| =$$

$$\lim_{\Phi \rightarrow \infty} \frac{1}{2\Phi} \sum_{n,m=0}^{\infty} \int_{-\Phi}^{\Phi} d\varphi e^{i\sigma(n-m)\varphi} \int_0^{\infty} dr r^{(n+m)\sigma+1} \frac{W_{\sigma}(r^2)}{\sqrt{n!m!}} \exp(-r^{2\sigma}) |n\rangle \langle m| = I, \quad (12)$$

where we used Klauder's "covering space formulation" ansatz [10] to perform the integration over $d\varphi$. Due to

$$\lim_{\Phi \rightarrow \infty} \frac{1}{2\Phi} \int_{-\Phi}^{\Phi} d\varphi e^{i\sigma(n-m)\varphi} = \delta_{m,n}, \quad (13)$$

we come to the following equation to find the function $W_{\sigma}(r^2)$

$$\sum_{n=0}^{\infty} \frac{1}{n!} \int_0^{\infty} dr r^{2n\sigma+1} W_{\sigma}(r^2) \exp(-r^{2\sigma}) |n\rangle \langle n| = I. \quad (14)$$

Hence we conclude that if a positive function $W_{\sigma}(r^2)$ satisfies the equation

$$\frac{1}{n!} \int_0^{\infty} dr r^{2n\sigma+1} W_{\sigma}(r^2) \exp(-r^{2\sigma}) = 1, \quad (15)$$

then due to completeness of orthonormal vectors $|n\rangle$

$$\sum_{n=0}^{\infty} |n\rangle \langle n| = I, \quad (16)$$

the resolution of unity expressed by Eq.(11) will hold. It is easy to see, that $W_{\sigma}(r^2)$ must be $W_{\sigma}(r^2) = 2\sigma r^{2(\sigma-1)}$ to satisfy Eq.(15), or

$$W_{\sigma}(|\varsigma|^2) = 2\sigma |\varsigma|^{2(\sigma-1)}. \quad (17)$$

Therefore, the resolution of unity is

$$2\sigma \int_{\mathbb{C}} d^2\varsigma |\varsigma\rangle_{\sigma} |\varsigma|^{2(\sigma-1)} {}_{\sigma}\langle \varsigma| = I, \quad (18)$$

which can be seen as a completeness condition for the stretched coherent states $|\varsigma\rangle_\sigma$.

To prove (iii), we note that if $|n\rangle$ is an eigenvector of the Hamiltonian operator $\hat{H} = \hbar\omega\hat{n} = \hbar\omega a^\dagger a$, where \hbar is Planck's constant, then the time evolution operator $\exp(-iHt/\hbar)$ results

$$\exp(-i\hat{H}t/\hbar)|n\rangle = e^{-i\omega nt}|n\rangle.$$

In other words, the time evolution of $|n\rangle$ results in appearance of the phase factor only. Let's consider time evolution of the stretched coherent state $|\varsigma\rangle_\sigma$ defined by Eq.(1). Since the stretched coherent state is not an eigenstate of \hat{H} , one would expect it to evolve to other states over time. However, we see that

$$\exp(-i\hat{H}t/\hbar)|\varsigma\rangle_\sigma = \exp\left(-\frac{|\varsigma|^{2\sigma}}{2}\right) \sum_{n=0}^{\infty} \frac{\varsigma^{\sigma n}}{\sqrt{n!}} e^{-i\omega nt} |n\rangle = |e^{-\frac{i\omega t}{\sigma}}\varsigma\rangle_\sigma, \quad (19)$$

which is just another coherent state belonging to a complex number $\varsigma e^{-\frac{i\omega t}{\sigma}}$. Hence, the time evolution of the stretched coherent state $|\varsigma\rangle_\sigma$ remains within the family of the coherent states $|\varsigma\rangle_\sigma$. The property embodied in Eq.(19) is called the temporal stability of coherent states $|\varsigma\rangle_\sigma$ under the action of the time evolution operator.

Thus, we conclude that stretched quantum coherent states $|\varsigma\rangle_\sigma$ satisfy Klauder's criteria (i) - (iii) for generalized coherent states [10].

3.1 Mandel parameter

The probability $P_\sigma(n, \varsigma)$ that the field represented by stretched coherent state $|\varsigma\rangle_\sigma$ is occupied by n photons is given by Eq.(9). The mean number of photons in the quantum state $|\varsigma\rangle_\sigma$ is

$$\sigma \langle \varsigma | \hat{n} | \varsigma \rangle_\sigma = \sigma \langle \varsigma | a^\dagger a | \varsigma \rangle_\sigma = \sum_{n=0}^{\infty} n P_\sigma(n, \varsigma) = |\varsigma|^{2\sigma}, \quad (20)$$

and the second order moment of the number of photons in the quantum state $|\varsigma\rangle_\sigma$ is

$$\sigma \langle \varsigma | \hat{n}^2 | \varsigma \rangle_\sigma = \sigma \langle \varsigma | (a^+ a)^2 | \varsigma \rangle_\sigma = \sum_{n=0}^{\infty} n^2 P_\sigma(n, \varsigma) = |\varsigma|^{2\sigma} + |\varsigma|^{4\sigma}, \quad (21)$$

where a^+ and a are photon field creation and annihilation operators. These equations allow us to calculate the Mandel parameter [11] using stretched coherent states. For one-mode quantum fields represented by the stretched coherent states the Mandel parameter Q_σ is given by

$$Q_\sigma = \frac{\sigma \langle \varsigma | (a^+ a)^2 | \varsigma \rangle_\sigma - (\sigma \langle \varsigma | a^+ a | \varsigma \rangle_\sigma)^2}{\sigma \langle \varsigma | a^+ a | \varsigma \rangle_\sigma} - 1. \quad (22)$$

Taking into account Eqs.(20) and (21) we conclude that $Q_\sigma = 0$. In other words, stretched quantum coherent states obey Poisson statistics.

4 Stretched displacement operator

We introduce the *stretched displacement operator* $D_\sigma(\varsigma)$ as follows

$$D_\sigma(\varsigma) = \exp\{\varsigma^\sigma a^+ - \varsigma^{\sigma*} a\}, \quad 0 < \sigma \leq 1, \quad (23)$$

where ς is a complex number, a^+ and a are photon field creation and annihilation operators.

When $\sigma = 1$ the operator $D_\sigma(\varsigma)|_{\sigma=1}$ becomes the well-known displacement operator $D(\varsigma)$ for the standard coherent states [1], [2], [9],

$$D(\varsigma) = D_\sigma(\varsigma)|_{\sigma=1} = \exp\{\varsigma a^+ - \varsigma^* a\}. \quad (24)$$

The stretched displacement operator $D_\sigma(\varsigma)$ is an unitary operator, i.e.

$$D_\sigma^+(\varsigma) D_\sigma(\varsigma) = 1, \quad (25)$$

where the sign "+" stands for Hermitian conjugation of the operator. Using the Baker-Campbell-Hausdorff formula for operators \hat{A} and \hat{B} ,

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{-\frac{1}{2}[\hat{A}, \hat{B}]}, \quad (26)$$

such that

$$[\hat{A}, [\hat{A}, \hat{B}]] = [\hat{B}, [\hat{A}, \hat{B}]] = 0, \quad (27)$$

we come to alternative representations for the stretched displacement operator $D_\sigma(\zeta)$,

$$D_\sigma(\zeta) = \exp\left(-\frac{|\zeta|^{2\sigma}}{2}\right) \exp(\zeta^\sigma a^+) \exp(-\zeta^{\sigma*} a), \quad (28)$$

or

$$D_\sigma(\zeta) = \exp\left(\frac{|\zeta|^{2\sigma}}{2}\right) \exp(-\zeta^{\sigma*} a) \exp(\zeta^\sigma a^+). \quad (29)$$

To show that

$$|\zeta \rangle_\sigma = D_\sigma(\zeta)|0 \rangle, \quad (30)$$

we perform the following chain of transformations

$$\begin{aligned} D_\sigma(\zeta)|0 \rangle &= \exp\left(-\frac{|\zeta|^{2\sigma}}{2}\right) \exp(\zeta^\sigma a^+) \exp(-\zeta^{\sigma*} a)|0 \rangle = \\ &= \exp\left(-\frac{|\zeta|^{2\sigma}}{2}\right) \exp(\zeta^\sigma a^+)|0 \rangle = \exp\left(-\frac{|\zeta|^{2\sigma}}{2}\right) \sum_{n=0}^{\infty} \frac{\zeta^{\sigma n} (a^+)^n}{n!} |0 \rangle = \\ &= \exp\left(-\frac{|\zeta|^{2\sigma}}{2}\right) \sum_{n=0}^{\infty} \frac{\zeta^{\sigma n}}{\sqrt{n!}} |n \rangle = |\zeta \rangle_\sigma, \end{aligned}$$

where Eq.(7) was used. Therefore, operator $D_\sigma(\zeta)$ generates stretched coherent state $|\zeta \rangle_\sigma$ from the vacuum state $|0 \rangle$.

It is easy to see that the following equations hold

$$[a, D_\sigma(\zeta)] = \zeta^\sigma D_\sigma(\zeta), \quad (31)$$

and

$$D_\sigma^+(\zeta) a D_\sigma(\zeta) = a + \zeta^\sigma, \quad D_\sigma(\zeta) a D_\sigma^+(\zeta) = a - \zeta^\sigma. \quad (32)$$

Multiplication law for the stretched displacement operators can be established using the Baker-Campbell-Hausdorff formula Eq.(26),

$$D_\sigma(\zeta) D_\sigma(\eta) = D(\zeta^\sigma + \eta^\sigma) \exp\left\{\frac{1}{2}(\zeta^\sigma \eta^{\sigma*} - \zeta^{\sigma*} \eta^\sigma)\right\}, \quad (33)$$

with $D(\zeta^\sigma + \eta^\sigma)$ being the well-known displacement operator defined by Eq.(24).

With help of Eqs.(33) and (26) it can be shown that the matrix element $\langle m|D_\sigma(\varsigma)|n\rangle$ of the stretched displacement operator in the number state representation $|n\rangle$ is expressed as

$$\langle m|D_\sigma(\varsigma)|n\rangle = \sqrt{\frac{n!}{m!}} \varsigma^{\sigma(m-n)} \exp\left(-\frac{|\varsigma|^{2\sigma}}{2}\right) L_n^{(m-n)}(|\varsigma|^{2\sigma}), \quad (34)$$

here $L_n^{(m-n)}(x)$ are the associated Laguerre polynomials, the generating function of which has form (see, Eq.(19), page 189, in [12])

$$\sum_{n=0}^{\infty} L_n^{(m-n)}(x) y^n = e^{-xy} (1+y)^m, \quad |y| < 1. \quad (35)$$

The diagonal matrix element $\langle n|D_\sigma(\varsigma)|n\rangle$ is

$$\langle n|D_\sigma(\varsigma)|n\rangle = \exp\left(-\frac{|\varsigma|^{2\sigma}}{2}\right) L_n(|\varsigma|^{2\sigma}), \quad (36)$$

where $L_n(x)$ is the Laguerre polynomial of order n , related to the associated Laguerre polynomial $L_n^{(m-n)}(x)$ as follows, $L_n(x) = L_n^{(0)}(x)$.

Finally note, that due to Eqs.(25), (30) and the orthonormality of the vectors $|n\rangle$, the scalar product ${}_\sigma\langle \eta|\varsigma\rangle_\sigma$ can be expressed as

$${}_\sigma\langle \eta|\varsigma\rangle_\sigma = \exp\left\{-\frac{|\eta|^{2\sigma}}{2} - \frac{|\varsigma|^{2\sigma}}{2} + \eta^{\sigma*} \varsigma^\sigma\right\}, \quad (37)$$

which is the *overcompleteness* relation for the stretched quantum coherent states $|\varsigma\rangle_\sigma$.

5 Stretched squeezed stretched coherent states

Let us introduce *stretched squeezed stretched coherent states* $|\varsigma, \xi\rangle_{\sigma, v}$ as follows,

$$|\varsigma, \xi\rangle_{\sigma, v} = D_\sigma(\varsigma) S_v(\xi) |0\rangle, \quad 0 < \sigma \leq 1, \quad 0 < v \leq 1, \quad (38)$$

with the stretched displacement operator $D_\sigma(\varsigma)$ defined by Eq.(23) and the *stretched squeezing operator* $S_v(\xi)$, which we introduce as follows,

$$S_v(\xi) = \exp\left\{\frac{1}{2}\xi^{v*} a^2 - \frac{1}{2}\xi^v a^{+2}\right\}, \quad 0 < v \leq 1, \quad (39)$$

where the squeeze parameter $\xi = \rho \exp(i\theta)$ is an arbitrary complex number, and a^+ and a are photon field creation and annihilation operators.

Quantum states $|\varsigma, \xi \rangle_{\sigma, v}$ represent the new family of coherent states, which includes the following members.

The well-known squeezed coherent states,

$$|\varsigma, \xi \rangle = D_\sigma(\varsigma)|_{\sigma=1}S_v(\xi)|_{v=1}|0 \rangle = D(\varsigma)S(\xi)|0 \rangle, \quad \sigma = 1, \quad v = 1, \quad (40)$$

where $D(\varsigma)$ is defined by Eq.(24) and $S(\xi)$ is given by

$$S(\xi) = \exp\left\{\frac{1}{2}\xi^*a^2 - \frac{1}{2}\xi a^{+2}\right\}. \quad (41)$$

The new *stretched squeezed coherent states*,

$$|\varsigma, \xi \rangle_v = D(\varsigma)|_{\sigma=1}S_v(\xi)|0 \rangle = D(\varsigma)S_v(\xi)|0 \rangle, \quad \sigma = 1, \quad 0 < v \leq 1. \quad (42)$$

The new *squeezed stretched coherent states*,

$$|\varsigma, \xi \rangle_\sigma = D_\sigma(\varsigma)S_v(\xi)|_{v=1}|0 \rangle = D_\sigma(\varsigma)S(\xi)|0 \rangle, \quad 0 < \sigma \leq 1, \quad v = 1. \quad (43)$$

The stretched squeezing operator $S_v(\xi)$ is unitary operator

$$S_v^+(\xi)S_v(\xi) = 1. \quad (44)$$

It is easy to see that the following transformations hold for the creation a^+ and annihilation a operators

$$S_v^+(\xi)a^+S_v(\xi) = a^+ \cosh \rho^v - a e^{-i v \theta} \sinh \rho^v, \quad (45)$$

and

$$S_v^+(\xi)aS_v(\xi) = a \cosh \rho^v - a^+ e^{i v \theta} \sinh \rho^v, \quad (46)$$

Let's show, as an example, the technique to calculate the expectation $\sigma < \varsigma, \xi | a | \varsigma, \xi \rangle_\sigma$ of annihilation operator in the stretched squeezed stretched coherent states $|\varsigma, \xi \rangle_{\sigma, v}$ basis. Using the definition Eq.(38) we write

$$\sigma, v < \varsigma, \xi | a | \varsigma, \xi \rangle_{\sigma, v} = \langle 0 | S_v^+(\xi) D_\sigma^+(\varsigma) a D_\sigma(\varsigma) S_v(\xi) | 0 \rangle = \quad (47)$$

$$\langle 0|S_v^+(\xi)(a + \varsigma^\sigma)S_v(\xi)|0 \rangle,$$

where the last transition took into account the first of Eq.(32). Further, using Eq.(46) we get

$${}_{\sigma,v} \langle \varsigma, \xi | a | \varsigma, \xi \rangle_{\sigma,v} = \varsigma^\sigma. \quad (48)$$

Similarly, one can calculate the following expectations

$${}_{\sigma,v} \langle \varsigma, \xi | a^2 | \varsigma, \xi \rangle_{\sigma,v} = |\varsigma|^{2\sigma} - e^{2iv\theta} \sinh \rho^v \cosh \rho^v, \quad (49)$$

and

$${}_{\sigma,v} \langle \varsigma, \xi | a^+ a | \varsigma, \xi \rangle_{\sigma,v} = |\varsigma|^{2\sigma} + \sinh^2 \rho^v. \quad (50)$$

5.1 Stretched squeezed stretched displaced number states

Using operators $S(\xi)$ and $D_\sigma(\varsigma)$ defined by Eqs.(39) and (23) respectively, we introduce the *stretched squeezed stretched displaced number states*,

$$|\varsigma, \xi, n \rangle_{\sigma,v} = D_\sigma(\varsigma)S_v(\xi)|n \rangle, \quad (51)$$

here $|n \rangle$ in the number state or Fock state and operators $D_\sigma(\varsigma)$ and $S_v(\xi)$ are defined by Eq.(23) and Eq.(39) respectively.

For $n = 0$, the stretched squeezed stretched displaced number state $|\varsigma, \xi, n \rangle_{\sigma,v} |_{n=0}$ becomes the stretched squeezed stretched coherent state $|\varsigma, \xi \rangle_{\sigma,v}$ defined by Eq.(38).

For $\varsigma = 0$, the stretched squeezed stretched displaced number state $|\varsigma, \xi, n \rangle_{\sigma,v} |_{\varsigma=0}$ becomes stretched squeezed displaced number state $|\xi, n \rangle_v$ defined by

$$|\xi, n \rangle_v = S_v(\xi)|n \rangle. \quad (52)$$

For $\xi = 0$, the stretched squeezed stretched displaced number state $|\varsigma, \xi, n \rangle_{\sigma,v} |_{\xi=0}$ becomes stretched displaced number state $|\varsigma, n \rangle_\sigma$ defined by

$$|\varsigma, n \rangle_\sigma = D_\sigma(\varsigma)|n \rangle. \quad (53)$$

With help of Eq.(34) the stretched displaced number state can be expressed as

$$|\varsigma, n \rangle_{\sigma} = D_{\sigma}(\varsigma)|n \rangle = \exp\left(-\frac{|\varsigma|^{2\sigma}}{2}\right) \sum_{m=0}^{\infty} \sqrt{\frac{n!}{m!}} \varsigma^{\sigma(m-n)} L_n^{(m-n)}(|\varsigma|^{2\sigma}) |m \rangle . \quad (54)$$

The stretched displaced number state $|\varsigma, n \rangle_{\sigma}$ can be expressed in terms of the stretched coherent state $|\varsigma \rangle_{\sigma}$ introduced by Eq.(1). Indeed, using Eq.(7) we have

$$\begin{aligned} |\varsigma, n \rangle_{\sigma} &= D_{\sigma}(\varsigma)|n \rangle = D_{\sigma}(\varsigma) \frac{(a^+)^n}{\sqrt{n!}} |0 \rangle = \quad (55) \\ &D_{\sigma}(\varsigma) \frac{(a^+)^n}{\sqrt{n!}} D_{\sigma}^+(\varsigma) D_{\sigma}(\varsigma) |0 \rangle = D_{\sigma}(\varsigma) \frac{(a^+)^n}{\sqrt{n!}} D_{\sigma}^+(\varsigma) |\varsigma \rangle_{\sigma} . \end{aligned}$$

Further, using the second of Eq.(32) we get

$$D_{\sigma}(\varsigma) \frac{(a^+)^n}{\sqrt{n!}} D_{\sigma}^+(\varsigma) = \frac{1}{\sqrt{n!}} (D_{\sigma}(\varsigma) a^+ D_{\sigma}^+(\varsigma))^n = \frac{1}{\sqrt{n!}} (a^+ - \varsigma^{\sigma*})^n . \quad (56)$$

Hence, combining Eqs.(55) and (56) yields

$$|\varsigma, n \rangle_{\sigma} = \frac{1}{\sqrt{n!}} (a^+ - \varsigma^{\sigma*})^n |\varsigma \rangle_{\sigma} . \quad (57)$$

This equation initiates the introduction of the *modified stretched displacement operator* $D_{\sigma}(\alpha, \varsigma)$,

$$D_{\sigma}(\alpha, \varsigma) = \exp\{\alpha^{\sigma} (a^+ - \varsigma^{\sigma*}) - \alpha^{\sigma*} (a - \varsigma^{\sigma})\}, \quad (58)$$

and *modified stretched coherent state* $|\alpha, \varsigma \rangle_{\sigma}$,

$$|\alpha, \varsigma \rangle_{\sigma} = D_{\sigma}(\alpha, \varsigma) |\varsigma \rangle_{\sigma} = \exp\{\alpha^{\sigma} (a^+ - \varsigma^{\sigma*}) - \alpha^{\sigma*} (a - \varsigma^{\sigma})\} |\varsigma \rangle_{\sigma}, \quad (59)$$

or

$$|\alpha, \varsigma \rangle_{\sigma} = \exp\{\alpha^{\sigma*} \varsigma^{\sigma} - \alpha^{\sigma} \varsigma^{\sigma*}\} D_{\sigma}(\alpha) |\varsigma \rangle_{\sigma}, \quad (60)$$

where $D_\sigma(\alpha)$ is the stretched displacement operator defined by Eq.(23).

Let us show that the modified stretched coherent state can be expressed in terms of the number states. From Eq.(60) we have

$$|\alpha, \varsigma \rangle_\sigma = \exp\{\alpha^{\sigma*} \varsigma^\sigma - \alpha^\sigma \varsigma^{\sigma*}\} \exp\left(-\frac{|\varsigma|^{2\sigma}}{2}\right) \sum_{n=0}^{\infty} \frac{\varsigma^{\sigma n}}{\sqrt{n!}} D_\sigma(\alpha) |n \rangle = \quad (61)$$

$$\exp\{\alpha^{\sigma*} \varsigma^\sigma - \alpha^\sigma \varsigma^{\sigma*}\} \exp\left(-\frac{|\varsigma|^{2\sigma}}{2}\right) \sum_{n=0}^{\infty} \frac{\varsigma^{\sigma n}}{\sqrt{n!}} |\alpha, n \rangle_\sigma,$$

where we used the definition given by Eq.(52). Then Eq.(54) gives us an expression for the modified stretched coherent state $|\alpha, \varsigma \rangle_\sigma$ in terms of the number states $|m \rangle$,

$$|\alpha, \varsigma \rangle_\sigma = e^{\alpha^{\sigma*} \varsigma^\sigma - \alpha^\sigma \varsigma^{\sigma*}} e^{-\frac{|\varsigma|^{2\sigma} + |\alpha|^{2\sigma}}{2}} \sum_{n,m=0}^{\infty} \frac{\varsigma^{\sigma n}}{\sqrt{m}} \alpha^{\sigma(m-n)} L_n^{(m-n)}(|\alpha|^{2\sigma}) |m \rangle,$$

where $L_n^{(m-n)}(x)$ are the associated Laguerre polynomials [12].

6 Quantum mechanical vector and operator representations based on stretched coherent states $|\varsigma \rangle_\sigma$

The resolution of unity condition Eq.(11), with $W_\sigma(|\varsigma|^2)$ given by Eq.(17), allows us to introduce the inner product of two quantum mechanical vectors.

1. *Inner Product* of quantum mechanical vectors $|\varphi \rangle$ and $|\psi \rangle$ defined as

$$\langle \varphi | \psi \rangle_\sigma = \frac{1}{\pi} \int_{\mathbb{C}} d^2\varsigma \langle \varphi | \varsigma \rangle_\sigma W_\sigma(|\varsigma|^2)_\sigma \langle \varsigma | \psi \rangle, \quad (62)$$

where $d^2\varsigma = d(\text{Re}\varsigma)d(\text{Im}\varsigma)$ and the integration extends over the entire complex plane \mathbb{C} , the vector representatives are wave functions $\langle \varphi | \varsigma \rangle_\sigma$ and ${}_\sigma \langle \varsigma | \psi \rangle$ given by

$$\langle \varphi | \varsigma \rangle_\sigma = \exp\left(-\frac{|\varsigma|^{2\sigma}}{2}\right) \sum_{n=0}^{\infty} \frac{\varsigma^{\sigma n}}{\sqrt{n!}} \langle \varphi | n \rangle, \quad (63)$$

$${}_{\sigma} \langle \varsigma | \psi \rangle = \exp\left(-\frac{|\varsigma|^{2\sigma}}{2}\right) \sum_{n=0}^{\infty} \langle n | \psi \rangle \frac{(\varsigma^{\sigma*})^n}{\sqrt{n!}}. \quad (64)$$

Having the inner product, we introduce the following transformation laws.

2. Vectors Transformation Law

$${}_{\sigma} \langle \varsigma | \mathcal{A} | \psi \rangle = \int_{\mathbb{C}} d^2 \varsigma'_{\sigma} \langle \varsigma | \mathcal{A} | \varsigma' \rangle_{\sigma} W_{\sigma}(|\varsigma'|^2)_{\sigma} \langle \varsigma' | \psi \rangle, \quad (65)$$

where ${}_{\sigma} \langle \varsigma | \mathcal{A} | \varsigma' \rangle_{\sigma}$ is the matrix element of quantum mechanical operator \mathcal{A} .

3. Operator Transformation Law

$${}_{\sigma} \langle \varsigma | \mathcal{A}_1 \mathcal{A}_2 | \varsigma' \rangle_{\sigma} = \int_{\mathbb{C}} d^2 \varsigma''_{\sigma} \langle \varsigma | \mathcal{A}_1 | \varsigma'' \rangle_{\sigma} W_{\sigma}(|\varsigma''|^2)_{\sigma} \langle \varsigma'' | \mathcal{A}_2 | \varsigma' \rangle_{\sigma}, \quad (66)$$

where \mathcal{A}_1 and \mathcal{A}_2 are two quantum mechanical operators.

Further, the inverse map from the functional Hilbert space representation of coherent states $|\varsigma \rangle_{\sigma}$ to the abstract one is provided by the following decomposition laws:

4. Vector Decomposition Law

$$|\psi \rangle = \int_{\mathbb{C}} d^2 \varsigma |\varsigma \rangle_{\sigma} W_{\sigma}(|\varsigma|^2)_{\sigma} \langle \varsigma | \psi \rangle. \quad (67)$$

5. Operator Decomposition Law

$$\mathcal{A} = \int_{\mathbb{C}} d^2 \varsigma_1 d^2 \varsigma_2 |\varsigma_1 \rangle_{\sigma} W_{\sigma}(|\varsigma_1|^2)_{\sigma} \langle \varsigma_1 | \mathcal{A} | \varsigma_2 \rangle_{\sigma} W_{\sigma}(|\varsigma_2|^2)_{\sigma} \langle \varsigma_2|. \quad (68)$$

Thus, we conclude that the resolution of unity Eq.(11) with $W_{\sigma}(|\varsigma|^2)$ given by Eq.(17), provides an appropriate inner product Eq.(62) and allows us to introduce the Hilbert space, Eqs.(65) - (68).

7 Conclusion

The properties of stretched coherent states were investigated. Proof that stretched coherent states are generalized coherent states was presented. It has been shown that stretched coherent states retain the fundamental properties of standard coherent states and generalize the resolution of unity or completeness condition, as well as the probability distribution that n photons are in a stretched coherent state. The stretched displacement and stretched squeezing operators are introduced and the multiplication law for stretched displacement operator is established.

Properties of coherent states resulting from an action of stretched displacement and stretched squeezing operators on the vacuum state and the Fock state are studied. Stretched squeezed stretched coherent states and stretched squeezed stretched displaced number states were introduced and their properties were studied.

The inner product of two quantum mechanical vectors was defined in terms of their stretched coherent state representations, and a functional Hilbert space was introduced.

The presented new concepts of quantum optics include two parameters, σ , $0 < \sigma \leq 1$ and ν , $0 < \nu \leq 1$. In the limiting case, when $\sigma = 1$ and $\nu = 1$, all our new results turn into the known equations of the theory of standard coherent states.

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