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Strict $\mathcal{C}^p$-triangulations of sets locally definable in o-minimal structures with an application to a $\mathcal{C}^p$-approximation problem

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Abstract
We show how to derive triangulations of sets locally definable in o-minimal structures from triangulations of compact definable sets. We give it in particular for strict $\mathcal{C}^p$-triangulations which has been recently studied by the author. This combined with a theorem of Fernando and Ghiloni implies that every continuous mapping defined on a locally compact subset B of $\mathbb{R}^m$ with values in any locally definable and locally compact subset A of $\mathbb{R}^n$ can be approximated by $\mathcal{C}^p$-mappings defined on B with values in A for any positive integer p.

Keywords O-minimal structure · Locally definable subset · $\mathcal{C}^p$-triangulation · Strict $\mathcal{C}^p$-triangulation

Mathematics Subject Classification Primary 32B25; Secondary 32S45 · 03C64 · 14P10 · 32B20 · 57R05

1 Introduction

The present article gives some generalization of results of our recent paper [9], which is important for an application in approximation theory (cf. [3, 4]). We will be working with an arbitrary fixed o-minimal expansion of the ordered field of real numbers $\mathbb{R}$. We will deal with subsets of spaces $\mathbb{R}^n$ $(n \in \mathbb{N})$ and mappings $f : A \longrightarrow \mathbb{R}^m$, where $A \subset \mathbb{R}^n$, which are *definable* in this structure (mapping f is called definable if its graph is a definable subset of $\mathbb{R}^{n+m}$).

We adopt the following general definitions. If $\mathcal{K}$ is any family of subsets of a set X, then by *a refinement of* $\mathcal{K}$ we understand any family $\mathcal{L}$ of subsets of X such that

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each $L \in \mathcal{L}$ is contained in some $K \in \mathcal{K}$ and each $K \in \mathcal{K}$ is the union $\bigcup \mathcal{L}'$ of some subfamily $\mathcal{L}' \subset \mathcal{L}$. If $\mathcal{K}$ is any family of subsets of a set X, then we will denote by $|\mathcal{K}|$ the union of all subsets K belonging to $\mathcal{K}$.

We adopt a standard definition of a *simplex of dimension* k in $\mathbb{R}^n$ as the convex hull of $k+1$ points $a_0, \dots, a_k$ affinely independent in $\mathbb{R}^n$; i.e.

$$\Delta = [a_0, \dots, a_k] := \Big\{ \sum_{i=0}^{k} \alpha_i a_i : \alpha_i \geq 0 \ (i \in \{0, \dots, k\}), \sum_{i=0}^{k} \alpha_i = 1 \Big\}.$$

If $0 \leq i_0 < i_1 < \cdots < i_l \leq k$, then the simplex $[a_{i_0}, \dots, a_{i_l}]$ is called a *face* of Δ of dimension l. The points $a_0, \dots, a_k$ are called *vertices* of Δ. The *boundary* $\partial\Delta$ of a simplex Δ of dimension k is the union of all faces of Δ of dimension $< k$. Its *relative interior* is by definition

$$\overset{\circ}{\Delta} := \Delta \setminus \partial\Delta = (a_0, \dots, a_k) := \Big\{ \sum_{i=0}^{k} \alpha_i a_i : \alpha_i > 0 \ (i \in \{0, \dots, k\}), \sum_{i=0}^{k} \alpha_i = 1 \Big\}.$$

Let p be a positive integer and let Ω be any open (not necessarily definable) subset of $\mathbb{R}^n$. We say that a subset $E \subset \Omega$ is *locally definable in* Ω if, for each point $a \in \Omega$, there exists a neighborhood U of a in Ω such that $E \cap U$ is definable. Similarly, we say that a *mapping* $f : E \longrightarrow \mathbb{R}^m$ *is locally definable* if for each point $a \in \Omega$, there exists a neighborhood U of a in Ω such that $f|U$ is definable. Moreover, if $f : E \longrightarrow \mathbb{R}^d$ is any locally definable mapping defined on a locally definable subset $E \subset \Omega$ we say that f is of class $\mathcal{C}^p$ if it admits a locally definable extension $\tilde{f} : U \longrightarrow \mathbb{R}^d$ of class $\mathcal{C}^p$ defined on an open locally definable neighborhood U of E in Ω.

By a *locally finite simplicial complex* in Ω we understand a locally finite in Ω family $\mathcal{K}$ of simplexes in Ω such that for each $K \in \mathcal{K}$ all faces of K belong to $\mathcal{K}$ and for each pair $K_1, K_2 \in \mathcal{K}$, $K_1 \cap K_2$ is empty or a common face of both K_1 and K_2. Then the set $|\mathcal{K}| := \bigcup \mathcal{K}$ is a locally definable subset of Ω. If $\mathcal{K}$ and $\mathcal{L}$ are two locally finite simplicial complexes in Ω, we say that $\mathcal{L}$ is a *subcomplex* of $\mathcal{K}$ if simply $\mathcal{L} \subset \mathcal{K}$.

Let A be any locally definable subset of Ω. By a *$\mathcal{C}^p$-triangulation* of A we understand a pair $(\mathcal{T}, h)$, where $\mathcal{T}$ is a locally finite simplicial complex in Ω and h is a locally definable homeomorphism of $|\mathcal{T}|$ onto A such that for each simplex $\Delta \in \mathcal{T}$ the restriction $h|\overset{\circ}{\Delta}$ is a $\mathcal{C}^p$-embedding of $\overset{\circ}{\Delta}$ into $\mathbb{R}^n$. If $\mathcal{E}$ is any locally finite family of locally definable subsets of Ω which are contained in A we say that a *triangulation* $(\mathcal{T}, h)$ *is compatible with* $\mathcal{E}$ if for each $E \in \mathcal{E}$ the inverse image $h^{-1}(E)$ is a union of a family of some $\overset{\circ}{\Delta}$, where $\Delta \in \mathcal{T}$.

A $\mathcal{C}^p$-triangulation $(\mathcal{T}, h)$ of A will be called a *strict $\mathcal{C}^p$-triangulation* of A if the mapping $h : |\mathcal{T}| \longrightarrow \mathbb{R}^n$ is of class $\mathcal{C}^p$ in the sense defined above.

We say that a $\mathcal{C}^p$-triangulation $(\mathcal{T}, h)$ of A, such that the restriction $h|\Delta$ to any simplex $\Delta \in \mathcal{T}$ is of class $\mathcal{C}^p$, is *orthogonally flat along simplexes* if for every pair of simplexes $\Lambda, \Delta \in \mathcal{T}$ such that Λ is a proper face of Δ, $\dim \Lambda = k$, $\dim \Delta = k + l$, and for vectors $v_1, \ldots, v_l \in \mathbb{S}^{n-1}$ pairwise orthogonal, parallel to Δ and orthogonal to Λ

$$\frac{\partial^{|\alpha|}(h|\Delta)}{\partial v_1^{\alpha_1} \ldots \partial v_l^{\alpha_l}} \equiv 0 \quad \text{on } \Lambda,$$

for every $\alpha \in \mathbb{N}^l$ such that $1 \leq |\alpha| \leq p$. The same definition of the orthogonality along simplexes applies to any locally definable mapping $g : |\mathcal{T}| \longrightarrow \mathbb{R}^d$ such that its restriction $g|\Delta$ to any simplex $\Delta \in \mathcal{T}$ is of class $\mathcal{C}^p$.

From main results of [9] we will derive the following local versions of these results. Notice that Theorem 1.2 below is a development and an improvement of Theorem 1.1.

Theorem 1.1 *Given any o-minimal structure expanding the ordered field of real numbers $\mathbb{R}$ and any open subset Ω of $\mathbb{R}^n$, let $f : A \longrightarrow \mathbb{R}^d$ be a locally definable and continuous mapping defined on a locally definable and closed subset A of Ω. Let $\mathcal{E}$ be a locally finite family of locally definable subsets of Ω contained in A. Let p be any positive integer.*

Then there exists a strict $\mathcal{C}^p$-triangulation $(\mathcal{T}, h)$ of A compatible with the family $\mathcal{E}$ and such that $f \circ h$ is of class $\mathcal{C}^p$.

Theorem 1.2 *Under the assumptions of Theorem 1.1, if additionally $\mathcal{K}$ is any locally finite simplicial complex in Ω and $g : |\mathcal{K}| \longrightarrow A$ is any locally definable homeomorphism, then there exists a strict $\mathcal{C}^p$-triangulation $(\mathcal{T}, h)$ of $|\mathcal{K}|$ such that $\mathcal{T}$ is a refinement of $\mathcal{K}$, $h(\Gamma) = \Gamma$, for each $\Gamma \in \mathcal{K}$, $(\mathcal{T}, g \circ h)$ is a strict $\mathcal{C}^p$-triangulation of A compatible with the family $\mathcal{E}$ and such that $f \circ g \circ h$ is of class $\mathcal{C}^p$.*

The both theorems were proved in [9] in the case where A is compact and all the involved simplicial complexes are finite. The main idea of the proof in [9] was inspired by the author's earlier paper [6] (common with B. Kocel-Cynk and A. Valette) on $\mathcal{C}^p$-parametrizations of sets definable in o-minimal structures, where definable sets are parametrized by $\mathcal{C}^p$-mappings defined on cubes and where injectivity of parametrization is not a point of interest. In [9] cubes are replaced by simplexes. The other important tools are: a variant of the Yomdin-Gromov Algebraic Lemma from [6](coming from Gromov [5] and Yomdin [13, 14]), special $\mathcal{C}^p$-functions of choice (called detectors) and decompositions into cells without vertical line segments in their boundaries (called capsules). An essential difficulty is to get a $\mathcal{C}^p$-triangulation $(\mathcal{T}, h)$ such that for each simplex $\Delta \in \mathcal{T}$ all partial derivatives of order $\leq p$ of the restriction $h|\mathring{\Delta}$ are bounded at the boundary $\partial\Delta$ of Δ. If this is attained the proof may be easily finished by substitution of a strict $\mathcal{C}^p$-triangulation orthogonally flat along simplexes according to the following theorem.

Theorem 1.3 *Let $\mathcal{M}$ and $\mathcal{N}$ be two finite simplicial complexes in $\mathbb{R}^n$ such that $|\mathcal{M}| = \overline{\operatorname{int}|\mathcal{M}|}$, $|\mathcal{N}| = \overline{\operatorname{int}|\mathcal{N}|}$ and the union $\mathcal{M} \cup \mathcal{N}$ is a simplicial complex.*

Then there exists a (semialgebraic) strict $\mathcal{C}^p$-triangulation $(\mathcal{M} \cup \mathcal{N}, \Phi)$ such that $\Phi(\Delta) = \Delta$, for each $\Delta \in \mathcal{M} \cup \mathcal{N}$, Φ is orthogonally flat along simplexes from $\mathcal{N}$ and $\Phi|\Delta = \mathrm{id}_\Delta$, for each simplex $\Delta \in \mathcal{M}$ such that $\Delta \cap |\mathcal{N}| = \emptyset$.

Proof See [9, Section 6]. □

The importance of existence of triangulations orthogonally flat along simplexes consists in the following observation.

Observation 1.4 *If $(\mathcal{T}, h)$ is a $\mathcal{C}^p$-triangulation of a compact definable subset A of $\mathbb{R}^n$ such that $\mathcal{T}$ is a finite simplicial complex in $\mathbb{R}^n$, $|\mathcal{T}| = \overline{\operatorname{int}|\mathcal{T}|}$, the restriction $h|\Delta$ to any simplex $\Delta \in \mathcal{T}$ is of class $\mathcal{C}^p$ and h is orthogonally flat along simplexes, then $(\mathcal{T}, h)$ is a strict $\mathcal{C}^p$-triangulation of A; i.e. h extends to a definable mapping to a definable neighborhood of $|\mathcal{T}|$ in $\mathbb{R}^n$ (see a definable version of Whitney's extension theorem in [7] and the main theorem in [11]). This applies also to other definable mappings $g : |\mathcal{T}| \longrightarrow \mathbb{R}^d$ orthogonally flat along simplexes and of class $\mathcal{C}^p$ on every simplex separately.*

Theorem 1.1 has interesting consequences in connection with the recent results of Fernando and Ghiloni concerning a $\mathcal{C}^p$-approximation problem (cf. [3, 4]). To present them in full generality we first recall some important definitions adopted in [4]. To this end we have to leave the realm of tame topology to deal with more general sets and mappings. If A and B are any subsets of $\mathbb{R}^n$ and of $\mathbb{R}^m$ respectively, we say that a *mapping* $f : B \longrightarrow A$ *is of class* $\mathcal{C}^p$ (or a $\mathcal{C}^p$*-mapping*) if it admits an extension $F : U \longrightarrow \mathbb{R}^n$ defined on an open neighborhood U of B in $\mathbb{R}^m$ and of class $\mathcal{C}^p$ in the usual sense. According to [4, Definition 1.5], a subset A of $\mathbb{R}^n$ is called *weakly* $\mathcal{C}^p$*-triangulable* if there exist a locally finite simplicial complex $\mathcal{T}$ in some space $\mathbb{R}^q$ and a triangulating homeomorphism $h : |\mathcal{T}| \longrightarrow A$ such that the restriction of h to each simplex of $\mathcal{T}$ is of class $\mathcal{C}^p$. Following [4, Definition 1.5], a locally compact subset A of $\mathbb{R}^n$ is called a $\mathcal{C}^p$*-approximation target space*, or a $\mathcal{C}^p$*-ats* for short, if for any locally compact subset B of any space $\mathbb{R}^m$, every continuous mapping $f : B \longrightarrow A$ can be approximated by $\mathcal{C}^p$-mappings $g : B \longrightarrow A$ with respect to the strong Whitney $\mathcal{C}^0$ topology.

Theorem 1.5 ([4, Theorem 1.6]) *If p is a positive integer, then every weakly $\mathcal{C}^p$-triangulable subset A of $\mathbb{R}^n$ is a $\mathcal{C}^p$-ats.*

Combining Theorem 1.5 with Theorem 1.1 leads almost immediately to the following result.

Theorem 1.6 *Let A be any locally compact subset of $\mathbb{R}^n$ and let p be any positive integer. If there exists an o-minimal structure expanding the field of real numbers $\mathbb{R}$ in which A is locally definable; i.e. each point $a \in A$ admits a neighborhood V in $\mathbb{R}^n$ such that $A \cap V$ is definable, then A is a $\mathcal{C}^p$-ats. In particular, every locally compact subanalytic set is a $\mathcal{C}^p$-ats.*

Proof Obviously, there exists an open subset Ω of $\mathbb{R}^n$ in which A is closed and locally definable. By Theorem 1.1 there exists a strict $\mathcal{C}^p$-triangulation $(\mathcal{T}, h)$ of A. Here $\mathcal{T}$ is a simplicial complex locally finite in Ω, but it is known that it is PL-isomorphic to a simplicial complex locally finite in $\mathbb{R}^{2n+1}$ (see [10, Chapter 3, Section 2.9]). □

Remark 1.7 The case $p = 1$ of Theorem 1.6 is due to Fernando and Ghiloni (see [4, Corollaries 1.11 and 1.13]), who used in his proof a result of Ohmoto and Shiota [8] (see [2] for a different proof).

A purely o-minimal variant of Theorem 1.6 is available (see [3] and [9, Section 9]).

2 Extension Lemma

In order to derive Theorem 1.1 and Theorem 1.2 from their compact versions we need some pasting procedure based on the following lemma.

Lemma 2.1 *Let $\mathcal{K}_1, \mathcal{K}_2$ be two finite simplicial complexes in $\mathbb{R}^n$ such that $\mathcal{K}_1$ is a subcomplex of $\mathcal{K}_2$. Let $(\mathcal{L}_1, h_1)$ be a $\mathcal{C}^p$-triangulation of $|\mathcal{K}_1|$ such that $\mathcal{L}_1$ is a refinement of $\mathcal{K}_1$ and $h_1(K_1) = K_1$, for each $K_1 \in \mathcal{K}_1$ and for each simplex $L_1 \in \mathcal{L}_1$ the restriction $h_1|L_1$ is of class $\mathcal{C}^p$.*

Then there exists a $\mathcal{C}^p$-triangulation $(\mathcal{L}_2, h_2)$ of $|\mathcal{K}_2|$ such that $\mathcal{L}_1$ is a subcomplex of $\mathcal{L}_2$, $\mathcal{L}_2$ is a refinement of $\mathcal{K}_2$, $h_2(K_2) = K_2$, for each $K_2 \in \mathcal{K}_2$, $h_2||\mathcal{K}_1| = h_1$ and if $K_2 \in \mathcal{K}_2$ and $K_2 \cap |\mathcal{K}_1| = \emptyset$, then $K_2 \in \mathcal{L}_2$ and $h_2|K_2 = \mathrm{id}_{K_2}$ and for each simplex $L_2 \in \mathcal{L}_2$ the restriction $h_2|L_2$ is of class $\mathcal{C}^p$.

Proof By induction, without any loss in generality, we can assume that $\mathcal{K}_2 = \mathcal{K}_1 \cup \{\Delta\}$, where $\Delta \notin \mathcal{K}_1$. By an affine change of coordinates we can also assume that

$$\Delta \subset \mathbb{R}^k = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_{k+1} = \dots = x_n = 0\},$$

where $k = \dim \Delta$ and that 0 is the center of mass of Δ.

If $\Delta \cap |\mathcal{K}_1| = \emptyset$, then Δ is a singleton and we put $\mathcal{L}_2 = \mathcal{L}_1 \cup \{\Delta\}$, $h_2|\Delta = \mathrm{id}_\Delta$ and $h_2||\mathcal{K}_1| = h_1$.

Assume now that $\Delta \cap |\mathcal{K}_1| \neq \emptyset$. Then, for every proper face Λ of Δ, $\Lambda \in \mathcal{K}_1$, so h_1 is defined and of class $\mathcal{C}^p$ when restricted to each $L_1 \in \mathcal{L}_1$ such that $L_1 \subset \partial\Delta$. Now we define $h_2|\Delta$ by the following formula

$$h_2(tu) := t^{p+1} h_1(u), \tag{2.1}$$

where $u \in \partial\Delta$ and $t \in [0, 1]$, and $\mathcal{L}_2$ by

$$\mathcal{L}_2 := \mathcal{L}_1 \cup \left\{L_1 \cdot [0, 1] : \ L_1 \in \mathcal{L}_1, L_1 \subset \partial\Delta\right\} \cup \{0\}$$

Obviously $h_2|\Delta : \Delta \longrightarrow \Delta$ is a homeomorphism extending $h_1|\partial\Delta$. Fix any $L_1 \in \mathcal{L}_1$ such that $L_1 \subset \partial\Delta$, $\dim L_1 = k - 1$ and put $L_2 := L_1 \cdot [0, 1]$. It is clear that $h_2|L_2$ is of class $\mathcal{C}^p$ on $L_2 \setminus \{0\}$. To check that it is of class $\mathcal{C}^p$ at 0 it suffices to check that

$$\lim_{\xi \to 0} D^\alpha (h_2|V)(\xi) = 0, \tag{2.2}$$

where $V := L_2 \setminus \{0\}$, $\xi = (x_1, \dots, x_k)$, $\alpha \in \mathbb{N}^k$ and $|\alpha| \le p$.

By a linear change of coordinates, we can assume that

$$L_1 \subset \{\xi = (x_1, \dots, x_k) \in \mathbb{R}^k : \ x_k = 1\}.$$

Then

$$h_2(\xi) = h_2(x_1, \dots, x_k) = {x_k}^{p+1} h_1\left(\frac{x_1}{x_k}, \dots, \frac{x_{k-1}}{x_k}, 1\right),$$

when checking of (2.2) is straightforward. □

Remark 2.2 Notice that a variant of the Extension Lemma, where $h_1 = \mathrm{id}_{|\mathcal{K}_1|}$, is true and then $h_2 = \mathrm{id}_{|\mathcal{K}_2|}$.

3 Proofs of Theorems 1.1 and 1.2

Before starting the proof we make the following quite obvious observation.

Proposition 3.1 *Let Ω be any open subset of $\mathbb{R}^n$ (not necessarily definable). Then there exists a locally finite simplicial complex $\mathcal{L}$ in Ω such that $|\mathcal{L}| = \Omega$.*

Proof Without any loss in generality, we can assume that Ω is connected. First we represent Ω as the union of a locally finite family of closed cubes

$$\Omega = \bigcup_{\nu=0}^{\infty} S_\nu,$$

where $S_\nu = [a_{1\nu} - \varepsilon_\nu, a_{1\nu} + \varepsilon_\nu] \times \ldots \times [a_{n\nu} - \varepsilon_\nu, a_{n\nu} + \varepsilon_\nu]$ $(\varepsilon_\nu > 0)$

$$(\mathrm{int} S_\nu) \cap (\mathrm{int} S_\mu) = \emptyset, \quad \text{if } \nu \neq \mu$$

and

$$(S_0 \cup \ldots \cup S_\nu) \cap S_{\nu+1} \neq \emptyset, \quad \text{if } \nu \in \mathbb{N}.$$

Now we define by induction a sequence of finite simplicial complexes $\{\mathcal{L}_\nu\}_{\nu\in\mathbb{N}}$ contained in Ω. Let $\mathcal{L}_0$ be any finite simplicial complex such that $S_0 = |\mathcal{L}_0|$. Assume $\mathcal{L}_0, \ldots, \mathcal{L}_\nu$ have already been defined such that $S_0 \cup \ldots \cup S_\mu = |\mathcal{L}_\mu|$, for $\mu \in \{0, \ldots, \nu\}$. Set

$$\mathcal{M} := \{S_\mu : \mu \leq \nu,\ S_\mu \cap S_{\nu+1} \neq \emptyset\}$$

Consider the polyhedron $Q := S_{\nu+1} \cup |\mathcal{M}|$ and a finite simplicial complex $\mathcal{P}$ such that $|\mathcal{P}| = Q$ and $\mathcal{P}$ is compatible with $S_{\nu+1}$ and

$$\mathcal{P}' := \{P \in \mathcal{P} : P \subset |\mathcal{M}|\}$$

is a refinement of the complex $\mathcal{N} := \{L \in \mathcal{L}_\nu : L \subset |\mathcal{M}|\}$. By the Extension Lemma and Remark 2.2, there exists a refinement $\mathcal{P}''$ of $\mathcal{L}_\nu$ such that $\mathcal{P}'$ is a subcomplex of $\mathcal{P}''$, and if $L \in \mathcal{L}_\nu$ and $L \cap |\mathcal{M}| = \emptyset$, then $L \in \mathcal{P}''$. Hence $\mathcal{L}_{\nu+1} := \mathcal{P}'' \cup \{P \in \mathcal{P} : P \subset S_{\nu+1}\}$ is a simplicial complex.

We check that the sequence $\{\mathcal{L}_\nu\}_\nu$ locally stabilizes. To see this, fix one cube S_ν. Then there exists $\kappa > \nu$ such that $S_\mu \cap S_\nu = \emptyset$, when $\mu \geq \kappa$. Next there exists $\lambda \geq \kappa$ such that $S_\sigma \cap S_\mu = \emptyset$ for any $\sigma \geq \lambda$ and any $\mu \in \{0, \ldots, \kappa\}$. By the definitions above we have that

$$\{L \in \mathcal{L}_\rho : L \subset S_\nu\}$$

does not depend on ρ, provided that $\rho > \lambda$; hence we can put

$$\mathcal{L} := \varinjlim \mathcal{L}_\nu$$

as a final locally finite simplicial complex partitioning Ω. □

Remark 3.2 Clearly, if $\mathcal{K}$ is any locally finite simplicial complex in Ω, then in Proposition 3.1 one can require that $\mathcal{L}||\mathcal{K}| := \{L \in \mathcal{L} : L \subset |\mathcal{K}|\}$ is a refinement of $\mathcal{K}$.

In the proof of Theorems 1.1 and 1.2 it will be convenient to make use of a locally definable version of the Tietze extension theorem.

Theorem 3.3 *(locally definable Tietze theorem) If Ω is any open subset of $\mathbb{R}^n$, A is a closed and locally definable subset of Ω and $f : A \longrightarrow \mathbb{R}$ is a locally definable continuous function, then there exists a locally definable continuous function $F : \Omega \longrightarrow \mathbb{R}$ such that $F|A = f$.*

Proof First consider a special case when $\Omega = \mathbb{R}^n$ and $f : A \longrightarrow (1, 2)$. Then we repeat after Aschenbrenner and Fischer [1, Lemma 6.6] the beautiful formula for F due to Riesz (1923):

$$F(x) := \begin{cases} \inf\limits_{a\in A} f(a)\dfrac{d(x,a)}{d(x,A)} & \text{, when } x \in \mathbb{R}^n \setminus A \\ f(x) & \text{, when } x \in A \end{cases}$$

giving a continuous locally definable extension $F : \mathbb{R}^n \longrightarrow (1, 2)$ of f.

The case when $\Omega = \mathbb{R}^n$ and $f : A \longrightarrow \mathbb{R}$ is arbitrary locally definable continuous function reduces to the previous one by the Nash isomorphism $\tau : \mathbb{R} \longrightarrow (1, 2)$ defined by the formula

$$\tau(t) := \frac{3}{2} + \frac{t}{2\sqrt{1+t^2}}.$$

Finally, the case when Ω is an arbitrary open subset is reduced to the case of A compact using gluing procedure by means of a partition of unity stated in a lemma below. □

Lemma 3.4 *Let Ω be any open subset of $\mathbb{R}^n$ and let $\{U_\nu\}_{\nu\in\mathbb{N}}$ be a locally finite covering of Ω such that, for each $\nu \in \mathbb{N}$, U_ν is open, definable with compact closure $\overline{U_\nu}$.*

Then there exists a family $\{\varphi_\nu\}_{\nu\in\mathbb{N}}$ of definable functions $\varphi_\nu : \mathbb{R}^n \longrightarrow [0, 1]$ such that, for each $\nu \in \mathbb{N}$, $\mathrm{supp}\varphi_\nu \subset U_\nu$ and

$$\sum_{\nu=0}^{\infty} \varphi_\nu(x) = 1 \quad \textit{for each } x \in \Omega.$$

Proof Standard (cf. [12, Chapter 6, Lemma 3.7]). □

Proof of Theorem 1.1 Theorem 3.3 allows us to reduce the general case to that when $A = \Omega$, arbitrary A including as a member of $\mathcal{E}$. Of course, we can assume that Ω is connected. By Proposition 3.1, $\Omega = |\mathcal{K}|$, where $\mathcal{K}$ is a locally finite simplicial complex in Ω. Let us arrange all simplexes in $\mathcal{K}$ of dimension n in a sequence $\{\Delta_\nu\}_{\nu\in\mathbb{N}}$ in such a way that $(\Delta_0 \cup \ldots \cup \Delta_\nu) \cap \Delta_{\nu+1} \neq \emptyset$, for each $\nu \in \mathbb{N}$.

Consider the finite simplicial complexes

$$\mathcal{M}_\nu := \{K \in \mathcal{K} : K \subset \Delta_0 \cup \ldots \cup \Delta_\nu\} \qquad (\nu \in \mathbb{N}).$$

We now construct inductively, for each $\nu \in \mathbb{N}$, a strict $\mathcal{C}^p$-triangulation $(\mathcal{T}_\nu, h_\nu)$ of the polyhedron $|\mathcal{M}_\nu|$ orthogonally flat along simplexes in such a way that $\mathcal{T}_\nu$ is a refinement of $\mathcal{M}_\nu$, $h_\nu(\Lambda) = \Lambda$, for each $\Lambda \in \mathcal{M}_\nu$, $f \circ h_\nu$ is of class $\mathcal{C}^p$ on $|\mathcal{M}_\nu|$ orthogonally flat along simplexes from $\mathcal{T}_\nu$ and finally $(\mathcal{T}_\nu, h_\nu)$ is compatible with the family

$$\mathcal{E}_\nu := \big\{E \cap \Delta_\mu : E \in \mathcal{E}, \ \mu \in \{0, \ldots, \nu\}\big\}.$$

A definition of $(\mathcal{T}_0, h_0)$ comes obviously from the Main Theorem and the Strict $\mathcal{C}^p$-Refinement Theorem in [9] combined with Theorem 1.3. Suppose now that $(\mathcal{T}_0, h_0), \ldots, (\mathcal{T}_\nu, h_\nu)$ have already been defined. Put

$$\mathcal{N}_\nu := \{\Gamma \in \mathcal{T}_\nu : \Gamma \subset \Delta_{\nu+1}\}.$$

Then $(\mathcal{N}_\nu, h_\nu||\mathcal{M}_\nu| \cap \Delta_{\nu+1})$ is a $\mathcal{C}^p$-triangulation of $|\mathcal{M}_\nu| \cap \Delta_{\nu+1}$ which according to Lemma 2.1 can be extended to a $\mathcal{C}^p$-triangulation $(\mathcal{L}_{\nu+1}, g_{\nu+1})$ of the simplex $\Delta_{\nu+1}$ in such a way that $\mathcal{L}_{\nu+1}$ is a refinement of the complex of all faces of $\Delta_{\nu+1}$, $\mathcal{N}_\nu$ is a subcomplex of $\mathcal{L}_{\nu+1}$, $g_{\nu+1}(K) = K$ for any face $K \in \mathcal{K}$ of $\Delta_{\nu+1}$ and $g_{\nu+1}$ is of class $\mathcal{C}^p$ when restricted to any simplex from $\mathcal{L}_{\nu+1}$.

Now by the Main Theorem and the Strict $\mathcal{C}^p$-Refinement Theorem in [9] there exists a strict $\mathcal{C}^p$-triangulation $(\mathcal{L}'_{\nu+1}, g'_{\nu+1})$ of $\Delta_{\nu+1}$ such that $\mathcal{L}'_{\nu+1}$ is a refinement of $\mathcal{L}_{\nu+1}$, $g'_{\nu+1}(L) = L$, for each $L \in \mathcal{L}_{\nu+1}$, $(\mathcal{L}'_{\nu+1}, g_{\nu+1} \circ g'_{\nu+1})$ is a strict $\mathcal{C}^p$-triangulation compatible with $\{E \cap \Delta_{\nu+1} : E \in \mathcal{E}\}$ and finally $f \circ g_{\nu+1} \circ g'_{\nu+1}$ is of class $\mathcal{C}^p$. By Lemma 2.1, there exists a $\mathcal{C}^p$-triangulation $(\mathcal{T}_{\nu+1}, h'_{\nu+1})$ of $|\mathcal{M}_\nu| \cup \Delta_{\nu+1} = \Delta_0 \cup \ldots \cup \Delta_\nu \cup \Delta_{\nu+1}$ such that $\mathcal{T}_{\nu+1}$ is a refinement of $\mathcal{T}_\nu \cup \mathcal{L}_{\nu+1}$, $\mathcal{L}'_{\nu+1}$ is a subcomplex of $\mathcal{T}_{\nu+1}$, $h'_{\nu+1}(\Lambda) = \Lambda$, for $\Lambda \in \mathcal{T}_\nu$, $h'_{\nu+1}|\Delta_{\nu+1} = g'_{\nu+1}$, for each $\Lambda \in \mathcal{T}_\nu$ such that $\Lambda \cap \Delta_{\nu+1} = \emptyset$, $\Lambda \in \mathcal{T}_{\nu+1}$ and $h'_{\nu+1}|\Lambda = \mathrm{id}_\Lambda$, and $h_{\nu+1}$ is of class $\mathcal{C}^p$ on every simplex from $\mathcal{T}_{\nu+1}$. To ensure orthogonality along simplexes we take a strict $\mathcal{C}^p$-triangulation $(\mathcal{T}_{\nu+1}, \Phi)$ of $|\mathcal{T}_{\nu+1}|$ orthogonally flat along simplexes following from Theorem 1.3 for the complex $\mathcal{T}_{\nu+1} = \mathcal{T}_\nu \cup \{\Lambda \in \mathcal{T}_{\nu+1} : \Lambda \cap \Delta_{\nu+1} \neq \emptyset\}$.

Finally we set

$$h_{\nu+1} = \begin{cases} h_\nu \circ h'_{\nu+1} \circ \Phi & , \mathrm{on} |\mathcal{M}_\nu| \\ g_{\nu+1} \circ h'_{\nu+1} \circ \Phi & , \mathrm{on} \Delta_{\nu+1} \end{cases}.$$

Since $h_{\nu+1}|K = h_\nu|K$, for any $K \in \mathcal{K}$ such that $K \cap \Delta_{\nu+1} = \emptyset$, the sequence $\{h_\nu\}_{\nu\in\mathbb{N}}$ locally stabilizes. For a similar reason $\{\mathcal{T}_\nu\}_{\nu\in\mathbb{N}}$ stabilizes, which allows us to define

$$\mathcal{T} := \varinjlim \mathcal{T}_\nu \quad \text{and} \quad h := \varinjlim h_\nu .$$

□

Proof of Theorem 1.2 By Proposition 3.1 and Remark 3.2, there exists a locally finite simplicial complex $\mathcal{L}$ in Ω such that $|\mathcal{L}| = \Omega$ and $\mathcal{L}\||\mathcal{K}|$ is a refinement of $\mathcal{K}$. By Theorem 3.3, $(g, f \circ g)$ can be extended to a locally definable and continuous mapping $F : \Omega \longrightarrow \mathbb{R}^{n+d}$. Now it suffices to repeat the argument the proof of Theorem 1.1, with $\mathcal{L}$ playing the role of $\mathcal{K}$ and F playing the role of f. □

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Declarations

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