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STRICT STATIONARITY OF GENERALIZED AUTOREGRESSIVE PROCESSES

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In this paper we consider the multivariate equation $X_{n+1} = A_{n+1}X_n + B_{n+1}$ with i.i.d. coefficients which have only a logarithmic moment. We give a necessary and sufficient condition for existence of a strictly stationary solution independent of the future. As an application we characterize the multivariate ARMA equations with general noise which have such a solution.

1. Introduction. In the last two decades, there has been a growing interest in various generalizations of autoregressive processes. Some classes of models are random coefficient models, dynamic models with a state space representation, multivariate autoregressive moving average (ARMA) models with nongaussian disturbances, bilinear models, stochastic difference equations, generalized autoregressive models with conditional heteroscedasticity (GARCH) processes [see, for instance, Nicholls and Quinn (1982), Caines (1988), Granger and Andersen (1978), Vervaat (1979), Engle and Bollerslev (1986)]. An overview of their main properties is presented in Priestley (1988). These processes are usually introduced to model stationary time series. Therefore their stationarity properties are to be studied carefully. General conditions ensuring existence and uniqueness of *second-order stationary* solutions are known and presented in the references given above. They are usually proved by Hilbert space techniques. However, it has been recognized that some important time series which appear in modelling are strictly stationary and non-square-integrable [we recall that a sequence $\{X_n\}$ is *strictly stationary* if, for all $n \in \mathbb{Z}$ and $m \in \mathbb{N}$, the law of $(X_n, X_{n+1}, \dots, X_{n+m})$ is independent of n]. These non-square-integrable strictly stationary processes were first introduced in modelling financial data by Mandelbrot (1963) [see also Granger and Orr (1972)]. Examples are ARMA processes with infinite variance [Stuck and Kleiner (1974), Hannan and Kanter (1977), Brockwell and Cline (1985)] and various ARCH and GARCH models [Engle and Bollerslev (1986), Nelson (1990)]. For instance, several statistical empirical studies have shown that the stationary GARCH processes associated with interest rates and exchange rates are typically non-square-integrable. The purpose of this paper is to give necessary and sufficient conditions for the existence of strictly stationary generalized autoregressive processes. Other recent results related to the stationarity properties of these models can be found in Feigin and Tweedie

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(1985), Liu and Brockwell (1988), Pham (1986), Tjøstheim (1986) and Pourahmadi (1988).

One can associate with each of these processes (sometimes through a so-called Markovian representation) a multivariate stochastic difference equation of the following type:

$$X_{n+1} = A_{n+1}X_n + B_{n+1}, \qquad n \in \mathbb{Z}, \tag{1}$$

where X_n and B_n are random vectors in $\mathbb{R}^d$, the A_n's are $d \times d$ matrices, and $\{(A_n, B_n),\ n \in \mathbb{Z}\}$ is a strictly stationary ergodic process. The stationarity properties of these processes are directly related to the stationarity properties of the solutions of (1). It is therefore sufficient to consider this general model and we will restrict ourselves to this situation.

Actually, a general sufficient condition ensuring the existence of a strictly stationary solution of (1) is already known. The following theorem is due to Brandt (1986). It is stated there for the one-dimensional case $d = 1$ but the proof is valid for all d (for the convenience of the reader we repeat the proof in the second part of the Proof of Theorem 2.5). Below, $\|\ \|$ is an operator norm on the set of $d \times d$ matrices.

THEOREM 1.1 (Brandt). *Let $\{(A_n, B_n),\ n \in \mathbb{Z}\}$ be a strictly stationary ergodic process such that both $\mathbb{E}(\log^+\|A_0\|)$ and $\mathbb{E}(\log^+\|B_0\|)$ are finite. Suppose that the top Lyapounov exponent γ defined by*

$$\gamma = \inf\left\{\mathbb{E}\left(\frac{1}{n+1}\log\|A_0A_{-1}\cdots A_{-n}\|\right),\ n \in \mathbb{N}\right\}$$

is strictly negative. Then, for all $n \in \mathbb{Z}$, the series

$$X_n = \sum_{k=0}^{+\infty} A_nA_{n-1}\cdots A_{n-k+1}B_{n-k}$$

converges a.s., and the process $\{X_n,\ n \in \mathbb{Z}\}$ is the unique strictly stationary solution of (1).

Our aim is to establish a converse to this theorem. We consider the case where the (A_n, B_n)'s are independent, identically distributed, and we look at strictly stationary solutions which are "independent of the future" (we call them nonanticipative; see Definition 2.2). Our main result (Theorem 2.5) is that under an irreducibility condition, if there is a nonanticipative strictly stationary solution of (1) then the top Lyapounov exponent γ is strictly negative. As an application, we study multivariate ARMA processes with arbitrary i.i.d. noise. We prove that the necessary and sufficient condition for the existence of a nonanticipative strictly stationary solution of an ARMA equation is the familiar condition on the polynomial matrix associated with the autoregressive part (see Theorem 4.1). We also consider dynamical models with a state space representation (Proposition 4.2). In that case our irreducibility condition reduces to controllability.

The plan of this paper is the following. In Section 2, we state our main results on the existence of strictly stationary solutions of (1). We consider both situations, with and without moment condition (Theorems 2.4 and 2.5). Section 3 is devoted to the proofs of these results. Applications to ARMA processes and dynamic models with a state space representation are presented in Section 4. The methods rest upon martingale and Markov chain theory and use the same approach as Bougerol (1987), where the case $B_n = 0$ was considered. Actually, this paper is independent of Bougerol (1987); the situation here is somewhat simpler (due to our irreducibility assumption) and can be described completely. This was not possible there, where, for instance, the A_n's had to be invertible.

Autoregressive processes with positive coefficients and GARCH processes are studied along the same lines in Bougerol and Picard (1992). Under moment conditions and in dimension 1, results similar to ours appeared in Elie (1982), Grincevicius (1981) and Vervaat (1979).

2. Statement of the general results. In this section we give the main definitions that we will use and we state our general results. Proofs are postponed until the next section. Below, $\mathbb{M}(d)$ denotes the set of $d \times d$ real matrices and $(\Omega, \mathscr{A}, \mathbb{P})$ is a given probability space.

DEFINITION 2.1. A *generalized autoregressive model with i.i.d. coefficients* is a model

$$X_{n+1} = A_{n+1} X_n + B_{n+1}, \qquad n \in \mathbb{Z}, \tag{2}$$

where $\{(A_n, B_n), n \in \mathbb{Z}\}$ is a given sequence of independent, identically distributed, random variables defined on $(\Omega, \mathscr{A}, \mathbb{P})$ with values in $\mathbb{M}(d) \times \mathbb{R}^d$. A solution of this equation is any sequence $\{X_n, n \in \mathbb{Z}\}$ of $\mathbb{R}^d$-valued random variables for which (2) holds.

In this paper we are only interested in stationary solutions which are independent of the future at any given time, in the sense of the following definition. We do not assume that they are causal nor measurable with respect to the past. In fact, it will be a consequence of our main theorem that this is actually true under an irreducibility condition.

DEFINITION 2.2. A *nonanticipative strictly stationary solution* of (2) is a strictly stationary process $\{X_n, n \in \mathbb{Z}\}$ which is a solution of (2) such that, for any $p \in \mathbb{Z}$, X_p is independent of the random variables $\{(A_n, B_n), n > p\}$.

Without an extra hypothesis the converse of Brandt's theorem is not true; for instance, if B_n is identically 0, then $X_n = 0$ is a stationary solution for any sequence $\{A_n\}$. In order to avoid such degenerate situations, we introduce the following irreducibility condition. Proposition 2.6 and Corollary 2.7 will shed some light on the role of this hypothesis. We recall that an affine subspace H of $\mathbb{R}^d$ is a translate $z + V$ of a linear subspace V. In general, we make no distinction between a matrix and the linear map which is associated with this matrix (in the canonical basis).

DEFINITION 2.3. An affine subspace H of $\mathbb{R}^d$ is said to be *invariant* under the model (2) if $\{A_0 x + B_0;\ x \in H\}$ is contained in H almost surely. The model (2) is called *irreducible* if $\mathbb{R}^d$ is the only affine invariant subspace.

Our first result does not require any integrability condition. It will be proved in Section 3.

THEOREM 2.4. *Consider a generalized autoregressive model* (2) *with i.i.d. coefficients. Suppose that this model is irreducible and that it has a nonanticipative strictly stationary solution* $\{X_n,\ n \in \mathbb{Z}\}$. *Then the following hold*:

(i) $A_0 A_{-1} \cdots A_{-k}$ *converges to* 0 *almost surely when* $k \to +\infty$.
(ii) *For any integer* n,

$$X_n = \sum_{k=0}^{+\infty} A_n A_{n-1} \cdots A_{n-k+1} B_{n-k}, \tag{3}$$

where the series converges almost surely.
(iii) *This solution is the unique strictly stationary solution of* (2).

We recall the definition of the top Lyapounov exponent. We choose any norm $\|\ \|$ on $\mathbb{R}^d$ and define an operator norm on $\mathbb{M}(d)$ by

$$\|M\| = \sup\left\{ \frac{\|Mx\|}{\|x\|};\ x \in \mathbb{R}^d,\ x \neq 0 \right\},$$

for M in $\mathbb{M}(d)$. The top Lyapounov exponent associated with the models (1) and (2) is defined, when $\mathbb{E}(\log^+\|A_0\|)$ is finite, by

$$\gamma = \inf\left\{ \mathbb{E}\left(\frac{1}{n+1} \log\|A_0 A_{-1} \cdots A_{-n}\| \right),\ n \in \mathbb{N} \right\}.$$

For instance, $\gamma \leq \mathbb{E}(\log\|A_0\|)$, with equality in dimension 1. When A_n is a constant matrix A, then γ is the logarithm of the spectral radius of A. It is known that a.s.,

$$\gamma = \lim_{n \to +\infty} \frac{1}{n} \log\|A_0 A_{-1} \cdots A_{-n}\| \tag{4}$$

[see Furstenberg and Kesten (1960) or Kingman (1973)]. This shows that γ is independent of the chosen norm. In general, this exponent can be difficult to compute but it is easily estimated by Monte Carlo simulations, using (4).

Our second main result is the following converse of Brandt's theorem.

THEOREM 2.5. *Suppose that the generalized autoregressive model* (2) *with i.i.d. coefficients is irreducible and that both* $\mathbb{E}(\log^+\|A_0\|)$ *and* $\mathbb{E}(\log^+\|B_0\|)$ *are* *finite. Then (2) has a nonanticipative strictly stationary solution if and only if the top Lyapounov exponent γ is strictly negative.*

The next proposition could be a first step in the study of the nonirreducible situation. We will use it in Section 4.

PROPOSITION 2.6. *Suppose that the model (2) has a nonanticipative strictly stationary solution $\{X_n,\ n \in \mathbb{Z}\}$. Let H be the minimal affine subspace of $\mathbb{R}^d$ such that $\mathbb{P}(X_0 \in H) = 1$. Then H is invariant under the model and any invariant subspace of H carries a nonanticipative strictly stationary solution.*

An immediate consequence of Theorem 1.1, Theorem 2.5 and Proposition 2.6 is the following corollary.

COROLLARY 2.7. *Consider a model (2) with $\mathbb{E}(\log^+\|A_0\|)$ and $\mathbb{E}(\log^+\|B_0\|)$ finite. Suppose that there exists a nonanticipative strictly stationary solution which is not carried by an affine hyperplane. Then the following three conditions are equivalent:*

(i) *The top Lyapounov exponent is strictly negative.*
(ii) *The model is irreducible.*
(iii) *There is a unique stationary solution.*

REMARK 2.8. Under the same hypotheses as above we may also consider the equation

$$X_{n+1} = A_{n+1}X_n + B_{n+1}, \qquad n \geq 0, \tag{5}$$

indexed by positive times. Suppose that $\{X_n,\ n \in \mathbb{N}\}$ is a nonanticipative strictly stationary solution of (5). It is easy to see that $\{(X_n, A_n, B_n),\ n \in \mathbb{N}\}$ is a strictly stationary process. It can be extended, as any stationary process indexed by the positive integers, to a new stationary process $\{(X_n, A_n, B_n),\ n \in \mathbb{Z}\}$. It is clear that $X_{n+1} = A_{n+1}X_n + B_{n+1}$, a.s., for all n in $\mathbb{Z}$, and that $\{X_n,\ n \in \mathbb{Z}\}$ will be a nonanticipative strictly stationary solution of this equation. This reduces the study of (5) to the study of (2).

3. Proofs of the general results. We consider a generalized autoregressive model with i.i.d. coefficients. One can associate with this model a Markov chain on $\mathbb{R}^d$: Starting from a deterministic point x in $\mathbb{R}^d$, the chain is at time n in the state X_n, where $\{X_n,\ n \in \mathbb{N}\}$ is the solution of (2) which satisfies $X_0 = x$. The transition probability of this Markov chain is the kernel P given by

$$P(x, C) = \mathbb{P}(A_0 x + B_0 \in C), \qquad x \in \mathbb{R}^d,$$

for any Borel set C of $\mathbb{R}^d$. A P-invariant distribution is a probability measure

m on $\mathbb{R}^d$ such that

$$\int P(x,C)\,dm(x) = m(C)$$

for any Borel set C. In other words, it is an invariant distribution of the Markov chain.

LEMMA 3.1. *There is a one-to-one correspondence between the nonanticipative strictly stationary solutions of* (2) *and the P-invariant distributions.*

PROOF. Let $\{X_n,\ n \in \mathbb{Z}\}$ be a nonanticipative strictly stationary solution of (2) and let m be the common law of the X_n's. By the definition of "nonanticipative," X_0 is independent of (A_1, B_1) and we may write, for any Borel set C of $\mathbb{R}^d$,

$$\begin{aligned} m(C) &= \mathbb{P}(X_1 \in C) \\ &= \mathbb{P}(A_1 X_0 + B_1 \in C) \\ &= \int \mathbb{P}(A_1 x + B_1 \in C)\, d\mathbb{P}_{X_0}(x) \\ &= \int P(x,C)\, dm(x). \end{aligned}$$

This shows that m is P-invariant. Conversely, let m be a P-invariant distribution. Consider a random variable X_0, with law m, independent of the sequence $\{(A_n, B_n),\ n \geq 1\}$. When n is positive, define a process by the formula $X_n = A_n X_{n-1} + B_n$. Then $\{X_n,\ n \in \mathbb{N}\}$ is a Markov chain with transition probability P. Since the law of X_0 is P-invariant, this is also a stationary process. It is clear that this process is a nonanticipative strictly stationary solution of (5). As seen in Remark 2.8, it can be extended to such a solution of (2). □

LEMMA 3.2. *Let m be a P-invariant distribution. Then the affine subspace H of minimal dimension such that* $m(H) = 1$ *is invariant under the model* (2).

PROOF. With the notation of Lemma 3.2, we have

$$\begin{aligned} 1 = m(H) &= \mathbb{P}(X_1 \in H) = \mathbb{P}(A_1 X_0 + B_1 \in H) \\ &= \mathbb{E}\left\{\int \mathbb{1}_H(A_1 x + B_1)\, dm(x)\right\}. \end{aligned}$$

Thus if $L = \{x \in \mathbb{R}^d;\ A_1 x + B_1 \in H,\ \text{a.s.}\}$, then $m(L) = 1$. The affine subspace $L \cap H$ is such that $m(L \cap H) = 1$. By minimality, H is contained in L. This shows that $\{A_1 x + B_1;\ x \in H\}$ is contained in H a.s., hence H is invariant. □

In order to prove Theorem 2.4 it is convenient to use affine maps. Let Aff(d) denote the set of affine maps from $\mathbb{R}^d$ into $\mathbb{R}^d$. Such a map f can be written in a unique way as

$$f(x) = Ax + b, \qquad x \in \mathbb{R}^d,$$

where A is in $\mathbb{M}(d)$ and b is in $\mathbb{R}^d$. The set Aff(d) is thus in one-to-one correspondence with $\mathbb{M}(d) \times \mathbb{R}^d$. It is a vector space of dimension $d(d+1)$. The composition of maps defines a product on Aff(d). Explicitly, if $f(x) = Ax + b$ and $g(x) = Cx + d$, then

$$(f \circ g)(x) = ACx + Ad + b.$$

With this product, Aff(d) is a topological semigroup.

PROOF OF THEOREM 2.4. We define random affine maps F_n $(= F_n^\omega)$ and Γ_n $(= \Gamma_n^\omega)$ by

$$F_n(x) = A_n x + B_n, \quad x \in \mathbb{R}^d, n \in \mathbb{Z},$$

and

$$\Gamma_n = F_0 \circ F_{-1} \circ \cdots \circ F_{-n}, \qquad n \in \mathbb{N}.$$

Let μ be the law of F_0, μ_n be the law of Γ_n, and $\nu = \sum_{n=0}^{\infty} 2^{-n-1}\mu_n$. These are probability measures on Aff(d). The topological support of ν, denoted by S, is a closed subsemigroup of Aff(d).

By assumption, the model is irreducible and (2) has a nonanticipative strictly stationary solution. Whence, it follows from the above lemmas that there exists a P-invariant distribution m which is not carried by an affine hyperplane.

The pairing which associates to the element (f, x) in Aff$(d) \times \mathbb{R}^d$ the vector $f(x)$ in $\mathbb{R}^d$ defines an action of Aff(d) on $\mathbb{R}^d$. In the terminology of Bougerol and Lacroix [(1985), I.3.3], m is a μ-invariant distribution. Thus we are in position to use Lemma II.2.1 of Bougerol and Lacroix (1985), due to Guivarc'h and Raugi (1985). By this lemma, there exists a measurable subset Ω_0 of Ω such that $\mathbb{P}(\Omega_0) = 1$ and such that for all ω in Ω_0 there is a probability measure m_ω on $\mathbb{R}^d$ with the following properties: For any bounded continuous function $\phi: \mathbb{R}^d \to \mathbb{R}$,

$$\lim_{n \to +\infty} \int \phi(\Gamma_n^\omega(x))\, dm(x) = \int \phi(x)\, dm_\omega(x) \tag{6}$$

and, for ν-almost all f in Aff(d),

$$\lim_{n \to +\infty} \int \phi((\Gamma_n^\omega \circ f)(x))\, dm(x) = \int \phi(x)\, dm_\omega(x). \tag{7}$$

Let H_ω be the smallest affine subspace of $\mathbb{R}^d$ such that $m_\omega(H_\omega) = 1$. We need the following lemma.

LEMMA 3.3. *If there exists a P-invariant distribution m which is not carried by an affine hyperplane, then, for all* $\omega \in \Omega_0$, *the following hold*:

(i) *The sequence* $\{\Gamma_n^\omega, n \geq 0\}$ *is bounded in the vector space* Aff(d).

(ii) *For any limit point* Γ_∞^ω *of this sequence*, (a) *the set* $\{\Gamma_\infty^\omega \circ f,\ f \in S\}$ *is bounded and* (b) *for all f in S*, $(\Gamma_\infty^\omega \circ f)(\mathbb{R}^d) = \Gamma_\infty^\omega(\mathbb{R}^d) = H_\omega$.

PROOF. We fix an element ω in Ω_0. Suppose that the sequence $\{\Gamma_n^\omega, n \geq 0\}$ is not bounded. We can find a subsequence $\{n_i, i \in \mathbb{N}\}$ and an affine map Γ^ω such that

$$\lim_{i \to +\infty} \left\| \Gamma_{n_i}^\omega \right\| = +\infty \quad \text{and} \quad \lim_{i \to +\infty} \frac{\Gamma_{n_i}^\omega}{\|\Gamma_{n_i}^\omega\|} = \Gamma^\omega$$

[where $\| \ \|$ denotes a norm on the vector space $\mathrm{Aff}(d)$]. Let $H = \{x \in \mathbb{R}^d; \Gamma^\omega(x) = 0\}$ and $\phi: \mathbb{R}^d \to \mathbb{R}$ be a continuous function with compact support. If x is not in H, then $\|\Gamma_{n_i}^\omega(x)\| \to +\infty$, and thus $\phi(\Gamma_{n_i}^\omega(x)) \to 0$, as $i \to +\infty$. This implies that

$$\lim_{i \to +\infty} \int \phi\left(\Gamma_{n_i}^\omega(x)\right) dm(x) = \lim_{i \to +\infty} \int \mathbb{1}_H(x) \phi\left(\Gamma_{n_i}^\omega(x)\right) dm(x).$$

Using (6) we obtain that

$$\left| \int \phi(x) \, dm_\omega(x) \right| \leq m(H) \sup_x |\phi(x)|.$$

As m_ω is a probability measure, $m(H) = 1$. Since m is not carried by an affine hyperplane, H must be equal to $\mathbb{R}^d$. Thus $\Gamma^\omega = 0$, but this is impossible since $\|\Gamma^\omega\| = 1$. Therefore the sequence $\{\Gamma_n^\omega, n \geq 0\}$ is bounded, which proves the first assertion.

Now, let Γ_∞^ω be a limit of this sequence. Then by (7), for ν-almost all f,

$$\int \phi\left(\left(\Gamma_\infty^\omega \circ f\right)(x)\right) dm(x) = \int \phi(x) \, dm_\omega(x), \tag{8}$$

for all bounded continuous ϕ. The set of affine maps f for which this holds is closed. Thus this equality holds for all f in the support S of ν. Using this property, the boundedness of $\{\Gamma_\infty^\omega \circ f, f \in S\}$ is then proved exactly as the first assertion. Now, it follows from (8) that, for any f in S,

$$m_\omega(C) = m\left\{x \in \mathbb{R}^d; \left(\Gamma_\infty^\omega \circ f\right)(x) \in C\right\}$$

for any Borel subset C of $\mathbb{R}^d$. Applying this relation with $C = H_\omega$, we see that m is carried by $\{x \in \mathbb{R}^d; (\Gamma_\infty^\omega \circ f)(x) \in H_\omega\}$. Since, by assumption, m is not carried by a proper affine subspace, this implies that $(\Gamma_\infty^\omega \circ f)(\mathbb{R}^d)$ is contained in H_ω. Using again this relation with $C = (\Gamma_\infty^\omega \circ f)(\mathbb{R}^d)$ and using the minimality of H_ω, we see that actually $(\Gamma_\infty^\omega \circ f)(\mathbb{R}^d) = H_\omega$. Since, by (6), (8) also holds when f is the identity transformation, we also have $\Gamma_\infty^\omega(\mathbb{R}^d) = H_\omega$, which proves (b). □

PROOF OF THEOREM 2.4 (Continued). Let $\Omega_1 = \{\omega \in \Omega_0; \Gamma_n^\omega \in S$, for all $n \in \mathbb{N}\}$. It is clear that $\mathbb{P}(\Omega_1) = 1$. We fix an ω in Ω_1. Let Γ_∞^ω be a limit point of the sequence $\{\Gamma_n^\omega, n \in \mathbb{N}\}$. By Lemma 3.3, the set $T = \{\Gamma_\infty^\omega \circ f, f \in S\}$ is bounded. Since Γ_∞^ω is in S, and since S is a subsemigroup of $\mathrm{Aff}(d)$, T is also a semigroup. Its closure K is a compact semigroup. This implies [see Hofmann and Mostert (1966), A.1.22] that there exists an affine map h in K such that $h \circ h = h$ and such that $G = \{h \circ f \circ h, f \in K\}$ is a compact group. Let λ be the Haar measure on G such that $\lambda(G) = 1$, and let z be the element of $\mathbb{R}^d$ defined by

$$z = \int_G g(0)\, d\lambda(g).$$

By invariance of the Haar measure under left translations, $g(z) = z$, for any g in G. In particular, for f in S,

$$(9) \qquad (h \circ \Gamma_\infty^\omega \circ f \circ h)(z) = z.$$

Let $g = h \circ \Gamma_\infty^\omega$ and V be the affine hull of the points $\{f(h(z));\ f \in S\}$. For all f in S, $f(V)$ is contained in V. The assumed irreducibility of the model implies that $V = \mathbb{R}^d$. Therefore, $g(\mathbb{R}^d) = \{z\}$. Since g is an element of S, we obtain, using Lemma 3.3(b),

$$H_\omega = (\Gamma_\infty^\omega \circ g)(\mathbb{R}^d) = \Gamma_\infty^\omega(\{z\}).$$

Let $Z(\omega) = \Gamma_\infty^\omega(z)$. By Lemma 3.3(b) again, $\Gamma_\infty^\omega(\mathbb{R}^d) = \{Z(\omega)\}$. In other words, Γ_∞^ω is the affine map which satisfies $\Gamma_\infty^\omega(x) = Z(\omega)$, for any $x \in \mathbb{R}^d$. This proves that the sequence $\{\Gamma_n^\omega,\ n \in \mathbb{N}\}$ converges to this map: For all ω in Ω_1, and thus almost surely, for all x in $\mathbb{R}^d$,

$$\lim_{k \to +\infty} A_0 A_{-1} \cdots A_{-k} x = \lim_{k \to +\infty} \{\Gamma_k(x) - \Gamma_k(0)\} = 0,$$

which proves the first claim of the theorem, and

$$\lim_{p \to +\infty} \sum_{k=0}^{p} A_0 A_{-1} \cdots A_{-k+1} B_{-k} = \lim_{p \to +\infty} \Gamma_p(0) = Z.$$

By stationarity, for any n, there exists a random vector Z_n such that, a.s.,

$$\lim_{p \to +\infty} \sum_{k=0}^{p} A_n A_{n-1} \cdots A_{n-k+1} B_{n-k} = Z_n.$$

Consider now an arbitrary strictly stationary solution Y_n, $n \in \mathbb{Z}$, of (2). Using (2) repeatedly, we have, for all n in $\mathbb{N}$,

$$Y_0 = F_0(Y_{-1}) = (F_0 \circ F_{-1} \circ \cdots \circ F_{-n})(Y_{-n-1}) = \Gamma_n(Y_{-n-1}),$$

so that

$$(10) \qquad Y_0 - Z = \lim_{n \to +\infty} \{\Gamma_n(Y_{-n-1}) - \Gamma_n(0)\}.$$

Note that, since we use an operator norm on $\mathbb{M}(d)$,

$$\begin{aligned} \|\Gamma_n(Y_{-n-1}) - \Gamma_n(0)\| &= \|A_0 A_{-1} \cdots A_{-n} Y_{-n-1}\| \\ &\le \|A_0 A_{-1} \cdots A_{-n}\|\, \|Y_{-n-1}\|. \end{aligned}$$

We have just proved that $\|A_0 A_{-1} \cdots A_{-n}\|$ converges a.s. to 0. Since the law of $\|Y_{-n-1}\|$ is constant, the left term above goes to 0 in probability, and thus $Y_0 = Z$ by (10). In a similar way, one shows that $Y_n = Z_n$ for all n. Therefore, there is a unique stationary solution. In particular, $X_n = Z_n$. This completes the proof. □

In order to prove Theorem 2.5, we will need the following more or less well known lemma, which is only a slight extension of Lemma 5.2 of Bougerol (1987).

LEMMA 3.4. *Let $\{M_n,\ n \in \mathbb{N}\}$ be an ergodic strictly stationary sequence of matrices in $\mathbb{M}(d)$. We suppose that $\mathbb{E}(\log^+\|M_0\|)$ is finite and that, almost surely,*

$$\lim_{n\to+\infty} \|M_n M_{n-1} \cdots M_1\| = 0.$$

Then the top Lyapounov exponent γ associated with this sequence is strictly negative.

PROOF. We may suppose that γ is not $-\infty$ (otherwise the result is obvious). In this case, we use the construction of Furstenberg and Kesten (1960). They have shown that one can, on an enlarged probability space, adjoin to the original sequence $\{M_n,\ n \in \mathbb{N}\}$ a sequence of matrices Z_n in $\mathbb{M}(d)$ such that $\{(M_n, Z_n),\ n \in \mathbb{N}\}$ is a strictly stationary sequence with the following properties, a.s.:

$$\lim_{n\to+\infty} \frac{1}{n}\log\|M_n M_{n-1} \cdots M_2 Z_1\| = \gamma,$$

$$\|M_{n+1}Z_n\| \neq 0,$$

and

$$Z_{n+1} = \frac{M_{n+1}Z_n}{\|M_{n+1}Z_n\|}.$$

We define a map Φ from $\mathbb{M}(d) \times \mathbb{M}(d)$ into $\mathbb{R}$ by

$$\Phi(M, Z) = \begin{cases} \log\dfrac{\|MZ\|}{\|Z\|}, & \text{if } \|Z\| \neq 0,\\ 1, & \text{otherwise}. \end{cases}$$

Then

$$\log\|M_n M_{n-1} \cdots M_2 Z_1\| - \log\|Z_1\| = \sum_{i=1}^{n-1} \Phi(M_{i+1}, Z_i), \quad \text{a.s.}$$

This implies that a.s.,

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{i=1}^{n-1} \Phi(M_{i+1}, Z_i) = \gamma,$$

and

$$\lim_{n\to+\infty} \sum_{i=1}^{n-1} \Phi(M_{i+1}, Z_i) \le \lim_{n\to+\infty} \log(\|M_n M_{n-1} \cdots M_2\|\,\|Z_1\|) - \log\|Z_1\| = -\infty.$$

From a general lemma of ergodic theory [see Guivarc'h and Raugi (1985), Lemma 3.6, or Bougerol and Lacroix (1985), Lemma 2.3] we conclude from these two equalities that γ is strictly negative. □

PROOF OF THEOREM 2.5. We first suppose that there exists a nonanticipative strictly stationary solution of (2). Let M_n be the transpose of the matrix A_{-n}. For the operator norm on $\mathbb{M}(d)$ associated with the Euclidean norm on $\mathbb{R}^d$, the norm of a matrix is equal to the norm of its transpose. Whence,

$$\|M_n M_{n-1} \cdots M_1\| = \|A_{-1} A_{-2} \cdots A_{-n}\|.$$

The top Lyapounov exponent associated with (M_n) is also γ. By Theorem 2.4, $A_0 A_{-1} \cdots A_{-k}$ converges a.s. to 0 and we conclude from Lemma 3.4 that γ is strictly negative. This proves the "only if" part of the theorem. The "if" part follows from Theorem 1.1. We give the proof of this theorem adapted from Brandt (1986) for the reader's convenience. Let n be an integer. Since $\mathbb{E}(\log^+\|B_0\|) < +\infty$,

$$\sum_{k=0}^{+\infty} \mathbb{P}\left(\log^+\|B_{n-k}\| > -\frac{k\gamma}{2}\right) < +\infty.$$

This implies, by the Borel–Cantelli lemma that a.s.,

$$\limsup_k \frac{1}{k}\log^+\|B_{n-k}\| \leq -\frac{\gamma}{2},$$

so that

$$\limsup_{k\to+\infty} \frac{1}{k}\log\|A_n A_{n-1} \cdots A_{n-k+1} B_{n-k}\|$$
$$\leq \limsup_{k\to+\infty} \frac{1}{k}\log(\|A_n A_{n-1} \cdots A_{n-k+1}\| \, \|B_{n-k}\|) \leq \frac{\gamma}{2}.$$

Therefore, when $\gamma < 0$, the series

$$Y_n = \sum_{k=0}^{+\infty} A_n A_{n-1} \cdots A_{n-k+1} B_{n-k}$$

converges a.s. It is clear that this sequence is a stationary nonanticipative solution of (2). Uniqueness is shown at the end of the proof of Theorem 2.4. □

PROOF OF PROPOSITION 2.6. The fact that H is invariant follows from Lemma 3.2. Let Aff(H) be the vector space of affine maps from H into H. Using the invariance of H, we may define maps F_n in Aff(H) by, if $x \in H$, $F_n(x) = A_n x + B_n$. Consider the model

$$Y_{n+1} = F_{n+1}(Y_n), \qquad n \in \mathbb{Z},$$

where $Y_n \in H$. Except for a change of notation, this is a generalized autoregressive model with i.i.d. coefficients in the sense of Definition 2.1. The process $\{X_n,\ n \in \mathbb{Z}\}$, which is carried by H, is a nonanticipative strictly stationary solution of this model. Moreover, by minimality of H, the law m of X_n is not carried by a proper affine subspace of H. Therefore Lemma 3.3 holds for this model. Let L be an invariant affine subspace of minimal dimension contained in H. Using this lemma, we may repeat the second part of the proof of Theorem 2.4 for the restrictions of the F_n's to L (everywhere in that proof L replaces $\mathbb{R}^d$, and the affine subspace V there is now equal to L). We obtain that, for any x in L, $(F_0 \circ F_{-1} \circ \cdots \circ F_{-n})(x)$ converges almost surely to a limit Z_0 independent of x. By stationarity, for any $k \in \mathbb{Z}$, $(F_k \circ F_{k-1} \circ \cdots \circ F_{-n})(x)$ converges a.s. to some random variable Z_k when n goes to $+\infty$. It is clear that (Z_k) is a stationary nonanticipative solution of (2) carried by L. □

4. Application to ARMA processes and state space models. In this section we give a necessary and sufficient condition for existence of strictly stationary solutions of multivariate ARMA equations and state space models, when the noise process is an arbitrary sequence of i.i.d. random vectors. We first consider multivariate ARMA models. Let F_i, $1 \le i \le p$, and G_j, $0 \le j \le q$, be given real matrices of dimension $d \times d$ and $d \times m$, respectively. A random process $\{Y_n,\ n \in \mathbb{Z}\}$ with values in $\mathbb{R}^d$ is solution of an ARMA equation if

$$Y_n = \sum_{i=1}^{p} F_i Y_{n-i} + \sum_{j=0}^{q} G_j \varepsilon_{n-j}, \qquad n \in \mathbb{Z}, \tag{11}$$

where ε_n, $n \in \mathbb{Z}$, are given independent, identically distributed, $\mathbb{R}^m$-valued random variables. Such a solution is called nonanticipative when, for any $r \in \mathbb{Z}$, $\{Y_k,\ k \le r\}$ is independent of $\{\varepsilon_n,\ n > r\}$.

Let us introduce the following matrices with polynomial coefficients:

$$F(x) = I_d - \sum_{n=1}^{p} F_n x^n, \qquad G(x) = \sum_{n=0}^{q} G_n x^n,$$

where I_d is the identity matrix in $\mathbb{M}(d)$. By definition, a $d \times d$ matrix with polynomial coefficients $D(x)$ is a common left divisor of $F(x)$ and $G(x)$ if there exist two matrices $P(x)$ and $Q(x)$, with polynomial coefficients, such that $F(x) = D(x)P(x)$ and $G(x) = D(x)Q(x)$. The matrix fraction $F(x)^{-1}G(x)$ is said to be irreducible if the determinant of every common left divisor of $F(x)$ and $G(x)$ is independent of x.

The following theorem is a generalization of the classical result on ARMA processes where the white noise process is assumed to be Gaussian [see e.g., Caines (1988)]. The "if" part is well known, and we give the proof for the reader's convenience. With an appropriate modification of the dimension m and of the G_i's, we can suppose without loss of generality that there is no affine hyperplane H of $\mathbb{R}^m$ such that $\mathbb{P}(\varepsilon_0 \in H) = 1$.

THEOREM 4.1. *We suppose that ε_0 is not carried by a fixed affine hyperplane of $\mathbb{R}^m$, that $\mathbb{E}(\log^+\|\varepsilon_0\|) < \infty$ and that the matrix fraction $F(x)^{-1}G(x)$ is* *irreducible. Then, there exists a nonanticipative strictly stationary solution of the ARMA equation* (11) *if and only if all the zeros of the polynomial* det $F(x)$ *lie outside the closed unit disk.*

PROOF. STEP 1. Consider the following matrices, written in block form:

$$\Psi = (F_1, \dots, F_{p-1}), \qquad \Gamma = (G_1, \dots, G_{q-1}),$$

$$A = \begin{bmatrix} \Psi & F_p & \Gamma & G_q \\ I_{(p-1)d} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_{m(q-1)} & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} G_0 \\ 0 \\ I_m \\ 0 \end{bmatrix};$$

for instance, A is a $(pd + qm) \times (pd + qm)$ matrix. Write ${}^tM$ for the transpose of a matrix M. The correspondence which associates with the sequence $\{Y_n,\ n \in \mathbb{Z}\}$ the random vectors

$$X_n = {}^t\left({}^tY_n, \dots, {}^tY_{n-p+1}, {}^t\varepsilon_n, \dots, {}^t\varepsilon_{n-q+1}\right) \tag{12}$$

is a bijection between the solutions of the ARMA equation (11) and the solutions of the generalized autoregressive model in $\mathbb{R}^{pd+qm}$,

$$X_{n+1} = AX_n + B\varepsilon_{n+1}, \qquad n \in \mathbb{Z}. \tag{13}$$

The inverse correspondence is given by

$$Y_{n+1} = CX_n + D\varepsilon_{n+1}, \qquad n \in \mathbb{Z}, \tag{14}$$

where $C = (\Psi, F_p, \Gamma, G_q)$ and $D = G_0$. Equations (13) and (14) are a state space representation of the ARMA model. Thus, by Caines [(1988), (8), Appendix 2],

$$F(x)^{-1}G(x) = C(x^{-1}I - A)^{-1}B + D. \tag{15}$$

STEP 2. We will now prove that if (13) has a strictly stationary nonanticipative solution $\{X_n,\ n \in \mathbb{Z}\}$, then the zeros of det $F(x)$ lie outside the closed unit disk. Let H be the affine subspace of minimal dimension of $\mathbb{R}^{pd+qm}$ such that $\mathbb{P}(X_0 \in H) = 1$ and let K be an affine subspace of minimal dimension of H which is invariant under the model (13). Write $K = z + V$, where V is a vector subspace and z is in $\mathbb{R}^{pd+qm}$. Since, a.s., for any $n \in \mathbb{N}$,

$$A(z + V) + B\varepsilon_n \subset z + V,$$

we have

$$Az - z + B\varepsilon_n \in V.$$

By subtraction, $B(\varepsilon_1 - \varepsilon_0)$ is in V. The ε_n's are not carried by a hyperplane. Therefore, $\varepsilon_1 - \varepsilon_0$ is not contained in a proper linear subspace of $\mathbb{R}^m$, which yields that

$$\operatorname{Im}(B) \subset V.$$

We conclude that $Az - z$ is in V and that AV is contained in V. We will use this information to write the system in block form. Let k be the dimension of V, and let W be the linear span of $\{e_i,\ 1 \le i \le k\}$, where (e_i) is the canonical basis of $\mathbb{R}^{pd+qm}$. Consider an invertible matrix M such that $M(V) = W$. Separating the first k coordinates from the others, we write any x in $\mathbb{R}^{pd+qm}$ as

$$x = \begin{bmatrix} x^{(1)} \\ x^{(2)} \end{bmatrix},$$

where $x^{(1)}$ are the first k coordinates of x, and $x^{(2)}$ the remaining ones. In a similar manner, since AV and Im B are contained in V, and since $Az - z$ is in V, we may write

$$MAM^{-1} = \begin{bmatrix} a^{(11)} & a^{(12)} \\ 0 & a^{(22)} \end{bmatrix}, \qquad MB = \begin{bmatrix} b^{(1)} \\ 0 \end{bmatrix},$$

$$CM^{-1} = (c^{(1)}, c^{(2)}), \qquad M(Az - z) = \begin{bmatrix} d^{(1)} \\ 0 \end{bmatrix}.$$

From Proposition 2.6, we know that there exists a nonanticipative strictly stationary solution $\{U_n,\ n \in \mathbb{Z}\}$ of (13) carried by K. Let $Z_n = M(U_n - z)$. Since $U_n - z$ is carried by V, we have $Z_{n+1}^{(2)} = 0$ and

$$Z_{n+1}^{(1)} = a^{(11)} Z_n^{(1)} + b^{(1)} \varepsilon_n + d^{(1)}. \tag{16}$$

Note that (16) is a generalized autoregressive model with i.i.d. coefficients. It is irreducible, by minimality of K. Hence, it follows from Theorem 2.5 that the spectral radius of $a^{(11)}$ is strictly smaller than 1. However, we deduce from (15) that

$$\begin{aligned} F(x)^{-1} G(x) &= CM^{-1}(x^{-1}I - MAM^{-1})^{-1} MB + D \\ &= c^{(1)}(x^{-1}I - a^{(11)})^{-1} b^{(1)} + D. \end{aligned}$$

Thus the poles of the matrix fraction $F(x)^{-1}G(x)$ are the inverses of the eigenvalues of $a^{(11)}$. Therefore, they lie outside the closed unit disk. On the other hand, the poles of $F(x)^{-1}G(x)$ are the roots of det $F(x) = 0$ because the matrix fraction is irreducible [see Kailath (1980), Section 6.5.3]. This proves that the zeros of det $F(x)$ lie outside the closed unit disk.

STEP 3. We now show that if (11) has a nonanticipative strictly stationary solution then so does (13) [notice that if we had supposed that $\{(Y_n, \varepsilon_n),\ n \in \mathbb{Z}\}$ were stationary, then it would have been obvious that (12) defines such a solution]. The process $\{X_n,\ n \in \mathbb{Z}\}$ defined by (12) is a Markov chain because X_n is independent of $\{\varepsilon_n,\ n > p\}$. Let P be the kernel of this chain and let m_n be the distribution of X_n. The family $\{m_n,\ n \in \mathbb{Z}\}$ is tight because $\{\varepsilon_n,\ n \in \mathbb{Z}\}$ and $\{Y_n,\ n \in \mathbb{Z}\}$ are stationary processes. Let m be a limit point of the sequence of probability measures $\{(1/n)\Sigma_{i=1}^{n} m_i,\ n \in \mathbb{N}\}$. As, for any bounded continuous function f,

$$\int f\,dm_{n+1} = \int Pf\,dm_n,$$

we see that m is a P-invariant distribution. Whence, by Lemma 3.1, there exists a nonanticipative strictly stationary solution of (13). This ends the proof of the "only if" statement.

STEP 4. Let us show the "if" statement. It is easily seen that

$$\det(\lambda I - A) = \lambda^{qm+dp} \det F(\lambda^{-1}).$$

If all the zeros of $\det F(x)$ lie outside the closed unit disk, the spectral radius of A is strictly less than 1, and hence the top Lyapounov exponent of (13) is strictly negative. Since $\mathbb{E}(\log^+ \|B\varepsilon_0\|)$ is finite, it follows from Theorem 1.1 that there exists a unique nonanticipative strictly stationary solution $\{X_n, n \in \mathbb{Z}\}$ of (13). The process defined by (14) is then a nonanticipative strictly stationary solution of (11). □

We now consider a general state space model. It is given by (13) and (14), where $\{\varepsilon_n, n \in \mathbb{Z}\}$ is a sequence of i.i.d. $\mathbb{R}^m$-valued random variables and A, B, C, and D are real matrices of size $d \times d$, $d \times m$, $p \times d$ and $p \times m$, respectively. A solution of this model is a sequence $\{Y_n, n \in \mathbb{Z}\}$ in $\mathbb{R}^p$ for which there exists a sequence $\{X_n, n \in \mathbb{Z}\}$ in $\mathbb{R}^d$ such that (13) and (14) hold. It is nonanticipative if, for each $k \in \mathbb{Z}$, $\{Y_k, k \leq r\}$ is independent of $\{\varepsilon_n, n > r\}$. The state space model is said to be *controllable* if the matrix $(B, AB, \ldots, A^{d-1}B)$ has rank d, and it is said to be *observable* if the matrix $({}^tC, {}^t(CA), \ldots, {}^t(CA^{d-1}))$ has rank d.

PROPOSITION 4.2. *Suppose that there is no affine hyperplane H of $\mathbb{R}^m$ such that $\mathbb{P}(\varepsilon_0 \in H) = 1$, that $\mathbb{E}(\log^+ \|B\varepsilon_0\|) < \infty$ and that the state space model is controllable and observable. Then there exists a nonanticipative strictly stationary solution if and only if the spectral radius of the matrix A is strictly smaller than* 1.

PROOF. We first suppose that there is a stationary nonanticipative solution. It follows from (13) and (14) that, for $k = 0, 1, \ldots, d-1$,

$$Y_{n-k} = CA^{d-k-1}X_{n-d} + \sum_{i=1}^{d-k-1} CA^{i-1}B\varepsilon_{n-k-i} + D\varepsilon_{n-k}.$$

By observability, this implies that X_{n-d} is a linear function of $Z_n = (Y_{n-d+1}, \ldots, Y_n, \varepsilon_{n-d+1}, \ldots, \varepsilon_n)$. Now, using (13), we see that X_n itself is a linear function of Z_n. Thus, on the one hand X_n is independent of $\{\varepsilon_m, m > n\}$ and therefore is a Markov chain, and on the other hand the family $\{X_n, n \in \mathbb{Z}\}$ is tight. Therefore one can show, exactly as in Step 3 of the proof of Theorem 4.1, that there exists a nonanticipative strictly stationary solution of the autoregressive equation (13). One easily verifies that the controllability condition is equivalent to the irreducibility of this model. Therefore, Theorem 2.5 implies that the spectral radius of A is strictly smaller than 1. The converse statement is clear. □

REMARK. One might be tempted to prove Theorem 4.1 by applying Proposition 4.2 to a minimal state space representation associated with the ARMA process. However, it is not known, a priori, that any strictly stationary solution of the ARMA model is also a solution of this associated dynamical model. Actually, it is a consequence of the proof given above that this property is true [see Picard (1990)].

REFERENCES

BOUGEROL, P. (1987). Tightness of products of random matrices and stability of linear stochastic systems. *Ann. Probab.* **15** 40–74.

BOUGEROL, P. and LACROIX, J. (1985). *Products of Random Matrices with Applications to Schrödinger Operators. Progress in Probability and Statistics.* Birkhäuser, Boston.

BOUGEROL, P. and PICARD, N. (1992). Stationarity of GARCH processes and of some nonnegative time series. *J. Econometrics* **52** 115–127.

BRANDT, A. (1986). The stochastic equation $Y_{n+1} = A_n Y_n + B_n$ with stationary coefficients. *Adv. in Appl. Probab.* **18** 211–220.

BROCKWELL, P. J. and CLINE, D. B. H. (1985). Linear prediction of ARMA processes with infinite variance. *Stochastic Process. Appl.* **19** 281–296.

CAINES, P. E. (1988). *Linear Stochastic Systems.* Wiley, New York.

ELIE, L. (1982). Comportement asymptotique du noyau potentiel sur les groupes de Lie. *Ann. Sci. École Norm. Sup.* **15** 257–354.

ENGLE, R. F. and BOLLERSLEV, T. (1986). Modelling the persistence of conditional variances. *Econometric Rev.* **5** 1–50.

FEIGIN, P. D. and TWEEDIE, R. L. (1985). Random coefficient autoregressive processes: A Markov chain analysis of stationarity and finiteness of moments. *J. Time Series Anal.* **6** 1–14.

FURSTENBERG, H. and KESTEN, H. (1960). Products of random matrices. *Ann. Math. Statist.* **31** 457–469.

GRANGER, W. J. and ANDERSEN, A. P. (1978). *An Introduction to Bilinear Time Series Analysis.* Vanderhoeck and Ruprecht, Göttingen.

GRANGER, W. J. and ORR, D. (1972). Infinite variance and research strategy in time series analysis. *J. Amer. Statist. Assoc.* **64** 275–285.

GRINCEVICIUS, A. K. (1981). A random difference equation. *Lithuanian Math. J.* **21** 302–306.

GUIVARC'H, Y. and RAUGI, A. (1985). Frontière de Furstenberg, propriétés de contraction et théorèmes de convergence. *Z. Wahrsch. Verw. Gebiete* **69** 187–242.

HANNAN, E. and KANTER, M. (1977). Autoregressive processes with infinite variance. *J. Appl. Probab.* **14** 411–415.

HOFMANN, K. H. and MOSTERT, P. S. (1966). *Elements of Compact Semigroups.* Merrill, Columbus, Ohio.

KAILATH, T. (1980). *Linear Systems.* Prentice-Hall, Englewood Cliffs, N.J.

KINGMAN, J. F. C. (1973). Subadditive ergodic theory. *Ann. Probab.* **1** 883–909.

LIU, J. and BROCKWELL, P. J. (1988). On the general bilinear time series model. *J. Appl. Probab.* **25** 553–564.

MANDELBROT, B. (1963). The variation of certain speculative prices. *J. Business* **36** 394–419.

NELSON, D. B. (1990). Stationarity and persistence in Garch(1, 1) model. *Econometric Theory* **6** 318–334.

NICHOLLS, D. F. and QUINN, B. G. (1982). *Random Coefficient Autoregressive Models: An Introduction. Lecture Notes in Statist.* **11.** Springer, Berlin.

PHAM, D. T. (1986). The mixing property of bilinear and generalized random coefficient autoregressive models. *Stochastic Processes. Appl.* **23** 291–300.

PICARD, N. (1990). Thèse. Université Nancy 1, France.

POURAHMADI, M. (1988). Stationarity of the solution of $X_t = A_t X_{t-1} + \varepsilon_t$ and analysis of non-gaussian dependent random variables. *J. Time Series Anal.* **9** 225–239.

PRIESTLEY, M. B. (1988). *Non Linear and Non Stationary Time Series.* Academic, New York.

STUCK, B. W. and KLEINER, B. (1974). A statistical analysis of telephone noise. *Bell Syst. Tech. J.* **53** 1263–1320.

TJØSTHEIM, D. (1986). Some doubly stochastic time series models. *J. Time Series Anal.* **7** 51–72.

VERVAAT, W. (1979). On a stochastic difference equation and a representative of non-negative infinitely divisible random variables. *Adv. in Appl. Probab.* **11** 750–783.

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