Strictly chained (p, q)-ary partitions

Laurent Imbert

Joint work with Fabrice Philippe, Montpellier, France

CNRS, PIMS, University of Calgary

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Outline

- Quick introduction to integer partitions
- Strictly chained (p, q)-ary partitions
  - Encoding
  - Generating
  - Counting
- Applications
- Shortest (p, q)-ary partitions
- Open problems

Integer partitions

A *partition* of an integer n is a nonincreasing sequence of positive integers $a_1, a_2, \ldots, a_k$ whose sum is n. Each a_i is called a *part*.

For example, here are the 5 partitions of the integer 4:

$$
\begin{aligned}
4 &= 4 \\
&= 3 + 1 \\
&= 2 + 2 \\
&= 2 + 1 + 1 \\
&= 1 + 1 + 1 + 1
\end{aligned}
$$

The partitions of n correspond to the set of solutions $(k_1, k_2, \ldots, k_n)$ in nonnegative integers to the Diophantine equation

$$1k_1 + 2k_2 + 3k_3 + \cdots + nk_n = n$$

Ferrers diagrams

A nice and useful way to visualize partitions:

$$
\begin{array}{rrl}
15 = & 6 & \bullet\ \bullet\ \bullet\ \bullet\ \bullet\ \bullet \\
 & +\,3 & \bullet\ \bullet\ \bullet \\
 & +\,3 & \bullet\ \bullet\ \bullet \\
 & +\,2 & \bullet\ \bullet \\
 & +\,1 & \bullet
\end{array}
$$

$p(n,k)$: The number of partitions of n whose largest part is k is equal to the number of partitions of n with k parts.

$p(n)$: The number of (unrestricted) partitions of n, where the order is not significant ($p(n) = 0$ for all $n < 0$ and $p(0) = 1$).

$\mathcal{P}(4) = \{(4), (3,1), (2,2), (2,1,1), (1,1,1,1)\}, \quad p(4) = 5.$

Euler's partition function

Consider the product

$$(1 + x + x^2 + x^3 + \cdots)(1 + x^2 + x^4 + x^6 + \cdots)(1 + x^3 + x^6 + \cdots)\cdots \quad (1)$$

What is the coefficient of x^n in (1)?

Each contribution (of 1) to the coefficient of x^n is of the form

$$x^{1k_1} \cdot x^{2k_2} \cdot x^{3k_3} \cdots = x^{1k_1 + 2k_2 + 3k_3 + \cdots}$$

Thus, the coefficient of x^n is the number of ways of writing n as $1k_1 + 2k_2 + 3k_3 + \cdots + nk_n$, where $k_i \geq 0$. This is exactly $p(n)$.

$$\sum_{n=0}^{\infty} p(n)x^n = \frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \cdots = \mathcal{E}(x)$$

Example 1

Let $f(n)$ denote the number of partitions of n with no part 1.

$$\begin{aligned}\sum_{n=0}^{\infty} f(n)x^n &= x^0 \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \cdots \\ &= \frac{1-x}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \cdots \\ &= (1-x)\mathcal{E}(x)\end{aligned}$$

This generating function yields the following result:

Lemma: $f(n) = p(n) - p(n-1)$.

Bijective proof: if a partition of n contains at least one part equal to 1, then removing one of these yields a partition of $n-1$. □

Example 2

$q(n)$ is the number of partitions of n with disctinct parts.

$$\sum_{n=0}^{\infty} q(n)x^n = (1+x)(1+x^2)(1+x^3)\cdots$$
$$= \frac{1-x^2}{1-x} \cdot \frac{1-x^4}{1-x^2} \cdot \frac{1-x^6}{1-x^3} \cdot \frac{1-x^8}{1-x^4} \cdots$$
$$= \frac{1}{1-x} \cdot \frac{1}{1-x^3} \cdot \frac{1}{1-x^5} \cdots$$

Theorem: The number of partitions of n with distinct parts is equal to the number of partitions with odd parts.

Bijective proof: uses the fact that each part can be written as a power of 2 times an odd number.

More examples

- Partitions into primes (Goldbach conjecture)
- m-ary partitions: partitions as a sum of powers of m for a fixed $m \geq 2$. (e.g. binary partitions)
- Partitions with parts occurring at most thrice [A. Fink, R. Guy, M. Krusemeyer 2008]

$$(1+x+x^2+x^3)(1+x^2+x^4+x^6)(1+x^3+x^6+x^{12})\cdots$$

$=$ Partitions with no part a multiple of 4
$=$ Partitions with no even parts repeated

- Chain, umbrella partitions: partitions constrained by divisibility conditions

Chain partitions

A *(strictly) chain partition* is a partition of the form $n = a_1 + a_2 + \cdots + a_k$ into (distinct) positive integers such that $a_k | a_{k-1} | \ldots | a_2 | a_1$.

$$\begin{aligned} 873 &= 512 + 256 + 64 + 32 + 8 + 1 \\ &= 720 + 120 + 24 + 6 + 2 + 1 \\ &= 696 + 174 + 3 \end{aligned}$$

[Erdös-Loxton 1979]

- # partitions of this type: $p(n) \geq \log_2 n$ for $n \geq 6$
- # partitions of this type whose smallest part is 1: $p_1(n) \geq \frac{1}{2} \log_2 n$ for $n \geq 27$ and $n - 1$ not a prime
- $P(x) = \sum_{1 \leq n \leq x} p(n) \approx cx^{\rho}$, where c is an unknown constant and ρ is the unique root of $\zeta(s) - 2$, where ζ is the Riemann zeta function.

Strictly chained (p, q)-ary partition

Strictly chained (p, q)-ary partitions are chain partitions with distinct parts of the form $p^a q^b$, where $p, q \geq 2$ and $(p, q) = 1$.

Notations:

- $\Omega(U)$: The set of all strictly chained (p, q)-ary partitions of U
- $\Omega^*(U)$: The subset of partitions $\omega \in \Omega(U)$ with no part 1
- $W(U) = \#\Omega(U)$
- $W^*(U) = \#\Omega^*(U)$

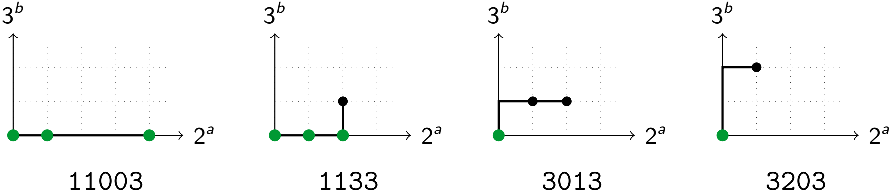
Special cases of interest:

- $\min(p, q) = 2$
- $(p, q) = (2, 3)$

Graphic representation and encoding

Example with $(p, q) = (2, 3)$.

$\Omega(19) = \{(16, 2, 1), (12, 4, 2, 1), (12, 6, 1), (18, 1)\}$



The couples of exponents (a, b) form a chain in $\mathbb{N}^2$. They can be encoded with words in $\{0, 1, 2, 3\}^*$. (Conventions: words end with '3', we go North before going East) If $\min(p, q) = 2$, the *binary amount* of a partition is equal to the sum of all its binary parts (● parts) or 0 if none.

Complete generation, maps

We define embeddings from subsets $\Omega \subset \Omega(U)$ to $\mathcal{P}$, the set of all unrestricted partitions (sequences of positive integers whose sum is finite)

Let $\omega = (a_1, a_2, \ldots, a_k) \in \Omega$.

- Mult. by p: $\omega \longmapsto (pa_1, pa_2, \ldots, pa_k) \in {}^p\Omega$
- Mult. by q: $\omega \longmapsto (qa_1, qa_2, \ldots, qa_k) \in {}^q\Omega$
- Add. 1: $\omega \longmapsto \begin{cases} (a_1, \ldots, a_k, 1) & \text{if } \min(p,q) > 2 \\ \text{binary amount} + 1 & \text{if } \min(p,q) = 2 \end{cases}$

  In both cases, the resulting set of partitions is denoted ${}^1\Omega$.

  Remark: the number of parts never increases by more than 1 and may be reduced due to carry propagation.

Complete generation, maps' properties

- ${}^{p}\Omega(U) \subset \Omega(pU)$
- ${}^{q}\Omega(U) \subset \Omega(qU)$
- ${}^{1}\Omega(U) \not\subset \Omega(U+1)$ in general

  If $\min(p, q) > 2$, the part 1 may appear twice in ${}^{1}\Omega$

  The strictly chained $(2, 3)$-ary partition $(6, 2, 1)$ is turned into $(6, 4) \notin \Omega(10)$

- If $\min(p, q) = 2$, the set $\Omega(U)$ contains at least the binary partition of U.
- By convention $\Omega(0) = \{()\}$

Some formulæ

Lemma: (+ denotes union of disjoint sets)

$$\Omega(U) = \Omega^*(U) + {}^1\Omega^*(U-1), \qquad \Omega^*(U) = {}^p\Omega(U/p) \cup {}^q\Omega(U/q)$$

Corollary:

$$\Omega(pqU) = {}^p\Omega(qU) + {}^q(\Omega(pU) \setminus {}^p\Omega(U)),$$
$$\Omega(pqU+1) = {}^{1p}\Omega(qU) + {}^{1q}(\Omega(pU) \setminus {}^p\Omega(U))$$

and for $1 < r < pq$

$$\Omega(pqU + r) = \Omega^*(pqU + r) + {}^1\Omega^*(pqU + r - 1) \tag{2}$$

Both sets Ω^* in the rhs of (2) are non empty if and only if:
$r = kp$ and $r - 1 = \ell q$, or $r = \ell q$ and $r - 1 = kp$.

Let $k_0 = p^{-1} \bmod q$ and $\ell_0 = q^{-1} \bmod p$. Then, $(k_0, p - \ell_0)$ is the unique positive solution to the equation $kp - \ell q = 1$. Therefore:

$$\text{if } r = k_0 p, \quad \Omega(pqU + r) = {}^p\Omega(qU + k_0) + {}^{1q}\Omega(pU + p - \ell_0)$$

Simpler relations

The complete formula:

$$\Omega(pqU+r) = \begin{cases} {}^{p}\Omega(qU+k_0) + {}^{1q}\Omega(pU+p-\ell_0) & \text{if } r = k_0 p \\ {}^{q}\Omega(pU+\ell_0) + {}^{1p}\Omega(qU+q-k_0) & \text{if } r = \ell_0 q \\ {}^{p}\Omega(qU+k) & \text{if } r = kp,\ k \neq k_0 \\ {}^{1p}\Omega(qU+k) & \text{if } r = kp+1,\ k \neq q-k_0 \\ {}^{q}\Omega(pU+\ell) & \text{if } r = \ell q,\ \ell \neq \ell_0 \\ {}^{1q}\Omega(pU+\ell) & \text{if } r = \ell q+1,\ \ell \neq p-\ell_0 \\ \emptyset & \text{otherwise.} \end{cases}$$

The case $(p,q) = (2,3)$ allows for some simplifications:

$$\begin{aligned} \Omega(3U) &= {}^{3}\Omega(U) + {}^{1}\Omega(3U-1) \\ \Omega(6U-1) &= {}^{12}\Omega(3U-1) \\ \Omega(6U+1) &= {}^{13}\Omega(2U) + {}^{11}\Omega(6U-1) \\ \Omega(6U+2) &= {}^{2}\Omega(3U+1) \\ \Omega(6U+4) &= {}^{13}\Omega(2U+1) + {}^{2}\Omega(3U+2) \end{aligned}$$

Examples

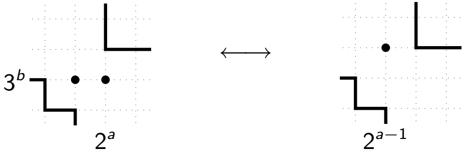
$$\Omega(217) = \{3000133,\ 30001003,\ 322033,\ 3220003,\\ 3200013,\ 10011013,\ 1001333,\ 10013003\}$$

$$\Omega(95) = \{1111103\}$$
$$\Omega(6143) = \{1111111111103\}$$

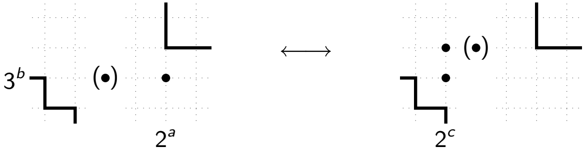
$$\Omega(575) = \{1111110003,\ 111111033\}$$
$$\Omega(959) = \{1111110113,\ 1111110303\}$$

Transitions

- $1 + 2 = 3$



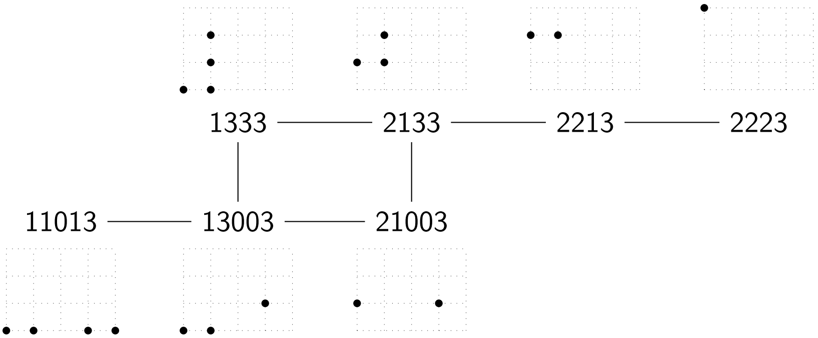
- $2(2^m - 1 + 2^{m+1}) = 3(2^{m+1} - 1) + 1$ (generalizes $4 = 3 + 1$)



Random walk

The transition graph is symmetric and connected.

E.g: $G(27)$ for $(p, q) = (2, 3)$



Computing $W(U)$

Let $W_p(U) \in \{0,1\}$ be the number of partitions of U with distinct parts taken in $\{p^n, n \in \mathbb{N}\}$. In other words, can U be written in base p with digits $\{0,1\}$ only?

$$W(U) = W_p(U) + W\left(\frac{U}{q}\right) + \sum_{c=0}^{\left\lfloor \log_p\left(\frac{U}{q+1}\right)\right\rfloor} \delta_{p,q}(c,U)\, W\left(\left\lfloor \frac{U}{p^c q}\right\rfloor\right),$$

$$\delta_{p,q}(c,U) = \begin{cases} 1 & \text{if } \lfloor U/p^c \rfloor \equiv 1 \pmod{q} \text{ and } W_p(U \bmod p^c) = 1 \\ 0 & \text{otherwise} \end{cases}$$

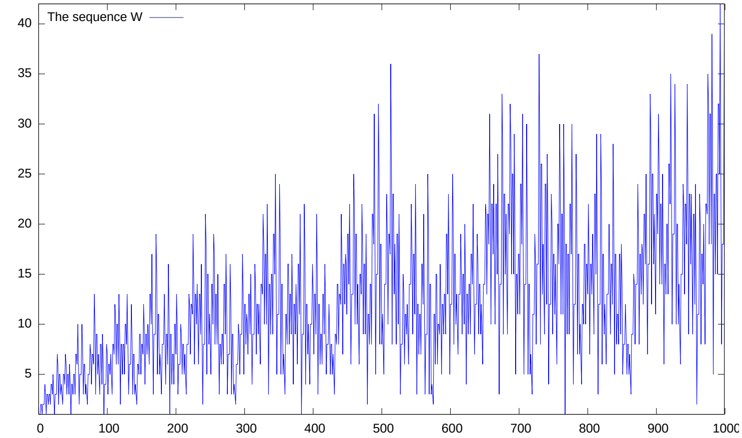
Sketch of proof: Order the partitions in $\Omega(U)$ w.r.t p-ary amount

$$W(U) = W(U/q) + \sum_{n=1}^{U} W_p(n)\, W\left(\frac{U-n}{p^{c_n} q}\right).$$

and remark that many summands vanish.

The sequence W

For any pair (p, q), the sequence W behaves rather irregularly.



Properties of W

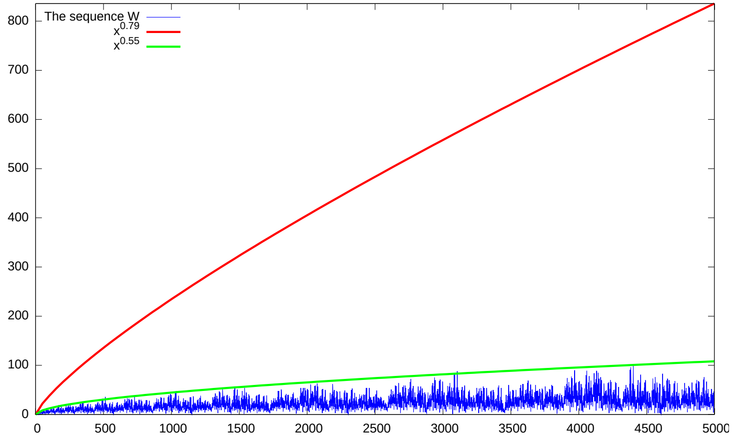
- W takes infinitely often the value 0
- If $\min(p, q) = 2$, then W takes infinitely often the value 1
- If $(p, q) = (2, 3)$, we have $W(U) = 1$ iff either $U \in \{0, 1\}$ of $U = 2^a 3 - 1$ for some $a \in \mathbb{N}$. Also, $W(U) = 2$ iff either $U \in \{3, 5, 6, 7\}$ or $U = 2^a 9 - 1$ or $U = 2^a 15 - 1$ for some $a \in \mathbb{N}$.

  Conjecture: all values in $\mathbb{N}$ are taken
- Theorem: The sequence W is either $\{0, 1\}$-valued or unbounded.

  Note: We are not aware of any pair (p, q) for which W is $\{0, 1\}$-valued.

Asymptotical behaviour of max W

Max value: Let $\beta \in (0,1)$ be the unique solution of $1/p^\beta + 1/q^\beta = 1$. Then $W(U) \leq U^\beta$ for $U \geq 1$. For $(p,q) = (2,3)$, we can (only) prove $W(U) \leq U^{0.79}$, whereas our numerical experiment suggest $U^{0.55}$.



Average value of W

Let $S(x) = \sum_{1 \leq U \leq \lfloor x \rfloor} W(U)$.

$$S(x) = \sum_{U=1}^{\lfloor x \rfloor} (W^*(U) + W^*(U-1))$$

$$= W^*(0) - W^*(\lfloor x \rfloor) + 2 \sum_{U=1}^{\lfloor x \rfloor} (W(U/p) + W(U/q) - W(U/pq))$$

Then, for all $x \in \mathbb{R}^+$ we have

$$S(x) = 2(S(x/p) + S(x/q) - S(x/pq)) + 1 - W^*(\lfloor x \rfloor)$$

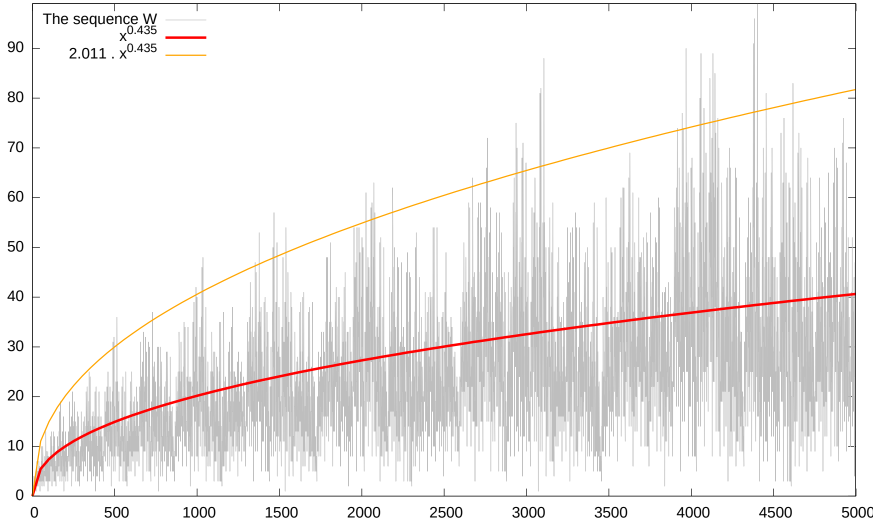
Therefore, if $S(x) \approx x^\alpha$, then α satisfies

$$1/p^\alpha + 1/q^\alpha - 1/(pq)^\alpha = 1/2$$

which also reads

$$(1 - p^{-\alpha})^{-1}(1 - q^{-\alpha})^{-1} = 2$$

Average value of W

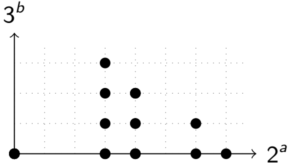


Applications

Fast exponentiation: given $g \in G$ and $e \geq 0$ compute g^e

$$g^{217} = (((g^2 \times g)^{2^2} \times g)^2 \times g)^{2^{2^2}} \times g$$

cost: 4 mults, 7 squares



$$g^{217} = g^{2^{2^{2^{3^{3^3}}}}} \times g$$

cost: 1 mult, 3 squares, 3 cubes

Requires: fast cubing (e.g. elliptic curves, quadratic fields), and a fast conversion algorithms into strictly chained $(2,3)$-ary partitions.

Conversion algorithms

- Right-to-left: divide by 3 and by 2 as much as possible; add or subtract 1 to make the resulting value divisible by 3

- Left-to-right: find the closest number of the form $2^a 3^b$ from e; subtract and continue until reaching 0

- None of these algorithm give a chain of minimal length.

- Can we find a shortest partition, or at least, compute its length?

Shortest partitions

Let $|w|$ the number of parts of a partition $w \in \Omega(U)$. We define $\sigma(U) = \min_{w \in \Omega(U)} |w|$, the length of a shortest partition in $\Omega(U)$.

$$\Omega(pqU) = {}^{p}\Omega(qU) + {}^{q}(\Omega(pU) \setminus {}^{p}\Omega(U)),$$
$$\Omega(pqU+1) = {}^{1p}\Omega(qU) + {}^{1q}(\Omega(pU) \setminus {}^{p}\Omega(U))$$

The mappings ${}^{p}\Omega$ and ${}^{q}\Omega$ do not change the number of parts.

$$\sigma(pqU) = \min(\sigma(qU), \sigma(pU))$$
$$\sigma(pqU+1) = 1 + \sigma(pqU)$$

Similarly, the relations in (2) can be adapted for numbers of the form $pqU + r$ for $1 < r < pq$.

Computing shortest partitions

For $(p, q) = (2, 3)$ the following Maple code can be used to compute the first 500000 values of σ in approximately 1 second.

```
s := proc(U)
option remember;
local r;
if U <= 2 then 1 else
r := irem(U,6);
if r=0 then min(s(U/3), s(U/2))
elif r=1 then 1 + s(U-1)
elif r=2 then s(U/2)
elif r=3 then min(s(U/3), 1+s((U-1)/2))
elif r=4 then min(s(U/2), 1+s((U-1)/3))
elif r=5 then 1 + s((U-1)/2)
fi: fi: end:
```

Remark: numercal experiments suggest $\sigma(U) \approx (\log_2 U)/4$ on average

Open questions

- When computing g^{-1} in G is easy, one may want to consider signed chained partitions, where the largest part in w is less than $f(U)$ for some function f (e.g $f(U) = U + 1$), while allowing the other parts to be either added or subtracted.

  Example: $314159 = \dots$
  Right-to-left: [1,9,6][-1,8,5][1,7,3][-1,5,2][-1,4,1][-1,0,0]
  Left-to-right: [1,4,9][-1,0,6][-1,0,3][-1,0,2][-1,0,1][-1,0,0]

  - Generating, random walk, etc?
  - How many are there?
  - Shortest signed partition?
  - Optimal choice of f?

- Many other questions related to numbers composed of small primes (density of various sequences)

Thanks!

http://www.lirmm.fr/~imbert