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Strictly Positive Real Systems Based on Reduced-Order Observers

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***Abstract*— We study the extension of the class of linear time-invariant open-loop systems that may be transformed into SPR systems introducing a reduced-order observer. It is shown that for open-loop stable systems a cascaded observer achieves the result. For open-loop unstable systems observer-based feedback is required to succeed. In general, any stabilizable and observable system may be transformed into an SPR system defining a new output based on the observer state. This overcomes the old conditions of minimum phase and relative degree one for the case of keeping the original output. The result is illustrated with some examples.**

Keywords: Strictly positive realness, SPR systems, observers, passivity, KYP lemma, Lyapunov functions

I. INTRODUCTION

The Kalman-Yakubovich-Popov (KYP) Lemma gives algebraic equations for a square transfer matrix $Z(s)$ to be Strictly Positive Real (SPR). These algebraic equations are equivalent to analytic conditions in the frequency domain which are not easy to test. The solution of these algebraic equations provides a practical way to verify that a given transfer function is SPR.

The original references [17], [31] and [25] express the algebraic equations in terms of a state space representation of $Z(s)$. The standard assumption on the state space representation is that it should be minimal, *i.e.*, it is controllable and observable. For a time, however, it was recognized that this condition may be weakened to stabilizability and observability. Indeed, Meyer explicitly pointed out this relaxation of the minimality assumption but did not provide a proof [21], [2]. Implicitly, Rantzer [26], in a novel proof based on convexity properties and linear algebra, did not require minimality of the state space representation of $Z(s)$. It was not until recently [8] that the minimality relaxation was explicitly proved in an algebraic fashion so that a state space representation can have noncontrollable modes and still satisfy the SPR conditions provided that the uncontrollable modes are stable. Other interesting properties of SPR systems and comparisons are presented in [29], [18].

This paper presents a technique to render SPR any stabilizable and observable linear time invariant system based on a reduced-order state observer and a feedback control law using the state estimate.

Molander and Willems [22] solved robust state-feedback problem using under the assumption that the original system has *relative degree one* and it is *minimum phase*. In the context of nonlinear systems, Byrnes *et al*. [7] presented a solution to the problem using smooth state feedback provided that the system has *relative degree one* and is *(weakly) minimum phase*. Furthermore, Kokotovic *et al*. [19], [20] addressed the problem of the stabilization of a linear system in cascade with a globally asymptotically stable nonlinear system. The proposed solutions also require the system to be *relative degree one* and *weakly minimum phase*. Another interesting solution has been presented by Sun *et al.* [28] based on output feedback. They established conditions to render the system Extended SPR (ESPR), requiring relative degree zero which means $D+D^T>0$. A closely related result was provided by Haddad and Bernstein [13] who arrived at a different pair of Riccati equations based on an auxiliary optimization problem.

Passification using constant output feedback was solved in [15], [4] with extensions to non-square systems [12]. Some approaches have been proposed to overcome the requirement of having a relative-degree-one open-loop system. Barkana introduced a 'parallel feedforward' configuration in the context of adaptive control [3]. Another related idea is passification by means of 'shunting' introduced by Fradkov [11]. Both approaches used dynamic extensions to obtain SPR loop transfer functions. The absolute stability problem with design of a nonminimal realization of the linear part was solved using a circle criterion design to obtain robustness properties [16].

This paper addresses the problem of design of a reduced-order observer and controller so that the modified system becomes SPR. Whereas the proposed method is described for the stable case, it is not detailed for the unstable case though it follows the same outline as for the full-order observer [16], [10]. In the case of stable open-loop systems the method reduces to the introduction of a reduced-order observer and the definition of a new output as a function of the estimated state only. In addition, for the unstable case, we have to introduce observer-state feedback to stabilize the system [16], [10]. The proposed approach *does not require* the original system to be neither *minimum phase* nor to have relative degree one. The SPR property is obtained with respect to the new output.

The paper is organized as follows: Section 2 presents some preliminaries and Section 3 deals with the open loop stable case. The open loop unstable case is addressed in Section 4. Some illustrative examples are given in Section 5 with concluding remarks in Section 6.

II. Preliminaries

Let us consider a linear time-invariant m-inputs p-outputs transfer matrix $Z(s)$ with a minimal realization given by

$$\begin{array}{rcl} \dot{x} & = & Ax+Bu, \qquad x\in\mathbb{R}^n \\ y & = & Cx, \end{array} \tag{1}$$

where $x\in\mathbb{R}^n$, $u\in\mathbb{R}^m$, $y\in\mathbb{R}^p$, $m\leq n$, $p\leq n$, and A,B,C are matrices of appropriate dimensions. Denote by $\mathbb{C},\mathbb{C}_-$ and $\mathbb{C}_-^\circ$, the complex plane, the closed left half complex plane and the open left half complex plane respectively. Denote by $\sigma(T)$ the set of eigenvalues of the square matrix T and let $\mathbb{R}_+$ represent the set of the positive real numbers.

Definition 1 ([1], [23]): The transfer matrix $Z(s)$ is said to be *positive real* (*PR*) if: i) All the elements of $Z(s)$ are analytical in $Re[s]>0$; and ii) $Z(s)+Z^T(-s)\geq 0$ for all $Re[s]>0$; $Z(s)$ is said to be *strictly positive real* (*SPR*) if $Z(s-\varepsilon)$ is *PR* for some $\varepsilon>0$.

For the scalar case, $m=1$, the classical interpretation of $Z(s)$ being *PR* (*SPR*) is that its Nyquist plot lies entirely in the right half complex plane (open right half complex plane). In the sequel, we will need the following version of the Kalman-Yakubovich-Popov (KYP) Lemma for strictly proper systems:

Lemma 2 (Kalman-Yakubovich-Popov [17], [31], [25]): Let $Z(s)=C(sI-A)^{-1}B$ be a $m\times m$ transfer matrix such that $Z(s)+Z^T(-s)$ has normal rank m, where A is Hurwitz, (A,B) is stabilizable, and (C,A) is observable. Then, $Z(s)$ is strictly positive real (SPR) if and only if there exist a positive definite symmetric matrices P, and Q, such that

$$\begin{array}{rcl} PA+A^TP & = & -Q \\ PB & = & C^T \end{array} \tag{2}$$

III. SPR Systems from Reduced-Order Observers

Previously, the stability and robustness results of Molander and Willems [22] and Kokotović and Sussman [19] were extended to a case with observer-based feedback control with resulting nonminimal loop-transfer functions [16]. A design procedure to full-state observers and Lyapunov functions was provided [16]. Modifications of the Kalman-Yakubovich-Popov Lemma for stabilizable systems using full-order observers were given in [8], [10]. Now, we turn attention to reduced-order observers as instruments in SPR design:

A. Main Result

Let us consider a linear time-invariant system described in standard state-space equations as

$$\Sigma_0: \quad \left\{ \begin{array}{rcl} \dot{x}_0 & = & A_0x_0+B_0u, \ x\in\mathbb{R}^n, u\in\mathbb{R}^m \\ y & = & C_0x_0, \ y\in\mathbb{R}^p \end{array} \right. \tag{3}$$

In order to avoid degenerate cases and to guarantee some matrix inverses required in the sequel—*e.g.*, Eq. (17)—we will assume that:

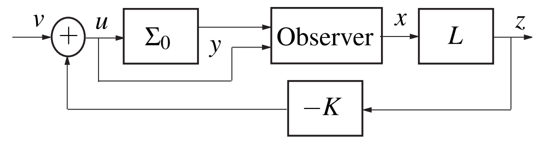

Fig. 1. Series and feedback compensators to transform open-loop unstable systems into SPR systems. The feedback matrix $K=0$ in the stable case.

Assumption I: The A matrix is stable—*i.e.*, $\sigma(A_0)\subset\mathbb{C}_-^\circ$, the spectrum of the matrix A_0 lies in the open left half complex plane [27] [18].

Assumption II: Assume that

$$\mathrm{rank}(C_0)=p$$

which means that the outputs are linearly independent.

A reduced-order observer (or a dynamic extension) for the system Σ_0 is given by

$$\Sigma_x: \quad \left\{ \begin{array}{rcl} \dot{x}_x & = & A_xx_x+B_uu+B_yy, \quad x\in\mathbb{R}^r \\ z & = & C_xx_x+C_yy, \qquad \sigma(A_x)\subset\mathbb{C}_-^\circ \end{array} \right. \tag{4}$$

where the output z is a linear combination of observer states x_x and measured output y, the system matrices A_x, B_u, B_y, C_x, C_y, to be determined.

The system (3) and the reduced-order observer (4) may be written compactly as

$$\Sigma: \quad \left\{ \begin{bmatrix} \dot{x}_0 \\ \dot{x}_x \end{bmatrix} = \begin{bmatrix} A_0 & 0 \\ B_yC_0 & A_x \end{bmatrix} \begin{bmatrix} x_0 \\ x_x \end{bmatrix} + \begin{bmatrix} B_0 \\ B_u \end{bmatrix} u \right. \tag{5}$$

or

$$\Sigma: \quad \left\{ \begin{array}{rcl} \begin{bmatrix} \dot{x}_0 \\ \dot{x}_x \end{bmatrix} & = & A\begin{bmatrix} x_0 \\ x_x \end{bmatrix}+Bu, \quad x=\begin{bmatrix} x_0 \\ x_x \end{bmatrix}\in\mathbb{R}^{n+r} \\ z & = & C\begin{bmatrix} x_0 \\ x_x \end{bmatrix} \end{array} \right. \tag{6}$$

where

$$A\triangleq\begin{bmatrix} A_0 & 0 \\ B_yC_0 & A_x \end{bmatrix} \tag{7a}$$

and

$$B\triangleq\begin{bmatrix} B_0 \\ B_u \end{bmatrix}, \ C\triangleq\begin{bmatrix} C_yC_0 & C_x \end{bmatrix} \tag{8}$$

Among the output available from the observer-extended system, one would choose a linear combination to be used for purposes of observer-feedback control. To this purpose, consider $L=C$ with

$$\Sigma_L: \quad \left[\begin{array}{c|c} A & B \\ \hline L & 0 \end{array}\right] = \left[\begin{array}{cc|c} A_0 & 0 & B_0 \\ B_yC_0 & A_x & B_u \\ \hline C_yC_0 & C_x & 0 \end{array}\right] \tag{9}$$

Since A_0 and A_x are stable by assumption, we investigate the conditions for matrix solution $P=P^T>0$ of the Kalman-Yakubovich-Popov equations

$$PA+A^TP=-Q, \ PB=L^T \tag{10}$$

Introduce the matrix decomposition

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}, \quad Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} > 0 \qquad (11)$$

The Kalman-Yakubovich-Popov equations (10) give

$$P_{11}A_0 + A_0^T P_{11} + P_{12}B_y C_0 + C_0^T B_y^T P_{12}^T = -Q_{11} \qquad (12)$$

$$P_{12}A_x + A_0^T P_{12} + C_0^T B_y^T P_{22} = -Q_{12} \qquad (13)$$

$$P_{22}A_x + A_x^T P_{22} = -Q_{22} \qquad (14)$$

$$P_{11}B_0 + P_{12}B_u = C_0^T C_y^T \qquad (15)$$

$$P_{12}^T B_0 + P_{22}B_u = C_x^T \qquad (16)$$

For a given matrix $Q > 0$ of Eq. (11), the matrix $Q_{22} > 0$ and a stable A_x of Eq. (4), the Lyapunov equation (14) has a solution $P_{22} > 0$. Now, choose

$$\begin{bmatrix} B_y \\ R^{-1} \end{bmatrix} = R_0[I_p - C_0(C_0^T C_0)^{-1} C_0^T], \qquad R \geq 0 \quad (17)$$

where R_0 an arbitrary matrix of appropriate dimensions and where I_p is the $p \times p$ identity matrix.

From [6], [30], it is known that the Sylvester equation $AX + XB = C$ has a unique solution if and only if A and $-B$ have no common eigenvalues. Thus, from the Sylvester equation (13), a unique solution $P_{12} \in \mathbb{R}^{n \times r}$ can be found if A_0 and A_x have no common eigenvalues.

Next, solve the Riccati equation

$$0 = A_0 X + X A_0^T + Q_{11} - X C_0^T R^{-1} C_0 X = 0, \quad X > 0 \quad (18)$$

$$\begin{aligned} 0 &= (A_0 - X C_0^T R^{-1} C_0)X + X(A_0 - X C_0^T R^{-1} C_0)^T \\ &+ Q_{11} + X C_0^T R^{-1} C_0 X \end{aligned} \qquad (19)$$

for $X > 0$. From Eqs. (17) and (12) follow that

$$\begin{bmatrix} P_{12} & C_0^T \end{bmatrix} \begin{bmatrix} B_y \\ R^{-1} \end{bmatrix} C_0 = 0 \Rightarrow P_{12}B_y C_0 = -C_0^T R^{-1} C_0$$

which means that $P_{12}B_y C_0 = -C_0^T R^{-1} C_0$ is a negative semidefinite form. Hence

$$P_{11}A_0 + A_0^T P_{11} - 2C_0^T R^{-1} C_0 = -Q_{11} \qquad (20)$$

or

$$P_{11}(A_0 - P_{11}^{-1} P_{12} B_y C_0) + (A_0 - P_{11}^{-1} P_{12} B_y C_0)^T P_{11} = -Q_{11}$$

from which is seen that

$$P_{11} = X^{-1} \text{ provided that } Q_{11} > 2C_0^T R^{-1} C_0 \qquad (21)$$

Summarizing, we have $P_{11} > 0$, P_{12}, $P_{22} > 0$, B_y

$$\begin{bmatrix} B_u \\ C_y^T \end{bmatrix} = -\begin{bmatrix} P_{12} & -C_0^T \end{bmatrix}^{-1} P_{11} B_0 \qquad (22)$$

$$C_x = B_0^T P_{12} + B_u^T P_{22} \qquad (23)$$

In order to assure that the matrix solution P is positive definite, it might be necessary to modify the matrix Q by choosing Q_{11} sufficiently large. Also, conditions for solution of Eq. (22) require that the state-space dimension $r = n - p$. Thus, the Kalman-Yakubovich-Popov equations

$$PA + A^T P = -Q, \qquad PB = L^T \qquad (24)$$

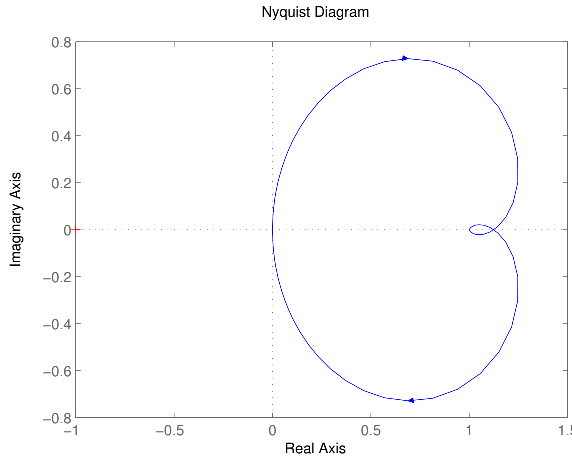

Fig. 2. Nyquist diagram of observer-supported SPR loop transfer function of Eq. (35).

have a solution and guarantee that the system of Eq. (9) is SPR

$$Z(s) = L(sI - A)^{-1} B \qquad \text{SPR} \qquad (25)$$

For the case of unstable systems, the system requires stabilizing feedback $u = -Lx$ suggested by Eq. (24) so that

$$Z(s) = L(sI - A + BL)^{-1} B \qquad \text{SPR} \qquad (26)$$

is SPR—or feedback-positive real (FPR) as suggested i [22], [19], [16].

B. Example—Output Feedback of Double-Integrator Dynamics

Consider a double integrator

$$\Sigma_0 : \left[\begin{array}{c|c} A_0 & B_0 \\ \hline C_0 & D_0 \end{array}\right] = \left[\begin{array}{cc|c} 0 & 0 & b \\ 1 & 0 & 0 \\ \hline 0 & 1 & 0 \end{array}\right]$$

where b is a constant. Whereas this system cannot be stabilized by static output feedback, it can be stabilized with the dynamic output feedback

$$\dot{x}_3 = r_0 u + s_0 y \qquad (27)$$

$$z = x_3 + s_1 y, \quad u = -z \qquad (28)$$

For example, $r_0 = 3$, $s_0 = 1/b$ and $s_1 = 3/b$ will accomplish pole assignment to $s = -1$ for all three closed-loop eigenvalues of

$$A_c = A - BL = \begin{bmatrix} 0 & -3 & -1 \\ 1 & 0 & 0 \\ 0 & -8 & -3 \end{bmatrix}, \quad \sigma(A_c) = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} \qquad (29)$$

The observer-extended system—*i.e.*, the open-loop system combined with the reduced-order observer in Eq. (27)—may be summarized as a state-space system where C, D define the output available from the observer-supported system. In this case, we have

$$\Sigma: \quad \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{ccc|c} 0 & 0 & 0 & b \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \tag{30}$$

For the feedback variable $z = x_3 + s_1 y = Lx$ to be used for output feedback, we have

$$\Sigma_L: \quad \left[\begin{array}{c|c} A & B \\ \hline L & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 0 & 0 & 0 & b \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ \hline 0 & 3 & 1/b & 0 \end{array}\right] \tag{31}$$

or specialized for $b = 1$

$$\Sigma_L: \quad \left[\begin{array}{c|c} A & B \\ \hline L & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ \hline 0 & 3 & 1 & 0 \end{array}\right] \tag{32}$$

The Kalman-Yakubovich-Popov equations (10) are satisfied by the matrix

$$P = \begin{bmatrix} 15.5000 & -0.500 & -5.167 \\ -0.5000 & 8.6666 & 1.1667 \\ -5.167 & 1.1667 & 2.0556 \end{bmatrix}, \ \sigma(P) = \begin{bmatrix} 0.194 \\ 8.689 \\ 17.34 \end{bmatrix} \tag{33}$$

with

$$\begin{aligned} PA_c + A_c^T P &= P(A-BL) + (A-BL)^T P \\ &= \begin{bmatrix} -1.0000 & 3.4999 & 1.1666 \\ 3.4999 & -15.6666 & -3.9444 \\ 1.1666 & -3.9444 & -2.0000 \end{bmatrix} \\ PB &= \begin{bmatrix} 0.0000 \\ 3.0000 \\ 1.0000 \end{bmatrix} = L^T \end{aligned}$$

for which

$$\sigma(PA_c + A_c P) = \begin{bmatrix} -17.5432 \\ -1.0000 \\ -0.1234 \end{bmatrix}$$

The feedback-transformed system

$$\Sigma_c: \quad \left[\begin{array}{c|c} A-BL & B \\ \hline L & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 0 & -3 & -1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -8 & -3 & 3 \\ \hline 0 & 3 & 1 & 0 \end{array}\right] \tag{34}$$

provides the SPR loop transfer function (Fig. 2)

$$Z(s) = L(sI - A + BL)^{-1}B = \frac{3s^2 + 3s + 1}{s^3 + 3s^2 + 3s + 1} \tag{35}$$

thus demonstrating that a reduced-order observer or dynamic extension is suitable to accomplish a feedback-transformed SPR system.

Remark 1: As described in [10], there exists a stabilizing feedback based on the observed state for the unstable cases.

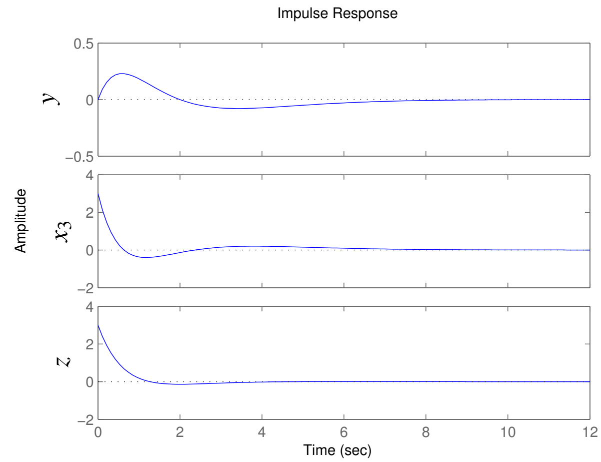

Fig. 3. Observer-supported SPR loop transfer function of Example 1 with impulse responses from u to y, x_3, and z, respectively.

IV. Discussion

The main contribution of this paper is a method to construct strictly positive real systems using reduced-order observers. The results are related to our previous results and methods based on full-order observers [16], [10]. As discussed in previous papers, there are two different approaches for stable and unstable systems relating the Kalman-Yacubovich-Popov equations to the Riccati equation

$$\begin{aligned} P(A-BL) + (A-BL)^T P + PBB^T P + Q &= 0 \qquad (36) \\ L &= B^T P \qquad (37) \\ L(sI - A + BL)^{-1}B \text{ is SPR} & \qquad (38) \end{aligned}$$

and the related problem of using P as a weighting matrix in a Lyapunov function for stability analysis. Whereas a solution always can be found under the conditions stated, the submatrices of the matrix Q are not entirely independent, where the matrix $Q_{11} > 0$ should be chosen sufficiently large as compared to $Q_{22} > 0$ for existence of a solution of the KYP equation (24). The Riccati equation of Eq. (36) makes this condition precise. Equation (36) also serves to link reduced-order observer design for the stable and unstable cases, respectively.

V. Conclusions

The main contribution of this paper is a method to construct strictly positive real systems using reduced-order observers. This paper has presented modification of a linear time-invariant system a reduced-order observer so that the modified system is SPR. The proposed method applies to stable open-loop systems as well as unstable systems and does not require the system to be minimum phase nor to have relative degree one. The original system may have a non-square transfer matrix, *i.e.*, the number of inputs can be different from the number of outputs. We have proved that the Kalman-Yakubovich-Popov Lemma holds for a series-connected system with a reduced-order observer for

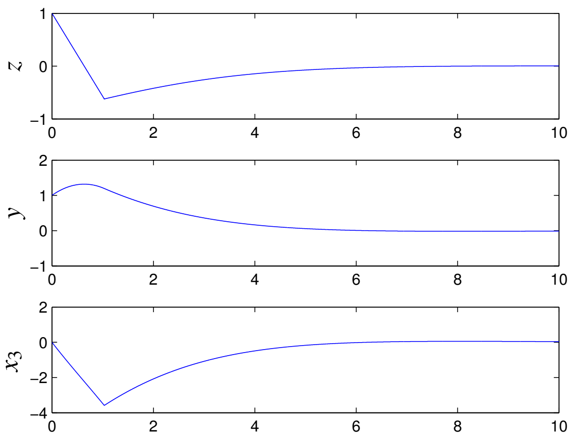

Fig. 4. Observer-supported SPR switching output feedback control based on loop transfer function of Example 1, Eq. (35), with switching SPR output feedback control responses vs. time in z, y, x_3, respectively.

stable open-loop systems. In the unstable case, observer state feedback is required in order to stabilize the system. Some examples with failing relative degree one have been given to illustrate the procedure. Future work in this area includes study of the robustness of the proposed method with respect to parametric uncertainties. As compared to previous results [10], [16], the results now extend the previous results for full-order observer to reduced-order observers.

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