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String Stability Analysis of a Vehicle Platoon with Communication Range 2 Using the Two-Dimensional Induced Operator Norm

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Abstract— **Vehicle strings can be modelled as continuous-discrete 2D systems. However, every such 2D system includes a structural nonessential singularity on the stability boundary and special care is needed to analyse BIBO stability of the system. A unidirectional vehicle string with communication range 2 is analysed and simulated.**

I. Introduction

The platooning problem investigates a line of vehicles (e.g. platoon) travelling, where each vehicle aims to maintain a prescribed separation to surrounding vehicles and the first vehicle follows a given trajectory.

In its simplest form, platoon control requires a constant distance between the vehicles and their direct predecessor e.g. [1]–[3]. To simplify communication requirements we consider the case where the automobiles can measure or have knowledge of the distance towards a limited number of preceding vehicles but do not consider the position error towards the following vehicles, i.e. *unidirectional control*. The number of vehicles ahead whose position is known is denoted as the *communication range*. We call the string *homogeneous* if the dynamics of the vehicle and controller are independent of the location in the string, with the possible exception of the initial vehicle or vehicles.

In most cases it is straight forward to design a suitable local controller to achieve a stable string in the conventional sense, i.e. the local error signals for every vehicle in the string are bounded and go to zero. However, in some system settings the peak in the error signals grows when traveling through the string. That means that the local error norm grows with the position in the string. This effect is referred to as string instability', [4], or 'slinky effect', [5], [6]

It has been shown that it is not possible to achieve string stability in a homogeneous string of strictly proper feedback control systems with nearest neighbour communications when using only linear systems with two integrators in the open loop and constant inter-vehicle spacing, [3], [7], independent of the particular linear controller design, [4]. However, string stability can be guaranteed using, for example, an appropriate speed dependent inter-vehicle spacing policy (also called 'time headway policy') introduced in [5], in contrast to a fixed separation.

Different methods have been employed to analyse string stability. Most commonly the Laplace transform with respect to time t is applied, [3], [4], [6]–[8]. In this context we define $\Gamma(s)$ as the transfer function describing how the error of the kth vehicle, $E_k(s)$, depends on $E_{k-1}(s)$. To guarantee string stability we require $|\Gamma(j\omega)| \leq 1$ for all ω. Other researchers applied the Z transform with respect to k, [2], while graph theory has been employed in [9] and a bidirectional string was approximated by a PDE in [10].

An alternative analysis tool for vehicle strings is to treat the platoon as a continuous-discrete two-dimensional (2D) system with two independent variables: the continuous time t and the discrete position within the string k.

Due to a wider field of applications, the related field of discrete 2D systems has been studied in more detail than that of continuous-discrete systems. One of the first sufficient conditions for bounded-input bounded-output (BIBO) stability in the frequency domain (using the 2D Z transform and requiring the poles of the system transfer function to lie inside the open stability bi-region) was presented in [11]. This lead to different stability tests such as those in [12], [13]. Similar results for BIBO stability of continuous systems (using the 2D Laplace transform) can be found in [14]–[17].

Most research in 2D systems, however, explicitly or implicitly exclude the case of a nonessential singularity of the second kind (NSSK) on the stability boundary. This NSSK is where there exists a set of (z_1,z_2) (in the discrete case), (s_1,s_2) (in the continuous case) or (s,z) (in the continuous-discrete case) such that both the denominator and the numerator of the transfer function go to zero at the same time. It was shown that while some transfer functions with an NSSK on the stability boundary are BIBO stable, others with an NSSK at the same position in the bi-plane are BIBO unstable in [18]. This was followed by the sufficient stability condition for discrete 2D systems in [19] that a system with finitely many NSSK on the stability boundary is BIBO stable if the transfer function can be continuously extended to the closed bi-disc. The author of [19] conjectures that this condition is also necessary. In [20] it was shown that BIBO stability cannot be achieved if the NSSK lies outside the boundary of the stability region.

In our context, that is studying vehicle platoons, NSSK on the stability boundary is generically the case: It will be shown in Section II that a 2D system description of a functioning vehicle platoon always exhibits an NSSK at $s = 0$ and $z = 1$. This is not due to string instability or poor design of the system but a necessary structural requirement of the vehicle string. (Note that another application where a NSSK on the stability boundary is desirable and necessary is discussed in [21].)

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This paper aims to show that 2D system BIBO stability analysis is a suitable tool to investigate string stability of vehicle platoons but does not deal with practical applications in the field of traffic control. The 2D L_2 signal norm and the corresponding induced operator norm will be discussed. It will be revealed that L_2 BIBO stability (i.e. boundedness of the induced operator norm) can be guaranteed even in some cases where the transfer function is discontinuous around the NSSK on the stability boundary. These results will be used in Section III to discuss string stability of a vehicle platoon with communication range 2.

II. BIBO Stability in the Presence of NSSK

A. Necessary Singularity on the Stability Boundary

Before discussing the BIBO stability and the induced operator norm of a continuous-discrete 2D system we will show that the 2D description of a class of vehicle platoon problems always includes at least one singularity on the stability boundary.

Consider the state space description of a continuous-discrete 2D system similar to the model proposed in [22] (where q denotes the shift operator with respect to k):

$$\dot{x}_1(t,k) := \frac{\mathrm{d}}{\mathrm{d}t} x_1(t,k) = A_{11}x_1(t,k) + A_{12}x_2(t,k), \quad (1)$$

$$qx_2(t,k) := x_2(t,k+1) = A_{21}x_1(t,k) + A_{22}x_2(t,k), \quad (2)$$

$$y(t,k) = c_1x_1(t,k) + c_2x_2(t,k), \quad (3)$$

with initial conditions $x_{10}(k) = x_1(0,k)$ and $x_{20}(t) = x_2(t,0)$, $x_i \in \mathbb{R}^{n_i}$, $y \in \mathbb{R}$ and the constant matrices A_{11}, A_{12}, A_{21}, A_{22}, c_1 and c_2 have appropriate dimensions. Assume that the 2D system (1)-(3) describes a vehicle platoon where $x_1(t,k)$ contains the local states of the kth vehicle (such as its position, velocity and controller states) and $x_2(t,k)$ contains the information of a group of preceding vehicles (such as the position of the two direct predecessors in case of a string with communication range 2). Assume that the initial conditions $x_{10}(k) = 0$ for all k. Since the vehicles need to be able to follow their predecessor and the first vehicle in the platoon a given trajectory, we require that for a vector of step responses as initial conditions $x_{20}(t) = f_2$ the output signal $y(t,k)$ tends to 1 for all k. Thus, f_2 has to be suitably chosen such that there exists a f_1 such that

$$0 = A_{11}f_1 + A_{12}f_2, \quad (4)$$

$$f_2 = A_{21}f_1 + A_{22}f_2, \quad (5)$$

$$1 = c_1f_1 + c_2f_2. \quad (6)$$

Note that if there is no solution, f_1, f_2, to (4)-(6), then it is impossible to have perfect tracking of a constant (step) reference. (This would, however, imply that the local errors due to a ramp reference signal would grow unbounded.) Under the given boundary conditions, we impose the tracking condition: $\forall k \lim_{t\to\infty} y(t,k) = 1$. Applying the Laplace transform with respect to t and the Z transform with respect to k and using the final value theorem this yields

$$\lim_{s\to 0} sY(s,z) = \frac{1}{1-z^{-1}}. \quad (7)$$

Applying the Laplace transform with respect to t and the Z transform with respect to k to (1)-(2) yields

$$X_1(s,z) = (sI - A_{11})^{-1} A_{12}X_2(s,z), \quad (8)$$

$$X_2(s,z) = (zI - A_{22})^{-1} \left(\frac{z}{s} f_2 + A_{21}X_1(s,z)\right). \quad (9)$$

Thus

$$X_2(s,z) = (zI - A_{22})^{-1} \left(\frac{z}{s} f_2 + A_{21}(sI - A_{11})^{-1} A_{12}X_2(s,z)\right)$$

$$= \left(I - z^{-1} \underbrace{\left(A_{22} + A_{21}(sI - A_{11})^{-1} A_{12}\right)}_{=:\Gamma(s)}\right)^{-1} \frac{1}{s} f_2 \quad (10)$$

with the transfer function $\Gamma(s) \in \mathbb{C}^{n_2\times n_2}$ describing how $X_2(s,k+1)$ depends on $X_2(s,k)$. Combining (8), (10) and the Laplace-Z transform of (3), $Y(s,z)$ gives

$$\begin{aligned} Y(s,z) &= c_1X_1(s,z) + c_2X_2(s,z) \\ &= \left(c_1(sI - A_{11})^{-1} A_{12} + c_2\right) X_2(s,z) \\ &= \left(c_1(sI - A_{11})^{-1}A_{12} + c_2\right)\left(I - z^{-1}\Gamma(s)\right)^{-1} \frac{f_2}{s}. \end{aligned} \quad (11)$$

Thus, with (11), (7) and assuming A_{11} is invertible, we have

$$\begin{aligned} \lim_{s\to 0} sY(s,z) &= \lim_{s\to 0}\left(c_1(sI - A_{11})^{-1}A_{12} + c_2\right)\left(I - z^{-1}\Gamma(s)\right)^{-1} f_2 \\ &= \left(-c_1A_{11}^{-1}A_{12} + c_2\right)\left(I - z^{-1}\Gamma(0)\right)^{-1} f_2. \end{aligned} \quad (12)$$

Since it is required in (7) that the right hand side of (12) is equal to $1/(1-z^{-1})$, the system description has to be chosen such that $\det(I - z^{-1}\Gamma(0))$ contains at least the factor $1 - z^{-1}$. This, however, also produces a singularity at $s = 0$ and $z = 1$.

Thus, every vehicle string set up such that the vehicles can follow a given trajectory (such as a step or a ramp signal) with the local error tending to 0 as $t \to \infty$, *must* include at least one singularity on the stability boundary.

B. BIBO Stability of Continuous-Discrete Two-Dimensional Systems with NSSK on the Stability Boundary

Consider the continuous-discrete 2D signal $x(t,k)$. If $x(t,k)$ does not grow faster than exponentially in both directions, its Laplace-Z transform $X(s,z)$ (a combination of the Laplace transform with respect to t and the Z transform with respect with k) exists. A formal definition and some properties of the Laplace-Z transform can be found in [23] as well as this 2D version of Parseval's Theorem:

Lemma 1 (Parseval's Theorem for 2D Cont.-Disc. Sys.): If the Laplace-Z transform $X(s,z)$ of $x(t,k)$ exists and there exist $a < 0$ and $0 < b < 1$ such that $|x(t,k)| \le ce^{at}b^k$, the L_2-norm in the time domain is bounded and equal to the L_2-norm in the frequency domain

$$\sum_{k=0}^{\infty} \int_0^{\infty} x^2(t,k)\mathrm{d}t = \|x(\cdot,\cdot)\|_2^2 = \|X(\cdot,\cdot)\|_2^2$$

$$= \frac{1}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} X^2\left(j\omega, \mathrm{e}^{j\theta}\right) \mathrm{d}\omega \mathrm{d}\theta. \quad \bullet$$

Now consider the continuous-discrete 2D system depicted in Figure 1 where $D(s,z)$ is the disturbance input of the

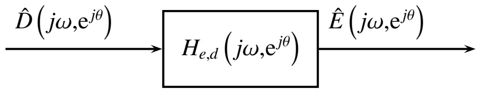

Fig. 1. Block diagram of a simple open loop system

system and $E(s,z)$ is the output (here the error), and $H_{e,d}$ is the disturbance to error transfer function.

Assume $H_{e,d}\left(j\omega,\mathrm{e}^{j\theta}\right)$ is continuous almost everywhere except at a finite number of points $\left(j\omega_{\mathrm{p}},\mathrm{e}^{j\theta_{\mathrm{p}}}\right)$ of discontinuity at nonessential singularities of the second kind. In addition, we require that for each such point $(j\omega_{\mathrm{p}},\mathrm{e}^{\theta_{\mathrm{p}}})$ there exists a neighbourhood around $(j\omega_{\mathrm{p}},\mathrm{e}^{\theta_{\mathrm{p}}})$ such that for every possible curve $\theta = \theta_i(\omega)$ in this neighbourhood the limit superior of the function $g_i(\omega) = \left|H_{e,d}\left(j\omega,\mathrm{e}^{j\theta_i(\omega)}\right)\right|$ exists, i.e. $\limsup_{\omega\to\omega_{\mathrm{p}}} g_i(\omega) = C_i$.

Then the induced operator norm of $H_{e,d}\left(j\omega,\mathrm{e}^{j\theta}\right)$ is

$$\left\|H_{e,d}(\cdot,\cdot)\right\|_{\mathrm{i}_2} = \operatorname*{ess\,sup}_{\omega,\theta}\left|H_{e,d}\left(j\omega,\mathrm{e}^{j\theta}\right)\right|. \tag{13}$$

Note that we include functions $H_{e,d}$ that are discontinuous at a finite number of points. This allows us to discuss BIBO stability of 2D transfer functions with discontinuities at the NSSK on the stability boundary such as fan filters, [21], and vehicle platoons, which are discussed here. Note also that we extend the sufficient stability condition given in [19]. [19] proves that a system with a finite number of NSSK on the stability boundary is stable if the transfer function can be continuously extended. The conjecture that this condition is sufficient and necessary (which was also used in [20]), however, is disproved since the transfer function of a 2D system describing a vehicle platoon is discontinuous around the NSSK at the origin ($\omega = 0$, $\theta = 0$) but can be chosen such that the system is BIBO stable. The full proof can be found in [23]. An outline is given below:

The induced Norm of $H_{e,d}\left(j\omega,\mathrm{e}^{j\theta}\right)$ is defined as

$$\left\|H_{e,d}(\cdot,\cdot)\right\|_{\mathrm{i}_2}^2 := \sup_{\|\hat{D}\|_2^2=1}\left\|H_{e,d}\left(j\omega,\mathrm{e}^{j\theta}\right)\hat{D}\left(j\omega,\mathrm{e}^{j\theta}\right)\right\|_2^2$$
$$= \sup_{\|\hat{D}\|_2^2=1}\left(\frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\infty}^{\infty}\left|H_{e,d}\left(j\omega,\mathrm{e}^{j\theta}\right)\hat{D}\left(j\omega,\mathrm{e}^{j\theta}\right)\right|^2\mathrm{d}\omega\mathrm{d}\theta\right). \tag{14}$$

Using Hölder's Inequality, [24, Theorem 3.8.2], it can be shown that the essential supremum of $\left|H_{e,d}\left(j\omega,\mathrm{e}^{j\theta}\right)\right|$ over all ω and θ is the upper bound of the induced operator norm.

$$\left\|H_{e,d}(\cdot,\cdot)\right\|_{\mathrm{i}_2}^2 \leq \operatorname*{ess\,sup}_{\omega,\theta}\left|H_{e,d}\left(j\omega,\mathrm{e}^{j\theta}\right)\right|^2. \tag{15}$$

To show that the essential supremum also is a lower bound we will use the following Lemma:

Lemma 2: [23, Lemma 2] Given a 2D operator $H_{e,d}\left(j\omega,\mathrm{e}^{j\theta}\right)$ which is continuous in $\left(j\omega_0,\mathrm{e}^{j\theta_0}\right)$ the induced operator norm of $H_{e,d}\left(j\omega,\mathrm{e}^{j\theta}\right)$ is always greater or equal to the magnitude of $H_{e,d}\left(j\omega_0,\mathrm{e}^{j\theta_0}\right)$. •

The essential supremum of $\left|H_{e,d}\right|$ in (15) can be achieved in three different ways:

(i) Suppose that the essential supremum of $\left|H_{e,d}\right|$ is achieved at $\overline{\omega}$ and $\overline{\theta}$ and $\left|H_{e,d}\right|$ is continuous in and around the supremum. Then it can be shown using Lemma 2 that the induced operator norm must always be greater or equal to $|H_{e,d}(\overline{\omega},\overline{\theta})|$.

(ii) However, it is also possible that the essential supremum is achieved at a point $(\omega_{\mathrm{p}},\theta_{\mathrm{p}})$ of discontinuity of $\left|H_{e,d}\left(j\omega,\mathrm{e}^{j\theta}\right)\right|$. We will use the assumptions made at the beginning of this section. We require that for each such point $(j\omega_{\mathrm{p}},\mathrm{e}^{\theta_{\mathrm{p}}})$ there exists a neighbourhood around $(j\omega_{\mathrm{p}},\mathrm{e}^{\theta_{\mathrm{p}}})$ such that for every possible curve $\theta = \theta_i(\omega)$ in this neighbourhood the limit superior of the function $g_i(\omega) = \left|H_{e,d}\left(j\omega,\mathrm{e}^{j\theta_i(\omega)}\right)\right|$ exists, i.e. $\limsup_{\omega\to\omega_{\mathrm{p}}} g_i(\omega) = C_i$. Given Lemma 2 above, for each $\epsilon_i > 0$ there exist a $\delta_i(\epsilon_i) > 0$ and a point $(j\omega_0,\mathrm{e}^{j\theta_i(\omega_0)})$ on g_i such that for all ω,θ in a circle with radius δ_i around $(\omega_0,\theta_i(\omega_0))$ (i.e. $|(\omega,\theta)-(\omega_0,\theta_i(\omega_0))| \leq \delta_i$) and $|(\omega_{\mathrm{p}},\theta_{\mathrm{p}})-(\omega_0,\theta_i(\theta_0))| = \epsilon_i$ we have $\|H_{e,d}(\cdot,\cdot)\|_{\mathrm{i}_2} \geq \operatorname{ess\,inf}_{|(\omega,\theta)-(\omega_0,\theta_i(\omega_0))|\leq\epsilon_i} g_i(\omega)$. Therefore, it must be true that $\|H_{e,d}(\cdot,\cdot)\|_{\mathrm{i}_2} \geq C_i$.

(iii) Finally, suppose the supremum of $\left|H_{e,d}\right|$ occurs as $\omega_0 \to \infty$ and θ_0. Using similar tricks as in the proof of Lemma 2 it can then be shown that $\|H_{e,d}\|_{\mathrm{i}_2} \geq \lim_{\omega\to\infty}\left|H_{e,d}\left(j\omega,\mathrm{e}^{j\theta_0}\right)\right|$.

Thus, $\left\|H_{e,d}(\cdot,\cdot)\right\|_{\mathrm{i}_2} \geq \operatorname{ess\,sup}_{\omega,\theta}\left|H_{e,d}\left(j\omega,\mathrm{e}^{j\theta}\right)\right|$. Together with (15) the induced L_2-norm of $H_{e,d}(j\omega,\mathrm{e}^{j\theta})$ is as given in (13).

III. String Stability of a Vehicle Platoon with Communication Range 2

A. System Description

Consider a simple platoon of vehicles where local controllers are used to regulate the local separation error. Suppose that these controllers are driven by

$$\begin{aligned}\hat{e}(t,k) =& (1-\alpha)\left(\hat{x}(t,k-1)-\hat{x}(t,k)\right)\\ &+\alpha\left(\hat{x}(t,k-2)-\hat{x}(t,k-1)\right)-h\hat{v}(t,k).\end{aligned} \tag{16}$$

Note that $\hat{x}(t,k)$ is the position of vehicle k at time t, $\hat{v}(t,k)$ its velocity and h the time headway. (In a 2D system description $x_1(t,k)$ could then include $\hat{x}(t,k)$, $\hat{v}(t,k)$ and controller states of vehicle k.) Thus, applying the Laplace transform with respect to t yields

$$\hat{E}(s,k) = (1-2\alpha)\hat{X}(s,k-1)+\alpha\hat{X}(s,k-2)-(\underbrace{1+hs}_{Q(s)}-\alpha)\hat{X}(s,k) \tag{17}$$

Choosing the parameter $\alpha = 0$ will lead to a simpler system with communication range 1. The selection of α and h to guarantee string stability, i.e. stability in the 2D sense, will be discussed in the next subsection.

The first vehicle in the platoon follows a given trajectory $\hat{x}(t,0)$. However, $\hat{x}(t,-1)$ does not exist. To ensure that the separation between the cars in steady state is equal, we will use the steady state separation $h\hat{v}(t,0)$ with $\hat{v}(t,0) = \frac{\mathrm{d}}{\mathrm{d}t}\hat{x}(t,0)$ instead of $(\hat{x}(t,k-2)-\hat{x}(t,k-1))$ in the equation for $\hat{e}(t,1)$:

$$\hat{e}(t,1) = (1-\alpha)\left(\hat{x}(t,0)-\hat{x}(t,1)\right)+\alpha h\hat{v}(t,0)-h\hat{v}(t,1). \tag{18}$$

Further assume that the plant transfer function and the controller transfer function are denoted by $P(s)$ and $C_{h,\alpha}(s)$ such that

$$\hat{X}(s,k) = P(s)\left(\hat{U}(s,k) + \hat{D}(s,k)\right), \tag{19}$$

$$\hat{U}(s,k) = C_{h,\alpha}(s)\hat{E}(s,k), \tag{20}$$

where $\hat{U}(s,k)$ is the Laplace transform of the actuator signal (the output of the controller) $\hat{u}(t,k)$ of vehicle k and $\hat{D}(s,k)$ the actuator disturbance signal in the frequency domain. Note that to preserve the local closed loop poles of the system with $h = 0$ and $\alpha = 0$, $C_{h,\alpha}(s)$ is chosen as $C_{h,\alpha}(s) = C(s)/(Q(s) - \alpha)$ where $C(s)$ is the controller transfer function for the system without time headway and communication range 1. Assume that $C(s)$ does not contain zeros with positive real parts and is chosen such that the system with closed loop transfer function $P(s)C(s)/(1 + P(s)C(s)) = T(s)$ is stable. Furthermore, we assume that the open loop transfer function of the time headway free system with communication range 1 has exactly two integrators, such that $T(s) = L(s)/(s^2 + L(s))$ with $L(0) \neq 0$. Thus, combining (16), (19) and (20) yields

$$\begin{aligned}\hat{E}(s,k) =& (1-2\alpha)P(s)\left(C_{h,\alpha}(s)\hat{E}(s,k-1) + \hat{D}(s,k-1)\right)\\ &+\alpha P(s)\left(C_{h,\alpha}(s)\hat{E}(s,k-2) + \hat{D}(s,k-2)\right)\\ &-(Q(s)-\alpha)P(s)\left(C_{h,\alpha}(s)\hat{E}(s,k) + \hat{D}(s,k)\right)\end{aligned} \tag{21}$$

Applying the Z transform with respect to k the disturbance to error transfer function yields

$$H_{e,d}(s,z) = \frac{(1-2\alpha)z^{-1} + \alpha z^{-2} - (Q(s)-\alpha)}{1-(1-2\alpha)\Gamma_\alpha(s)z^{-1} - \alpha\Gamma_\alpha(s)z^{-2}}\Gamma_\alpha(s)C^{-1}_{\mathrm{h},\alpha}(s) \tag{22}$$

with $\Gamma_\alpha(s) = T(s)/(Q(s) - \alpha)$.

B. Pole Location and NSSK

To guarantee BIBO stability of the continuous-discrete 2D system with transfer function (22) $\left|H_{e,d}\left(j\omega, \mathrm{e}^{j\theta}\right)\right|$ has to be bounded for all ω and θ. As discussed in Section II-A the system includes a singularity at $\omega = 0$ and $\theta = 0$. This can also be seen from (22) since $\Gamma_\alpha(0) = \frac{1}{1-\alpha}$. Note that the numerator of (22) also tends to zero for $\omega,\theta \to 0$. Thus, the system contains an NSSK at the origin, $\omega = 0$ and $\theta = 0$. Before discussing the poles for $\omega \neq 0$, let us show that the transfer function is bounded around the NSSK.

We will show that $\limsup_{\omega,\theta\to 0}\left|H_{e,d}\left(j\omega, \mathrm{e}^{j\theta}\right)\right|$ is bounded:

$$\begin{aligned}\left|H_{e,d}(s,z)\right| &= \left|\frac{(1-2\alpha)z^{-1} + \alpha z^{-2} - (Q(s)-\alpha)}{1-(1-2\alpha)\Gamma_\alpha(s)z^{-1} - \alpha\Gamma_\alpha(s)z^{-2}}\Gamma_\alpha(s)C^{-1}_{\mathrm{h},\alpha}(s)\right|\\ &= \left|\frac{1-(Q(s)-\alpha)\Gamma_\alpha(s)}{1-((1-2\alpha)z^{-1} + \alpha z^{-2})\Gamma_\alpha(s)} - 1\right|\left|C^{-1}_{\mathrm{h},\alpha}(s)\right|\\ &\le \left(\left|\frac{1-(Q(s)-\alpha)\Gamma_\alpha(s)}{1-((1-2\alpha)z^{-1} + \alpha z^{-2})\Gamma_\alpha(s)}\right| + 1\right)\left|C^{-1}_{\mathrm{h},\alpha}(s)\right|.\end{aligned} \tag{23}$$

Since it is true that

$$\begin{aligned}&\left|1-((1-2\alpha)z^{-1} + \alpha z^{-2})\Gamma_\alpha(s)\right|\\ &\ge 1 - \left|(1-2\alpha)z^{-1} + \alpha z^{-2}\right|\left|\Gamma_\alpha(s)\right|\end{aligned} \tag{24}$$

and $\left|(1-2\alpha)z^{-1} + \alpha z^{-2}\right| \le (1-2\alpha)\left|z^{-1}\right| + \alpha\left|z^{-2}\right| \le 1-\alpha$, we can rewrite (23) and get

$$\begin{aligned}&\limsup_{\omega,\theta\to 0}\left|H_{e,d}\left(j\omega, \mathrm{e}^{j\theta}\right)\right|\\ &\le \limsup_{\omega\to 0}\left(\frac{\left|1-(Q(j\omega)-\alpha)\Gamma_\alpha(j\omega)\right|}{1-(1-\alpha)\left|\Gamma_\alpha(j\omega)\right|} + 1\right)\left|C^{-1}_{\mathrm{h},\alpha}(j\omega)\right|.\end{aligned} \tag{25}$$

Using the fact that $\Gamma_\alpha(s) = \frac{T(s)}{Q(s)-\alpha}$, $T(s) = \frac{L(s)}{s^2+L(s)}$ and the following approximation for $L(j\omega)$ for small ω, $L(j\omega) = a_0 + a_2\omega^2 + a_4\omega^4 + \ldots (a_1\omega + a_3\omega^3 + \ldots)j$, we can simplify the first term of the right hand side of (25), use l'Hôspital's Rule and get

$$\begin{aligned}&\limsup_{\omega\to 0}\frac{\left|1-(Q(j\omega)-\alpha)\Gamma_\alpha(j\omega)\right|}{1-(1-\alpha)\left|\Gamma_\alpha(j\omega)\right|}\\ &= \limsup_{\omega\to 0}\frac{\frac{\omega^2}{\left|-\omega^2+L(j\omega)\right|}}{1-(1-\alpha)\frac{1}{\sqrt{h^2\omega^2+(1-\alpha)^2}}\frac{|L(j\omega)|}{\left|-\omega^2+L(j\omega)\right|}}\\ &= \frac{1}{\frac{h^2a_0}{2(1-\alpha)^2} - 1}.\end{aligned} \tag{26}$$

For any $h > h_{0,2} = (1-\alpha)\sqrt{2/|L(0)|}$, (26) is bounded. Note that when choosing $\alpha = 1/2$ the system admits a second NSSK at $s = 0$ and $z = -1$. The same argument as above can be followed to assure $\left|H_{e,d}\left(j\omega, \mathrm{e}^{j\theta}\right)\right|$ is bounded for $\omega = 0$ and $\theta = \pi$. Therefore, with $h > h_{0,2}$ and $0 \le \alpha \le 1/2$, $H_{e,d}\left(j\omega, \mathrm{e}^{j\theta}\right)$ is bounded for $\omega = 0$ and $\theta = 0$.

To guarantee that the magnitude of the transfer function is bounded for all ω and θ we will examine the poles of $H_{e,d}$.

Given the fact that $\Gamma_\alpha(s)$ and $C^{-1}_{\mathrm{h},\alpha}(s)$ do not have poles with real parts greater or equal to zero, string stability of the system will depend on the zeros of the denominator of (22).

The zeros of $p(s,z) = z^2 - (1-2\alpha)\Gamma_\alpha(s)z - \alpha\Gamma_\alpha(s)$ for $\alpha < 1/2$ are

$$z_{1,2}(s) = \frac{1-2\alpha}{2}\Gamma_\alpha(s) \pm \sqrt{\frac{(1-2\alpha)^2}{4}\Gamma^2_\alpha(s) + \alpha\Gamma_\alpha(s)}. \tag{27}$$

To guarantee string stability both poles have to lie within the closed unit circle around the origin in the z-plane for $s = j\omega$ for all $\omega \neq 0$. That means that we have to guarantee $|z_{1,2}(j\omega)| < 1$ for all $\omega \neq 0$. We will now write $\Gamma_\alpha(j\omega)$ as $T(j\omega)/(Q(j\omega) - \alpha)$ where $T(j\omega) = r(\omega)\exp(j\phi(\omega))$. The magnitude of pole $z_1(j\omega)$ can be bounded by

$$\begin{aligned}|z_1(j\omega)| \le& \frac{1-2\alpha}{2}\frac{r(\omega)}{\sqrt{h^2\omega^2+(1-\alpha)^2}}\\ &+\left|\sqrt{\frac{(1-2\alpha)^2}{4}\frac{r^2(\omega)\mathrm{e}^{2j\left(\phi - \arctan\frac{h\omega}{1-\alpha}\right)}}{h^2\omega^2+(1-\alpha)^2} + \frac{\alpha r(\omega)\mathrm{e}^{j\left(\phi - \arctan\frac{h\omega}{1-\alpha}\right)}}{\sqrt{h^2\omega^2+(1-\alpha)^2}}}\right|.\end{aligned} \tag{28}$$

Using the fact that $\left|\mathrm{e}^{j\phi}\right| = 1$ for all ϕ, the bound yields

$$\begin{aligned}|z_1(j\omega)| \le& \frac{1-2\alpha}{2}\frac{r(\omega)}{\sqrt{h^2\omega^2+(1-\alpha)^2}}\\ &+\sqrt{\frac{(1-2\alpha)^2}{4}\frac{r^2(\omega)}{h^2\omega^2+(1-\alpha)^2} + \frac{\alpha r(\omega)}{\sqrt{h^2\omega^2+(1-\alpha)^2}}}.\end{aligned} \tag{29}$$

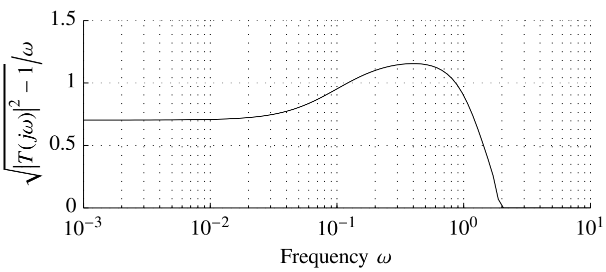

Fig. 2. Curve to determine the infimal time headway h_0

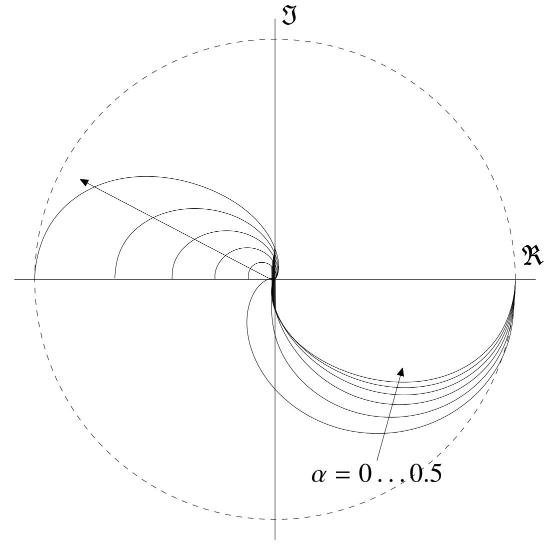

Fig. 3. Pole location for $0 \leq \omega \leq \infty$ and $h = 1$

Since it is required that $|z_{1,2}(j\omega)| < 1$ for all ω we can derive from (29)

$$\frac{(1-2\alpha)^2}{4}\frac{r^2(\omega)}{h^2\omega^2+(1-\alpha)^2}+\frac{\alpha r(\omega)}{\sqrt{h^2\omega^2+(1-\alpha)^2}} < \left(1-\frac{1-2\alpha}{2}\frac{r(\omega)}{\sqrt{h^2\omega^2+(1-\alpha)^2}}\right)^2 \qquad (30)$$

and thus

$$\frac{\alpha r(\omega)}{\sqrt{h^2\omega^2+(1-\alpha)^2}} < 1-\frac{(1-2\alpha)\,r(\omega)}{\sqrt{h^2\omega^2+(1-\alpha)^2}}. \qquad (31)$$

Simplifying even further yields for the infimal headway for a communication range of two for $0 \leq \alpha \leq 1/2$

$$h_{0,2} = (1-\alpha)\sup_{\omega}\frac{\sqrt{r^2(\omega)-1}}{\omega} = (1-\alpha)\sup_{\omega}\frac{\sqrt{|T(\omega)|^2-1}}{\omega}. \qquad (32)$$

Note that for $\alpha > 1/2$ the infimal time headway becomes $h_{0,2} = \sup_{\omega}\sqrt{(3\alpha-1)^2 r^2(\omega)-(1-\alpha)^2)}/\omega$. For $\omega \to 0$ the absolute value of $|T(j\omega)| = r(\omega)$ will approach 1 and square root will go to a constant value $c(\alpha) \neq 0$. Therefore, $h_{0,2} \to \infty$ and string stability cannot be guaranteed for $\alpha > 1/2$.

C. Simulation Results

To illustrate our results and string stability discussion above, we will analyse a string of vehicles with $P(s) = 1/(s^2 + 2C_{\mathrm{d}}v_0 s)$ and the PID controller $C(s) = k_{\mathrm{p}} + k_{\mathrm{i}}/s + sk_{\mathrm{d}}/(Ts{+}1)$ where $C_{\mathrm{d}} = 7{\cdot}10^{-4}$, $v_0 = 30$, $k_{\mathrm{p}} = 1.66$, $k_{\mathrm{i}} = 0.17$, $k_{\mathrm{d}} = 4.1$ and $T = 1/30$. Examining the curve in Figure 2 we find that $h_{0,2} = (1-\alpha)1.18$. The location of the two poles $z_{1,2}$ for $h = 1$ are displayed in Figure 3 and Figure 4. Choosing a time headway of $h = 1$ string stability can be guaranteed if alpha is at least $\alpha_{\min} \approx 0.15$. The magnitude of $H_{e,d}$ for $h = 1$ and $\alpha = 0.3$ is displayed in Figure 5. Note that $H_{e,d}$ is discontinuous but bounded around the NSSK.

As the simulations of a string of forty vehicles in Figure 6 and Figure 7 show the 2D system is BIBO stable for $h = 1$ and $\alpha = 0.3$ but BIBO unstable for $h = 1$ and $\alpha = 0.1$. Thus, the vehicle platoon is string stable for $h = 1$ and $\alpha = 0.3$ but string unstable for $\alpha = 0.1$.

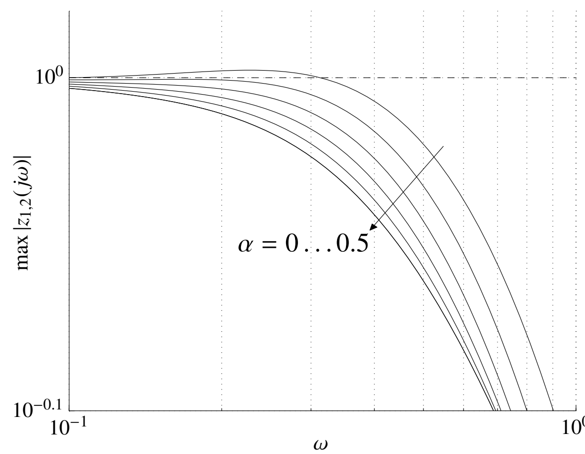

Fig. 4. Magnitude of maximal pole for $h = 1$

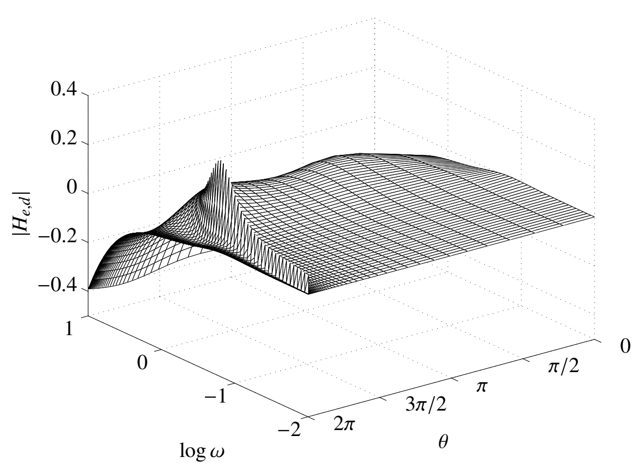

Fig. 5. $|H_{e,d}(j\omega,\mathrm{e}^{j\theta})|$ for $h = 1$ and $\alpha = 0.3$

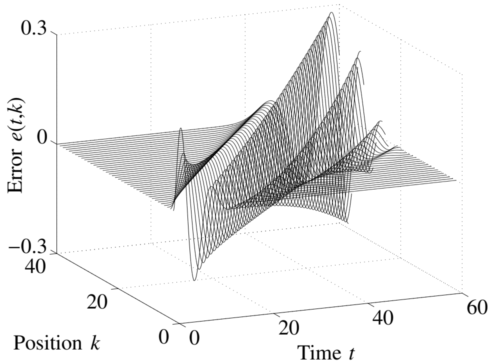

Fig. 6. Unidirectional string with communication range 2, $h = 1$, $\alpha = 0.1$

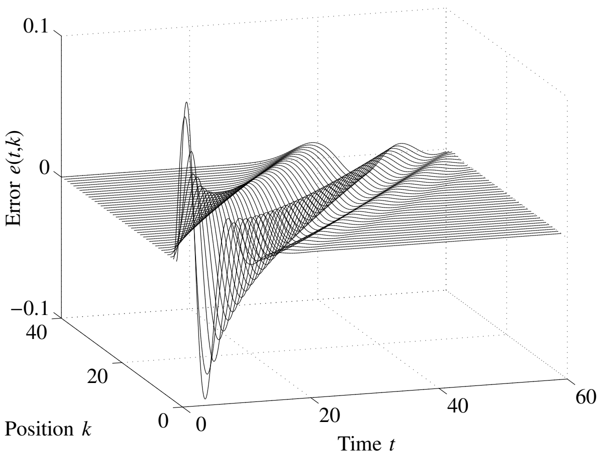

Fig. 7. Unidirectional string with communication range 2, $h = 1$, $\alpha = 0.3$

IV. Conclusion

We have shown how vehicle platoons can be modelled as 2D systems with one continuous variable (i.e. time t) and one discrete variable (i.e. position within the string k). However, every such vehicle string includes an unavoidable singularity on the stability boundary. In case not only the denominator but also the numerator of the resulting transfer function of the 2D system in the frequency domain tends to zero at this singularity, it is called a nonessential singularity of the second kind (NSSK). This is due to structural setup of the platoon and not a result of poor design or string instability.

BIBO stability of the resulting continuous-discrete 2D system can be analysed using the induced operator norm. Since the induced operator norm can be applied if the transfer function in the frequency domain is discontinuous (but yet bounded) this result is less conservative than the sufficient stability condition presented in [19] that the system with a finite number of NSSK on the stability boundary is stable if the transfer function can be continuously extended to the closed stability bi-region.

A unidirectional vehicle string with communication range 2 was modelled as a 2D system and the transfer function in the 2D frequency domain was derived. Analysing the pole loci of the transfer function, BIBO stability of the 2D system and thus string stability of the vehicle platoon can be guaranteed for a sufficiently large time headway and an appropriate choice of the parameter α.

It should be noted, however, that analysing the pole loci requires in theory an infinite number of poles as the roots $z_{1,2}$ are functions of the variable ω. Thus, more advanced stability conditions testing the pole location are required. Note also that the 2D system description is not suitable to analyse bidirectional or heterogeneous vehicle strings.

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